Macroeconomic Dynamics, **7**, 2003, 42–62. Printed in the United States of America.
DOI: 10.1017.S1365100501010331

COORDINATION ON SADDLE-PATH SOLUTIONS: THE EDUCTIVE VIEWPOINT—LINEAR UNIVARIATE MODELS

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We investigate local strong rationality (LSR) in a one-step-forward-looking univariate model with memory one. Eductive arguments are used to determine when common knowledge (CK) that the solution is near some perfect-foresight path is sufficient to trigger complete coordination on that path (i.e., the path is LSR). Coordination of expectations is shown to depend on three factors: the nature of the CK initial beliefs, the degree of structural heterogeneity, and the information structure. Our sufficient conditions for LSR precisely reflect these features and provide basic consistent justifications for the choice of the saddle-path solution.

Keywords: Eductive Stability, Coordination, Heterogeneity, Strong Rationality

1. INTRODUCTION

Take a linear one-dimensional system of the form $y_t - \delta y_{t-1} - \beta y^e_{t+1} = 0$, where y^e_{t+1} denotes the expectation of y_{t+1}. This dynamical system is one-step-forward looking and has memory one. It has a single steady state, $y_t = 0$, and for some values of the parameters displays a saddle-path configuration: Starting from any y_0, there are two constant-growth-rate *perfect-foresight solutions*, one converging to zero, the so-called *saddle-path stable solution*, and the other one exploding away from zero. There are also many other solutions, indeed a continuum, with variable growth rates, going ultimately to infinity.

The conventional wisdom among economists is that, among the infinity of perfect-foresight solutions that exist in this context, the right one to select, for economic modeling purposes, is the saddle-path stable solution.

The first author acknowledges financial support from the National Science Foundation. An early version of this paper was circulated under the title "Strong Rationality in Dynamic Expectations Models" (1997). Address correspondence to: Professor George W. Evans, Department of Economics, 1285 University of Oregon, Eugene, OR 97403-1285, USA; e-mail: gevans@oregon.uoregon.edu.

 1365-1005/03 $12.00

It seems that such conventional wisdom arose first from considerations of convenience: The selected solution is the only one that does not go to infinity, a rather desirable feature for descriptive realism in most situations. However, more convincing arguments have been given, and we underline two of them. The first elaborates on the convenience argument by putting emphasis on "determinacy" ideas: For given initial conditions, the saddle-point stable path is not just the only perfect-foresight equilibrium that converges to the steady state, but it is also, since in this case there are no stationary sunspot equilibria in its neighbourhood, the only rational-expectations equilibrium that does not go to infinity. The other argument is based on "evolutive" or "adaptive" learning, that is, learning in real time based on rules for revising expectations, and gives conditions for the asymptotic convergence of such learning rules.[1]

The purpose of this paper is to revisit the justifications of the saddle-path stable solution by taking the somewhat more basic perspective of "eductive learning," which refers to considerations that have a game-theoretical flavor and explicitly refer to common-knowledge (CK) considerations.

Specifically, the viewpoint we take, the "strong rationality viewpoint,"[2] proceeds as follows. We start from restrictions on the possible paths of the system, which themselves reflect restrictions on individual strategies. These restrictions, tentatively supposed to be CK, trigger a mental process that, when rationality is itself "commonly known," mimics the process of determination of rationalizable strategies (from the initial set of restricted strategies). When such a process converges to the candidate equilibrium, the equilibrium is said to be *strongly rational*. Actually, as in the following, the CK initial restrictions will always be taken locally, so that we shall be concerned only with a weaker variant of the test that selects *locally* strongly rational equilibria. The question treated in this paper can then be more compactly reformulated: When is it the case that the *saddle-path stable solution* of a dynamical system *is* a good candidate for expectational coordination, in the sense just introduced of being *locally strongly rational*, for restrictions to be made precise?

As the reader will easily guess, the question, as raised, is almost meaningless if we refer to the standard reduced forms of dynamical systems, such as the one alluded to earlier. To make sense of the question, we must, as we did in Evans and Guesnerie (1993) in a different context, imbed the model in a framework in which agents and their strategies are well defined. This is indeed what we do in Section 2, where the strategic imbedding is presented in the more general framework of a one-step-forward-looking, one-period memory nth-dimensional system. Sections 3, 4, and 5 focus on the analysis of expectational coordination along the lines just presented in the one-dimensional linear version of the model: Section 3 considers a benchmark case, Section 4 is concerned with an extension of the so-called CK restriction, and Section 5 considers a variant of the model where the formation of expectations is subject to different institutional constraints. The conclusion follows.

A companion paper, Evans and Guesnerie (2000), extends the analysis to the nth dimensional version of the paper.

