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Coutinho's Method for the Altitude

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In the first aerial crossing of the South Atlantic, by Gago Coutinho and Sacadura Cabral in 1922, several methods of astronomical maritime navigation were used with adaptions to aerial navigation. In order to apply these methods, the navigator needed to know the approximate altitude of the aircraft so that its position could be determined. The instrument available at that time, the altimeter, did not give reliable values for altitude. Therefore, Coutinho had to devise a method that enabled the navigator to determine the altitude quickly and efficiently. The method Coutinho devised is based on a mathematical and geometrical procedure. In this paper, we study in detail Coutinho's method to determine altitude, with diagrams to aid understanding of the deductions and calculations. We also present a real example of how this method would be used during the flight.

KEYWORDS

1. Altitude. 2. Aviation. 3. Gago Coutinho. 4. History. 5. Navigation.

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1. INTRODUCTION.

> Navigation is the means whereby the mariner or aviator ascertains his position on the surface of the earth, and determines the exact direction in which he must head his craft in order to reach its destination. (Brown, 1920, p. 93)

Until 1922 there were no scientific methods of astronomical navigation that permitted an aircraft to fly out of sight of land safely. There was little need for such methods, since most of the flights that were made were of short distance and within sight of land. In this situation the navigator could check the progress of the aircraft towards the desired destination simply by observation of landmarks and the use of very rudimentary instruments. Although such equipment was capable of providing basic data such as direction (compass), distance (speedometer) and altitude (altimeter), this information was not sufficient to achieve the objective of flying long distances over the sea. For this reason, it was essential to develop air navigation procedures which allowed an aircraft to fly out of sight of land safely. The first methods developed in the beginnings of aerial navigation were inspired by the procedures used in maritime navigation.

> In short journeys over land by aeroplane or airship the duties of a navigator are light, so long as he can see the ground and check his progress towards the objective by observation and a suitable map. But for long distance flights, especially over the ocean and under circumstances whereby the ground cannot be seen, the navigator of the air borrows much from the navigator of the sea. (Brown, 1920, p. 94)

Nevertheless, Brown highlights the fact that there are also some distinctions between air and sea navigation, resulting from differences in the characteristics of the elements in which each type of navigation is carried out, namely, water or air.

> The chief differences between the navigation of aircraft and the navigation of seacraft are occasioned by:
> (a) The vastly greater speed of aircraft, necessitating more frequent observations and quicker methods of calculation.
> (b) The serious drift caused by the wind. This may take aircraft anything up to forty or more miles off the course in each hour's flying, according to the direction and strength of the wind. In cloudy weather, or at night, a change in the wind can alter the drift without the knowledge of the navigator. Hence, special precautions must be taken to observe the drift at all possible times.
> (c) The absence of need for extreme accuracy of navigation in the air, since a ten or even twenty mile error from the destination in a long journey is permissible. Another favorable point is that rocks, reefs and shoals need not be avoided. This permits the aerial navigator to use short cuts and approximations in calculation, which would be criminal in marine navigation. (Brown, 1920, p. 95)

In 1919 Hawker and Grieve attempted to cross the Atlantic Ocean but unfortunately they were obliged to ditch their plane at sea. From their experience, however, they concluded that accurate methods of aerial navigation needed to be developed in order to guarantee the success of long flights over the ocean: 'in a flight such that which we attempted, a non-stop journey of over 2000 miles, accurate navigation is of absolutely prime importance' (Hawker and Grieve, 1919, p. 60).

In 1922 Gago Coutinho and Sacadura Cabral, two officers of the Portuguese Navy, completed the first air crossing of the South Atlantic from Lisbon to Rio de Janeiro. This crossing was different from that of Hawker and Grieve in that the latter only depended on the navigator's ability to reach the west coast of Ireland, but Coutinho and Cabral needed to reach Saint Paul's Rocks, a tiny point in the immensity of the Atlantic Ocean.

> The highest point of St. Paul's Rocks is sixty feet above water; for the most part they are only twenty feet in height, and they are less than a hundred yards long. To hit this tiny spot after flying $11\frac{1}{2}$ hours over the open Atlantic is the greatest example of air navigation yet performed. (Jones, 1931, p. 255)

The success of this flight was due to the fact that the Portuguese expedition conceived and developed the first scientific methods of aerial navigation: 'This crossing, having been done in the most accurate and scientific manner, is an unparalleled stride towards the practical solution of the problems concerning navigation through the air.' (Anonymous, 1922, p. 361).

