

EXPLORING THE LANDSCAPE OF RELATIONAL SYLLOGISTIC LOGICS

ALEX KRUCKMAN

Wesleyan University
and

LAWRENCE S. MOSS

Indiana University Bloomington

Abstract. This paper explores relational syllogistic logics, a family of logical systems related to reasoning about relations in extensions of the classical syllogistic. These are all decidable logical systems. We prove completeness theorems and complexity results for a natural subfamily of relational syllogistic logics, parametrized by constructors for terms and for sentences.

§1. Introduction. This paper explores several fragments of *relational syllogistic logic* and aims to provide completeness and complexity results for them. These are among the simplest of all logical systems. To set notation and terminology by example, let us consider the absolutely simplest logical system \mathcal{A} , the one for sentences “all p are q ” introduced in [3]. The syntax begins with a set \mathbf{N} , called the *nouns*. Then the sentences of \mathcal{A} are simply the expressions $(\text{all } p \ q)$, where p and q are nouns. The semantics is equally straightforward. A *model* \mathcal{M} is a set M together with an interpretation function giving a subset $\llbracket p \rrbracket \subseteq M$ for each noun p . We then define:

$$\mathcal{M} \models \text{all } p \ q \quad \text{iff} \quad \llbracket p \rrbracket \subseteq \llbracket q \rrbracket. \quad (1)$$

We employ standard model-theoretic notation and terminology. We say that \mathcal{M} *satisfies* a sentence φ , or that \mathcal{M} is a *model of* φ , when $\mathcal{M} \models \varphi$. A *theory* is a set of sentences. Given a theory Γ , we write $\mathcal{M} \models \Gamma$ to mean that \mathcal{M} satisfies every sentence in Γ ; naturally we say that \mathcal{M} *satisfies* Γ , or that \mathcal{M} is a *model of* Γ . We then have the usual notion of logical consequence: given a theory Γ and a sentence φ , we write $\Gamma \models \varphi$ if every model of Γ satisfies φ .

We match the semantics with a proof system. Our system has two rules of inference, shown below:

$$\frac{}{\text{all } x \ x} \text{AX} \qquad \frac{\text{all } x \ y \quad \text{all } y \ z}{\text{all } x \ z} \text{BARBARA.}$$

In these rules, the material above the line is the set of *premises*, and the sentence below is the *conclusion*. So (AX) has no premises, and (BARBARA) has two. A *substitution instance* of a rule is obtained by substituting nouns in \mathbf{N} for the variables x , y , and z .

Received: September 3, 2018.

2020 *Mathematics Subject Classification*: 03B65.

Key words and phrases: natural logic, syllogistic logic, proof systems, completeness, computational complexity.

We can then define the provability relation $\Gamma \vdash \varphi$: this means that there is a tree whose root is φ , and every node in the tree is either a leaf and belongs to Γ , or else it is the conclusion, and its children are the premises, of a substitution instance of one of our rules. The soundness/completeness theorem for this system states that $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. For the proof, see [3].

1.1. Syntax and semantics of the logics in this paper. We are concerned not with \mathcal{A} but rather with a family of extensions of it. We start with a set \mathbf{N} of nouns (just as above), and also a set \mathbf{V} , the *verbs*. Then we define *terms* and *sentences* via the syntax below, where p is any noun in \mathbf{N} , r is any verb in \mathbf{V} , and x and y are any terms:

$$\begin{array}{ll} \text{terms} & p \mid r \text{ all } x \mid r \text{ some } x \mid \bar{x} \\ \text{sentences} & \text{all } x \ y \mid \text{some } x \ y. \end{array} \tag{2}$$

Note that we have recursion here, so terms can be nested, e.g. $(r \text{ all } (\bar{s} \text{ some } \bar{p}))$.

For the semantics, we start with a model in our previous sense (to interpret the nouns), and add interpretations of the verbs as binary relations: $\llbracket r \rrbracket \subseteq M \times M$ for all $r \in \mathbf{V}$. Then we interpret our terms in a given model by recursion as follows:

$$\begin{aligned} \llbracket r \text{ all } x \rrbracket &= \{m \in M : \text{for all } n \in \llbracket x \rrbracket, m \llbracket r \rrbracket n\} \\ \llbracket r \text{ some } x \rrbracket &= \{m \in M : \text{there is } n \in \llbracket x \rrbracket \text{ such that } m \llbracket r \rrbracket n\} \\ \llbracket \bar{x} \rrbracket &= M \setminus \llbracket x \rrbracket. \end{aligned}$$

Thus, the interpretation of every term is a subset of M . We define the truth-relation for sentences and models by generalizing (1) above:

$$\begin{aligned} \mathcal{M} \models \text{all } x \ y &\text{ iff } \llbracket x \rrbracket \subseteq \llbracket y \rrbracket \\ \mathcal{M} \models \text{some } x \ y &\text{ iff } \llbracket x \rrbracket \cap \llbracket y \rrbracket \neq \emptyset. \end{aligned}$$

The basic languages in this paper are all sublanguages of the language just presented, which we call $\mathcal{L}_{5,5}$. And they are all extensions of the language we call \mathcal{L}_1 , which only has the sentence former (all $x \ y$) and the term former ($r \text{ all } x$). There are three features in $\mathcal{L}_{5,5}$ which are absent from \mathcal{L}_1 : the sentence former (some $x \ y$), the term former ($r \text{ some } x$), and term complementation. We thus explore $2^3 = 8$ logical systems, obtained by all possible combinations of these features. Those languages are listed in the chart in Figure 1. We organize matters by studying \mathcal{L}_n and $\mathcal{L}_{n,5}$ in Section n . Note that \mathcal{L}_1 and \mathcal{L}_2 are related in the same way as \mathcal{L}_n and $\mathcal{L}_{n,5}$ for $n > 2$, namely by adding the sentence former (some $x \ y$).

We are interested in complete proof systems and the complexity of the consequence relation for each of these languages. By this we mean the computational complexity of the following decision problem: given a finite theory Γ and a sentence φ , output “yes” if $\Gamma \models \varphi$ and “no” otherwise. Our results, which are summarized in the last two columns of Figure 1, will be explained in more detail in §1.5 below.

1.2. Related work. This paper is similar in spirit to [10], which also considered completeness and complexity results for decidable languages extending the basic syllogistic logic \mathcal{A} . In fact, the largest language of this paper, $\mathcal{L}_{5,5}$, is essentially equivalent to the largest language considered in [10], called $\mathcal{R}^{*\dagger}$ there (see Propositions 5.1 and 5.2 below). But there are several differences between the other languages in [10] and in this paper. First, [10] allowed for complemented *verbs*, contrary to what we do here. Second, [10] also explored logical systems where the two terms in sentences

language	term former(s)	sentence former(s)	additions to syllogistic logic	complexity of the consequence relation
\mathcal{L}_1	r all x	all x y	none	P _{TIME}
\mathcal{L}_2	r all x	all x y , some x y	(CASES), or (CHAINS), or extra syntax	P _{TIME}
\mathcal{L}_3	r all x , r some x	all x y	(CASES') & (CASES' ₂)	Co-NP _{TIME} complete
$\mathcal{L}_{3,5}$	r all x , r some x	all x y , some x y	(CASES) & (CASES' ₂)	Co-NP _{TIME} complete
\mathcal{L}_4	r all x , \bar{x}	all x y	extra syntax & schematic rules	Co-NP _{TIME} hard
$\mathcal{L}_{4,5}$	r all x , \bar{x}	all x y , some x y	extra syntax & (RAA) & schematic rules	EXP _{TIME}
\mathcal{L}_5	r all x , r some x , \bar{x}	all x y		EXP _{TIME} complete
$\mathcal{L}_{5,5}$	r all x , r some x , \bar{x}	all x y , some x y	individual variables	EXP _{TIME} complete

Fig. 1. Languages in this paper, given by section.

like (all x y) were not treated the same. For example, in two systems there, the subject noun phrase x was required to be (negated) atomic. Third, we allow nested terms, which are not part of the syntax in [10]. The upshot is that there is no overlap in the technical results from [10] and this paper, except for the result on $\mathcal{L}_{5,5}$ which we quote in §5.

In [2], McAllester and Givan considered a language which is a slight extension of our language $\mathcal{L}_{3,5}$. In §3, we essentially provide a new proof of their result that the consequence relation for this language is Co-NP_{TIME} complete. Our version of this result is slightly sharper, since we prove that the weaker language \mathcal{L}_3 is already Co-NP_{TIME} hard, and we also provide complete (though nonsyllogistic) proof systems for \mathcal{L}_3 and $\mathcal{L}_{3,5}$.

We began this paper with \mathcal{A} , which was introduced in [3]. But the smallest language in Figure 1 is \mathcal{L}_1 , which is the extension of \mathcal{A} by the term former (r all x). As a proof system for \mathcal{L}_1 , we take the rules (AX) and (BARBARA) above (but we allow terms, not just nouns, to be substituted for the variables x , y , and z), together with a new rule:

$$\frac{\text{all } x \ y}{\text{all } (r \ \text{all } y) \ (r \ \text{all } x)} \text{ ANTI.}$$

It is easy to see that this proof system is sound (if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$). Completeness was shown in [8], which also contains completeness and complexity results for a number of related languages. The completeness proof originated in [7], where it is also shown that the consequence relation for \mathcal{L}_1 is in P_{TIME}. We reprove these results in §1.4 below, but using a more general framework described in §1.3. This framework unifies the P_{TIME} results for \mathcal{L}_1 and \mathcal{L}_2 in §2.5, and allows us to obtain more precise negative results for the other languages in the paper, as we shall see.

1.3. Syllogistic proof systems and bounded completeness. At this point, we wish to formally state what we mean by a syllogistic proof system. To state rules, we employ a language with *noun variables* p, q, \dots , *verb variables* r, s, \dots , *term variables* x, y, \dots , and *sentence variables* φ, ψ, \dots . (In practice, none of our rules will make use of noun variables, and very few will make use of sentence variables.)

A *term template* is defined as in (2), but using noun and verb variables in place of nouns and verbs, and with an additional base case: a term variable is a term template. A *sentence template* is defined as in (2), but using term templates in place of terms, and with an addition option: a sentence variable is a sentence template.

A *sylogistic rule* ρ consists of finitely many (possibly none) sentence templates as premises, and a single sentence template as a conclusion. A *sylogistic proof system* is a finite set of sylogistic rules.

We use \vdash to denote a sylogistic proof system. Given a sylogistic proof system, we also use the symbol \vdash for the (standard) provability relation, defined shortly. A *substitution instance* of a rule ρ is obtained by substituting nouns, verbs, terms, and sentences for all noun variables, verb variables, term variables, and sentence variables, respectively. A *proof tree over a theory* Γ is a tree labeled by sentences, such that each node is either a leaf and belongs to Γ , or else it is the conclusion, and its children are the premises, of a substitution instance of one of the rules. We write $\Gamma \vdash \varphi$ if there is a proof tree over Γ whose root is φ .

We should mention that a sylogistic proof system is subject to some important limitations. First, the system cannot include rules which allow for the withdrawal of assumptions, as in *reductio ad absurdum*, or proof by cases. (On the other hand, *ex falso quodlibet* is a sylogistic rule; see [3].) Second, the premises of each rule must be a fixed finite set of sentence templates; the set of premises cannot be listed as a *schema*.

Later in this paper, we shall see proof systems that are not sylogistic in our sense: To obtain completeness theorems for various logics, we need to add the rule (CASES) and its variants (CASES₂), (CASES'), and (CASES'₂) in §§2.3 and 3, the rule (RAA) in §4, and the schema (CHAINS) in §2.4.

Next, we introduce a strengthening of the notion of sylogistic proof system, which ensures that the consequence relation $\Gamma \models \varphi$ is efficiently decidable, for any finite theory Γ and any sentence φ .

DEFINITION 1.1. Let \mathcal{L} be a language, equipped with a sylogistic proof system \vdash , and let A be any set of sentences in \mathcal{L} . We write $\Gamma \vdash_A \varphi$ if $\Gamma \vdash \varphi$ via a proof tree \mathcal{T} with the property that all sentences in \mathcal{T} belong to A .

EXAMPLE 1.2. This example is based on the observation that

$$\{\text{all } x \ y, \text{all } y \ z\} \vdash \text{all } (r \ \text{all } z) \ (r \ \text{all } x). \tag{3}$$

For example, here is a proof tree:

$$\frac{\frac{\text{all } y \ z}{\text{all } (r \ \text{all } z) \ (r \ \text{all } y)} \text{ ANTI} \quad \frac{\text{all } x \ y}{\text{all } (r \ \text{all } y) \ (r \ \text{all } x)} \text{ ANTI}}{\text{all } (r \ \text{all } z) \ (r \ \text{all } x)} \text{ BARBARA.}$$

Let A be the set of all sentences ψ such that every subterm of ψ is in the set $\{x, y, z, r \ \text{all } x, r \ \text{all } z\}$ (this is the set of subterms of the sentences appearing in (3)).

Our tree above does not show that

$$\{\text{all } x \ y, \text{all } y \ z\} \vdash_A \text{all } (r \ \text{all } z) \ (r \ \text{all } x). \tag{4}$$

The problem is that the term $(r \ \text{all } y)$ is not in A . But the tree below does show (4):

$$\frac{\frac{\text{all } x \ y \quad \text{all } y \ z}{\text{all } x \ z} \text{ BARBARA}}{\text{all } (r \ \text{all } z) \ (r \ \text{all } x)} \text{ ANTI.}$$

DEFINITION 1.3. A *boundedly complete syllogistic proof system* for \mathcal{L} is a syllogistic proof system \vdash for \mathcal{L} which is sound, and such that there exists a PTIME-computable $f : \mathcal{P}_{\text{fin}}(\mathcal{L}) \rightarrow \mathcal{P}_{\text{fin}}(\mathcal{L})$ such that whenever $\Gamma \models \varphi$, then $\Gamma \vdash_{f(\Gamma \cup \{\varphi\})} \varphi$.

EXAMPLE 1.4. Suppose we are working with the language \mathcal{L}_1 . For any finite set of sentences Δ , let $f(\Delta)$ be the set of all sentences φ such that every subterm of φ is a subterm of a sentence in Δ . Note that f is computable in polynomial time by enumerating the subterms of sentences in Δ , and then forming all sentences $(\text{all } u \ v)$ such that u and v are on the list. In connection with Example 1.2, the set A there is exactly $f(\{\text{all } x \ y, \text{all } y \ z, \text{all } (r \ \text{all } z) (r \ \text{all } x)\})$. In Proposition 1.6 below, we are going to show that the proof system with rules (AX), (BARBARA), and (ANTI) is boundedly complete for \mathcal{L}_1 . The function f could serve as a witness. But in order to simplify the proof, we use a slightly different function g .

THEOREM 1.5. Fix a language \mathcal{L} . Let \vdash be a boundedly complete syllogistic proof system for a language \mathcal{L} . Then \vdash is complete, and for any finite theory Γ and any sentence φ , the problem of deciding whether $\Gamma \models \varphi$ is in PTIME.

Proof. This is essentially Appendix A of McAllester [1]. Here is a sketch, based on this, and also on a parallel result in [10] which was proved in the easier setting of syllogistic logics without complex terms.

Let $f : \mathcal{P}_{\text{fin}}(\mathcal{L}) \rightarrow \mathcal{P}_{\text{fin}}(\mathcal{L})$ be a PTIME-computable function such that $\Gamma \vdash_{f(\Gamma \cup \{\varphi\})} \varphi$ whenever $\Gamma \models \varphi$. In particular, $\Gamma \models \varphi$ implies that $\Gamma \vdash \varphi$. So the proof system is complete.

For the PTIME decidability, first compute $f(\Gamma \cup \{\varphi\})$. Call this set A . Let $X_0 = \Gamma \cap A$. We compute an increasing sequence of subsets of A by induction. Given X_n , take each of the finitely many rules ρ of the logic, and do the following: compute the set of all substitution instances of ρ whose premises are all in X_n ; for each such substitution instance, if the conclusion ψ belongs to A , then add ψ to X_{n+1} . Continue until the first n^* such that $X_{n^*+1} = X_{n^*}$. Since all the X_n are subsets of A , we have $n^* \leq 1 + |A|$. And $\Gamma \vdash_A \varphi$ iff $\varphi \in X_{n^*}$. We take it as standard that all of this can be done in PTIME. \square

1.4. Example: Completeness and PTIME decidability for \mathcal{L}_1 . We illustrate the application of Theorem 1.5 to \mathcal{L}_1 with the syllogistic proof system consisting of the rules (AX), (BARBARA), and (ANTI).

For any set of sentences Δ , let $T(\Delta)$ be the set of subterms of sentences in Δ . Let $T^+(\Delta)$ be $T(\Delta)$ together with the terms $(r \ \text{all } w)$ where $w \in T(\Delta)$ and where r is a verb which appears in Δ .

Let $g(\Delta)$ be the set of all sentences $(\text{all } u \ v)$, where $u \in T(\Delta)$ and $v \in T^+(\Delta)$. Note that when Δ is finite, g is computable in PTIME.

PROPOSITION 1.6 [7]. If $\Gamma \models \varphi$, then $\Gamma \vdash_{g(\Gamma \cup \{\varphi\})} \varphi$. Hence the consequence relation for \mathcal{L}_1 is in PTIME.

Proof. Fix a finite theory Γ and a sentence φ . We are going to save on some notation in this proof by writing T for $T(\Gamma \cup \{\varphi\})$, T^+ for $T^+(\Gamma \cup \{\varphi\})$, and A for $g(\Gamma \cup \{\varphi\})$.

We make a model \mathcal{M} as follows. The domain M of the model is T . The structure of the model is given by

$$\begin{aligned} t \in \llbracket p \rrbracket & \text{ iff } \Gamma \vdash_A \text{all } t \ p \\ t \llbracket r \rrbracket u & \text{ iff } \Gamma \vdash_A \text{all } t (r \ \text{all } u) \end{aligned}$$

(Recall that the set A is fixed as $g(\Gamma \cup \{\varphi\})$. When we write $\Gamma \vdash_A \psi$ in this proof, we are not changing the meaning of A to $g(\Gamma \cup \{\psi\})$.)

CLAIM 1.7 (Truth Lemma). *For all $a \in T$,*

$$t \in \llbracket a \rrbracket \text{ iff } \Gamma \vdash_A \text{all } t \text{ } a. \tag{5}$$

Proof. The proof is by induction on a . When a is a noun, the assertion in (5) is part of the definition of the model. Assume (5) for a , and consider $(r \text{ all } a)$. Since $(r \text{ all } a) \in T$, then $a \in T = M$ as well. Note also that the sentence $(\text{all } a \text{ } a)$ belongs to A , and so $\Gamma \vdash_A \text{all } a \text{ } a$ by (AX). By induction, $a \in \llbracket a \rrbracket$.

Let $t \in \llbracket r \text{ all } a \rrbracket$. Since $a \in \llbracket a \rrbracket$, we have $t \llbracket r \rrbracket a$, and hence $\Gamma \vdash_A \text{all } t \text{ } (r \text{ all } a)$ by definition of $\llbracket r \rrbracket$.

Conversely, assume that $\Gamma \vdash_A \text{all } t \text{ } (r \text{ all } a)$. Then $t \in T = M$. We show that $t \in \llbracket r \text{ all } a \rrbracket$. For this, let $b \in \llbracket a \rrbracket$, so by induction, $\Gamma \vdash_A \text{all } b \text{ } a$. We must show that $t \llbracket r \rrbracket b$, equivalently $\Gamma \vdash_A \text{all } t \text{ } (r \text{ all } b)$. Note that $b \in M = T$, so $(r \text{ all } b) \in T^+$. The key point is that the sentence $\text{all } (r \text{ all } a) \text{ } (r \text{ all } b)$ belongs to A , since $(r \text{ all } a) \in T$ and $(r \text{ all } b) \in T^+$. Using (ANTI) and $\Gamma \vdash_A \text{all } b \text{ } a$, we see that $\Gamma \vdash_A \text{all } (r \text{ all } a) \text{ } (r \text{ all } b)$. Then by (BARBARA), $\Gamma \vdash_A \text{all } t \text{ } (r \text{ all } b)$. \square

We continue by showing that $\mathcal{M} \models \Gamma$. For this, take any sentence $(\text{all } u \text{ } v)$ in Γ . Let $t \in \llbracket u \rrbracket$. By (5), $\Gamma \vdash_A \text{all } t \text{ } u$. Now t, u , and v are in T , so the sentences $(\text{all } u \text{ } v)$ and $(\text{all } t \text{ } v)$ are in A . By (BARBARA), $\Gamma \vdash_A \text{all } t \text{ } v$. By (5) again, $t \in \llbracket v \rrbracket$. Since t was arbitrary, we have shown that $\llbracket u \rrbracket \subseteq \llbracket v \rrbracket$, and $\mathcal{M} \models \text{all } u \text{ } v$.

Since $\mathcal{M} \models \Gamma$ and $\Gamma \models \varphi$, the sentence φ holds in \mathcal{M} . Let us write φ as $(\text{all } a \text{ } b)$. Then $a \in T = M$ and the sentence $(\text{all } a \text{ } a)$ belongs to A , and so $\Gamma \vdash_A \text{all } a \text{ } a$ by (AX). Thus, $a \in \llbracket a \rrbracket$ by (5). So $a \in \llbracket b \rrbracket$. By (5) again, $\Gamma \vdash_A \text{all } a \text{ } b$. This concludes the proof. \square

1.5. Results. Our two main themes are trade-offs between expressive power and computational complexity, and also the variety of devices that one can add on top of pure syllogistic logic in order to obtain sound and complete proof systems. Our results are summarized in Figure 1.

We begin with a negative result: \mathcal{L}_2 has no sound and complete syllogistic proof system. The argument for this is combinatorial, and in outline it is based on a similar result in [10] for \mathcal{R} . Nevertheless, there are proof systems which capture the consequence relation of \mathcal{L}_2 . Most of §2 is devoted to several such logics, each of which extends syllogistic logic in a different way. One way is to add a rule (CASES) which enables proof by cases, another uses a schema of rules called (CHAINS) (thus there are infinitely many rules, but the set of rules is an easily-defined set), and the last is to extend the syntax in such a way that the extension, called \mathcal{L}_2^+ , does have a sound and complete syllogistic proof system. In fact, this system is boundedly complete, and by Theorem 1.5, the consequence relation of \mathcal{L}_2^+ , and hence of its sublanguage \mathcal{L}_2 , is in PTIME.

Notice that what we will show is that \mathcal{L}_2 has no sound and complete syllogistic proof system, but the larger language \mathcal{L}_2^+ does have such a proof system. The first example of this phenomenon is in [9].

The lower bounds on complexity established in the rest of the paper show that (assuming $P \neq NP$) none of the other languages admit boundedly complete syllogistic proof systems.

The languages \mathcal{L}_3 and $\mathcal{L}_{3,5}$ extend \mathcal{L}_1 and \mathcal{L}_2 with the term former (r some x). We show that the smaller language \mathcal{L}_3 has a consequence relation which is CO-NPTIME hard, via a reduction from the one-in-three positive 3-SAT problem. On the other hand, a finite countermodel construction shows that the consequence relation of the larger language $\mathcal{L}_{3,5}$ is in CO-NPTIME. It follows that the consequence relation for both languages are CO-NPTIME complete. We give sound and complete proof systems for these logics which are not syllogistic: they use variants of the rule (CASES).

The languages \mathcal{L}_4 and $\mathcal{L}_{4,5}$ extend \mathcal{L}_1 and \mathcal{L}_2 with term complementation. As with \mathcal{L}_3 , we show that the consequence relation for \mathcal{L}_4 is CO-NPTIME hard, via a reduction from 3-SAT. But this time we leave open the question of CO-NPTIME completeness; the upper bound of EXPTIME comes from the known complexity of the larger language $\mathcal{L}_{5,5}$. We also leave open the problem of formulating proof systems for \mathcal{L}_4 and $\mathcal{L}_{4,5}$ in their original syntax. Instead, we present a completeness result for an extension $\mathcal{L}_{4,5}^+$ of $\mathcal{L}_{4,5}$. The proof system and the completeness result are comparatively simple, though the proof system is decidedly nonsyllogistic: it includes a form of *reductio ad absurdum* (RAA), as well as several rule schemas, and the syntax is not even finitary, since there is an infinite family of sentence formers. This proof system restricts, by dropping (RAA), to a sound and complete proof system for the corresponding extension \mathcal{L}_4^+ of \mathcal{L}_4 .

The largest languages in this paper are \mathcal{L}_5 and $\mathcal{L}_{5,5}$. We have the least to say about them, mostly because $\mathcal{L}_{5,5}$ has been studied (in a different but equivalent formulation, called $\mathcal{R}^{*\dagger}$) in [10], and the complexity result for it from [10] also holds for \mathcal{L}_5 , as we shall see. The sound and complete proof system for $\mathcal{R}^{*\dagger}$ from [10] (which is nonsyllogistic due to its use of individual variables) can be adapted to a similar proof system for $\mathcal{L}_{5,5}$, but we do not make that explicit here. We leave open the problem of formulating a proof system for \mathcal{L}_5 .

One might guess that when we explore a partial order of logics, stronger logics are harder to work with and to prove completeness for. But this is not always the case. One reason: as the logics get stronger, they include more and more features of first-order logic and are thus easier to analyze, due to our experience with first-order logic. A second reason: sometimes adding to the syntax of a logic restores harmony in some way, thereby making it easier for us to work with. The examples of \mathcal{L}_2^+ , \mathcal{L}_4^+ , and $\mathcal{L}_{4,5}^+$ emphasize the fact that there is not a monotone relationship between the strength of logical systems and their elegance, or between their strength and the difficulty of proving completeness. It is an open problem to develop a general theory which could explain this phenomenon. For example, why it is that some logical systems have (boundedly complete) syllogistic proof systems (\mathcal{L}_2^+), while others do not (\mathcal{L}_2)? What we have at present are some ad hoc results, and much remains to be done.

§2. \mathcal{L}_2 : Adding the sentence former (some x y) to \mathcal{L}_1 . The main results on \mathcal{L}_2 are (1) it has no finite sound and complete syllogistic proof system; (2) nevertheless there are several nonsyllogistic devices which allow us to obtain sound and complete proof systems; (3) alternatively, we can extend the syntax of \mathcal{L}_2 to a larger language \mathcal{L}_2^+ in such a way that \mathcal{L}_2^+ has a boundedly complete syllogistic proof system; and (4) as a result of this last point, the consequence relation $\Gamma \models \varphi$ for \mathcal{L}_2^+ (and hence \mathcal{L}_2) is in PTIME.

2.1. The base system \vdash_0 . We begin with a proof system in Figure 2 that will be used in this section and beyond.

$$\begin{array}{c}
 \frac{}{\text{all } x \ x} \text{ AX} \qquad \frac{\text{all } x \ y \quad \text{all } y \ z}{\text{all } x \ z} \text{ BARBARA} \qquad \frac{\text{all } x \ y}{\text{all } (r \ \text{all } y) \ (r \ \text{all } x)} \text{ ANTI} \\
 \\
 \frac{\text{some } x \ y}{\text{some } x \ \bar{x}} \text{ SOME}_1 \qquad \frac{\text{some } x \ y}{\text{some } y \ \bar{x}} \text{ SOME}_2 \qquad \frac{\text{some } x \ y \quad \text{all } y \ z}{\text{some } x \ z} \text{ DARI}
 \end{array}$$

Fig. 2. The proof rules of \vdash_0 .

It is easy to check that the rules in Figure 2 are sound. Theorem 2.3 below shows that they are also complete for conclusions of the form $(\text{all } x \ y)$. Since the rules (AX), (BARBARA), and (ANTI) are complete for \mathcal{L}_1 , this result can be interpreted as saying that \mathcal{L}_2 is a conservative extension of \mathcal{L}_1 . It is also possible to give a direct model-theoretic proof of this conservativity result.

In many places in this paper, we will work with a set \mathbb{T} of terms. We will always assume that such a set \mathbb{T} is closed under subterms. We could usually take \mathbb{T} to be the set of all terms in the language under study. But when we use \mathbb{T} to build a model (as we do just below), that model will be infinite when \mathbb{T} is infinite. Working with a finite set \mathbb{T} allows us to build finite models in many situations.

We write $\Gamma \vdash_0 \varphi$ if there is a proof of the sentence φ from the theory Γ using the rules in Figure 2. We also write

$$x \leq y$$

to mean that $\Gamma \vdash_0 \text{all } x \ y$. We use this notation because \leq is a preorder, due to (AX) and (BARBARA). Please note that Γ is left off of this notation.

The first canonical model of a theory Γ . Let Γ be a theory, let \mathbb{T} be a set of terms (closed under subterms as usual), and let M be the set of unordered pairs $\{t, u\}$ of terms from \mathbb{T} . This includes singletons $\{t\} = \{t, t\}$. We define a model $\mathcal{M}(\Gamma, \mathbb{T})$ with domain M by setting

$$\begin{aligned}
 \{t, u\} \in \llbracket p \rrbracket & \text{ iff } t \leq p \text{ or } u \leq p \\
 \{t, u\} \llbracket r \rrbracket \{v, w\} & \text{ iff for some } a \in \{t, u\} \text{ and } b \in \{v, w\}, a \leq r \ \text{all } b.
 \end{aligned} \tag{6}$$

LEMMA 2.1 (Truth Lemma). *In $\mathcal{M}(\Gamma, \mathbb{T})$, for all terms $x \in \mathbb{T}$,*

$$\llbracket x \rrbracket = \{\{t, u\} \in M : t \leq x \text{ or } u \leq x\}.$$

Proof. By induction on x . For a noun $p \in \mathbb{N}$, this is by definition of the model. For a term of the form $(r \ \text{all } x) \in \mathbb{T}$, note that $x \in \mathbb{T}$, since \mathbb{T} is closed under subterms.

Suppose that $\{t, u\} \in \llbracket r \ \text{all } x \rrbracket$. Since $x \leq x$ by (AX), the induction hypothesis implies $\{x\} \in \llbracket x \rrbracket$. So $\{t, u\} \llbracket r \rrbracket \{x\}$. By the definition of $\llbracket r \rrbracket$, either $t \leq r \ \text{all } x$, or else $u \leq r \ \text{all } x$.

Conversely, fix $\{t, u\} \in M$, and suppose that (without loss of generality) $t \leq r \ \text{all } x$. Let $\{v, w\} \in M$ be an element of $\llbracket x \rrbracket$. By the induction hypothesis, we have (without loss of generality) $v \leq x$. By (ANTI), $r \ \text{all } x \leq r \ \text{all } v$. By (BARBARA), $t \leq r \ \text{all } v$. So $\{t, u\} \llbracket r \rrbracket \{v, w\}$, and hence $\{t, u\} \in \llbracket r \ \text{all } x \rrbracket$. \square

LEMMA 2.2. *If $(\text{all } x \ y) \in \Gamma$, then $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{all } x \ y$. If x and y are any terms in \mathbb{T} , we have $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{some } x \ y$. As a consequence, if \mathbb{T} contains all subterms of sentences in Γ , then $\mathcal{M}(\Gamma, \mathbb{T}) \models \Gamma$.*

Proof. Suppose $(\text{all } x \ y) \in \Gamma$. If $\{t, u\} \in \llbracket x \rrbracket$ in $\mathcal{M}(\Gamma, \mathbb{T})$, then by Lemma 2.1 (without loss of generality), $t \leq x$. But then $t \leq y$ by (BARBARA), so $\{t, u\} \in \llbracket y \rrbracket$ by Lemma 2.1, and $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{all } x \ y$.

Now suppose $x, y \in \mathbb{T}$. Since $x \leq x$ and $y \leq y$, we have $\{x, y\} \in \llbracket x \rrbracket \cap \llbracket y \rrbracket$ by Lemma 2.1, so $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{some } x \ y$.

If \mathbb{T} contains all subterms of sentences in Γ , then $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{all } x \ y$ whenever $(\text{all } x \ y) \in \Gamma$ by the first assertion, and $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{some } x \ y$ whenever $(\text{some } x \ y) \in \Gamma$ by the second assertion. □

THEOREM 2.3. *If $\Gamma \models \text{all } x \ y$, then $\Gamma \vdash_0 \text{all } x \ y$. Moreover, the proof only uses the rules (AX), (BARBARA), and (ANTI).*

Proof. Choose \mathbb{T} so that it contains x, y , and all terms in Γ . If $\Gamma \models \text{all } x \ y$, then $\mathcal{M}(\Gamma, \mathbb{T}) \models \text{all } x \ y$ by Lemma 2.2. Then $\{x\} \in \llbracket x \rrbracket \subseteq \llbracket y \rrbracket$ by (AX) and Lemma 2.1. By Lemma 2.1 again, $\Gamma \vdash_0 \text{all } x \ y$.

To see that the proof uses only the rules (AX), (BARBARA), and (ANTI), we just need to examine the rules in the proof system and note that no rule which produces a conclusion of the form $(\text{all } x \ y)$ has a premise of the form $(\text{some } a \ b)$. □

The proof of Theorem 2.3 shows that if $\Gamma \not\vdash_0 \text{all } x \ y$, then there is a countermodel of size $O(n^2)$, where n is the complexity of $\Gamma \cup \{\text{all } x \ y\}$. However, as noted in Lemma 2.2, $\mathcal{M}(\Gamma, \mathbb{T})$ satisfies every sentence $(\text{some } x \ y)$ with $x, y \in \mathbb{T}$. To obtain a countermodel for sentences of this form, we will look at a submodel of $\mathcal{M}(\Gamma, \mathbb{T})$.

The second canonical model of a theory Γ . Let M' be the set of unordered pairs $\{t, u\}$ of terms in \mathbb{T} such that $\Gamma \vdash_0 \text{some } t \ u$. Note that we allow $t = u$. Define a model $\mathcal{M}'(\Gamma, \mathbb{T})$ with domain M' just as in (6).

The proof system \vdash_0 is not complete for sentences of the form $(\text{some } x \ y)$, but we can prove a partial completeness result under the following additional hypothesis on Γ , a form of which was first introduced by McAllester and Givan [2].

DEFINITION 2.4. We say that Γ *determines existentials* for \mathbb{T} if, for all verbs $r \in \mathbf{V}$ and all terms $x, y \in \mathbb{T}$, either $\Gamma \vdash_0 \text{some } x \ x$ or $\Gamma \vdash_0 \text{all } y \ (r \ \text{all } x)$.

The intuition behind this definition is that in any model \mathcal{M} , for any term x , either $\mathcal{M} \models \text{some } x \ x$, or $\llbracket x \rrbracket = \emptyset$, in which case $\llbracket r \ \text{all } x \rrbracket = M$ for any verb r , and $\mathcal{M} \models \text{all } y \ (r \ \text{all } x)$ for any term y .

LEMMA 2.5. *Suppose Γ determines existentials for \mathbb{T} . Then:*

- (1) *In $\mathcal{M}'(\Gamma, \mathbb{T})$, for all terms $x \in \mathbb{T}$, $\llbracket x \rrbracket = \{\{t, u\} \in M' : t \leq x \text{ or } u \leq x\}$.*
- (2) *If $(\text{all } x \ y) \in \Gamma$, then $\mathcal{M}'(\Gamma, \mathbb{T}) \models \text{all } x \ y$.*
- (3) *If $x, y \in \mathbb{T}$ and $(\text{some } x \ y) \in \Gamma$, then $\mathcal{M}'(\Gamma, \mathbb{T}) \models \text{some } x \ y$.*

As a consequence, if Γ is any theory, \mathbb{T} contains all subterms of sentences in Γ , and $\Gamma^ \supseteq \Gamma$ is a theory which determines existentials for \mathbb{T} , then $\mathcal{M}'(\Gamma^*, \mathbb{T}) \models \Gamma$.*

Proof. The proof of (1) is exactly like the proof of Lemma 2.1, with the following adjustment: If $\{t, u\} \in \llbracket r \ \text{all } x \rrbracket$ and $\{x\} \in M'$, the proof in Lemma 2.1 goes through as written. But if $\{x\} \notin M'$, then $\Gamma \not\vdash_0 \text{some } x \ x$, so $t \leq r \ \text{all } x$ (and also $u \leq r \ \text{all } x$), since Γ determines existentials for \mathbb{T} .

The proof of (2) is exactly as in the proof of Lemma 2.2.

The proof of (3) is also exactly as in the proof of Lemma 2.2, with the following adjustment: We need to use the fact that $(\text{some } x \ y) \in \Gamma$ to see that $\{x, y\} \in M'$.

Putting this together, suppose Γ is any theory, \mathbb{T} contains all subterms of sentences in Γ , and $\Gamma^* \supseteq \Gamma$ determines existentials for \mathbb{T} . If $(\text{all } x \ y) \in \Gamma \subseteq \Gamma^*$, then $\mathcal{M}'(\Gamma^*, \mathbb{T}) \models \text{all } x \ y$. And if $(\text{some } x \ y) \in \Gamma \subseteq \Gamma^*$, then since $x, y \in \mathbb{T}$, $\mathcal{M}'(\Gamma^*, \mathbb{T}) \models \text{some } x \ y$. \square

THEOREM 2.6. *Suppose that \mathbb{T} contains x, y , and all subterms of sentences in Γ , and $\Gamma^* \supseteq \Gamma$ is a theory which determines existentials for \mathbb{T} . If $\Gamma \models \text{some } x \ y$, then $\Gamma^* \vdash_0 \text{some } x \ y$.*

Proof. Since $\Gamma \subseteq \Gamma^*$, $\mathcal{M}'(\Gamma^*, \mathbb{T}) \models \Gamma$ by Lemma 2.5, so $\mathcal{M}'(\Gamma^*, \mathbb{T}) \models \text{some } x \ y$. Suppose $\{t, u\} \in \llbracket x \rrbracket \cap \llbracket y \rrbracket$. Since $\{t, u\} \in M'$, $\Gamma^* \vdash_0 \text{some } t \ u$. And by Lemma 2.5, $\Gamma^* \vdash_0 \text{all } v \ x$ and $\Gamma^* \vdash_0 \text{all } w \ y$ for some $v, w \in \{t, u\}$. By analyzing the four cases and using (SOME₁), (SOME₂), and (DARII), we see that $\Gamma^* \vdash_0 \text{some } x \ y$. \square

COROLLARY 2.7. *If \mathbb{T} contains all subterms of sentences in $\Gamma \cup \{\varphi\}$ and Γ determines existentials for \mathbb{T} , then $\Gamma \models \varphi$ if and only if $\Gamma \vdash_0 \varphi$.*

Proof. The implication $\Gamma \vdash_0 \varphi$ implies $\Gamma \models \varphi$ is just soundness of the rules in \vdash_0 . The converse follows immediately from Theorems 2.3 and 2.6, taking $\Gamma^* = \Gamma$ in Theorem 2.6. \square

We conclude this section with two proof-theoretic observations about the system \vdash_0 . If $\vec{r} = r_1, \dots, r_n$ is a sequence of verbs and x is a term, we use the notation $(\vec{r} \text{ all } x)$ for the term $(r_1 \text{ all } (r_2 \text{ all } (\dots (r_n \text{ all } x))))$. When \vec{r} is the empty sequence, $(\vec{r} \text{ all } x) = x$.

If $\psi = \text{all } u \ v$, we define

$$\text{Anti}(\vec{r}, \psi) = \begin{cases} \text{all } (\vec{r} \text{ all } u) \ (\vec{r} \text{ all } v) & \text{if the length of } \vec{r} \text{ is even} \\ \text{all } (\vec{r} \text{ all } v) \ (\vec{r} \text{ all } u) & \text{if the length of } \vec{r} \text{ is odd.} \end{cases}$$

Note that $\{\psi\} \vdash_0 \text{Anti}(\vec{r}, \psi)$ by repeated applications of (ANTI).

DEFINITION 2.8. Let Γ be a theory. A Γ -sequence is a finite sequence of terms t_1, \dots, t_n , such that for all $1 \leq i < n$ there is a sentence $\psi = (\text{all } a \ b) \in \Gamma$ and a sequence of verbs \vec{r} such that $(\text{all } t_i \ t_{i+1}) = \text{Anti}(\vec{r}, \psi_i)$.

LEMMA 2.9. $\Gamma \vdash_0 \text{all } x \ y$ if and only if there is a Γ -sequence of terms t_1, \dots, t_n such that $x = t_1$ and $y = t_n$.

Proof. Suppose $x = t_1, \dots, t_n = y$ is a Γ -sequence. If $n = 1$, then $x = y$, and $\Gamma \vdash_0 \text{all } x \ y$ by (AX). If $n > 1$, then for all $1 \leq i < n$, $\Gamma \vdash_0 \text{all } t_i \ t_{i+1}$ by repeated applications of (ANTI). And by repeated applications of (BARBARA), $\Gamma \vdash_0 \text{all } x \ y$.

We prove the converse by induction on the height of the proof tree. In the base case, $(\text{all } x \ y) \in \Gamma$, and x, y is a Γ -sequence as desired.

Case 1: If the root of the proof tree is

$$\frac{}{\text{all } x \ x} \text{ AX}$$

then x is a Γ -sequence from x to x .

Case 2: If the root of the proof tree is

$$\frac{\text{all } x \ y \quad \text{all } y \ z}{\text{all } x \ z} \text{ BARBARA}$$

then by induction we have Γ -sequences t_1, \dots, t_n and t'_1, \dots, t'_m with $x = t_1, y = t_n = t'_1$, and $z = t'_m$. Then $t_1, \dots, t_{n-1}, t'_1, \dots, t'_m$ is a Γ -sequence from x to z .

Case 3: If the root of the proof tree is

$$\frac{\text{all } x \ y}{\text{all } (r \ \text{all } y) \ (r \ \text{all } x)} \text{ ANTI}$$

then by induction we have a Γ -sequence t_1, \dots, t_n with $x = t_1$ and $y = t_n$. Then $(r \ \text{all } t_n), \dots, (r \ \text{all } t_1)$ is a Γ -sequence from $(r \ \text{all } y)$ to $(r \ \text{all } x)$. \square

LEMMA 2.10. $\Gamma \vdash_0 \text{some } x \ y$ if and only if there is a sentence $(\text{some } t_1 \ t_2) \in \Gamma$ such that $\Gamma \vdash_0 \text{all } t_i \ x$ and $\Gamma \vdash_0 \text{all } t_j \ y$, for some $i, j \in \{1, 2\}$.