The solutions devised by the two officers include the invention of two instruments, namely the path corrector and the precision sextant – a sextant with an artificial horizon. Besides these instruments, Coutinho also developed the first methods of aerial astronomic navigation, which include a method to determine the aircraft's position as well as its altitude. We shall describe these solutions briefly.

The determination of wind direction and speed was a problem that needed to be solved so that the navigator could estimate the drift caused by the wind. To deal with this difficulty, Coutinho and Cabral invented an instrument, called the path corrector, which enabled the navigator to determine the direction and speed of the wind. Furthermore, the path corrector could also be used to determine the new direction of the aircraft in order to compensate for wind drift. For a detailed study of the construction and use of the path corrector, the reader is referred to Coutinho and Cabral (1927) and Canas et al. (2019): 'Among the instruments used for the purpose the course corrector must have a prominent place, because perfectly accurate dead reckoning can be done with it.' (Anonymous, 1922, p. 361)

The sextant used in maritime navigation could not be used in aerial navigation due to the difficulty of the definition of the sea horizon at a normal flight altitude. Nevertheless, Coutinho believed that its accuracy would be essential to determine the position of the aircraft during flight. Therefore, he developed a new model of sextant with an artificial horizon, called a precision sextant, which could be used to measure the height of a star without the need of the sea horizon. For details about the precision sextant, the reader is referred to Coutinho and Cabral (1927) and Pereira (2015).

> The sextant, used for observations in the air, with its own artificial horizon, makes it practical to estimate the altitudes with the accuracy required in aerial navigation, in cases where it is impossible or inconvenient to descend in order to observe the sea horizon. (Anonymous, 1922, p. 361)

Another important achievement accomplished by Coutinho and Cabral was the development of methods of astronomic navigation which could be used effectively on board of aircrafts.

> The use of points of reference through the line of the intended crossing, and the able and skilled modifications of the formulae of nautical astronomy, by which the observer may, before starting, prepare the greatest part of his calculations, in this way leaving only another quite minor part to be done in the air; are the two conceptions by which astronomical navigation with a sextant can be done in the air with accuracy and comfort, as happens on board ship, together with the quickness exacted in virtue of the great speed of aeroplanes. (Anonymous, 1922, pp. 361–362)

A crucial aspect for the success and effectiveness of the methods of astronomic navigation developed and used by Coutinho and Cabral was the determination of the aircraft's altitude. 'The calculation of the position, through observations of the stars, requires the approximate knowledge of our altitude'[1] (Coutinho, 1923, p. 10). To solve this problem, Coutinho developed a geometrical procedure which allowed the navigator to determine the aircraft's altitude quickly during flight. For this, the navigator would have to measure the aircraft's shadow on the sea surface and then, with the help of a table which had been pre-calculated before the flight, he could easily determine the aircraft's altitude.

> [...] the shadow of the aeroplane on the surface of the sea is sufficiently clear for measuring with the sextant or telemetric binoculars. Evidently, one must, [. . .] have a small table calculated with the length of the wings of the aeroplane to give the coefficient K in the formula
>
> $$\text{height} = K \cot.\text{angle of shadow}$$

[1] 'O cálculo da posição, por meio de observações dos astros, exige o conhecimento aproximado da nossa altitude [...]' (Coutinho, 1923, p. 10).

> a formula which gives the height with an error of a few feet, equivalent to an error of less than a minute of depression. (Coutinho and Cabral, 1922, p. 374)

In Section 3, we study in detail Coutinho's geometrical procedure to determine the aircraft's altitude[2] and how to determine the coefficient K in the above formula. Having determined the mathematical expression to calculate the coefficient K, we will then explain how Coutinho calculated in advance the table he referred to. We will include various diagrams to understand better the mathematical construction that underlies the procedure. We conclude the paper with an example of how the procedure would have been used during the flight.

2. THE NEED TO DETERMINE THE AIRCRAFT'S ALTITUDE IN AERIAL NAVIGATION.

The first scientific methods of aerial navigation were inspired by the techniques used in maritime navigation, however, aerial navigation involved specific problems that did not arise in maritime navigation. One of those problems was the need to determine the aircraft's altitude.

> Then again, height has to be allowed for. On board a ship this remains pretty constant, so that corrections are easily made, whereas in an aeroplane, as in our case, one's height can vary within an hour or two or less from 15,000 feet to 1000 feet. Height is taken from the barometer, but as the barometer reading alters according to atmospheric conditions as well as being affected by height, a very big error can enter into this part of the calculation. (Hawker and Grieve, 1919, pp. 61–62)

An alternative would be the use of an altimeter; however, readings from the altimeter were also not reliable: 'Hence an altimeter will commonly read 21,000 feet when the height is really 20,000 feet.' (Wimperis, 1920, pp. 40–41). As we can see, the altimeter was not reliable and efficient for aerial navigation. Hence, determining the altitude of the aircraft was a new problem that needed to be solved. 'Until we know our exact height above the sea we cannot plot our exact position.' (Maitland, 1920, p. 44).

Knowledge of the exact altitude of the aircraft was important when determining its exact position. The exact position of the aircraft could be known using methods of astronomic navigation, therefore the knowledge of the aircraft's altitude was important in order to account for corrections related to dip.

> **Dip Correction**. —The second correction to be applied is the **correction for the dip** of the horizon. Viewed from an airplane, the horizon is below the eye. The greater the height of the airplane above the water, the greater is the angle of the horizon below a horizontal line through the observer's eye. This angle is called the **angle of dip**. (Jones, 1931, pp. 130–131)

Therefore, it was necessary to develop an efficient and accurate procedure to determine the altitude of the aircraft. Jones (1931) describes a method to determine the altitude.

> One method of finding the altitude above the water was to drop plaster of Paris eggs and count time from the instant they were released until they could be seen shattering on the

[2] The vertical (geometrical) distance of the aircraft from mean sea level (MSL) is called true altitude. Altitude (in current air navigation) is a barometric measure, it is the vertical distance in ISA (International Standard Atmosphere) between the static pressure of the isobaric surface the aircraft is flying and the QNH pressure of the isobaric surface at the MSL, at a certain temperature. True altitude is different from altitude by a correction which depends on the real atmosphere deviation from ISA pressure and temperature. Coutinho's method measures the true altitude. In the early days of air navigation, however, there was no distinction between altitude and true altitude. Therefore, we shall use the term altitude instead of true altitude throughout the paper.

> water. At night a searchlight with a conical beam of light was pointed vertically downwards. The size of the circle of light on the water was dependent upon the altitude, so that by measuring the diameter of this circle by means of stadia wires in a telescope mounted alongside the searchlight the altitude could be obtained. (Jones, 1931, p. 40)

The latter method and Coutinho's method are based on the same principle. Coutinho's method can only be used during the day, however, while the second method described above can only be used at night. Coutinho's method will be described in detail in Section 3.

Jones's book was written in 1931. Therefore, some of the methods Jones described might have been developed after Coutinho and Cabral's 1922 flight or even developed independently by other navigators. Furthermore, Jones does not describe the flights where the methods were used, although he mentions that they were practical and used on actual flights. 'The methods described in this book are practical; they all have been used on actual flights.' (Jones, 1931, p. vii).

Another method that we found in the literature was the method used by H.M. Airship R34 in its successful east-to-west Atlantic crossing in 1919. Very briefly, the method consisted of measuring the angle shadow of the airship on the sea surface and, with the knowledge of that angle, the navigator would determine the value of the altitude.

> Scott works out our height above the water in the following way – The airship is throwing a very dark shadow on the surface of the sea on starboard side – almost immediately underneath the ship. By taking with a sextant the angle subtended by length of the shadow, and knowing the length of the shadow to be 640 feet, he gets the true height. In this case the height works out at 2100 feet, whilst the aneroid gives us only 1200 feet – a variation of 900 feet. (Maitland, 1920, p. 44)

Unfortunately, Maitland (1920) does not explain how the method worked, although he mentions later in the text that the method is simple and reliable.

> The sun is high, so Cooke is able to get a good idea of any barometric changes by observing the angle the ship's shadow on the water subtends with a sextant, thus calculating the distance of the shadow from the observer, and comparing with height recorded in the altimeter. This is only possible at low altitudes, *i.e*. about 1500 feet. (It sounds a bit complicated, but is quite effective!) (Maitland, 1920, p. 104)

Coutinho also faced the same problem of determining altitude. Therefore, he had to conceive a solution to this problem that could be used effectively during the flight. His method is similar to that used in 1919 by H.M. Airship R34 since it is based on the measurement of the shadow of the aeroplane's wingspan on the sea surface.

In the next section, we study in detail the method devised by Coutinho to determine the aircraft's altitude during flight. His procedure relies on a geometrical and mathematical procedure to determine a specific constant K that would be multiplied by the cotangent of the angle of the plane's wingspan shadow on the sea surface. Our study of the mathematical construction will be complemented with numerous diagrams and pictures so that the reader can follow the construction easily.