Proof. Suppose there is a sentence $(\text{some } t_1 \ t_2) \in \Gamma$ such that $\Gamma \vdash_0 \text{all } t_i \ x$ and $\Gamma \vdash_0 \text{all } t_j \ y$, for some $i, j \in \{1, 2\}$. Then applying (SOME₁) or (SOME₂) if necessary, $\Gamma \vdash_0 \text{some } t_i \ t_j \ \text{some } t_i \ t_j$. By two applications of (DARII) and (SOME₂), $\Gamma \vdash_0 \text{some } x \ y$.

We prove the converse by induction on the height of the proof tree. In the base case, when $(\text{some } x \ y) \in \Gamma$, we have $\Gamma \vdash_0 \text{all } x \ x$ and $\Gamma \vdash_0 \text{all } y \ y$ by (AX).

Case 1: If the root of the proof tree is

$$\frac{\text{some } x \ y}{\text{some } x \ x} \text{ SOME}_1$$

then by induction there is a sentence $(\text{some } t_1 \ t_2) \in \Gamma$ such that $\Gamma \vdash_0 \text{all } t_i \ x$ for some $i \in \{1, 2\}$.

Case 2: If the root of the proof tree is

$$\frac{\text{some } x \ y}{\text{some } y \ x} \text{ SOME}_2$$

then by induction there is a sentence $(\text{some } t_1 \ t_2) \in \Gamma$ such that $\Gamma \vdash_0 \text{all } t_i \ y$ and $\Gamma \vdash_0 \text{all } t_j \ x$, for some $i, j \in \{1, 2\}$.

Case 3: If the root of the proof tree is

$$\frac{\text{some } x \ y \ \text{all } y \ z}{\text{some } x \ z} \text{ DARII}$$

then by induction there is a sentence $(\text{some } t_1 \ t_2) \in \Gamma$ such that $\Gamma \vdash_0 \text{all } t_i \ x$ and $\Gamma \vdash_0 \text{all } t_j \ y$, for some $i, j \in \{1, 2\}$. But also $\Gamma \vdash_0 \text{all } y \ z$, so $\Gamma \vdash_0 \text{all } t_j \ z$ by (BARBARA). \square

2.2. No sound and complete syllogistic proof system for \mathcal{L}_2 . In this section, we prove that there is no sound and complete syllogistic proof system for \mathcal{L}_2 . This suggests that we need nonsyllogistic devices like those which we shall see in coming sections of this paper. But this talk of rules being “needed” is not precise, and at the end of the day, it is not quite what we shall prove. At the same time, what we do prove is in a real way stronger than the statement above. So we need to make all of this precise.

We return to our discussion of syllogistic rules in §1.3. Every syllogistic proof system defines a *provability relation* between theories and sentences. In this section, we write this relation as $\Gamma \vdash^* \varphi$. To be *sound*, we require that if $\Gamma \vdash^* \varphi$, then $\Gamma \models \varphi$. To be *complete*, we require that if $\Gamma \models \varphi$, then also $\Gamma \vdash^* \varphi$.

The *degree k consequence relation* \models_k is the relation between sets Γ and sentences φ defined as follows: $\Gamma \models_k \varphi$ if there is a finite tree with nodes labeled by sentences, such that each node is either a leaf and in Γ , or else is a sentence φ with children ψ_1, \dots, ψ_j

for some $j \leq k$, and such that $\{\psi_1, \dots, \psi_j\} \models \varphi$. If we have a sound syllogistic proof system \vdash^* , then since it has only finitely many rules, each with finitely many premises, there is a number k (the maximum number of premises in any rule in \vdash^*) such that whenever $\Gamma \vdash^* \varphi$, we also have $\Gamma \models_k \varphi$.

THEOREM 2.11. *For all n , there is a theory Γ_{n+1} and a sentence φ such that $\Gamma_{n+1} \models \varphi$, and $\Gamma_{n+1} \not\models_n \varphi$. As a consequence, there is no sound and complete syllogistic proof system for \mathcal{L}_2 .*

Note that the first statement does not refer to proof systems in any way. It is completely semantic. But it immediately implies the negative result about proof systems.

The sets Γ_n . For all n , let $\mathbf{N} = \{a, b\}$, let $\mathbf{V} = \{r_1, \dots, r_n\}$, and let Γ_n be the following theory:

$$\begin{aligned} \alpha &= \text{some } (r_1 \text{ all } (r_1 \text{ all } a)) (r_1 \text{ all } (r_1 \text{ all } a)) \\ \varphi_1 &= \text{all } (r_1 \text{ all } b) (r_2 \text{ all } (r_2 \text{ all } a)) \\ \varphi_2 &= \text{all } (r_2 \text{ all } b) (r_3 \text{ all } (r_3 \text{ all } a)) \\ &\vdots \\ \varphi_i &= \text{all } (r_i \text{ all } b) (r_{i+1} \text{ all } (r_{i+1} \text{ all } a)) \\ &\vdots \\ \varphi_{n-1} &= \text{all } (r_{n-1} \text{ all } b) (r_n \text{ all } (r_n \text{ all } a)) \\ \omega &= \text{all } (r_n \text{ all } b) a. \end{aligned}$$

We will use Γ_n as a recurring example in the forthcoming sections, to demonstrate proof systems.

If r is a verb, then an r -king in a model \mathcal{M} is an element $x \in M$ such that for all $y \in M$, $x \llbracket r \rrbracket y$.

LEMMA 2.12. $\Gamma_n \models \text{some } a a$.

Proof. Let $\mathcal{M} \models \Gamma_n$. If $\llbracket a \rrbracket \neq \emptyset$, we are done. So we shall assume that $\llbracket a \rrbracket = \emptyset$. Then for any verb r , $\llbracket r \text{ all } a \rrbracket = M$. By α , \mathcal{M} contains an r_1 -king x . Then φ_1 implies that x is also an r_2 -king. Continuing by induction, φ_i implies that x is an r_{i+1} -king. Finally, x is an r_n -king, and ω implies that $x \in \llbracket a \rrbracket$, which is a contradiction. \square

LEMMA 2.13. *If $\Gamma_n \models \text{all } u v$, then either $u = v$, or $(\text{all } u v) = \text{Anti}(\vec{r}, \psi)$, for some sequence \vec{r} and some sentence $\psi \in \Gamma_n$.*

Proof. By Theorem 2.3, $\Gamma_n \vdash_0 \text{all } u v$, and by Lemma 2.9, there is a Γ_n -sequence $u = t_1, \dots, t_m = v$ of length m . If $m = 1$, then $u = v$ and we are done. If $m = 2$, then $(\text{all } u v) = \text{Anti}(\vec{r}, \psi)$ for some sequence \vec{r} and some $\psi \in \Gamma_n$; we are again done.

It remains to prove a contradiction from $m \geq 3$. Suppose that we have $\psi, \theta \in \Gamma_n$ and sequences of verbs \vec{r} and \vec{s} such that $(\text{all } t_1 t_2) = \text{Anti}(\vec{r}, \psi)$ and $(\text{all } t_2 t_3) = \text{Anti}(\vec{s}, \theta)$. Notice that t_2 must contain either a or b . Assume that t_2 contains a . The argument when t_2 contains b is similar. Let m be the number of verbs in t_2 , the second term of $\text{Anti}(\vec{r}, \psi)$. Since it is a rather than b which occurs in the second term of $\text{Anti}(\vec{r}, \psi)$, m is even. We see this by examining the sentences in Γ_n . And let n be the number of verbs

in t_2 , the first term of $\text{Anti}(\vec{s}, \theta)$. This time, we see that n is odd. But $m = n$, since the second term of $\text{Anti}(\vec{r}, \psi)$ is the first term of $\text{Anti}(\vec{s}, \theta)$. This is a contradiction. \square

The sets $\Delta_{n,i}$. For all n and all $1 \leq i \leq n$, let the theory $\Delta_{n,i}$ be given by

$$\Delta_{n,i} = \begin{cases} (\Gamma_n \setminus \{\varphi_i\}) & \text{if } 1 \leq i < n \\ (\Gamma_n \setminus \omega) & \text{if } i = n. \end{cases}$$

LEMMA 2.14. Suppose that for some $1 \leq i \leq n$ and some terms u and v ,

$$\Delta_{n,i} \models \text{some } u \text{ } v.$$

Then $u = v = (r_1 \text{ all } (r_1 \text{ all } a))$, so that $(\text{some } u \text{ } v) = \alpha$.

Proof. It suffices to show that for every term $t \neq (r_1 \text{ all } (r_1 \text{ all } a))$, there is a model of $\Delta_{n,i}$ in which $\llbracket t \rrbracket = \emptyset$. Indeed, then $\Delta_{n,i} \not\models \text{some } u \text{ } v$ when $u = t$ or $v = t$.

We proceed by cases. In every case except $t = a$, we actually obtain a model of Γ_n in which $\llbracket t \rrbracket = \emptyset$. Since the models \mathcal{M}_4 of Γ_n constructed in Case 4 have $\mathcal{M}_4 \not\models \text{some } a \text{ } (r_1 \text{ all } (r_1 \text{ all } a))$, this implies the additional result that if $\Gamma_n \models \text{some } u \text{ } v$, then $u = v = (r_1 \text{ all } (r_1 \text{ all } a))$ or $u = v = a$.

Case 1: $t = (\vec{s} \text{ all } b)$ or $t = (\vec{s} \text{ all } a)$, where the length of \vec{s} is odd. Let $M_1 = \{*\}$, and define the model \mathcal{M}_1 with domain M_1 by $\llbracket a \rrbracket = \llbracket b \rrbracket = \{*\}$, and $\llbracket r_i \rrbracket = \emptyset$ for all i . In \mathcal{M}_1 , we have

$$\llbracket \vec{s} \text{ all } a \rrbracket = \llbracket \vec{s} \text{ all } b \rrbracket = \begin{cases} \{*\} & \text{if the length of } \vec{s} \text{ is even} \\ \emptyset & \text{if the length of } \vec{s} \text{ is odd,} \end{cases}$$

so $\mathcal{M}_1 \models \Gamma_n$.

Case 2: $t = (\vec{s} \text{ all } b)$, where the length of \vec{s} is even. Define \mathcal{M}_2 in the same way as \mathcal{M}_1 , but with $\llbracket b \rrbracket = \emptyset$. This time, we have

$$\begin{aligned} \llbracket \vec{s} \text{ all } a \rrbracket &= \begin{cases} \{*\} & \text{if the length of } \vec{s} \text{ is even} \\ \emptyset & \text{if the length of } \vec{s} \text{ is odd} \end{cases} \\ \llbracket \vec{s} \text{ all } b \rrbracket &= \begin{cases} \emptyset & \text{if the length of } \vec{s} \text{ is even} \\ \{*\} & \text{if the length of } \vec{s} \text{ is odd,} \end{cases} \end{aligned}$$

so again $\mathcal{M}_2 \models \Gamma_n$.

Case 3: $t = a$. Fix $1 \leq i \leq n$, let $M_3 = \{*\}$, and define the model $\mathcal{M}_3(i)$ with domain M_3 by $\llbracket a \rrbracket = \emptyset$, $\llbracket b \rrbracket = \{*\}$, and

$$\llbracket r_j \rrbracket = \begin{cases} \{(*, *)\} & \text{for } j \leq i \\ \emptyset & \text{for } j > i \end{cases}$$

In $\mathcal{M}_3(i)$, we have

$$\begin{aligned} \llbracket r_j \text{ all } a \rrbracket &= \llbracket b \rrbracket = \{*\} \\ \llbracket r_j \text{ all } (r_j \text{ all } a) \rrbracket &= \llbracket r_j \text{ all } b \rrbracket = \begin{cases} \{*\} & \text{if } j \leq i \\ \emptyset & \text{if } j > i, \end{cases} \end{aligned}$$

so $\mathcal{M}_3(i) \models \Delta_{n,i}$.

Case 4: $t = (\vec{s} \text{ all } a)$, where $\vec{s} = (s_1, \dots, s_k)$, k is even and nonzero, and $\vec{s} \neq (r_1, r_1)$. Let $M_4 = \{w, x, y, z\}$, and define a model \mathcal{M}_4 with domain M_4 by $\llbracket a \rrbracket = \{w\}$ and $\llbracket b \rrbracket = \{w, x, y, z\}$. To define the verb interpretations $\llbracket r_i \rrbracket$, we break into subcases.

Subcase 4a: $s_k \neq r_1$. Set $x \llbracket r_1 \rrbracket w$, $y \llbracket r_1 \rrbracket x$, $z \llbracket s_k \rrbracket w$, and no other instances of verbs:

$$z \xrightarrow{s_k} w \xleftarrow{r_1} x \xleftarrow{r_1} y$$

Subcase 4b: $s_k = r_1$ and $s_{k-1} \neq r_1$. Set $x \llbracket r_1 \rrbracket w$, $y \llbracket r_1 \rrbracket x$, and no other instances of verbs.

$$z \qquad w \xleftarrow{r_1} x \xleftarrow{r_1} y$$

Subcase 4c: $s_k = r_1$, $s_{k-1} = r_1$, and $k > 2$. Set $x \llbracket r_1 \rrbracket w$, $y \llbracket r_1 \rrbracket x$, $z \llbracket s_{k-2} \rrbracket y$, and no other instances of verbs.

$$w \xleftarrow{r_1} x \xleftarrow{r_1} y \xleftarrow{s_{k-2}} z$$

In all three subcases, $\llbracket r_1 \text{ all } a \rrbracket = \{x\}$ and $\llbracket r_1 \text{ all } (r_1 \text{ all } a) \rrbracket = \{y\}$, so $\mathcal{M}_4 \models \alpha$. And for all $1 \leq i \leq n$, $\llbracket r_i \text{ all } b \rrbracket = \emptyset$, so $\mathcal{M}_4 \models \Gamma_n$. It remains to show that $\llbracket \vec{s} \text{ all } a \rrbracket = \emptyset$. We introduce some notation: write $\vec{s}_{\geq j}$ for the sequence (s_j, \dots, s_k) .

In subcase 4a, $\llbracket s_k \text{ all } a \rrbracket = \{z\}$, and $\llbracket s_{\geq k-1} \text{ all } a \rrbracket = \emptyset$. Since k is even we have

$$\llbracket \vec{s}_{\geq j} \text{ all } a \rrbracket = \begin{cases} \emptyset & \text{if } j \text{ is odd} \\ M_4 & \text{if } j \text{ is even,} \end{cases}$$

and $\llbracket \vec{s} \text{ all } a \rrbracket = \llbracket \vec{s}_{\geq 1} \text{ all } a \rrbracket = \emptyset$.

In subcase 4b, $\llbracket s_k \text{ all } a \rrbracket = \{x\}$, and $\llbracket s_{\geq k-1} \text{ all } a \rrbracket = \emptyset$. As in subcase 4a, $\llbracket \vec{s} \text{ all } a \rrbracket = \emptyset$.

In subcase 4c, $\llbracket \vec{s}_{\geq k-2} \text{ all } a \rrbracket = \{z\}$. So $\llbracket \vec{s}_{\geq k-3} \text{ all } a \rrbracket = \emptyset$. And then since k is even and $k \geq 4$, $\llbracket \vec{s}_{\geq 1} \text{ all } a \rrbracket = \emptyset$ by the same argument which we have seen above.

This completes the proof. □

LEMMA 2.15. *For any natural number k , and any $n \geq k + 1$, if $\Gamma_n \models_k \text{some } u v$, then $u = v = (r_1 \text{ all } (r_1 \text{ all } a))$, so that $(\text{some } u v) = \alpha$.*

Proof. By induction on the depth of the tree witnessing $\Gamma_n \models_k \text{some } u v$. In the base case, $(\text{some } u v)$ is a leaf in Γ_n . Since α is the only sentence of the form $(\text{some } u v)$ in Γ_n , we are done.

Now suppose the root of the tree is $(\text{some } u v)$ with children $\{\psi_1, \dots, \psi_j\}$, where $j \leq k$, $\Gamma_n \models_k \psi_i$ for all $1 \leq i \leq j$, and $\{\psi_1, \dots, \psi_j\} \models \text{some } u v$.

We claim that for all ψ_i , there is a single sentence $\chi_i \in \Gamma_n$ such that $\{\chi_i\} \models \psi_i$. By induction, if ψ_i has the form $(\text{some } x y)$, then $\psi_i = \alpha$, and we can take $\chi_i = \alpha$. On the other hand, if ψ_i has the form $(\text{all } x y)$, then since $\Gamma_n \models \psi_i$, by Lemma 2.13 either $\psi_i = (\text{all } x x)$ and we can take χ_i to be any sentence in Γ_n , or $\psi_i = \text{Anti}(\vec{r}_i, \chi_i)$ for some $\chi_i \in \Gamma_n$, and $\{\chi_i\} \models \psi_i$.

By the claim, $\{\chi_k : k \leq j\} \models \text{some } u v$. And since $j \leq k \leq n - 1$, there is some $1 \leq i^* \leq n$ such that $\{\chi_k : k \leq j\} \subseteq \Delta_{n,i^*}$. For this i^* , we see that $\Delta_{n,i^*} \models \psi_k$ for all $k \leq j$, so $\Delta_{n,i^*} \models \text{some } u v$. Our result follows from Lemma 2.14. □

Proof of Theorem 2.11. By Lemma 2.12, $\Gamma_{n+1} \models \text{some } a a$. And by Lemma 2.15, $\Gamma_{n+1} \not\models_n \text{some } a a$.

Suppose that \vdash^* is a sound and complete syllogistic proof system for \mathcal{L}_2 . Let n be the maximum number of premises in any rule in \vdash^* . Then since $\Gamma_{n+1} \models \text{some } a \ a$, we have $\Gamma_{n+1} \vdash^* \text{some } a \ a$ by completeness, and $\Gamma_{n+1} \not\models_n \text{some } a \ a$ by soundness. This is a contradiction. \square

We have shown that the full semantic entailment relation \models for \mathcal{L}_2 is not \models_n for any n . This contrasts with logics like the one for \mathcal{L}_1 ; for that, \models coincides with \models_2 , since there is a sound and complete syllogistic proof system in which every rule has at most two premises.

REMARK. Our work in this section used the fact that our set \mathbf{V} of verbs can be arbitrarily large. It is open whether the negative result holds when \mathbf{V} is a fixed finite set.

2.3. Completeness using the (CASES) rule. We have just seen that \mathcal{L}_2 has no logical system which is syllogistic, sound, and complete. The rest of this section rectifies this, in three different ways.

In this section, we add a single rule. It is called (CASES), and while it is not *syllogistic* as we defined the term in §1.3, it is simple and natural. Here is a statement of it.

$$\begin{array}{c}
 \text{some } x \ x \quad \text{all } y \ (r \ \text{all } x) \\
 \vdots \quad \quad \quad \vdots \\
 \varphi \quad \quad \quad \varphi \\
 \hline
 \varphi \quad \text{CASES}
 \end{array} \tag{7}$$

The nonsyllogistic feature is that premises are withdrawn in derivations.¹ Let us explain how the (CASES) rule is used. To prove φ from a set Γ , it is sufficient to take a term x , prove φ from $\Gamma \cup \{\text{some } x \ x\}$, and also prove φ from $\Gamma \cup \{\text{all } y \ (r \ \text{all } x)\}$. Here y can be any term and r can be any verb.

In the logic itself, we take two derivations of φ , and then in one we withdraw a sentence (*some* $x \ x$), while in the other we withdraw a sentence of the form (*all* $y \ (r \ \text{all } x)$) (for this same term x). We may withdraw zero occurrences or more than one. The overall conclusion is φ .

In this subsection, we write \vdash for provability in the \vdash_0 system from Figure 2.1, together with (CASES).

LEMMA 2.16 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

Proof. By induction on the number n of uses of (CASES) in derivations. For $n = 0$, this is just soundness of \vdash_0 . Assume our result for n , and let $\Gamma \vdash \varphi$ via a derivation with $n + 1$ uses of (CASES). We may assume that the last use of (CASES) is at the root of the proof tree. So we have

- (1) $\Gamma \cup \{\text{some } x \ x\} \vdash \varphi$
- (2) $\Gamma \cup \{\text{all } y \ (r \ \text{all } x)\} \vdash \varphi$,

where both derivations have at most n uses of (CASES). By our induction hypotheses, (1) and (2) hold when \vdash is replaced by \models . Let $\mathcal{M} \models \Gamma$. Then we have two cases. If

¹ The reader might wonder why we are indicating withdrawal of premises using a large “X” rather than the standard notation of square brackets. The reason is that later in the paper we use square brackets in our syntax, and we will thus need a different notation later to indicate withdrawals.

$\llbracket x \rrbracket \neq \emptyset$, then $\mathcal{M} \models \text{some } x x$, so $\mathcal{M} \models \varphi$. And when $\llbracket x \rrbracket = \emptyset$, we have $\llbracket r \text{ all } x \rrbracket = M$. So $\mathcal{M} \models \text{all } y (r \text{ all } x)$, and thus $\mathcal{M} \models \varphi$. \square

EXAMPLE 2.17. Here is a sample derivation.

$$\{\text{some } c d, \text{all } a x, \text{all } a y, \text{all } (r \text{ all } a) x, \text{all } (r \text{ all } a) y\} \vdash \text{some } x y.$$

Let Γ be the theory on the left. We show that

1. $\Gamma \cup \{\text{some } a a\} \vdash \text{some } x y$.
2. $\Gamma \cup \{\text{all } c (r \text{ all } a)\} \vdash \text{some } x y$.

The first is easy from $(\text{all } a x)$ and $(\text{all } a y)$. The second comes from

$$\frac{\frac{\frac{\text{some } c d}{\text{some } c c} \text{ SOME}_1 \quad \frac{\text{all } c (r \text{ all } a) \quad \text{all } (r \text{ all } a) x}{\text{all } c x} \text{ DARI}}{\text{some } c x}{\text{some } x c} \text{ SOME}_2 \quad \frac{\text{all } c (r \text{ all } a) \quad \text{all } (r \text{ all } a) y}{\text{all } c y} \text{ DARI}}{\text{some } x y} \text{ BARB}$$

(Here (BARB) abbreviates (BARBARA).) Note that the premise $(\text{all } c (r \text{ all } a))$ was used twice.

EXAMPLE 2.18. We show that $\Gamma_n \vdash \text{some } a a$, where Γ_n is the theory in §2.2.

First, note that $\Delta \cup \{\text{some } a a\} \vdash \text{some } a a$ for any theory Δ . So by n applications of (CASES), to show that $\Gamma_n \vdash \text{some } a a$, it suffices to show that $\Gamma_n \cup \{\text{all } b (r_i \text{ all } a) : 1 \leq i \leq n\} \vdash \text{some } a a$. Let Γ_n^* be the theory on the left.

Let $\psi_i = \text{all } (r_i \text{ all } (r_i \text{ all } a)) (r_i \text{ all } b)$. By (ANTI), $\Gamma_n^* \vdash \psi_i$ for all $1 \leq i \leq n$. Repeatedly applying (BARBARA) to the sequence $\psi_1, \varphi_1, \psi_2, \varphi_2, \dots, \psi_n, \omega$, we find that $\Gamma_n^* \vdash \text{all } (r_1 \text{ all } (r_1 \text{ all } a)) a$. So by α , (DARI), and (SOME₂), $\Gamma_n^* \vdash \text{some } a a$.

THEOREM 2.19 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

Proof. By Theorem 2.3, if $\Gamma \models \text{all } x y$, then already $\Gamma \vdash_0 \text{all } x y$. So we may assume that φ has the form $(\text{some } x y)$. We prove the contrapositive, so assume $\Gamma \not\models \varphi$. Let \mathbb{T} be a set of terms, closed under subterms, which contains x, y , and all subterms of sentences in Γ . By Zorn’s Lemma², let $\Gamma^* \supseteq \Gamma$ be a maximal extension, such that $\Gamma^* \not\models \varphi$. The sentences in Γ^* may contain any terms in the language.

Assume for contradiction that Γ^* does not determine existentials for \mathbb{T} . Then there are terms $x, y \in \mathbb{T}$ and a verb $r \in \mathbf{V}$ such that $\Gamma^* \not\models_0 \text{some } x x$ and $\Gamma^* \not\models_0 \text{all } y (r \text{ all } x)$. In particular, Γ^* does not contain either of these sentences. By maximality, we have $\Gamma^* \cup \{\text{some } x x\} \vdash \varphi$ and $\Gamma^* \cup \{\text{all } y (r \text{ all } x)\} \vdash \varphi$. By (CASES), $\Gamma^* \vdash \varphi$, contradiction. Thus Γ^* determines existentials for \mathbb{T} .

Now since $\Gamma^* \not\models \varphi$, we clearly have $\Gamma^* \not\models_0 \varphi$. So by Theorem 2.6, $\Gamma \not\models \varphi$. This is what was to be shown. \square

The proofs of Theorem 2.3 and Theorem 2.6 show that if $\Gamma \not\models \varphi$, then either $\mathcal{M}(\Gamma, \mathbb{T})$ or $\mathcal{M}'(\Gamma^*, \mathbb{T})$ are countermodels, depending on the form of φ . Both of these models have size $O(n^2)$, where n is the complexity of $\Gamma \cup \{\varphi\}$.

REMARK. Since \vdash_0 is already complete for sentences of the form $(\text{all } a b)$, we only need to use (CASES) in proofs of sentences of the form $(\text{some } a b)$. Using Lemma 2.10,

² Of course, we do not actually need the Axiom of Choice when the set of all sentences in the language is countable.

it is possible to show that (CASES) is equivalent over \vdash_0 to the following rule:

$$\frac{\overline{\text{all } y} \text{ } (\overline{r} \text{ all } x)}{\dots} \frac{\text{all } x \text{ } a \quad \text{all } x \text{ } b \quad \text{some } a \text{ } b}{\text{some } a \text{ } b} \text{ CASES}^*$$

So the system with rules (AX), (BARBARA), (ANTI), (SOME₁), (SOME₂), (DARII), and (CASES*) is also sound and complete for \mathcal{L}_2 . We chose to emphasize (CASES) rather than (CASES*) because it made the completeness proof quicker, and because we will use (CASES) again in §3.2.

2.4. Completeness using the (CHAINS) schema. In Theorem 2.11, we proved that there are no syllogistic proof systems for \mathcal{L}_2 which are sound and complete. We have just seen that \mathcal{L}_2 has a sound and complete proof system, but one which is not syllogistic in our sense. In this section, we give another proof system, which this time makes use of a schema of rules with arbitrarily long (but finite) premise sets.

DEFINITION 2.20. Let a and b be nouns. A *chain linking a to b* is a sequence C of sentences

$$C = (\text{all } a \text{ } u_1, \text{all } v_1 \text{ } u_2, \dots, \text{all } v_i \text{ } u_{i+1}, \dots, \text{all } v_m \text{ } b),$$

such that for all $1 \leq i \leq m$, either

1. $u_i = (\vec{r} \text{ all } z_i)$ and $v_i = (\vec{r} \text{ all } (r \text{ all } t_i))$, where \vec{r} is a sequence of even length, or
2. $u_i = (\vec{r} \text{ all } (r \text{ all } t_i))$ and $v_i = (\vec{r} \text{ all } z_i)$, where \vec{r} is a sequence of odd length.

We say that this chain has length $(m + 1)$, and the terms t_1, \dots, t_m are the *missing link terms* in C .

Note that a chain of length 1 linking a to b is just the single sentence (all a b) and has no missing link terms. Here are two chains of length 2 linking a to b :

$$\begin{aligned} &(\text{all } a \text{ } z, \text{all } (r \text{ all } t) \text{ } b) \\ &(\text{all } a \text{ } (s \text{ all } (r \text{ all } t)), \text{all } (s \text{ all } z) \text{ } b). \end{aligned}$$

In both of these chains, t is the missing link term.

Returning to the definition, we emphasize that the terms denoted z_1, \dots, z_m may be arbitrary (they need not be nouns) and are not missing link terms. The sequence \vec{r} is also arbitrary.

LEMMA 2.21. *Suppose $C = (\text{all } a \text{ } u_1, \text{all } v_1 \text{ } u_2, \dots, \text{all } v_m \text{ } b)$ is a chain linking a to b , $\mathcal{M} \models C$, and $\llbracket t \rrbracket = \emptyset$ for every missing link term t in C . Then $\llbracket a \rrbracket \subseteq \llbracket b \rrbracket$.*

Proof. Since \mathcal{M} satisfies all the sentences in C , $\llbracket a \rrbracket \subseteq \llbracket u_1 \rrbracket$, $\llbracket v_m \rrbracket \subseteq \llbracket b \rrbracket$, and $\llbracket v_i \rrbracket \subseteq \llbracket u_{i+1} \rrbracket$ for all i . So it suffices to show that $\llbracket u_i \rrbracket \subseteq \llbracket v_i \rrbracket$ for all i .

Let t_i be the missing link term for u_i and v_i . Since $\llbracket t_i \rrbracket = \emptyset$, $\llbracket r \text{ all } t_i \rrbracket = \mathcal{M}$. So $\llbracket z_i \rrbracket \subseteq \llbracket r \text{ all } t_i \rrbracket$ for any term z_i . This is the desired inclusion when \vec{r} is the empty sequence. The result then follows by induction on the length of \vec{r} , using the fact that if $\llbracket x \rrbracket \subseteq \llbracket y \rrbracket$, then $\llbracket r \text{ all } y \rrbracket \subseteq \llbracket r \text{ all } x \rrbracket$. □

DEFINITION 2.22. Let x and y be terms. An (x, y) *chain system* is a sequence of chains C_1, \dots, C_l such that for every missing link term t in every chain C_n , there exist $m, m' < n$ such that C_m links t to x and $C_{m'}$ links t to y . When $x = y$, we may take $m = m'$.

If chains C_m and $C_{m'}$ link t to x and y , respectively, we can think of these chains as witnessing that we are allowed to use t as a missing link term later in the (x, y) chain system. Of course, the first chain in an (x, y) chain system must have length 1, since there are no available missing link terms from previous chains.

LEMMA 2.23. *Let C_1, \dots, C_l be an (x, y) chain system. Suppose $\mathcal{M} \models C_i$ for all i , and suppose that $\llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$ in \mathcal{M} . Then $\llbracket a \rrbracket \subseteq \llbracket b \rrbracket$ whenever some C_i links a to b .*

Proof. By induction on l . When $l = 0$, there are no chains, so the conclusion is vacuously satisfied. Now suppose C_1, \dots, C_{l+1} is an (x, y) chain system. By induction, the conclusion holds for the (x, y) chain system C_1, \dots, C_l . So suppose C_{l+1} links a to b . For any missing link term t in C_{l+1} , there are chains C_i and C_j with $i, j \leq l$, linking t to x and t to y . So $\llbracket t \rrbracket \subseteq \llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$. By Lemma 2.21, $\llbracket a \rrbracket \subseteq \llbracket b \rrbracket$, as desired. \square

We introduce the new rule schema

$$\frac{\text{some } a \ b \ C_1 \ \dots \ C_l}{\text{some } x \ y} \text{ CHAINS,}$$

where C_1, \dots, C_l is an (x, y) chain system, some C_m links a to x , and some C_n links b to y .

In this section, we write \vdash for the proof system \vdash_0 augmented by the rule schema (CHAINS).

THEOREM 2.24. *The (CHAINS) schema is sound.*

Proof. Let (some $a \ b$), C_1, \dots, C_l be the premises of an instance of (CHAINS), and suppose that \mathcal{M} satisfies these premises. Suppose towards a contradiction that $\llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$, so Lemma 2.23 applies. Since some C_m links a to x and some C_n links b to y , we use Lemma 2.23 to see that $\llbracket a \rrbracket \cap \llbracket b \rrbracket \subseteq \llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$. But this contradicts the assumption that $\mathcal{M} \models \text{some } a \ b$. \square

EXAMPLE 2.25. We show that $\Gamma_n \vdash \text{some } a \ a$, where Γ_n is the theory in §2.2. We will find an (a, a) chain system which contains a chain linking $(r_1 \text{ all } (r_1 \text{ all } a))$ to a .

$C_1 = (\text{all } a \ a)$ is a chain of length 1 linking a to a . This allows a to be used as a missing link term in C_2 . Let β be the sentence

$$\text{all } (r_1 \text{ all } (r_1 \text{ all } a)) \ (r_1 \text{ all } (r_1 \text{ all } a)).$$

Then

$$C_2 = (\beta, \varphi_1, \dots, \varphi_{n-1}, \omega)$$

is a chain linking $(r_1 \text{ all } (r_1 \text{ all } a))$ to a , in which the only missing link term is a .

Let's check that C_2 is a chain. First, $u_1 = (r_1 \text{ all } (r_1 \text{ all } a))$ and $v_1 = (r_1 \text{ all } c)$, so this is alternative 2 in Definition 2.20, with $t = a, z = c, r = r_1$, and $\vec{r} = r_1$. All of the rest of the links from u_i to v_i are justified in the same way.

Then we have a derivation:

$$\frac{\alpha \ C_1 \ C_2}{\text{some } a \ a} \text{ CHAINS.}$$

This shows that $\Gamma_n \vdash \text{some } a \ a$, because $\alpha \in \Gamma_n$, and each sentence in C_1, C_2 is either β , which is an instance of (AX), or an element of Γ_n .

THEOREM 2.26 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

Proof. By Theorem 2.3, if $\Gamma \models \text{all } x \ y$, then already $\Gamma \vdash_0 \text{all } x \ y$. So we may assume that φ has the form (some $x \ y$). Let \mathbb{T} be a set of terms, closed under subterms, which contains x, y , and all subterms of sentences in Γ .

For any term $t \in \mathbb{T}$, let $\Delta_t = \{\text{all } z \ (r \ \text{all } t) : z \in \mathbb{T}, r \in \mathbf{V}\}$. In order to extend Γ to a theory which determines existentials for \mathbb{T} , we define an increasing sequence of theories by induction. Set $\Gamma_0 = \Gamma$, and given Γ_n , define

$$\Gamma_{n+1} = \Gamma_n \cup \{\text{all } z \ (r \ \text{all } t) : t, z \in \mathbb{T}, r \in \mathbf{V}, \text{ and } \Gamma_n \cup \{\text{some } t \ t\} \vdash_0 \varphi\}.$$

So Γ_{n+1} includes Δ_t for all terms $t \in \mathbb{T}$ such that $\Gamma_n \cup \{\text{some } t \ t\} \vdash_0 \varphi$. Let $\Gamma_\omega = \bigcup_{n \in \omega} \Gamma_n$, and let

$$\Gamma^* = \Gamma_\omega \cup \{\text{some } t \ t : t \in \mathbb{T} \text{ and } \Delta_t \not\subseteq \Gamma_\omega\}.$$

CLAIM 2.27. *There is some n such that $\Gamma_n \vdash_0 \varphi$.* □

Proof. Γ^* determines existentials for \mathbb{T} , and by Theorem 2.6, $\Gamma^* \vdash_0 \varphi$. By Lemma 2.10, there is a sentence (some $a_1 \ a_2$) $\in \Gamma^*$ such that $\Gamma^* \vdash_0 \text{all } a_i \ x$ and $\Gamma^* \vdash_0 \text{all } a_j \ y$ for some $i, j \in \{1, 2\}$. Since \vdash_0 -proofs of all-sentences are finite and never contain some-sentences, there is already some n such that $\Gamma_n \vdash_0 \text{all } a_i \ x$ and $\Gamma_n \vdash_0 \text{all } a_j \ y$, and we have $\Gamma_n \cup \{\text{some } a_1 \ a_2\} \vdash_0 \varphi$.

It remains to show that the sentence (some $a_1 \ a_2$) belongs to $\Gamma \subseteq \Gamma_n$, since then $\Gamma_n \vdash_0 \varphi$. Suppose not. Then (some $a_1 \ a_2$) is not in Γ_ω , since the sets Γ_n only add sentences of the form (all $z \ (r \ \text{all } t)$) to Γ . So (some $a_1 \ a_2$) is a sentence (some $t \ t$) such that $\Delta_t \not\subseteq \Gamma_\omega$. But then $\Gamma_n \cup \{\text{some } t \ t\} \vdash_0 \varphi$, so $\Delta_t \subseteq \Gamma_{n+1} \subseteq \Gamma_\omega$, which is a contradiction. □

CLAIM 2.28. *If there is some n such that $\Gamma_n \vdash_0 \varphi$, then $\Gamma \vdash \varphi$.*

Proof. Assume $\Gamma \not\vdash \varphi$. Then we will prove the following two claims for all n , by induction:

- (1) _{n} For every $k \geq 1$ and every Γ_n -sequence of terms t_1, \dots, t_k , there is an (x, y) chain system C_1, \dots, C_ℓ , such that C_ℓ links t_1 to t_k , and such that for all $1 \leq i \leq \ell$ and all $\psi \in C_i$, $\Gamma \vdash \psi$.
- (2) _{n} $\Gamma_n \not\vdash_0 \varphi$.

So assume (1) _{m} and (2) _{m} hold for all $m < n$. We will first prove (1) _{n} by induction on k .

In the base case, when $k = 1$, we have $t_1 = t_k$, and $\Gamma \vdash \text{all } t_1 \ t_k$ by (AX). The chain (all $t_1 \ t_k$) has no missing link terms and links t_1 to t_k . So we have the required (x, y) chain system, consisting of just this one chain.

Now suppose $k > 1$, and fix a Γ_n -sequence t_1, \dots, t_k . Let C_1, \dots, C_ℓ be the (x, y) chain system obtained by induction for the Γ_n -sequence t_1, \dots, t_{k-1} . Then C_ℓ links t_1 to t_{k-1} , so the last sentence in C_i is (all $c \ t_{k-1}$) for some term c . By the definition of Γ_n -sequence, there is a sentence (all $d \ e$) $\in \Gamma_n$ and a sequence of verbs \vec{r} such that (all $t_{k-1} \ t_k$) = Anti(\vec{r} , (all $d \ e$)).

If (all $d \ e$) $\in \Gamma$, then by repeated applications of (ANTI), $\Gamma \vdash \text{all } t_{k-1} \ t_k$. Since also $\Gamma \vdash \text{all } c \ t_{k-1}$ by induction, we have $\Gamma \vdash \text{all } c \ t_k$ by (BARBARA). Replacing the last sentence of C_ℓ with (all $c \ t_k$), we are done.

If (all $d \ e$) $\notin \Gamma$, then (all $d \ e$) $\in \Gamma_{m+1} \setminus \Gamma_m$ for some $0 \leq m < n$. It follows that $e = (r \ \text{all } t)$ for some term t such that $\Gamma_m \cup \{\text{some } t \ t\} \vdash_0 \varphi$. By Lemma 2.10, there

is a sentence (some $p q$) in $\Gamma_m \cup \{\text{some } t t\}$ such that for all $w \in \{x, y\}$ there is some $v \in \{p, q\}$ such that $\Gamma_m \vdash_0 v w$. If (some $p q$) $\in \Gamma_m$, then $\Gamma_m \vdash_0 \varphi$, contradicting $(2)_m$. Otherwise, the sentence (some $p q$) must be (some $t t$). That is, $p = q = t$, so $\Gamma_m \vdash_0$ all $t x$ and $\Gamma_m \vdash_0$ all $t y$.

By Lemma 2.9, there are Γ_m -sequences linking t to x and t to y . By $(1)_m$, there are (x, y) chain systems $C'_1, \dots, C'_{\ell'}$ and $C''_1, \dots, C''_{\ell''}$ such that $C'_{\ell'}$ links t to x , $C''_{\ell''}$ links t to y , and for every sentence ψ in every chain, $\Gamma \vdash \psi$. Let C_ℓ^* be the chain C_ℓ with the sentence (all $t_k t_k$) appended. Then C_ℓ^* links t_1 to t_k . We either have $t_{k-1} = (\vec{r} \text{ all } d)$ and $t_k = (\vec{r} \text{ all } (r \text{ all } t))$ where \vec{r} has even length, or $t_{k-1} = (\vec{r} \text{ all } (r \text{ all } t))$ and $t_k = (\vec{r} \text{ all } d)$ where \vec{r} has odd length, so the missing link terms in C_ℓ^* are those in C_ℓ , together with t . Also $\Gamma \vdash \psi$ for every sentence ψ in C_ℓ^* , by our assumption about C_ℓ and (AX). So $C'_1, \dots, C'_{\ell'}, C''_1, \dots, C''_{\ell''}, C_1, \dots, C_{\ell-1}, C_\ell^*$ is our desired (x, y) chain system.

Having established $(1)_n$, we prove $(2)_n$. Assume for contradiction that $\Gamma_n \vdash_0 \varphi$. By Lemma 2.10, there is a sentence (some $a_1 a_2$) in Γ_n such that $\Gamma_n \vdash_0$ all $a_i x$ and $\Gamma_n \vdash_0$ all $a_j y$ for some $i, j \in \{1, 2\}$. By Lemma 2.10, we thus have Γ_n -sequences from a_i to x and from a_j to y . Apply $(1)_n$ to these sequences to obtain (x, y) chain systems C_1, \dots, C_ℓ and $C'_1, \dots, C'_{\ell'}$ such that C_ℓ links a_i to x and $C'_{\ell'}$ links a_j to y , and all sentences in all chains are \vdash -provable from Γ . Then we have an instance of chains

$$\frac{\text{some } a_i a_j \quad C_1 \quad \dots \quad C_\ell \quad C'_1 \quad \dots \quad C'_{\ell'}}{\text{some } x y} \text{ CHAINS.}$$

The sentence (some $a_1 a_2$) belongs to Γ_n , hence to Γ . Applying (SOME₁) or (SOME₂) as needed, $\Gamma \vdash$ some $a_i a_j$. So this deduction is the root of a proof tree proving (some $x y$) from Γ . □

Claims 2.27 and 2.28 complete the proof of Theorem 2.26.

2.5. Completeness and PTIME decidability for the extended language \mathcal{L}_2^+ . We have seen that \mathcal{L}_2 has no sound and complete syllogistic proof system. And we have seen proof systems which go beyond the “purely syllogistic” in earlier sections. But this section goes in a different direction. We show that if we enhance the syntax of \mathcal{L}_2 in a certain way, then we are able to find a boundedly complete syllogistic proof system for the larger language.