3. COUTINHO'S GEOMETRICAL METHOD FOR THE AIRCRAFT'S ALTITUDE.

One of the problems of early aerial navigation was the determination of altitude in order to account for dip correction, which is necessary to determine the exact position of the aircraft. Therefore, it was necessary to find a way to determine the flight altitude as accurately as possible. As we have seen above, the altitude could be obtained using the value

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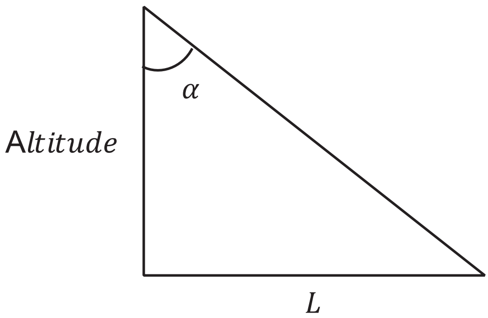

Figure 1. Theoretical example to determine altitude.

of the atmospheric pressure given by the barometer. However, this result would not be the most accurate and would introduce errors in the calculations which could compromise the success of the flight. Therefore, Coutinho needed to find a good solution to solve this problem.

> With clear sky, and with the sun at an altitude of more than 30 degrees, the shadow of the aeroplane on the surface of the sea is sufficiently clear to be measured using the sextant or the telemetric binoculars. Evidently, one must, as we did for *Lusitania*, have a small table, calculated with the length of the wings of the aeroplane to give the coefficient K in the formula:
>
> $$\text{height} = K\ \text{cot.angle of shadow}$$
>
> a formula which gives the height with an error of a few feet, equivalent to an error of less than a minute of depression. (Coutinho and Cabral, 1922, p. 374)

First, we have to observe that Coutinho is perfectly aware that this formula is not exact; however, as he states, the error is sufficiently small and can be ignored. Second, note that Coutinho states that in order to measure the shadow of the aeroplane's wingspan on the sea surface, the sun's height must be at least at 30°. This requirement is essential; in fact, if the sun's height is less than 30° then the plane's shadow is stretched and projected far away, making it extremely hard to make accurate measurements in this case. Another important requirement is that the sun must be at the aircraft's beam, so in order to make this measurement, Coutinho had to steer the aircraft so that the sun was at its beam.

In the rest of this section, we will explain in detail how Coutinho arrived at the equation:

$$\text{height} = K\ \cot\,(\text{angle of shadow}) \tag{1}$$

and how this equation can be used to determine altitude in flight. We start with a simple theoretical example. Let us observe the situation presented in Figure 1.

In this case, we have

$$\tan\alpha = \frac{L}{\text{Altitude}} \Leftrightarrow \text{Altitude} = \frac{L}{\tan\alpha} \Leftrightarrow \text{Altitude} = L\cot\alpha \tag{2}$$

Therefore, by knowing the angle α and L, we can easily determine the altitude. However, this is not the situation that we want to study, rather a theoretical example that illustrates the problem of determining the altitude of an aircraft in flight. To solve this problem, Coutinho developed an easy method which gives the altitude of the aircraft with an error of a few

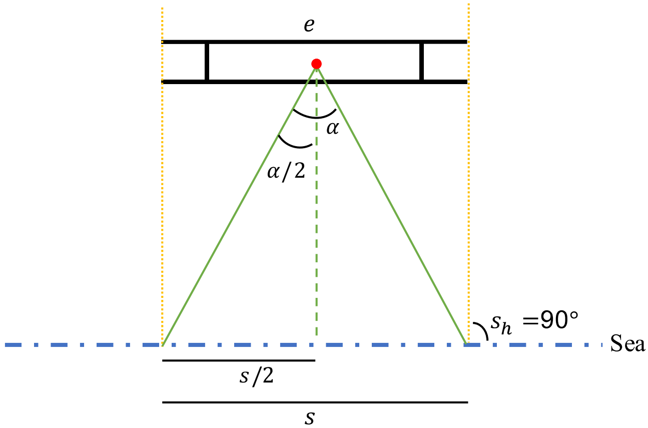

Figure 2. Geometrical scheme to determine altitude with the sun's height at exactly 90°.

metres. Furthermore, Coutinho's procedure could be performed quickly during the flight, which is crucial in aerial navigation. In the rest of this section, we will study in detail how Coutinho arrived at his solution for determining altitude during flight and we will conclude with a real example.