We add to \mathcal{L}_2 a new four-place sentence former

$$(\text{all } a b) \vee (\text{some } x y)$$

with the evident semantics

$$\mathcal{M} \models (\text{all } a b) \vee (\text{some } x y) \quad \text{iff} \quad \llbracket a \rrbracket \subseteq \llbracket b \rrbracket \text{ or } \llbracket x \rrbracket \cap \llbracket y \rrbracket \neq \emptyset.$$

We call the larger language \mathcal{L}_2^+ . Note that we do not allow the disjunction of arbitrary sentences. Rather, there is a new kind of sentence, which is the disjunction of exactly one sentence (all $a b$) and one sentence (some $x y$). For a proof system, we take the rules in Figure 3, and we write \vdash for provability in this system.

LEMMA 2.29. *The proof system is sound.*

Proof. We argue soundness for (EMPTY₁), (EMPTY₂), and (DD), since soundness of the other rules is clear from the meaning of disjunction. Fix a model \mathcal{M} .

For (EMPTY₁), either $\llbracket a \rrbracket \neq \emptyset$, or $\emptyset = \llbracket a \rrbracket \subseteq \llbracket b \rrbracket$. In either case, $\mathcal{M} \models (\text{all } a b) \vee (\text{some } a a)$.

$$\begin{array}{c}
 \frac{}{(all\ a\ b) \vee (some\ a\ a)}\ \text{EMPTY}_1 \qquad \frac{(all\ a\ b) \vee (some\ x\ y)}{(all\ a\ b) \vee (some\ x\ x)}\ \text{SOME}'_1 \qquad \frac{all\ a\ b}{(all\ a\ b) \vee (some\ x\ y)}\ \text{WL} \\
 \\
 \frac{}{(all\ b\ (r\ all\ a)) \vee (some\ a\ a)}\ \text{EMPTY}_2 \qquad \frac{(all\ a\ b) \vee (some\ x\ y)}{(all\ a\ b) \vee (some\ y\ x)}\ \text{SOME}'_2 \qquad \frac{some\ x\ y}{(all\ a\ b) \vee (some\ x\ y)}\ \text{WR} \\
 \\
 \frac{(all\ a\ b) \vee (some\ x\ y) \quad (all\ b\ c) \vee (some\ x\ y)}{(all\ a\ c) \vee (some\ x\ y)}\ \text{BARB}' \qquad \frac{(all\ a\ b) \vee (some\ x\ y)}{(all\ (r\ all\ b)\ (r\ all\ a)) \vee (some\ x\ y)}\ \text{ANTI}' \\
 \\
 \frac{some\ t\ u \quad (all\ t\ x) \vee (some\ x\ y) \quad (all\ u\ y) \vee (some\ x\ y)}{some\ x\ y}\ \text{DD} \\
 \\
 \frac{(all\ a\ b) \vee (some\ t\ u) \quad (all\ t\ x) \vee (some\ x\ y) \quad (all\ u\ y) \vee (some\ x\ y)}{(all\ a\ b) \vee (some\ x\ y)}\ \text{DD}'
 \end{array}$$

Fig. 3. Rules in §2.5. These rules are added on top of the rules in Figure 2.

For (EMPTY₂), either $\llbracket a \rrbracket \neq \emptyset$, or $\llbracket a \rrbracket = \emptyset$. In the latter case, $\llbracket b \rrbracket \subseteq \llbracket r\ all\ a \rrbracket = M$. In either case, $\mathcal{M} \models (all\ b\ (r\ all\ a)) \vee (some\ a\ a)$.

For (DD), suppose that \mathcal{M} satisfies the premises of the rule, and assume for contradiction that $\llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$. Then $\mathcal{M} \models all\ t\ x$ and $\mathcal{M} \models all\ u\ y$. So $\llbracket t \rrbracket \cap \llbracket u \rrbracket \subseteq \llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$, contradicting $\mathcal{M} \models some\ t\ u$. \square

EXAMPLE 2.30. The rules (DD) and its companion (DD') stand for “double darii”. To see the connection between these rules and (DARII), we note that (DARII) is redundant in this system:

$$\frac{some\ x\ y \quad \frac{all\ x\ x}{(all\ x\ x) \vee (some\ x\ z)}\ \text{AX} \quad \frac{all\ y\ z}{(all\ y\ z) \vee (some\ x\ z)}\ \text{WL}}{some\ x\ z}\ \text{DD}$$

Note also that (DD) is the only new rule in our proof system which produces a conclusion in the original syntax of \mathcal{L}_2 . It is responsible, together with (EMPTY₂), for the reasoning captured by (CASES) and (CHAINS) in the previous two sections.

EXAMPLE 2.31. We show that $\Gamma_n \vdash some\ a\ a$, where once again Γ_n is from §2.2. For each $1 \leq i < n$, we have the derivation:

$$\frac{\frac{(all\ b\ (r_i\ all\ a)) \vee (some\ a\ a)}{(all\ (r_i^2\ all\ a)\ (r_i\ all\ b)) \vee (some\ a\ a)}\ \text{EMPTY}_2 \quad \frac{\varphi_i}{(all\ (r_i\ all\ b)\ (r_{i+1}^2\ all\ a)) \vee (some\ a\ a)}\ \text{WL}}{(all\ (r_i^2\ all\ a)\ (r_{i+1}^2\ all\ a)) \vee (some\ a\ a)}\ \text{BARB}'$$

and similarly, we have:

$$\frac{\frac{(all\ b\ (r_n\ all\ a)) \vee (some\ a\ a)}{(all\ (r_n^2\ all\ a)\ (r_n\ all\ b)) \vee (some\ a\ a)}\ \text{EMPTY}_2 \quad \frac{\omega}{(all\ (r_n\ all\ b)\ a) \vee (some\ a\ a)}\ \text{WL}}{(all\ (r_n^2\ all\ a)\ a) \vee (some\ a\ a)}\ \text{BARB}'$$

By n applications of (BARB'), we obtain $(\text{all } (r_1^2 \text{ all } a) a) \vee (\text{some } a a)$. And then we conclude:

$$\frac{\alpha \quad (\text{all } (r_1^2 \text{ all } a) a) \vee (\text{some } a a)}{\text{some } a a} \text{ DD.}$$

Completeness and PTIME-decidability. At this point, we turn to the completeness and PTIME-decidability of the logic. We are going to apply Theorem 1.5. For any set Δ , let $T(\Delta)$ be the set of subterms of sentences in Δ . Let $T^+(\Delta)$ be $T(\Delta)$ together with the terms $(r \text{ all } w)$ where $w \in T(\Delta)$ and where r occurs in Δ .

Let $g(\Delta)$ be the set consisting of

- (i) All sentences $(\text{all } x y)$, where $x \in T(\Delta)$ and $y \in T^+(\Delta)$.
- (ii) All sentences $(\text{some } u v)$, where $u, v \in T(\Delta)$.
- (iii) All sentences $(\text{all } x y) \vee (\text{some } u v)$, where $x, u, v \in T(\Delta)$ and $y \in T^+(\Delta)$.

Note that g is computable in PTIME.

THEOREM 2.32. *If $\Gamma \models \varphi$, then $\Gamma \vdash_{g(\Gamma \cup \{\varphi\})} \varphi$. Hence the consequence relation for \mathcal{L}_2^+ is in PTIME.*

Proof. As in §1.4, we are going to save on some notation below by writing T for $T(\Gamma \cup \{\varphi\})$, T^+ for $T^+(\Gamma \cup \{\varphi\})$, and A for $g(\Gamma \cup \{\varphi\})$.

We are going to do this entire proof in two parts. The first part handles the case that φ is either $(\text{all } a b) \vee (\text{some } x y)$ or else $(\text{some } x y)$. After that, we handle the relatively simpler case that φ is $(\text{all } a b)$.

So until further notice, we are in the first part of this theorem. Please note that x, y, a , and b are fixed throughout the rest of this proof.

Let $M = M_{xy}$ be the set of unordered pairs $\{t, u\}$ of terms from T such that there is some $z \in \{x, y\}$ such that for all $v \in \{t, u\}$, $\Gamma \not\vdash_A (\text{all } v z) \vee (\text{some } x y)$.

We allow $t = u$, and it follows that whenever M contains $\{t, u\}$, then it also contains $\{t\} = \{t, t\}$. The point of the definition of M will become clearer after we see the Truth Lemma and Claim 2.35 below: we are building a model which is guaranteed to have $\llbracket x \rrbracket \cap \llbracket y \rrbracket = \emptyset$.

We define a model \mathcal{M} with domain M by setting

$$\begin{aligned} \{t, u\} \in \llbracket p \rrbracket & \text{ iff either } \Gamma \vdash_A (\text{all } t p) \vee (\text{some } x y), \text{ or } \Gamma \vdash_A (\text{all } u p) \vee (\text{some } x y) \\ \{t, u\} \llbracket r \rrbracket \{v, w\} & \text{ iff for some } c \in \{t, u\} \text{ and } d \in \{v, w\}, \\ & \Gamma \vdash_A (\text{all } c (r \text{ all } d)) \vee (\text{some } x y). \end{aligned}$$

CLAIM 2.33 (Truth Lemma). *In \mathcal{M} , we have the following for all terms $z \in T$,*

$$\llbracket z \rrbracket = \{ \{t, u\} \in M : \Gamma \vdash_A (\text{all } t z) \vee (\text{some } x y) \text{ or } \Gamma \vdash_A (\text{all } u z) \vee (\text{some } x y) \}. \quad \square$$

Proof. By induction on z . For a noun in \mathbf{N} , this is by definition of the model. So we assume our statement for z and prove it for $(r \text{ all } z) \in T$. Note that $z \in T$, since T is closed under subterms.

Fix $\{t, u\} \in M$, and suppose (without loss of generality) $\Gamma \vdash_A (\text{all } t (r \text{ all } z)) \vee (\text{some } x y)$. We show that $\{t, u\} \in \llbracket r \text{ all } z \rrbracket$. Let $\{v, w\} \in M$ be an element of $\llbracket z \rrbracket$. By the induction hypothesis, we have (without loss of generality) $\Gamma \vdash_A (\text{all } v z) \vee (\text{some } x y)$. Then by (ANTI'), $\Gamma \vdash_A (\text{all } (r \text{ all } z) (r \text{ all } v)) \vee (\text{some } x y)$. Note that the sentence

$(\text{all } (r \text{ all } z) (r \text{ all } v)) \vee (\text{some } x y)$ belongs to A because $(r \text{ all } z)$, x , and y belong to T and $r \text{ all } v$ to T^+ . Moreover, $(\text{all } t (r \text{ all } v)) \vee (\text{some } x y)$ again belongs to A . By (BARB'), $\Gamma \vdash_A (\text{all } t (r \text{ all } v)) \vee (\text{some } x y)$, and hence $\{t, u\} \Vdash r \{v, w\}$. So $\{t, u\} \in \llbracket r \text{ all } z \rrbracket$.

Conversely, suppose $\{t, u\} \in \llbracket r \text{ all } z \rrbracket$.

Case 1: $\{z\} \in M$. Notice that $\Gamma \vdash_A (\text{all } z z) \vee (\text{some } x y)$ by (AX) and (WL). By induction, $\{z\} \in \llbracket z \rrbracket$. Hence $\{t, u\} \Vdash r \{z\}$. So by the definition of the model, we have either $\Gamma \vdash_A (\text{all } t (r \text{ all } z)) \vee (\text{some } x y)$ or $\Gamma \vdash_A (\text{all } u (r \text{ all } z)) \vee (\text{some } x y)$, as desired.

Case 2: $\{z\} \notin M$. Then $\Gamma \vdash_A (\text{all } z x) \vee (\text{some } x y)$ and $\Gamma \vdash_A (\text{all } z y) \vee (\text{some } x y)$. So we have a proof from Γ :

$$\frac{\frac{\text{EMPTY}_2}{(\text{all } t (r \text{ all } z)) \vee (\text{some } z z)} \quad \begin{array}{c} \vdots \\ (\text{all } z x) \vee (\text{some } x y) \end{array} \quad \begin{array}{c} \vdots \\ (\text{all } z y) \vee (\text{some } x y) \end{array}}{(\text{all } t (r \text{ all } z)) \vee (\text{some } x y)} \text{DD}'$$

As before, all sentences shown belong to A . We are done. Incidentally, the same argument shows that also $\Gamma \vdash_A (\text{all } u (r \text{ all } z)) \vee (\text{some } x y)$. □

We conclude the first part of our proof of Theorem 2.32 with two claims. Together with the assumption that $\Gamma \models \varphi$, they show that $\Gamma \vdash_A \varphi$, where φ is the sentence in the statement of our theorem. In this part of the proof, recall that φ is either $(\text{all } a b) \vee (\text{some } x y)$ or $(\text{some } x y)$.

CLAIM 2.34. *Either $\Gamma \vdash_A \varphi$, or $\mathcal{M} \models \Gamma$.*

Proof. Let ψ be a sentence in Γ . We check that either $\Gamma \vdash_A \varphi$, or $\mathcal{M} \models \psi$.

Case 1: ψ is $(\text{all } c d)$. By (WL), $\Gamma \vdash_A (\text{all } c d) \vee (\text{some } x y)$. For any $\{t, u\} \in \llbracket c \rrbracket$, by the Truth Lemma (without loss of generality) $\Gamma \vdash_A (\text{all } t c) \vee (\text{some } x y)$. By (BARB'), $\Gamma \vdash_A (\text{all } t d) \vee (\text{some } x y)$. So $\{t, u\} \in \llbracket d \rrbracket$ by the Truth Lemma again. So in this case, we have $\mathcal{M} \models \psi$.

Case 2: ψ is $(\text{some } c d)$. There are two subcases, depending on whether or not $\{c, d\}$ belongs to M . If it does, then $\{c, d\} \in \llbracket c \rrbracket \cap \llbracket d \rrbracket$, so $\mathcal{M} \models \psi$. So we assume that $\{c, d\} \notin M$. Then there are $e, f \in \{c, d\}$ such that $\Gamma \vdash_A (\text{all } e x) \vee (\text{some } x y)$ and $\Gamma \vdash_A (\text{all } f y) \vee (\text{some } x y)$. Applying (SOME₁) and (SOME₂) as needed, $\Gamma \vdash_A \text{some } e f$. Then by (DD), $\Gamma \vdash_A \text{some } x y$. If φ is $(\text{some } x y)$, then we are immediately done. And if φ is $(\text{all } a b) \vee (\text{some } x y)$, then we are done after applying (WR).

Case 3: ψ is $(\text{all } s t) \vee (\text{some } c d)$. This is a combination of the two previous arguments.

Again there are two subcases, depending on whether or not $\{c, d\}$ belongs to M . If it does, then $\{c, d\} \in \llbracket c \rrbracket \cap \llbracket d \rrbracket$, so $\mathcal{M} \models \psi$. So we assume that $\{c, d\} \notin M$. Then there are $e, f \in \{c, d\}$ such that $\Gamma \vdash_A (\text{all } e x) \vee (\text{some } x y)$ and $\Gamma \vdash_A (\text{all } f y) \vee (\text{some } x y)$. Applying (SOME'₁) and (SOME'₂) as needed, $\Gamma \vdash_A (\text{all } s t) \vee (\text{some } e f)$. Then by (DD'), $\Gamma \vdash_A (\text{all } s t) \vee (\text{some } x y)$. For any $\{u, v\} \in \llbracket s \rrbracket$, by the Truth Lemma (without loss of generality) $\Gamma \vdash_A (\text{all } u s) \vee (\text{some } x y)$. By (BARB'), $\Gamma \vdash_A (\text{all } u t) \vee (\text{some } x y)$. So $\{u, v\} \in \llbracket t \rrbracket$ by the Truth Lemma again. So we have $\mathcal{M} \models \text{all } s t$, and $\mathcal{M} \models \psi$. □

CLAIM 2.35. *If $\mathcal{M} \models \varphi$, then $\Gamma \vdash_A \varphi$.*

Proof. Case 1: φ is $(\text{some } x y)$. In this case, we claim that $\mathcal{M} \not\models \varphi$. The reason is that by the Truth Lemma and the definition of M ,

$$\{t, u\} \in M \quad \text{iff} \quad \{t, u\} \notin \llbracket x \rrbracket \cap \llbracket y \rrbracket.$$

Case 2: φ is $(\text{all } a b) \vee (\text{some } x y)$. By Case 1, we assume that $\mathcal{M} \models \text{all } a b$. Consider $\{a\}$. If $\{a\} \notin M$, then $\Gamma \vdash_A (\text{all } a x) \vee (\text{some } x y)$ and $\Gamma \vdash_A (\text{all } a y) \vee (\text{some } x y)$. Then we have the following proof of φ from Γ :

$$\frac{\overline{(\text{all } a b) \vee (\text{some } a a)} \quad \text{EMPTY}_1 \quad \begin{array}{c} \vdots \\ (\text{all } a x) \vee (\text{some } x y) \end{array} \quad \begin{array}{c} \vdots \\ (\text{all } a y) \vee (\text{some } x y) \end{array}}{(\text{all } a b) \vee (\text{some } x y)} \text{DD}'$$

All sentences shown belong to A . So we have the desired conclusion $\Gamma \vdash_A \varphi$. On the other hand, if $\{a\} \in M$, then since $\{a\} \in \llbracket a \rrbracket$ and $\mathcal{M} \models \text{all } a b$, we have $\{a\} \in \llbracket b \rrbracket$. By the Truth Lemma, we again have $\Gamma \vdash_A (\text{all } a b) \vee (\text{some } x y)$. \square

This concludes the first part of the proof of Theorem 2.32. The second part is when φ is a sentence $(\text{all } c d)$. In this part, we repeat the construction and proof above, with the following adjustments:

1. We let M be the set of all unordered pairs of terms from T , with no restriction.
2. We drop the disjunct $\vee(\text{some } x y)$ from all sentences which appear in the proof, including the definition of \mathcal{M} and the statement of the Truth Lemma.
3. In the proof of the Truth Lemma, Case 2 does not occur, since $\{z\} \in M$.
4. In the proof of Claim 2.34, the subcases where $\{c, d\} \notin M$ do not occur.
5. We replace the proof of Claim 2.35 with the following argument. Recall that φ is $(\text{all } c d)$. We assume that $\llbracket c \rrbracket \subseteq \llbracket d \rrbracket$, and we need to show that $\Gamma \vdash_A \text{all } c d$. By the Truth Lemma, $\{c\} \in \llbracket c \rrbracket$. Thus, $\{c\} \in \llbracket d \rrbracket$. By the Truth Lemma again, $\Gamma \vdash_A \text{all } c d$.

§3. \mathcal{L}_3 and $\mathcal{L}_{3.5}$: Adding the term former $(r \text{ some } x)$ to \mathcal{L}_1 and \mathcal{L}_2 . In this section, we study \mathcal{L}_3 , the language with term formers $(r \text{ all } x)$ and $(r \text{ some } x)$, and with sentence former $(\text{all } x y)$. We also study the larger language $\mathcal{L}_{3.5}$ which adds the sentence former $(\text{some } x y)$.

The language $\mathcal{L}_{3.5}$ has essentially already been studied by McAllester and Givan in [2], but that paper is primarily concerned with complexity results rather than completeness results. What McAllester and Givan would call a *quantifier-free atomic formula without constants* is exactly what we call a sentence of $\mathcal{L}_{3.5}$. What they would call a *quantifier-free literal without constants* is either an $\mathcal{L}_{3.5}$ sentence φ or its negation $\neg\varphi$. They show that the satisfiability problem for sets Γ of literals which determine existentials is in PTIME. And from this, they derive that the satisfiability problem for sets Γ of literals (which perhaps do not determine existentials) of literals is in NPTIME. Thus, their result implies that the consequence relation for $\mathcal{L}_{3.5}$ is in Co-NPTIME. They prove a matching hardness result as well, and so the consequence relation for $\mathcal{L}_{3.5}$ is Co-NPTIME complete.

We show that the consequence relation for \mathcal{L}_3 is Co-NPTIME hard. Our proof is based on the one in [2], and the result here is a slight improvement on [2] because \mathcal{L}_3 is a little weaker than their language. As a corollary, if $P \neq NP$, then there is no boundedly complete syllogistic proof system for \mathcal{L}_3 or any language larger than it.

In §§3.3 and 3.2 we formulate proof systems and obtain completeness results for \mathcal{L}_3 and $\mathcal{L}_{3.5}$. We also reprove the Co-NPTIME decidability of $\mathcal{L}_{3.5}$ by a polynomial-size countermodel construction.

3.1. CO-NPTIME *hardness of the consequence relation of \mathcal{L}_3 .*

THEOREM 3.1. *The problem of deciding whether $\Gamma \models \varphi$, for $\Gamma \cup \{\varphi\}$ a finite set of \mathcal{L}_3 -sentences, is NPTIME hard.*

Proof. We use a reduction from the *one-in-three positive 3-SAT* problem first studied by Schaefer [11]. This problem is defined as follows. We are given a set \mathcal{S} of clauses of the form $U \vee V \vee W$, where U, V , and W are distinct. (Note that negation is not used.) The problem is to find a truth assignment f to the variables making exactly one variable in each clause T and the other two variables F. We call this a *1-valued* assignment for \mathcal{S} . This problem was shown to be NPTIME complete in Schaefer [11].