For the sake of simplicity, we will start by considering the simplest case and then proceed to the general case. Let us start with the case when the sun's height is exactly 90°, that is, the sun is vertical relative to the aircraft. Let s_h denote the sun's height, e the wingspan of the biplane and let α denote the angle, measured with a sextant, of the biplane's shadow on the sea surface. Finally, let s denote the length of the biplane's wingspan shadow on the sea surface. In this particular situation, since $s_h = 90°$ we clearly have $s = e$. (see Figure 2)

In this case, we have

$$\text{Altitude} = \frac{s}{2}\cot\left(\frac{\alpha}{2}\right) \tag{3}$$

Since the previous situation only occurs when the sun's declination is equal to the latitude of the location, which only happens in inter-tropical regions, Coutinho needed to develop a procedure to determine altitude which could be used in any given situation, requiring only that the sun's height is at least 30°. As noted above, Coutinho formulated Equation (1), where K is a coefficient that will be listed in a table. We will now explain how the coefficient K is calculated.

Coutinho's aircraft was a biplane, that is, an aircraft with two main wings, one above the other and supported by struts. It was necessary to calculate the total length of the biplane's shadow on the surface of the sea, and the vertical distance between the two wings will increase the length of the biplane's shadow on the sea surface by a quantity that will depend on the sun's height (see Figure 3). Therefore, we first need to determine how the vertical distance between the wings will increment the value of s. Recall that s denotes the length of the biplane's wingspan shadow on the sea surface. Let P denote the vertical distance between the two wings and let x denote the quantity that we need to add to the length of the wingspan to determine the length of the biplane's shadow on the sea surface. That is,

$$s = e + x \tag{4}$$

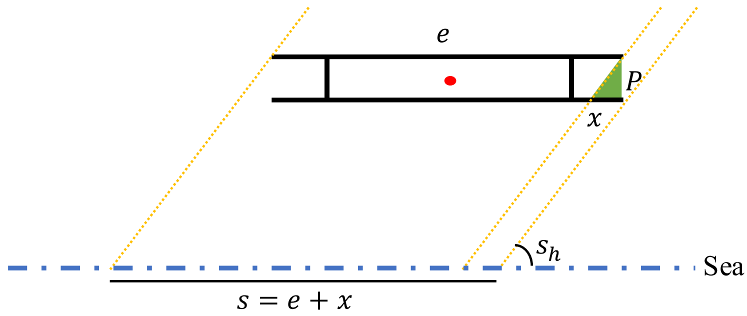

Figure 3. Scheme explaining how the sun's height affects the length of the biplane's shadow on the sea surface.

where

$$x = P\cot(s_h) \tag{5}$$

After determining how the distance between the biplane wings will affect the dimension of its shadow on the sea surface, we will now proceed with the detailed explanation of Coutinho's method of determining the altitude. We recommend the reader to follow the diagram in Figure 4.

Recall that the objective is to determine the value of the altitude, here denoted by H. However, as noted above,

$$H = K\cot(\text{angle of shadow}) \tag{6}$$

where K is a coefficient that will depend only on known values, such as the sun's height, the wingspan of the biplane and the vertical distance between the two wings. We will now present a detailed explanation of how Coutinho determined the value of the coefficient K. We will first determine the distance d. Consider the right triangle with right sides y, d and angle $\alpha/2$. To be absolutely precise, the measure of the angle is not exactly $\alpha/2$, as can be seen easily in the diagram depicted in Figure 4, nevertheless, for simplicity and without any effect on the calculation, we can assume the measure of the angle is exactly $\alpha/2$. We have

$$d = y\cot\left(\frac{\alpha}{2}\right) \tag{7}$$

The value of α will be at most 14°. This upper bound was deduced by the authors from a planar model of the problem, that is, not considering the effect of the altitude. In fact, for a fixed altitude we can easily see that the largest value of α is attained when the sun's height is exactly 90°. Furthermore, if the altitude increases then the value of α will decrease. Coutinho's observations were made at an altitude of around 90 metres, for which the value of α will be at most 14°. 'The best plan, when more reliable observations are desired, is to lower the aeroplane till the line of the sea horizon becomes sufficiently clear (a height of 300 feet generally ensuring this), and to make observations upon it' (Coutinho and Cabral, 1922, p. 373).