We define a set $\Gamma = \Gamma(\mathcal{S})$ below, in two steps. We use nouns which correspond to the variables of \mathcal{S} , writing u for the noun corresponding with U , etc. Γ also uses a number of other nouns and verbs. It is defined as follows:

- (1) For each clause $c \in \mathcal{S}$, say $c = U \vee V \vee W$, put the following sentences in Γ :

$$\begin{array}{ll} \text{all start } (r_c^1 \text{ all } u) & \text{all } (r_c^1 \text{ some } u) y_c \\ \text{all } y_c (r_c^2 \text{ all } v) & \text{all } (r_c^2 \text{ some } v) z_c \\ \text{all } z_c (r_c^3 \text{ all } w) & \text{all } (r_c^3 \text{ some } w) \text{ finish.} \end{array}$$

Here *start* and *finish* are new nouns (not varying with the clause), y_c and z_c are also new nouns (these do vary with c), and r_c^1, r_c^2 , and r_c^3 are new verbs.

- (2) Let P and Q be any two distinct variables which occur together in some clause c . Then add to Γ the sentence $\varphi_{p,q}$:

$$\text{all } (r_{p,q} \text{ all } p) (r'_{p,q} \text{ some } q).$$

Here $r_{p,q}$ and $r'_{p,q}$ are new verbs. (By symmetry, we also add $\varphi_{q,p}$.)

So if \mathcal{S} has k clauses, then the first point will add $2 + 2k$ new nouns and $3k$ new verbs. The second clause will add at most $2 \cdot \binom{3k}{2} < 18k^2$ new verbs.

CLAIM 3.2. *\mathcal{S} has a 1-valued assignment iff $\Gamma \models \text{all start finish}$.* □

Proof. In one direction, assume that $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models \text{all start finish}$. Define a truth assignment f by $f(U) = \text{T}$ iff $\llbracket u \rrbracket \neq \emptyset$. Consider a clause $c = U \vee V \vee W$ of \mathcal{S} . If $f(U) = f(V) = f(W) = \text{T}$, then $\llbracket u \rrbracket, \llbracket v \rrbracket$, and $\llbracket w \rrbracket$ are all nonempty. By the sentences in (1),

$$\llbracket \text{start} \rrbracket \subseteq \llbracket y_c \rrbracket \subseteq \llbracket z_c \rrbracket \subseteq \llbracket \text{finish} \rrbracket.$$

But this contradicts that $\mathcal{M} \not\models \text{all start finish}$. Thus we know that at least one variable in c is assigned the value T by f . We claim that only one variable can be T. For suppose towards a contradiction that (for example) $U \neq V$ but $f(U) = f(V) = \text{T}$. Then $\llbracket u \rrbracket = \llbracket v \rrbracket \neq \emptyset$. So $\llbracket r_{p,q} \text{ all } u \rrbracket = M$ and $\llbracket r'_{p,q} \text{ some } v \rrbracket \neq \emptyset$. By the sentence $\varphi_{p,q}$ in point (2), M is empty. But this is impossible, since $\mathcal{M} \models \text{all start finish}$.

Conversely, suppose f is 1-valued on \mathcal{S} . We must find a model $\mathcal{M} \models \Gamma$ where $\mathcal{M} \models \text{all start finish}$. Let M be the set of variables U such that $f(U) = \text{T}$, together with *start* and *finish*. For a variable X , define $\llbracket x \rrbracket = \emptyset$ if $f(X) = \text{T}$, and $\llbracket x \rrbracket = \{x\}$ if $f(X) = \text{F}$. We also take $\llbracket \text{start} \rrbracket = \{\text{start}\}$, and $\llbracket \text{finish} \rrbracket = \{\text{finish}\}$.

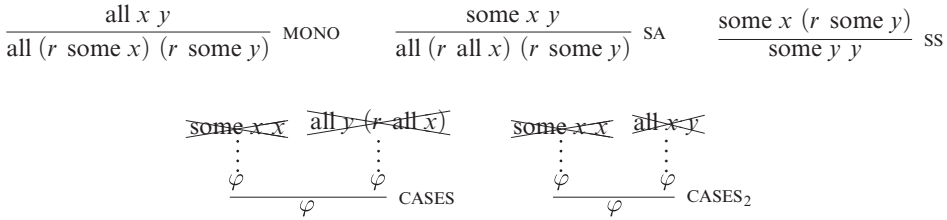


Fig. 4. The logic of §3.2. We also use the rules in Figure 2.

We still need to define $\llbracket y_c \rrbracket$, $\llbracket z_c \rrbracket$, $\llbracket r_c^1 \rrbracket$, $\llbracket r_c^2 \rrbracket$ for all clauses c , and also $\llbracket r_{p,q} \rrbracket$ and $\llbracket r'_{p,q} \rrbracket$ when P and Q are distinct variables in the same clause.

Suppose that P and Q are distinct variables which happen to belong to the same clause. We must arrange that $\mathcal{M} \models \varphi_{p,q}$. Set $\llbracket r_{p,q} \rrbracket = \emptyset$ and $\llbracket r'_{p,q} \rrbracket = M \times M$. We know that either $f(P) = F$ or $f(Q) = F$ (or both). In the first case, $\llbracket r_{p,q} \text{ all } p \rrbracket = \emptyset$, so $\mathcal{M} \models \varphi_{p,q}$. In the second case, $\llbracket r'_{p,q} \text{ some } q \rrbracket = M$ so again $\mathcal{M} \models \varphi_{p,q}$.

Finally, we consider the sentences in (1). There are three cases.

If $f(U) = T$, $f(V) = F$, and $f(W) = F$, then we have $\llbracket u \rrbracket = \emptyset$, $\llbracket v \rrbracket = \{v\}$, and $\llbracket w \rrbracket = \{w\}$. We set $\llbracket y_c \rrbracket = \emptyset$, $\llbracket z_c \rrbracket = \emptyset$, $\llbracket r_c^1 \rrbracket = M \times M$, $\llbracket r_c^2 \rrbracket = \emptyset$, and $\llbracket r_c^3 \rrbracket = \emptyset$.

If $f(U) = F$, $f(V) = T$, and $f(W) = F$, we have $\llbracket u \rrbracket = \{u\}$, $\llbracket v \rrbracket = \emptyset$, and $\llbracket w \rrbracket = \{w\}$. We set $\llbracket y_c \rrbracket = M$, $\llbracket z_c \rrbracket = \emptyset$, $\llbracket r_c^1 \rrbracket = M \times M$, $\llbracket r_c^2 \rrbracket = \emptyset$, and $\llbracket r_c^3 \rrbracket = \emptyset$.

If $f(U) = F$, $f(V) = F$, and $f(W) = T$, we have $\llbracket u \rrbracket = \{u\}$, $\llbracket v \rrbracket = \{v\}$, and $\llbracket w \rrbracket = \emptyset$. We set $\llbracket y_c \rrbracket = M$, $\llbracket z_c \rrbracket = M$, $\llbracket r_c^1 \rrbracket = M \times M$, $\llbracket r_c^2 \rrbracket = M \times M$, and $\llbracket r_c^3 \rrbracket = \emptyset$.

In all cases, the resulting model \mathcal{M} satisfies all sentences in (1), hence all sentences in Γ . And in all cases, $\mathcal{M} \not\models$ all start finish. □

The claim concludes the proof Theorem 3.1.

3.2. Completeness and Co-NP_{TIME} decidability for $\mathcal{L}_{3,5}$. We first present a sound and complete proof system for $\mathcal{L}_{3,5}$, because it is actually a bit simpler than \mathcal{L}_3 , and mirrors more closely our work from §2. The rules are in Figure 4. We return to \mathcal{L}_3 in §3.3 below.

Since the consequence relations for \mathcal{L}_3 and $\mathcal{L}_{3,5}$ are Co-NP_{TIME} hard, by Theorem 1.5 we cannot hope for a boundedly complete syllogistic proof system for these languages (unless P = NP). We regard it as unlikely that they admit any sound and complete syllogistic proof system. Instead, we settle for a proof system with (CASES) from §2.3, as well as a variant, (CASES₂). Figure 4 gives proof rules for this logic.

In this section, we write \vdash for provability in the system with rules (AX), (BARBARA), (ANTI), (SOME₁), (SOME₂), (DARII), (MONO), (SA), (SS), (CASES), and (CASES₂). Given a theory Γ , we write $x \leq y$ when $\Gamma \vdash$ all $x y$. We say that Γ determines existentials for a set of terms \mathbb{T} if for every $x \in \mathbb{T}$, either $\Gamma \vdash$ some $x x$, or else $\Gamma \vdash$ all $x y$ and $\Gamma \vdash$ all $y (r \text{ all } x)$ for all terms $y \in \mathbb{T}$ and all verbs r .

The canonical model. Let Γ be a theory, and let \mathbb{T} be a set of terms, closed under subterms as usual. Define

$$M = \{ \langle x, y, Q \rangle \in \mathbb{T} \times \mathbb{T} \times \{ \forall, \exists \} : \Gamma \vdash \text{some } x y \}.$$

We define a model $\mathcal{M}(\Gamma, \mathbb{T})$ with domain M by setting

$$\begin{aligned} \langle x, y, Q \rangle \in \llbracket p \rrbracket & \text{ iff } x \leq p \text{ or } y \leq p \\ \langle x_1, y_1, Q_1 \rrbracket \llbracket r \rrbracket \langle x_2, y_2, Q_2 \rangle & \text{ iff for some } z_1 \in \{x_1, y_1\} \text{ and } z_2 \in \{x_2, y_2\}, \text{ either} \\ & \text{(a) } z_1 \leq (r \text{ all } z_2), \text{ or} \\ & \text{(b) } Q_2 = \exists, x_2 = y_2, \text{ and } z_1 \leq (r \text{ some } z_2). \end{aligned}$$

LEMMA 3.3 (Truth Lemma). *If Γ determines existentials for \mathbb{T} , then in $\mathcal{M}'(\Gamma, \mathbb{T})$, for all $t \in \mathbb{T}$,*

$$\llbracket t \rrbracket = \{ \langle x, y, Q \rangle \in M : x \leq p \text{ or } y \leq p \}. \tag{8}$$

Proof. The proof is by induction on t . For a noun p , this is by definition of the model.

The induction step for $(r \text{ all } t)$. Since \mathbb{T} is closed under subterms, $t \in \mathbb{T}$. Take some $\langle c_1, c_2, Q \rangle$ such that $c_i \leq (r \text{ all } t)$ for some $i \in \{1, 2\}$. We show that $\langle c_1, c_2, Q \rangle \in \llbracket r \text{ all } t \rrbracket$. For this, suppose $\langle d_1, d_2, Q' \rangle \in \llbracket t \rrbracket$. By induction hypothesis, $d_j \leq t$ for some $j \in \{1, 2\}$. Using (ANTI), $(r \text{ all } t) \leq (r \text{ all } d_j)$, so $c_i \leq (r \text{ all } d_j)$. Thus, $\langle c_1, c_2, Q \rrbracket \llbracket r \rrbracket \langle d_1, d_2, Q' \rangle$. Since $\langle d_1, d_2, Q' \rangle$ was arbitrary, we have shown that $\langle c_1, c_2, Q \rangle \in \llbracket r \text{ all } t \rrbracket$, as desired.

In the other direction, suppose that $\langle c_1, c_2, Q \rangle \in \llbracket r \text{ all } t \rrbracket$. If $\Gamma \vdash \text{some } t t$, then $\langle t, t, \forall \rangle \in M$. By induction, since $c_i \leq t$, $\langle t, t, \forall \rangle \in \llbracket t \rrbracket$, so $\langle c_1, c_2, Q \rrbracket \llbracket r \rrbracket \langle t, t, \forall \rangle$. Since the last component in $\langle t, t, \forall \rangle$ is \forall rather than \exists , case (a) in the definition of $\llbracket r \rrbracket$ holds, and $c_i \leq (r \text{ all } t)$ for some $i \in \{1, 2\}$.

It remains to consider the case when $\Gamma \not\vdash \text{some } t t$. But since Γ determines existentials, $\Gamma \vdash \text{all } y (r \text{ all } t)$ for all terms y , and in particular $c_1 \leq (r \text{ all } t)$.

The induction step for $(r \text{ some } t)$. Again, since \mathbb{T} is closed under subterms, $t \in \mathbb{T}$. Take some $\langle c_1, c_2, Q \rangle$ such that $c_i \leq (r \text{ some } t)$. The fact that $\langle c_1, c_2, Q \rangle \in M$ implies that $\Gamma \vdash \text{some } c_1 c_2$. By (SOME₂) and (SOME₁), $\Gamma \vdash \text{some } c_i c_i$, and by (DARII), $\Gamma \vdash \text{some } c_i (r \text{ some } t)$. But then by (SS), $\Gamma \vdash \text{some } t t$, and hence $\langle t, t, \exists \rangle \in M$. By case (b) in the definition of $\llbracket r \rrbracket$, $\langle c_1, c_2, Q \rrbracket \llbracket r \rrbracket \langle t, t, \exists \rangle$. By induction $\langle t, t, \exists \rangle \in \llbracket t \rrbracket$, so $\langle c_1, c_2, Q \rangle \in \llbracket r \text{ some } t \rrbracket$.

In the other direction, suppose that $\langle c_1, c_2, Q \rangle \in \llbracket r \text{ some } t \rrbracket$. Then we have $\langle c_1, c_2, Q \rrbracket \llbracket r \rrbracket \langle d_1, d_2, Q' \rangle$ for some $\langle d_1, d_2, Q' \rangle \in \llbracket t \rrbracket$. Since $\langle d_1, d_2, Q' \rangle \in M$, $\Gamma \vdash \text{some } d_1 d_2$. And also $d_k \leq t$ for some $k \in \{1, 2\}$, by induction.

We first consider case (a) in the definition of $\llbracket r \rrbracket$: there are i and j so that $c_i \leq (r \text{ all } d_j)$. From $\Gamma \vdash \text{some } d_1 d_2$ and $d_k \leq t$, using (DARII), (SOME₁), and (SOME₂), $\Gamma \vdash \text{some } d_j t$. By (SA), $(r \text{ all } d_j) \leq (r \text{ some } t)$, so by (BARBARA), $c_i \leq (r \text{ some } t)$.

In case (b), $Q' = \exists$, $d_1 = d_2$, and for some i , $c_i \leq (r \text{ some } d_1)$. By (MONO), $(r \text{ some } d_1) \leq (r \text{ some } t)$, so $c_i \leq (r \text{ some } t)$. □

LEMMA 3.4. *Suppose that \mathbb{T} contains all subterms of sentences in Γ and $\Gamma^* \supseteq \Gamma$ is a theory which determines existentials for \mathbb{T} . Then $\mathcal{M}(\Gamma^*, \mathbb{T}) \models \Gamma$.*

Proof. For a sentence $(\text{all } x y) \in \Gamma$, we have $\Gamma^* \vdash \text{all } x y$. We are going to apply Lemma 3.3 to $\mathcal{M}(\Gamma^*, \mathbb{T})$ (that is, the \leq symbol here is for provability in Γ^*). If $\langle c_1, c_2, Q \rangle \in \llbracket x \rrbracket$ in $\mathcal{M}(\Gamma^*, \mathbb{T})$, then $c_i \leq x$ for some $i \in \{1, 2\}$. But then also $c_i \leq y$, so $\langle c_1, c_2, Q \rangle \in \llbracket y \rrbracket$.

For a sentence $(\text{some } x y) \in \Gamma$, we have $\Gamma^* \vdash \text{some } x y$, so $\langle x, y, \forall \rangle \in M$. And $\langle x, y, \forall \rangle \in \llbracket x \rrbracket \cap \llbracket y \rrbracket$ in $\mathcal{M}(\Gamma^*, \mathbb{T})$. □

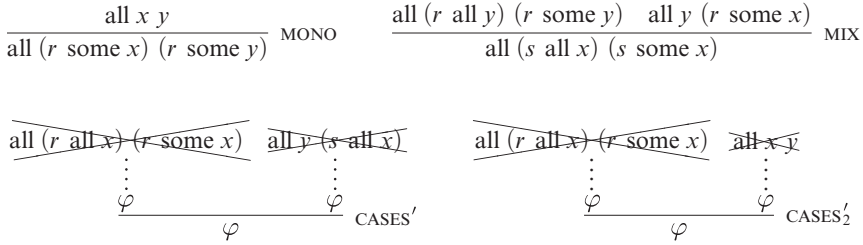


Fig. 5. The logic of §3.3. We also use (AX), (BARBARA), (ANTI) from Figure 2.

THEOREM 3.5. $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.

Proof. The soundness is easy, with soundness of (CASES₂) following just as in the proof of Lemma 2.16.

The argument for completeness is the same as we saw in the proof of Theorem 2.19. Let \mathbb{T} be the set of all subterms of sentences in $\Gamma \cup \{\varphi\}$. We assume that $\Gamma \not\vdash \varphi$ and show that $\Gamma \not\models \varphi$. We may pass to a maximal extension $\Gamma^* \supseteq \Gamma$ with the property that $\Gamma^* \not\vdash \varphi$. It follows from (CASES) and (CASES₂) that Γ^* determines existentials for \mathbb{T} , just as in the proof of Theorem 2.19. By Lemma 3.4, $\mathcal{M}(\Gamma^*, \mathbb{T}) \models \Gamma$. We claim that φ is false in this model.

Case 1: φ is (all $x \ y$). Since Γ^* determines existentials for \mathbb{T} and $\Gamma^* \not\vdash \text{all } x \ y$, we have $\Gamma^* \vdash \text{some } x \ x$. So $\langle x, x, \forall \rangle \in M$. By Lemma 3.4, $\langle x, x, \forall \rangle \in \llbracket x \rrbracket \setminus \llbracket y \rrbracket$, since $x \leq x$ but $x \not\leq y$.

Case 2: φ is (some $x \ y$). Suppose towards a contradiction that $\mathcal{M}(\Gamma^*, \mathbb{T}) \models \varphi$. Specifically, let $\langle d_1, d_2, Q \rangle \in \llbracket x \rrbracket \cap \llbracket y \rrbracket$. Then $\Gamma^* \vdash \text{some } d_1 \ d_2$, and also there are i and j such that $d_i \leq x$ and $d_j \leq y$. Using (SOME₁), (SOME₂), and (DARII), $\Gamma^* \vdash \text{some } x \ y$. □

The proof shows that if $\Gamma \not\vdash \varphi$, then there is a countermodel of size $O(n^2)$, where n is the complexity of $\Gamma \cup \{\varphi\}$. Since $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models \varphi$ can be checked in time polynomial in the size of \mathcal{M} and the complexity of $\Gamma \cup \{\varphi\}$, this shows that the consequence relation for $\mathcal{L}_{3,5}$ is in CO-NPTIME.

3.3. Completeness for \mathcal{L}_3 . In \mathcal{L}_3 , we do not have the sentence former (some $x \ y$), which was used in the previous section to formulate the condition that Γ determines existentials. Nevertheless, we are able to follow the same strategy as for $\mathcal{L}_{3,5}$ to prove a completeness theorem. The key observation is that for a term x and a nonempty model \mathcal{M} , $\llbracket x \rrbracket \neq \emptyset$ in \mathcal{M} if and only if $\mathcal{M} \models \text{all } (r \text{ all } x) \ (r \text{ some } x)$ for every verb r . We use the collection of all sentences of this form as a replacement for (some $x \ x$).

Figure 5 gives our proof system for this logic. The soundness of (MONO) is immediate. For (MIX), note that if $\mathcal{N} \models \text{all } (r \text{ all } y) \ (r \text{ some } y)$, then either $N = \emptyset$, or $\llbracket y \rrbracket \neq \emptyset$. If $N = \emptyset$, the conclusion of the rule holds. And if $\llbracket y \rrbracket \neq \emptyset$ and $\mathcal{N} \models \text{all } y \ (r \text{ some } x)$, then $\llbracket x \rrbracket \neq \emptyset$ also. Again, the conclusion of the rule follows. The soundness of the (CASES) variants follow as in Lemma 2.16, using the above observation about sentences of the form $\text{all } (r \text{ all } y) \ (r \text{ some } y)$.

Note that in (MIX), the verb s appearing in the conclusion may be different than the verb r appearing in the premises. And in CASES', the withdrawn premises are (all $(r \text{ all } x) \ (r \text{ some } x)$) and (all $y \ (s \text{ all } x)$), where the verbs r and s may be different.

As usual, we write \vdash for provability in this system, and given a theory Γ , we write $x \leq y$ when $\Gamma \vdash \text{all } x \ y$.

DEFINITION 3.6. Let Γ be a theory and \mathbb{T} a set of terms. A term $x \in \mathbb{T}$ is *effectively nonempty* (for Γ) if for all verbs r , $\Gamma \vdash \text{all } (r \ \text{all } x) \ (r \ \text{some } x)$. And x is *effectively empty* (for Γ) if for all $y \in \mathbb{T}$, $\Gamma \vdash \text{all } x \ y$ and $\Gamma \vdash \text{all } y \ (r \ \text{all } x)$ for all verbs r .

Γ *effectively determines existentials* for \mathbb{T} if every $x \in \mathbb{T}$ is either effectively empty or effectively nonempty for Γ .

The canonical model. Let Γ be a theory, and let \mathbb{T} be a set of terms. Define

$$M = \{ \langle x, Q \rangle \in \mathbb{T} \times \{ \forall, \exists \} : x \text{ is effectively nonempty for } \Gamma \}.$$

We define a model $\mathcal{M}(\Gamma, \mathbb{T})$ with domain M by setting

$$\begin{aligned} \langle x, Q \rangle \in \llbracket p \rrbracket & \text{ iff } x \leq p \\ \langle x, Q \rrbracket \llbracket r \rrbracket \langle y, Q' \rangle & \text{ iff either (a) } x \leq (r \ \text{all } y), \\ & \text{ or (b) } Q' = \exists, \text{ and } x \leq (r \ \text{some } y). \end{aligned}$$

LEMMA 3.7 (Truth Lemma). *Assume that Γ effectively determines existentials for \mathbb{T} . Then for all $x \in \mathbb{T}$,*

$$\llbracket x \rrbracket = \{ \langle y, Q \rangle \in M : y \leq x \}.$$

Proof. By induction on x . When x is a noun, this follows immediately from the definition.