Using simple trigonometric formulas, we can easily deduce that

$$2\cot(\alpha) = \cot\left(\frac{\alpha}{2}\right)\left[1 - \tan^2\left(\frac{\alpha}{2}\right)\right] \tag{8}$$

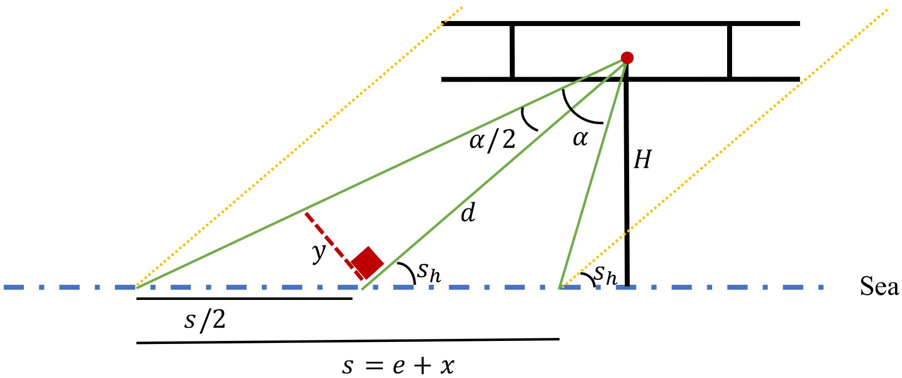

Figure 4. Diagram explaining how to determine the variable d.

Since $\alpha \leq 14^\circ$ we conclude that the term $[1 - \tan^2(\alpha/2)]$ is close to 1, that is, in this case, we have $\cot(\alpha/2) \approx 2\cot(\alpha)$. Therefore, it follows that

$$d = y \cot\left(\frac{\alpha}{2}\right) \approx 2y \cot(\alpha) \tag{9}$$

Our next step will be to determine the value of y.

Since the sum of the measures of the interior angles of a triangle equals 180°, we must have $\gamma = s_h - \alpha/2$, where γ is the angle marked in Figure 5 (left) and $\theta = 90^\circ + \alpha/2$, where θ is the angle marked in Figure 5 (right). Because the value of α is small, we can approximate θ by a right angle, therefore we have

$$\cos(90^\circ - s_h) = \frac{y}{s/2} \Leftrightarrow y = \frac{s}{2}\cos(90^\circ - s_h) \Leftrightarrow y = \frac{s}{2}\sin s_h \tag{10}$$

Recall that our aim was to determine the distance d. In fact, by using all the results we have obtained so far, we have

$$d = y \cot\left(\frac{\alpha}{2}\right) \quad \text{(by Equation (7))}$$

$$d \approx 2y \cot(\alpha) \quad \text{(by Equation (9))}$$

$$d \approx 2\frac{s}{2}\sin(s_h)\cot(\alpha) = s\sin(s_h)\cot(\alpha) \quad \text{(by Equation (10))}$$

$$d \approx (e + P\cot(s_h))\sin(s_h)\cot(\alpha) \quad \text{(by Equations (4) and (5))}$$

Therefore,

$$d \approx (e\sin(s_h) + P\cos(s_h))\cot(\alpha) \tag{11}$$

To conclude, consider the right triangle with side H and angle s_h as highlighted in Figure 6.

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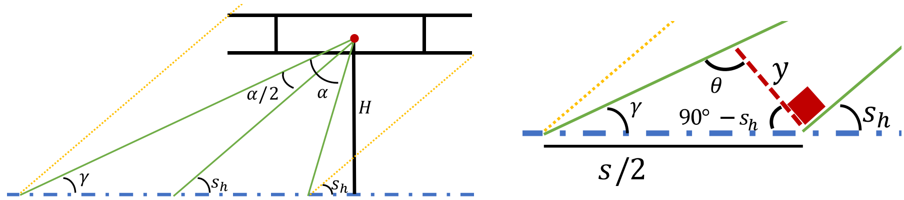

Figure 5. Diagram to determine the angle γ (left) and the variable y (right).

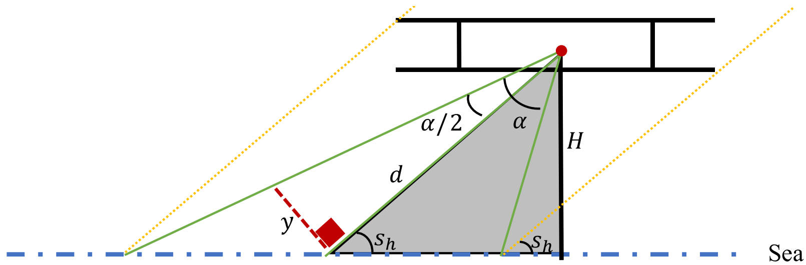

Figure 6. Diagram of the triangle to find the variable H.

In this case, we have

$$\sin(s_h) = \frac{H}{d} \tag{12}$$

That is,

$$H = d \sin(s_h) \tag{13}$$

Therefore,

$$\begin{aligned} H &\approx (e \sin(s_h) + P \cos(s_h)) \cot(\alpha) \sin(s_h) \quad \text{(by Equation (10))} \\ &\approx \left(e \sin^2(s_h) + P \cos(s_h) \sin(s_h)\right) \cot(\alpha) \end{aligned} \tag{14}$$

Thus, by letting

$$K = e \sin^2(s_h) + P \cos(s_h) \sin(s_h) \tag{15}$$

we have

$$H = K \cot(\alpha) \tag{16}$$

as required.