Here is the induction step for $(r \ \text{all } x)$. Suppose that $\langle y, Q \rangle \in \llbracket r \ \text{all } x \rrbracket$. If x is effectively empty, then $y \leq (r \ \text{all } x)$, and so we are done. If x is effectively nonempty, then $\langle x, \forall \rangle \in M$. Since $x \leq x$, by induction $\langle x, \forall \rangle \in \llbracket x \rrbracket$, and $\langle y, Q \rrbracket \llbracket r \rrbracket \langle x, \forall \rangle$. Then case (a) holds and again $y \leq (r \ \text{all } x)$.

In the other direction, suppose that $y \leq (r \ \text{all } x)$. We show that for any $\langle y, Q \rangle \in M$, $\langle y, Q \rrbracket \llbracket r \rrbracket \langle z, Q' \rangle$ for all $\langle z, Q' \rangle \in \llbracket x \rrbracket$. By induction, $z \leq x$. Using (ANTI), $(r \ \text{all } x) \leq (r \ \text{all } z)$. Thus, $y \leq (r \ \text{all } z)$, and so indeed $\langle y, Q \rrbracket \llbracket r \rrbracket \langle z, Q' \rangle$.

Finally, we have the induction step for $(r \ \text{some } x)$. Suppose that $\langle y, Q \rangle \in \llbracket r \ \text{some } x \rrbracket$. Then there is some $\langle z, Q' \rangle \in \llbracket x \rrbracket$ such that $\langle y, Q \rrbracket \llbracket r \rrbracket \langle z, Q' \rangle$. So z is effectively nonempty, and by induction $z \leq x$.

Case 1: $Q' = \exists$. Then $y \leq (r \ \text{some } z)$. By (MONO), $(r \ \text{some } z) \leq (r \ \text{some } x)$, so also $y \leq (r \ \text{some } x)$, as was to be shown.

Case 2: $Q' = \forall$. Then $y \leq (r \ \text{all } x)$. By (ANTI), $(r \ \text{all } x) \leq (r \ \text{all } z)$, so $y \leq (r \ \text{all } z)$. The fact that z is effectively nonempty means that $(r \ \text{all } z) \leq (r \ \text{some } z)$. And by (MONO), as observed above, $(r \ \text{some } z) \leq (r \ \text{some } x)$. So again we have $y \leq (r \ \text{some } x)$.

In the other direction, let $y \in \mathbb{T}$ be such that $y \leq (r \ \text{some } x)$. Suppose that $\langle y, Q \rangle \in M$. Then y is effectively nonempty. In particular, $(r \ \text{all } y) \leq (r \ \text{some } y)$, and by (MX), for every verb s , $(s \ \text{all } x) \leq (s \ \text{some } x)$, so x is effectively nonempty. Then $\langle x, \exists \rangle \in M$. By case (b), $\langle y, Q \rrbracket \llbracket r \rrbracket \langle x, \exists \rangle$, and by induction $\langle x, \exists \rangle \in \llbracket x \rrbracket$, so $\langle y, Q \rangle \in \llbracket r \ \text{some } x \rrbracket$. \square

LEMMA 3.8. *Suppose \mathbb{T} contains all subterms of sentences in Γ and $\Gamma^* \supseteq \Gamma$ is a theory which effectively determines existentials for \mathbb{T} . Then $\mathcal{M}(\Gamma^*, \mathbb{T}) \models \Gamma$.*

Proof. For a sentence $(\text{all } x \ y) \in \Gamma$, we have $\Gamma^* \vdash \text{all } x \ y$, so $x \leq y$. If $\langle z, Q \rangle \in \llbracket x \rrbracket$ in $\mathcal{M}(\Gamma^*, \mathbb{T})$, then $z \leq x$. But then also $z \leq y$, so $\langle z, Q \rangle \in \llbracket y \rrbracket$. \square

THEOREM 3.9. $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.

Proof. We discussed the soundness when we introduced the rules.

For completeness, let \mathbb{T} be the set of all subterms of sentences in $\Gamma \cup \{\varphi\}$. We assume that $\Gamma \not\vdash \varphi$ and show that $\Gamma \not\models \varphi$. We may pass to a maximal extension $\Gamma^* \supseteq \Gamma$ with the property that $\Gamma^* \not\vdash \varphi$. It follows from (CASES') and (CASES'_2) that Γ^* effectively determines existentials for \mathbb{T} , just as in the proof of Theorem 2.19. By Lemma 3.8, $\mathcal{M}(\Gamma^*, \mathbb{T}) \models \Gamma$.

We claim that φ is false in this model. Write φ as $(\text{all } x \ y)$. Since Γ^* effectively determines existentials for \mathbb{T} and $\Gamma^* \not\vdash \text{all } x \ y$, x is effectively nonempty for Γ^* . So $\langle x, \forall \rangle \in M$. By Lemma 3.7, $\langle x, \forall \rangle \in \llbracket x \rrbracket \setminus \llbracket y \rrbracket$, since $x \leq x$ but $x \not\leq y$. \square

The proof shows that if $\Gamma \not\vdash \varphi$, then there is a countermodel of size $O(n)$, where n is the complexity of $\Gamma \cup \{\varphi\}$.

§4. \mathcal{L}_4 and $\mathcal{L}_{4.5}$: Adding term complementation to \mathcal{L}_1 and \mathcal{L}_2 . This section extends \mathcal{L}_1 and \mathcal{L}_2 by adding term complements. That is, we extend the syntax of \mathcal{L} so that whenever t is a term, \bar{t} is a term, and we extend the semantics so that in a model \mathcal{M} , $\llbracket \bar{t} \rrbracket = M \setminus \llbracket t \rrbracket$. If we add term complements to \mathcal{L}_1 , we get \mathcal{L}_4 , and if we add term complements to \mathcal{L}_2 , we get $\mathcal{L}_{4.5}$.

4.1. CO-NPTIME hardness of the consequence relation of \mathcal{L}_4 . We begin with a negative complexity-theoretic result, reducing 3-SAT to the relation $\Gamma \not\models \varphi$ in \mathcal{L}_4 .

Let $BV = \{P_i : i \in N\}$ be a set of boolean variables. Suppose we have an instance of 3-SAT, $c_1 \wedge \dots \wedge c_k$, where each clause c_i has the form $U_i \vee V_i \vee W_i$, where U_i, V_i and W_i are literals: variables in BV or their negations.

Then we consider the language with nouns

$$\{p : P \in BV\} \cup \{q\} \cup \{y_i, z_i : 1 \leq i \leq k\}$$

and verbs $\{r_i : 1 \leq i \leq k\}$. Notice that we write p for the noun corresponding to the variable P . We will also write u for the literal U , where if U is a variable P , then u is the noun p , and if U is a negated variable $\neg P$, then u is the term \bar{p} .

For each clause $c_i = U_i \vee V_i \vee W_i$, we define the following sentences:

$$\begin{aligned} \psi_1^i &= \text{all } \bar{u}_i \ (r_i \ \text{all } y_i) \\ \psi_2^i &= \text{all } \bar{v}_i \ (r_i \ \text{all } \bar{y}_i) \\ \psi_3^i &= \text{all } (r_i \ \text{all } z_i) \ w_i. \end{aligned}$$

And we define

$$\begin{aligned} \Gamma &= \{\psi_1^i, \psi_2^i, \psi_3^i : 1 \leq i \leq k\} \\ \varphi &= \text{all } q \ \bar{q}. \end{aligned}$$

LEMMA 4.1. $\Gamma \not\models \varphi$ if and only if Γ has a nonempty model.

Proof. Suppose Γ has a model \mathcal{M} with nonempty domain M . Let \mathcal{M}' be a model with domain M and the same interpretations of all of the nouns and verbs, except for q , which we interpret as all of M . Since the sentences in Γ do not mention q , we still have $\mathcal{M}' \models \Gamma$. And for any $x \in M$, we have $x \in \llbracket q \rrbracket = M$ and $x \notin \llbracket \bar{q} \rrbracket = \emptyset$, so $\Gamma \not\models \varphi$.

Conversely, suppose every model of Γ is empty. Then in any model \mathcal{M} of Γ , we have $\llbracket q \rrbracket = \emptyset \subseteq \llbracket \bar{q} \rrbracket = \emptyset$, so $\Gamma \models \varphi$. □

THEOREM 4.2. $\Gamma \not\models \varphi$ if and only if $c_1 \wedge \dots \wedge c_k$ is satisfiable.

Proof. Suppose $\Gamma \not\models \varphi$. By Lemma 4.1, Γ has a model \mathcal{M} with nonempty domain M . Let $x \in M$. Define a truth assignment f for the proposition letters, by

$$f(P) = \begin{cases} \text{T} & \text{if } x \in \llbracket P \rrbracket \\ \text{F} & \text{if } x \notin \llbracket P \rrbracket. \end{cases}$$

The truth assignment extends to literals in the natural way: $f(\neg P) = \text{T}$ if $f(P) = \text{F}$, and $f(\neg P) = \text{F}$ if $f(P) = \text{T}$. Note that if U is a literal with corresponding term u , then we have $x \in \llbracket u \rrbracket$ if and only if $f(U) = \text{T}$.

We check that each clause $c_i = U_i \vee V_i \vee W_i$ is satisfied. If $f(U_i) = \text{T}$ or $f(V_i) = \text{T}$, then c_i is satisfied. So suppose that $f(U_i) = f(V_i) = \text{F}$. Then, since $\llbracket z_i \rrbracket \subseteq \llbracket y_i \rrbracket \cup \llbracket \bar{y}_i \rrbracket$,

$$x \in \llbracket \bar{u}_i \rrbracket \cap \llbracket \bar{v}_i \rrbracket \subseteq \llbracket r_i \text{ all } y_i \rrbracket \cap \llbracket r_i \text{ all } \bar{y}_i \rrbracket \subseteq \llbracket r_i \text{ all } z_i \rrbracket \subseteq \llbracket w_i \rrbracket.$$

So $x \in \llbracket w_i \rrbracket$, and $f(w_i) = \text{T}$, and c_i is satisfied.

Conversely, suppose f is a truth assignment such that $c_1 \wedge \dots \wedge c_k$ is satisfied. By Lemma 4.1, it suffices to build a nonempty model of Γ . Let $\mathcal{M} = \{x\}$, and for each proposition letter P , set

$$\llbracket P \rrbracket = \begin{cases} \{x\} & \text{if } f(P) = \text{T} \\ \emptyset & \text{if } f(P) = \text{F}. \end{cases}$$

Note that again, if U is a literal with corresponding term u , we have $x \in \llbracket u \rrbracket$ if and only if $f(U) = \text{T}$.

Consider a clause $c_i = U_i \vee V_i \vee W_i$. We must define the interpretations of y_i, z_i , and r_i , so that ψ_1^i, ψ_2^i , and ψ_3^i are satisfied for $1 \leq i \leq k$.

Case 1: $x \in \llbracket w_i \rrbracket$. Set $\llbracket r_i \rrbracket = \{(x, x)\}$. The interpretations of y_i and z_i are irrelevant. Indeed, ψ_1^i, ψ_2^i , and ψ_3^i are satisfied, since $\llbracket r_i \text{ all } y_i \rrbracket = \llbracket r_i \text{ all } \bar{y}_i \rrbracket = \llbracket w_i \rrbracket = M$.

Case 2: $x \notin \llbracket w_i \rrbracket$. Set $\llbracket z_i \rrbracket = \{x\}$ and $\llbracket r_i \rrbracket = \emptyset$. Then ψ_3^i is satisfied, since $\llbracket r_i \text{ all } z_i \rrbracket = \llbracket w_i \rrbracket = \emptyset$. Since the clause $c_i = U_i \vee V_i \vee W_i$ is satisfied, x must be in at least one of $\llbracket u_i \rrbracket$ or $\llbracket v_i \rrbracket$. If it's in both, we're done (and the interpretation of y_i is irrelevant), since $\llbracket \bar{u}_i \rrbracket = \llbracket \bar{v}_i \rrbracket = \emptyset$. Otherwise, set

$$\llbracket y_i \rrbracket = \begin{cases} \emptyset & \text{if } x \notin \llbracket u_i \rrbracket, x \in \llbracket v_i \rrbracket \\ \{x\} & \text{if } x \in \llbracket u_i \rrbracket, x \notin \llbracket v_i \rrbracket. \end{cases}$$

In the first case, $\mathcal{M} \models \psi_1^i$, since $\llbracket r_i \text{ all } y_i \rrbracket = M$, and $\mathcal{M} \models \psi_2^i$, since $\llbracket \bar{v}_i \rrbracket = \emptyset$. In the second case, $\mathcal{M} \models \psi_2^i$, since $\llbracket \bar{u}_i \rrbracket = \emptyset$, and $\mathcal{M} \models \psi_3^i$, since $\llbracket r_i \text{ all } \bar{y}_i \rrbracket = M$. □

4.2. Completeness for the extended languages \mathcal{L}_4^+ and $\mathcal{L}_{4.5}^+$. We enlarge our syntax of $\mathcal{L}_{4.5}$ from sentences (all $x y$) and (some $x y$) to expressions of the form $[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_n \rangle$ for $n \geq 1$. Note that this is a departure from the languages we have studied previously, since here we have infinitely many sentence formers, of arbitrary finite length. We emphasize that $[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_n \rangle$ are sentences, not terms. The terms of $\mathcal{L}_{4.5}^+$ are the same as the terms of $\mathcal{L}_{4.5}$.

$$\begin{array}{c}
 \frac{}{[x, \bar{x}]} \text{AX} \qquad \frac{[x, y_1, \dots, y_n] \quad [\bar{x}, z_1, \dots, z_m]}{[y_1, \dots, y_n, z_1, \dots, z_m]} \text{RES} \qquad \frac{[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, x_n]}{[r \text{ all } x_1, r \text{ all } x_2, \dots, r \text{ all } x_{n-1}, r \text{ all } x_n]} \text{REL} \\
 \\
 \frac{[x_1, \dots, x_m]}{[y_1, \dots, y_n]} \text{STR} \qquad \frac{\langle y_1, \dots, y_m \rangle \quad [y_1, \dots, y_m]}{[x_1, \dots, x_n]} \text{EFQ} \qquad \frac{\begin{array}{c} \langle x_1, \dots, x_n \rangle \\ \vdots \\ \langle x_1, \dots, x_n \rangle \end{array} \quad \begin{array}{c} \langle x_1, \dots, x_n \rangle \\ \vdots \\ \langle x_1, \dots, x_n \rangle \end{array}}{\langle y_1, \dots, y_m \rangle \quad [y_1, \dots, y_m]} \text{RAA}
 \end{array}$$

Fig. 6. Rules of the logical system for $\mathcal{L}_{4,5}^+$. The side condition on the structural rule (STR) is that each x_i must be identical to some y_j .

The semantics of this new language $\mathcal{L}_{4,5}^+$ is:

$$\begin{array}{l}
 \mathcal{M} \models [x_1, \dots, x_n] \quad \text{iff} \quad \bigcap_{i=1}^n \llbracket x_i \rrbracket = \emptyset \\
 \mathcal{M} \models \langle x_1, \dots, x_n \rangle \quad \text{iff} \quad \bigcap_{i=1}^n \llbracket x_i \rrbracket \neq \emptyset.
 \end{array}$$

It is clear that $\langle x_1, \dots, x_n \rangle$ generalizes (some $x y$), since the latter sentence can be translated into $\mathcal{L}_{4,5}^+$ as $\langle x, y \rangle$. To see that $[x_1, \dots, x_n]$ generalizes (all $x y$), note that $\bigcap_{i=1}^n \llbracket x_i \rrbracket = \emptyset$ if and only if for some j (equivalently, for all j), $\bigcap_{i \neq j} \llbracket x_i \rrbracket \subseteq \llbracket \bar{x}_j \rrbracket$. So the sentence (all $x y$) can be translated into $\mathcal{L}_{4,5}^+$ as $[x, \bar{y}]$.

Our proof rules are shown in Figure 6. (AX) is a version of the axiom rule as we have seen it throughout the paper. (RES) is named for *resolution*. (But please note that the sentence $[x_1, \dots, x_n]$ is not interpreted disjunctively as in resolution.) The name (REL) stands for *relational*, since it is the only rule of the system that mentions relations. We name (STR) after *structural rules* of sequent calculi. The side condition on this rule is that each x_i appears in the list y_1, \dots, y_n . It implies the usual rules of weakening, contraction, and exchange. (EFQ) and (RAA) are our formulations of *ex falso quodlibet* and *reductio ad absurdum*. In this system, $[y_1, \dots, y_n]$ and $\langle y_1, \dots, y_n \rangle$ are contradictories. Given two derivations with contradictory conclusions, one may use (EFQ) to put these two together and conclude any sentence of the form $[x_1, \dots, x_n]$. Alternatively, one may use (RAA) to withdraw all occurrences of any one assumption of the form $[x_1, \dots, x_n]$, and conclude the contradictory of that assumption, $\langle x_1, \dots, x_n \rangle$. Note the asymmetry between $[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_n \rangle$; this is arranged so as to allow an easy proof-theoretic argument (Corollary 4.12) that (AX), (RES), (REL), and (STR) give a sound and complete proof system for the smaller language \mathcal{L}_4^+ , which only has sentences of the form $[x_1, \dots, x_n]$.

EXAMPLE 4.3. Here is how the translations of (BARBARA) and (SOME₁) are derived in this system:

$$\frac{\frac{[x, \bar{y}]}{[\bar{y}, x]} \text{STR} \quad [y, \bar{z}]}{[x, \bar{z}]} \text{RES} \qquad \frac{\langle x, y \rangle \quad \frac{\langle x, y \rangle \quad [x, y]}{[x, y]} \text{STR}}{\langle x, x \rangle} \text{RAA}$$

And here are (DARII) and (ANTI):

$$\frac{\frac{\frac{\cancel{[x, \bar{x}]}}{[z, x]} \text{ STR} \quad \frac{[y, \bar{z}]}{[\bar{z}, y]} \text{ STR}}{[x, y]} \text{ RES}}{\langle x, z \rangle} \text{ RAA} \qquad \frac{[x, \bar{y}]}{[\bar{y}, x]} \text{ STR}}{[r \text{ all } y, \bar{r} \text{ all } x]} \text{ REL}$$

LEMMA 4.4 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

Proof. The soundness of (AX) and (STR) are clear. For (RES), (REL), and (EFQ), fix a model \mathcal{M} .

For (RES), assume that $\llbracket x \rrbracket \cap (\bigcap_{i \leq n} \llbracket y_i \rrbracket) = \emptyset$ and $\llbracket \bar{x} \rrbracket \cap (\bigcap_{j \leq m} \llbracket z_j \rrbracket) = \emptyset$. Suppose there is some $\ell \in (\bigcap_{i \leq n} \llbracket y_i \rrbracket) \cap (\bigcap_{j \leq m} \llbracket z_j \rrbracket)$. Then either $\ell \in \llbracket x \rrbracket$ or $\ell \in \llbracket \bar{x} \rrbracket$, which is a contradiction in either case.

For (REL), assume that $(\bigcap_{i < n} \llbracket \bar{x}_i \rrbracket) \cap \llbracket x_n \rrbracket = \emptyset$, so $\bigcap_{i < n} \llbracket \bar{x}_i \rrbracket \subseteq \llbracket \bar{x}_n \rrbracket$. Suppose there is some $\ell \in (\bigcap_{i < n} \llbracket r \text{ all } x_i \rrbracket) \cap \llbracket r \text{ all } x_n \rrbracket$. Then there is some $m \in \llbracket x_n \rrbracket$ such that it is not true that $\ell \llbracket r \rrbracket m$. For all $i < n$, since $\ell \in \llbracket r \text{ all } x_i \rrbracket$, $m \in \llbracket \bar{x}_i \rrbracket$. But then $m \in \llbracket \bar{x}_n \rrbracket$, and this is a contradiction.

For (EFQ), it is vacuously true that if $\mathcal{M} \models \langle y_1, \dots, y_m \rangle$ and $\mathcal{M} \models [y_1, \dots, y_m]$, then $\mathcal{M} \models [x_1, \dots, x_n]$, since $\langle y_1, \dots, y_m \rangle$ and $[y_1, \dots, y_m]$ are contradictory.

The soundness of (RAA) is by induction on the number of instances of (RAA) in the proof tree. Assume that the last use of (RAA) is at the root of the proof tree showing $\Gamma \vdash \langle x_1, \dots, x_n \rangle$. Then we have $\Gamma \cup \{[x_1, \dots, x_n]\} \vdash \langle y_1, \dots, y_m \rangle$ and $\Gamma \cup \{[x_1, \dots, x_n]\} \vdash [y_1, \dots, y_m]$. By induction, these deductions are sound, so every model of $\Gamma \cup \{[x_1, \dots, x_n]\}$ satisfies both $\langle y_1, \dots, y_m \rangle$ and $[y_1, \dots, y_m]$. But these sentences are contradictory, so $\Gamma \cup \{[x_1, \dots, x_n]\}$ has no models. In other words, every model of Γ satisfies $\langle x_1, \dots, x_n \rangle$, as was to be shown. \square

DEFINITION 4.5. Let Γ be a theory, and let \mathcal{S} be a set of terms.

1. Γ is *inconsistent* if it proves both $[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_n \rangle$ for some list of terms x_1, \dots, x_n . Otherwise, Γ is *consistent*.
2. \mathcal{S} is Γ -*inconsistent* if there is a list of terms x_1, \dots, x_n from \mathcal{S} such that $\Gamma \vdash [x_1, \dots, x_n]$. Otherwise, \mathcal{S} is Γ -*consistent*.

LEMMA 4.6. *Let \mathcal{S} be Γ -consistent. Then for all x , either $\mathcal{S} \cup \{x\}$ or $\mathcal{S} \cup \{\bar{x}\}$ is Γ -consistent.*

Proof. Suppose not. Then by (STR), there are $y_1, \dots, y_n \in \mathcal{S}$ such that $\Gamma \vdash [x, y_1, \dots, y_n]$, and there are $z_1, \dots, z_m \in \mathcal{S}$ such that $\Gamma \vdash [\bar{x}, z_1, \dots, z_m]$. By (RES), $\Gamma \vdash [y_1, \dots, y_n, z_1, \dots, z_m]$. This contradicts the Γ -consistency of \mathcal{S} . \square

LEMMA 4.7. *Let \mathcal{S} be a maximal Γ -consistent set of terms. Then for all x , exactly one of x or \bar{x} belongs to \mathcal{S} .*

Proof. By Lemma 4.6 and maximality, either x or \bar{x} belongs to \mathcal{S} . Both cannot belong to \mathcal{S} , since this would contradict Γ -consistency, due to (AX). \square

LEMMA 4.8. *Let \mathcal{S} be maximal Γ -consistent, and suppose that $(r \text{ all } x) \notin \mathcal{S}$. Then there is some maximal Γ -consistent \mathcal{T} such that $x \in \mathcal{T}$; and whenever $(r \text{ all } y) \in \mathcal{S}$, we have $\bar{y} \in \mathcal{T}$.*

Proof. Let $\mathcal{T}_0 = \{x\} \cup \{\bar{y} : (r \text{ all } y) \in \mathcal{S}\}$. If \mathcal{T}_0 is Γ -inconsistent, then by (STR) there are y_1, \dots, y_n such that $(r \text{ all } y_i) \in \mathcal{S}$ for all i and $\Gamma \vdash [\bar{y}_1, \dots, \bar{y}_n, x]$. By (REL),

$$\Gamma \vdash [r \text{ all } y_1, \dots, r \text{ all } y_n, \overline{r \text{ all } x}].$$

Since \mathcal{S} contains each term $(r \text{ all } y_i)$, by Γ -consistency it does not contain $(\overline{r \text{ all } x})$. So by Lemma 4.7, Γ contains $(r \text{ all } x)$. This is a contradiction.

So \mathcal{T}_0 is Γ -consistent. By Zorn's Lemma, \mathcal{T}_0 has a maximal Γ -consistent extension, say \mathcal{T} . □

The canonical model of Γ . Let M be the set of all maximal Γ -consistent sets of terms. We define a model $\mathcal{M}(\Gamma)$ with domain M by defining:

$$\begin{aligned} \mathcal{S} \in \llbracket p \rrbracket & \text{ iff } p \in \mathcal{S} \\ \mathcal{S} \llbracket r \rrbracket \mathcal{T} & \text{ iff for some } z \in \mathcal{T}, (r \text{ all } z) \in \mathcal{S}. \end{aligned}$$

LEMMA 4.9 (Truth Lemma). *In $\mathcal{M}(\Gamma)$, for any term x , $\llbracket x \rrbracket = \{\mathcal{S} \in M : x \in \mathcal{S}\}$.*

Proof. By induction on x . When x is a noun, this is by the definition of the model.

Induction step for $(r \text{ all } x)$: Suppose $(r \text{ all } x) \in \mathcal{S}$, and suppose $\mathcal{T} \in \llbracket x \rrbracket$. By induction $x \in \mathcal{T}$, so by the definition of the model, $\mathcal{S} \llbracket r \rrbracket \mathcal{T}$. Thus $\mathcal{S} \in \llbracket r \text{ all } x \rrbracket$. Conversely, suppose $\mathcal{S} \in \llbracket r \text{ all } x \rrbracket$, and assume for contradiction that $(r \text{ all } x) \notin \mathcal{S}$. By Lemma 4.8, there is some $\mathcal{T} \in M$ such that $x \in \mathcal{T}$, and whenever $(r \text{ all } y) \in \mathcal{S}$, we have $\bar{y} \in \mathcal{T}$, so $y \notin \mathcal{T}$ by Lemma 4.7. By induction, $\mathcal{T} \in \llbracket x \rrbracket$, but it is not the case that $\mathcal{S} \llbracket r \rrbracket \mathcal{T}$, which is a contradiction.

Induction step for \bar{x} : For any $\mathcal{S} \in M$, we have $\mathcal{S} \in \llbracket \bar{x} \rrbracket$ if and only if $\mathcal{S} \notin \llbracket x \rrbracket$. By induction, this is equivalent to $x \notin \mathcal{S}$. And by Lemma 4.7, $x \notin \mathcal{S}$ if and only if $\bar{x} \in \mathcal{S}$. □

LEMMA 4.10. *If Γ is consistent, then $\mathcal{M}(\Gamma) \models \Gamma$.*

Proof. Let $\varphi \in \Gamma$. First, suppose $\varphi = [x_1, \dots, x_n]$. Suppose for contradiction that there exists $\mathcal{S} \in \bigcap_i \llbracket x_i \rrbracket$. Then by the Truth Lemma, $x_i \in \mathcal{S}$ for all i , contradicting Γ -consistency of \mathcal{S} .

Now suppose $\varphi = \langle x_1, \dots, x_n \rangle$. We claim that the set $\{x_1, \dots, x_n\}$ is Γ -consistent. If not, then using (STR), $\Gamma \vdash [x_1, \dots, x_n]$. So Γ is inconsistent, contradicting our assumption.

Since $\{x_1, \dots, x_n\}$ is Γ -consistent, we can extend it to a maximal Γ -consistent set \mathcal{S} , and by the Truth Lemma, $\mathcal{S} \in \bigcap_{i=1}^n \llbracket x_i \rrbracket$. □

THEOREM 4.11 (Completeness). *For any sentence φ , if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

Proof. Suppose $\varphi = [x_1, \dots, x_n]$. If Γ is inconsistent, then by (EFQ), $\Gamma \vdash \varphi$, and we are done. So we may assume that Γ is consistent. Assume for contradiction that $\Gamma \not\vdash [x_1, \dots, x_n]$. Consider the canonical model $\mathcal{M}(\Gamma)$. By Lemma 4.10, $\mathcal{M}(\Gamma) \models \Gamma$, so $\mathcal{M}(\Gamma) \models \varphi$. By (STR), the set $\{x_1, \dots, x_n\}$ is Γ -consistent. So we can extend it to a maximal Γ -consistent set \mathcal{S} . By the Truth Lemma, $\mathcal{S} \in \bigcap_{i=1}^n \llbracket x_i \rrbracket$, so in $\mathcal{M}(\Gamma) \not\models [x_1, \dots, x_n]$, which is a contradiction.

Now suppose $\varphi = \langle x_1, \dots, x_n \rangle$. If $\Gamma \models \varphi$, then $\Gamma \cup \{[x_1, \dots, x_n]\}$ has no models, so by Lemma 4.10, $\Gamma \cup \{[x_1, \dots, x_n]\}$ is inconsistent. This means that $\Gamma \cup \{[x_1, \dots, x_n]\}$ proves both $\langle y_1, \dots, y_m \rangle$ and $[y_1, \dots, y_m]$, so by (RAA), $\Gamma \vdash \langle x_1, \dots, x_n \rangle$, as was to be shown. □

We have just seen the completeness theorem for $\mathcal{L}_{4.5}^+$. Let \mathcal{L}_4^+ be the generalization of \mathcal{L}_4 obtained by adding sentences of the form $[x_1, \dots, x_n]$ (but not $\langle x_1, \dots, x_n \rangle$). Then we can restrict our logical system for $\mathcal{L}_{4.5}^+$ to all the rules except for (EFQ) and (RAA). These rules are still sound for \mathcal{L}_4^+ , and we will show that they are complete as well.

COROLLARY 4.12. *Let \vdash_0 be the proof system consisting of the rules (AX), (STR), (RES), and (REL). Then \vdash_0 is sound and complete for \mathcal{L}_4^+ .*

Proof. We have already observed the soundness. So suppose Γ is a theory in \mathcal{L}_4^+ and φ is a sentence in \mathcal{L}_4^+ such that $\Gamma \models \varphi$. Moving up to the larger language $\mathcal{L}_{4.5}^+$, we have $\Gamma \vdash \varphi$ by Theorem 4.11. Our goal is to show that $\Gamma \vdash_0 \varphi$.

We first claim that if Γ is any theory in \mathcal{L}_4^+ and ψ is any sentence in $\mathcal{L}_{4.5}^+$ such that $\Gamma \vdash \psi$, then (RAA) is not used in the proof. The argument is by induction on height of the proof tree. Suppose for contradiction that (RAA) is used. We may assume that the root of the proof tree is an application of (RAA):

$$\frac{\begin{array}{c} \frac{[x_1, \dots, x_n]}{\vdots} \\ \langle y_1, \dots, y_m \rangle \end{array} \quad \begin{array}{c} \frac{[x_1, \dots, x_n]}{\vdots} \\ [y_1, \dots, y_m] \end{array}}{\langle x_1, \dots, x_n \rangle} \text{RAA.}$$

The left subtree is a proof of $\langle y_1, \dots, y_m \rangle$ from $\Gamma \cup \{[x_1, \dots, x_n]\}$. Since $\Gamma \cup \{[x_1, \dots, x_n]\}$ is a theory in \mathcal{L}_4^+ , by induction (RAA) is not used in this proof. But none of the other rules produce consequences of the form $\langle y_1, \dots, y_m \rangle$, so this is a contradiction.

Now it is easy to see that if Γ is any theory in \mathcal{L}_4^+ , then no \vdash proof from Γ uses (EFQ), since none of our rules other than (RAA) allow us to produce or introduce a premise of the form $\langle y_1, \dots, y_m \rangle$.

Therefore, if $\Gamma \vdash \varphi$, then $\Gamma \vdash_0 \varphi$. □

4.3. Open problems concerning \mathcal{L}_4 and $\mathcal{L}_{4.5}$. We began this section with the result that the consequence relation for \mathcal{L}_4 is CO-NPTIME hard. This implies that, assuming $P \neq NP$, there is no boundedly complete syllogistic proof system for either \mathcal{L}_4 or $\mathcal{L}_{4.5}$. Instead, we added to the syntax and formulated a proof system which went beyond the “purely syllogistic”; in that it used schematic rules and also (RAA). But we did not find proof systems of any kind for the original languages \mathcal{L}_4 and $\mathcal{L}_{4.5}$. We leave this as an open problem. (There is a result of possible relevance in [5]: a syllogistic system for sentences of the form (all p x), where p is a (complemented) noun, and x is either a (complemented) noun or a term (r all q), where q is a (complemented) noun.) For that matter, we also leave open the question of determining the exact complexities of the consequence relations for \mathcal{L}_4 and $\mathcal{L}_{4.5}$.

§5. \mathcal{L}_5 and $\mathcal{L}_{5.5}$: Putting it all together. The largest logics in this paper are \mathcal{L}_5 and $\mathcal{L}_{5.5}$, as described in Figure 1. We have much less to say about them than about their sublanguages because a notational variant of $\mathcal{L}_{5.5}$ has already been studied. This is the language $\mathcal{R}^{*\dagger}$ in [10]. Here is the syntax of this language. We begin with a set \mathbf{N} of nouns a set \mathbf{V} of verb atoms. A verb literal is either a verb r or its complement \bar{r} . We

define *terms* and *sentences* via the syntax below:

terms x, y, \dots $p \in \mathbf{N} \mid \bar{p} \mid r \text{ all } p \mid \bar{r} \text{ all } p \mid r \text{ some } p \mid \bar{r} \text{ some } p$
 sentences φ, ψ, \dots $\text{all } x \ y \mid \text{some } x \ y.$

Note that we do *not* have recursion for terms. In the semantics, we interpret complemented verbs using relational complement:

$$\llbracket \bar{r} \rrbracket = (M \times M) \setminus \llbracket r \rrbracket.$$

The rest of the semantics is clear.

PROPOSITION 5.1. *There is a translation map $\varphi \mapsto \varphi^*$ of sentences in $\mathcal{R}^{*\dagger}$ to sentences in $\mathcal{L}_{5.5}$ with the following properties:*

1. $\Gamma \models \varphi$ iff $\Gamma^* \models \varphi^*$, where $\Gamma^* = \{\psi^* : \psi \in \Gamma\}$.
2. $\varphi \mapsto \varphi^*$ is computable in PTIME.

Proof. It is sufficient to translate terms x of $\mathcal{R}^{*\dagger}$. For this, we use

$$\begin{aligned} (\bar{r} \text{ all } x)^* &= \overline{r \text{ some } x} \\ (\bar{r} \text{ some } x)^* &= r \text{ all } x. \end{aligned}$$

The point is that the left sides of the equations above have the same interpretations as the right side in every model. □

The translation in the other direction is more complicated due to the complex terms in the languages of this paper. We use the standard technique of *flattening*.

PROPOSITION 5.2. *There is a translation map $(\Gamma, \varphi) \mapsto (\Gamma^*, \varphi^*)$ from assertions in $\mathcal{L}_{5.5}$ to assertions in $\mathcal{R}^{*\dagger}$ with the following properties:*

1. $\Gamma \models \varphi$ iff $\Gamma^* \models \varphi^*$.
2. If Γ is finite, so is Γ^* . In this case, $(\Gamma, \varphi) \mapsto (\Gamma^*, \varphi^*)$ is computable in PTIME.

Proof. Fix Γ and φ in $\mathcal{L}_{5.5}$. Let \mathbb{T} be the set of all terms in $\Gamma \cup \{\varphi\}$, including subterms. For each $t \in \mathbb{T}$, let x_t be a new noun. Let Δ_1 be the set of all sentences of $\mathcal{R}^{*\dagger}$ below, for $x \in \mathbb{T}$:

all $x_p \ p$	all $p \ x_p$
all $x_{(r \text{ all } t)} \ (r \text{ all } x_t)$	all $(r \text{ all } x_t) \ x_{(r \text{ all } t)}$
all $x_{(r \text{ some } t)} \ (r \text{ some } x_t)$	all $(r \text{ some } x_t) \ x_{(r \text{ some } t)}$
all $x_{\bar{p}} \ \bar{p}$	all $\bar{p} \ x_{\bar{p}}$
all $x_{\overline{r \text{ all } t}} \ (\bar{r} \text{ some } x_t)$	all $(\bar{r} \text{ some } x_t) \ x_{\overline{r \text{ all } t}}$
all $x_{\overline{r \text{ some } t}} \ (\bar{r} \text{ all } x_t)$	all $(\bar{r} \text{ all } x_t) \ x_{\overline{r \text{ some } t}}$
all $x_{\bar{t}} \ x_t$	all $x_t \ x_{\bar{t}}$

An easy induction shows that for all $t \in \mathbb{T}$, and all models $\mathcal{M} \models \Delta_1$, $\llbracket x_t \rrbracket = \llbracket t \rrbracket$. We translate the sentences of $\mathcal{L}_{5.5}$ to those of $\mathcal{R}^{*\dagger}$ (with the new nouns) in the obvious way: $(\text{all } t \ u)^* = \text{all } x_t \ x_u$, and $(\text{some } t \ u)^* = \text{some } x_t \ x_u$. Let $\Delta_2 = \{\psi^* : \psi \in \Gamma\}$. Finally, we take Γ^* to be $\Delta_1 \cup \Delta_2$, and φ^* to be the translation that we just saw.

We check point (1): $\Gamma \models \varphi$ iff $\Gamma^* \models \varphi^*$. Assume that $\Gamma \models \varphi$, and let $\mathcal{M} \models \Gamma^*$. Due to Δ_1 , we have our key fact: for all relevant terms t , $\llbracket t \rrbracket = \llbracket x_t \rrbracket$. Using this and the fact that $\mathcal{M} \models \Delta_2$, it follows that $\mathcal{M} \models \Gamma$. And so $\mathcal{M} \models \varphi$. But then using our key fact again, $\mathcal{M} \models \varphi^*$. The converse is similar. □

As shown in [10], the consequence relation for $\mathcal{R}^{*\dagger}$ is EXPTIME complete. Moreover, there are no proof systems which are finite, sound, and complete for the logic, even allowing *reductio ad absurdum*. (Nevertheless, there are logical systems for $\mathcal{R}^{*\dagger}$. For example, Fitch-style system may be found in [4, 6]. That system uses individual variables, as in first-order logic, but in a controlled way.)

It follows from the translations in Propositions 5.1 and 5.2 that the consequence relation for $\mathcal{L}_{5,5}$ is EXPTIME complete. Moreover, there can be no sound and complete syllogistic proof system for $\mathcal{L}_{5,5}$, even allowing all of the (CASES) rules in this paper. Indeed, any rule allowing proof by cases would correspond to a rule in the language of $\mathcal{R}^{*\dagger}$ which is derivable from *reductio ad absurdum*.

We would like to point out that the EXPTIME hardness result for $\mathcal{L}_{5,5}$ extends to the weaker logic \mathcal{L}_5 . To see this, we must recall the outline of the argument in [10]. The starting point is Spaan's theorem [12] that the satisfiability problem for \mathcal{L}_U , modal logic with the universal modality, is EXPTIME hard. One takes a sentence φ in \mathcal{L}_U and translates it to a finite set S_φ of sentences of $\mathcal{R}^{*\dagger}$ with the property that φ and S_φ are equisatisfiable. By our translation, S_φ may be taken to be a set of sentences in $\mathcal{L}_{5,5}$. By examining the details, S_φ is a set S_φ^* of sentences in \mathcal{L}_5 , together with one additional sentence of the form (some $x \bar{x}$). The upshot is that φ is unsatisfiable iff $S_\varphi^* \models \text{all } x \bar{x}$. Note that (all $x \bar{x}$) is a sentence in \mathcal{L}_5 . In this way, the consequence relation for \mathcal{L}_5 is at least as hard as the (un)satisfiability problem for \mathcal{L}_U .

This paper also explored extensions of syllogistic logic using schemes like (CHAINS). It is possible that there is a schematic extension of $\mathcal{L}_{5,5}$, and it is also possible that extensions to the syntax will help. We have not explored this. The logical system for $\mathcal{R}^{*\dagger}$ which uses individual variables adapts to $\mathcal{L}_{5,5}$ in a straightforward way, and we expect that the completeness and finite model properties which were shown in [4] hold in the adapted system.

BIBLIOGRAPHY

- [1] McAllester, D. A. (1993). Automatic recognition of tractability in inference relations. *Journal of the ACM (JACM)*, **40**(2), 284–303.
- [2] McAllester, D. A., & Givan, R. (1992). Natural language syntax and first-order inference. *Artificial Intelligence*, **56**, 1–20.
- [3] Moss, L. S. (2008). Completeness theorems for syllogistic fragments. In Hamm, F., & Kepser, S., editors. *Logics for Linguistic Structures*. Berlin: de Gruyter Mouton, pp. 143–173.
- [4] ———. (2010). Logics for two fragments beyond the syllogistic boundary. *Fields of Logic and Computation: Essays Dedicated to Yuri Gurevich on the Occasion of His 70th Birthday*, volume 6300 of Lecture Notes in Computer Science. Berlin, Germany: Springer-Verlag, pp. 538–563.
- [5] ———. (2010). Syllogistic logics with verbs. *Journal of Logic and Computation*, **20**(4), 947–967.
- [6] ———. (2015). Natural logic. In *Handbook of Contemporary Semantic Theory* (second edition, chapter 18). Hoboken, NJ: John Wiley & Sons.
- [7] ———. (2020). *Logic from Language*. Bloomington, IN: Indiana University.
- [8] Moss, L. S., & Kruckman, A. (2017). All and only. In *Partiality and Underspecification in Natural Language Processing*. Cambridge: Cambridge Scholars Publishers.

- [9] Pratt-Hartmann, I. (2014). The relational syllogistic revisited. *Linguistic Issues in Language Technology*, **9**, 195–227.
- [10] Pratt-Hartmann, I., & Moss, L. S. (2009). Logics for the relational syllogistic. *Review of Symbolic Logic*, **2**(4), 647–683.
- [11] Thomas, J. S. (1978). The complexity of satisfiability problems. *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing*. San Diego, CA: San Diego, pp. 216–226.
- [12] Spaan, E. (1993). Complexity of Modal Logics. Ph.D. Thesis, ILLC, University of Amsterdam.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
WESLEYAN UNIVERSITY
MIDDLETOWN, CT, USA
E-mail: akruckman@wesleyan.edu

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY BLOOMINGTON
BLOOMINGTON, IN, USA
E-mail: lmoss@indiana.edu