Note that the coefficient K depends only on known values, such as the sun's height, the wingspan of the biplane and the height between the two wings, denoted by s_h, e and P, respectively. Observe that e and P are fixed values for a given aircraft, in fact, in Coutinho's biplane, we have $e = 19{\cdot}2$ metres and $P = 1{\cdot}6$ metres. With this deduction for the coefficient K Coutinho constructed a table as follows. In the left-hand side, we have the sun's height s_h, varying from 90° to 20° and on the right-hand side we have the values of $\log K$ calculated using Equation (14). All logarithms considered here are base 10 logarithms (see Figure 7).

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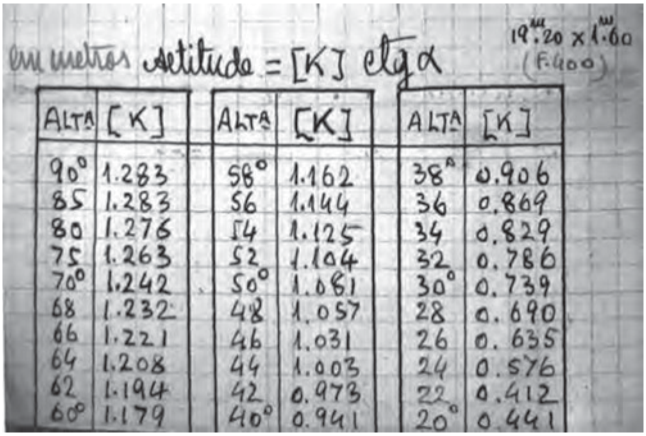
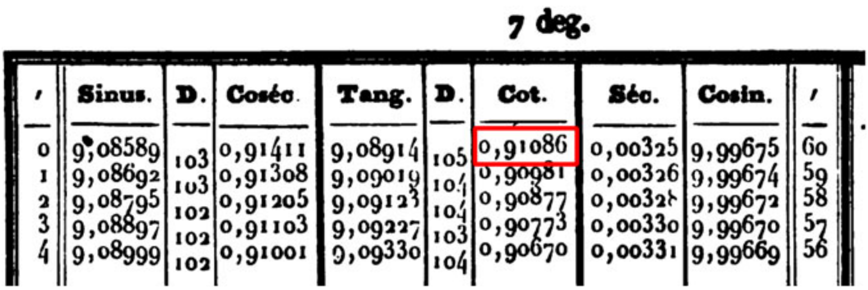


em metros Altitude = [K] ctg α

19m.20 x 1m.60 (F.400)

ALTA	[K]	ALTA	[K]	ALTA	[K]
90°	1.283	58°	1.162	38°	0.906
85	1.283	56	1.144	36	0.869
80	1.276	54	1.125	34	0.829
75	1.263	52	1.104	32	0.786
70°	1.242	50°	1.081	30°	0.739
68	1.232	48	1.057	28	0.690
66	1.221	46	1.031	26	0.635
64	1.208	44	1.003	24	0.576
62	1.194	42	0.973	22	0.412
60°	1.179	40°	0.941	20°	0.441

Figure 7. Gago Coutinho's table to determine altitude during the flight (Pereira, 2015, p. 312).



7 deg.

′	Sinus.	D.	Coséc.	Tang.	D.	Cot.	Séc.	Cosin.	′
0	9,08589	103	0,91411	9,08914	105	0,91086	0,00325	9,99675	60
1	9,08692	103	0,91308	9,09019	104	0,90981	0,00326	9,99674	59
2	9,08795	102	0,91205	9,09123	104	0,90877	0,00328	9,99672	58
3	9,08897	102	0,91103	9,09227	103	0,90773	0,00330	9,99670	57
4	9,08999	102	0,91001	9,09330	104	0,90670	0,00331	9,99669	56

Figure 8. Value of log cot 7° (Hoüel, 1914, p. 48).

We will conclude this section with a practical example of how Coutinho determined the altitude during the flight. Suppose that at the time of the measurement the sun's height is $s_h = 50^\circ$ and $\alpha = 7^\circ$. Then, by Equation (14) we have

$$\begin{aligned} \log K &= \log(e \sin^2(s_h) + P\cos(s_h)\sin(s_h))e\sin^2(s_h)) \\ &= \log(19{\cdot}2\sin^2(50^\circ) + 1{\cdot}6\cos(50^\circ)\sin(50^\circ)) = 1{\cdot}081162 \end{aligned} \tag{17}$$

Observe that this is exactly the value that appears on the table in Figure 7 immediately to the right of 50°. Having the value for log K Coutinho would consult the book *Table de Logarithmes*, by J. Hoüel, to obtain the value for $\log(\cot\alpha)$. In this particular case, $\log(\cot(7^\circ)) = 0{\cdot}91086$ (see Figure 8). Thus,

$$\log H = \log K + \log(\cot(\alpha)) = 1{\cdot}992$$

This implies that

$$H = 10^{1{\cdot}992}$$

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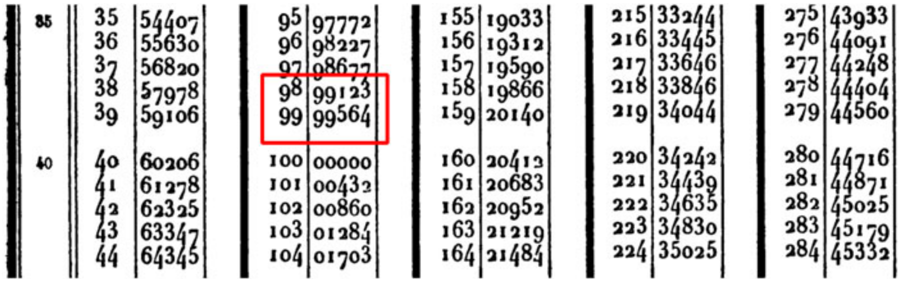

35	35	54407	95	97772	155	19033	215	33244	275	43933
	36	55630	96	98227	156	19312	216	33445	276	44091
	37	56820	97	98677	157	19590	217	33646	277	44248
	38	57978	98	99123	158	19866	218	33846	278	44404
	39	59106	99	99564	159	20140	219	34044	279	44560
40	40	60206	100	00000	160	20412	220	34242	280	44716
	41	61278	101	00432	161	20683	221	34439	281	44871
	42	62325	102	00860	162	20952	222	34635	282	45025
	43	63347	103	01284	163	21219	223	34830	283	45179
	44	64345	104	01703	164	21484	224	35025	284	45332

Figure 9. Value of $10^{1 \cdot 992}$ (Hoüel, 1914, p. 2).

Finally, we need to find the value of $10^{1 \cdot 992}$. This value can also be found in Hoüel's logarithm tables. The value of H is between 10 and 100, closer to 100 than to 10. Thus, searching for the value in Hoüel's tables, one has to look for numbers on the table between 10 and 100 that best approximate 992, in this particular case the numbers are 98 and 99 (See Figure 9). Therefore, we conclude that the value of H would be between 98 and 99 metres, that is, the aircraft's altitude would be approximately 98 metres.

To conclude, we should observe that for this procedure to be done easily and quickly during flight, all the necessary tables should be at hand for the navigator. As it was previously emphasised, it was crucial that the navigator could be able to calculate the altitude quickly and efficiently. Therefore, Coutinho had also pre-prepared Hoüel's logarithm tables by attaching markers at the margins of the relevant pages in order to find the necessary page easily.

> The tables of logarithms, which were those of Hoüel, had also been prepared in such a way that they could be immediately opened on the page necessary, for which there were glued markers at the margin, like the indices of commercial books.[3] (Coutinho, 1923, p. 17)

4. CONCLUSIONS.

Gago Coutinho and Sacadura Cabral successfully completed the first air crossing of the South Atlantic from Lisbon to Rio de Janeiro in 1922. In the crossing, they used, for the first time, scientific methods of astronomic navigation. The methods and procedures they developed and used were adapted from and inspired by the methods used in maritime navigation. For these new methods to be used successfully, however, it was essential to know the approximate value of the aircraft's altitude. The use of the altimeter was not reliable since it gave readings with large errors. Therefore, Coutinho had to conceive a method that allowed the navigator to determine the approximate value of the altitude quickly and efficiently during the flight.

Coutinho's method is based on a mathematical geometrical procedure. This research aimed to study in detail Coutinho's method to determine the altitude. The paper contains several diagrams so that the reader can easily follow all the mathematical deductions. We conclude our work with a real example of how this method would have been used in a real flight.

[3] 'As tábuas de logaritmos, que eram as de Hoüel, tinham também sido preparadas de modo a poderem ser abertas imediatamente na folha necessária para o que lhe foram coladas chamadas á margem, como usam os índices dos livros comerciais.' (Coutinho, 1923, p. 17).

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