2. FRAMEWORK

2.1. Dynamic Expectations Models

We are interested in models of the following kind:

$$Q\left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}^e_{t+1}\right) = O,$$

where t is a time index, $\boldsymbol{y}$ is a finite dimensional vector, and Q is a temporary equilibrium map that relates $\boldsymbol{y}_t$ to its lagged values and to expectations. The quantity $\boldsymbol{y}^e_{t+1}$ denotes the expectation of $\boldsymbol{y}_{t+1}$ formed by agents at time t. In this formulation, we assume that agents are able to observe $\boldsymbol{y}_t$ when forming their expectations or, if not, that they can condition their actions on the values $\boldsymbol{y}_t$ that are realized.

Since, depending on the problem, this may or may not be a realistic assumption, we also consider a second variation in which the strategies of agents at t cannot be conditioned on $\boldsymbol{y}_t$:

$$\tilde{Q}\left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}^*_t, \boldsymbol{y}^*_{t+1}\right) = O.$$

In this case the actions of agents, and hence $\boldsymbol{y}_t$, depend not only on the expectations of $\boldsymbol{y}_{t+1}$, as before but now denoted by $\boldsymbol{y}^*_{t+1}$, but also on their expectations of $\boldsymbol{y}_t$ itself, which we denote by $\boldsymbol{y}^*_t$. The change in notation is used to emphasize the change in the information assumptions of the model.

In either case, as announced above, we restrict attention to deterministic systems, though it would be possible to generalize the argument to allow for exogenous stochastic shocks, and we are interested in perfect-foresight equilibria, in which expectations are correct, and in the coordination issues associated with such equilibria.

We need to be more precise on the strategic aspects of the coordination problem. To do so we will adopt a very simple strategic interpretation of the model which makes explicit the decision-theoretic aspects of the model and the aggregation of those decisions into a temporary equilibrium map.

2.2. Strategic Expectations Model

2.2.1. Basic structure. We now embed the dynamic model in a dynamic game, along lines that are somewhat similar to those of Evans and Guesnerie (1993). In this section we consider the first version of the model:

$$Q\left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}^e_{t+1}\right) = O.$$

We assume that, at each period t, there exists a continuum of agents, a part of whose strategies are not reactive to expectations (in an OLG context, these are the agents who are at the last period of their lives), and a part of which "react to expectations." The latter agents are denoted ω_t and belong to a convex segment of R, endowed with Lebesgue measure $d\omega_t$. It is assumed that an agent of period t is different from any other agent of period t', $t' \neq t$.[3] More precisely, agent ω_t has a (possibly indirect) utility function that depends upon

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(i) his own strategy $s(\omega_t)$;
(ii) sufficient statistics of the strategies played by others, that is, on $\boldsymbol{y}_t = F(\Pi_{\omega_t}\{s(\omega_t)\}, *)$, where F in turn depends first upon the strategies of all agents who at time t react to expectations, and second upon $(*)$, which is here supposed to be sufficient statistics of the strategies played by those who do not react to expectations, and that includes but is not necessarily identified with—see below—$\boldsymbol{y}_{t-1}$;
(iii) finally upon the sufficient statistics for time $t+1$, as perceived at time t: that is, on $\boldsymbol{y}_{t+1}(\omega_t)$, which *may be random* and, now directly, upon the sufficient statistics $\boldsymbol{y}_{t-1}$.

In this version of the model, we assume that the strategies played at time t can be made conditional on the equilibrium value of the t sufficient statistics $\boldsymbol{y}_t$. Now, let $(\bullet)$ denote both (the product of) $\boldsymbol{y}_{t-1}$ and the probability distribution of the random variable $\tilde{\boldsymbol{y}}_{t+1}(\omega_t)$ (the random expectation held by ω_t of $\boldsymbol{y}_{t+1}$). Then, let $G(\omega_t, \boldsymbol{y}_t, \bullet)$ be the best response function of agent ω_t. Under these assumptions, the sufficient statistics for the strategies of agents who do not react to expectations is $(*) = (\boldsymbol{y}_{t-1}, \boldsymbol{y}_t)$.

The equilibrium equations at time t are written

$$\boldsymbol{y}_t = F\left[\Pi_{\omega_t}\{G(\omega_t, \boldsymbol{y}_t, \boldsymbol{y}_{t-1}, \tilde{\boldsymbol{y}}_{t+1}(\omega_t))\}, \boldsymbol{y}_{t-1}, \boldsymbol{y}_t\right]. \quad \textbf{(1)}$$

Note that when all agents have the same point expectations denoted $\boldsymbol{y}^e_{t+1}$, the equilibrium equations determine what we called earlier the temporary equilibrium mapping:

$$Q\left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}^e_{t+1}\right) = \boldsymbol{y}_t - F\left[\Pi_{\omega_t}\left\{G\left(\omega_t, \boldsymbol{y}_t, \boldsymbol{y}_{t-1}, \boldsymbol{y}^e_{t+1}\right)\right\}, \boldsymbol{y}_{t-1}, \boldsymbol{y}_t\right].$$

2.2.2. Linearization. Let us return to the basic expression (1). The right-hand side is a rather complex term, but under regularity assumptions,[4] it has, through two different channels, derivatives with respect to $\boldsymbol{y}_t$, and with respect to $\boldsymbol{y}_{t-1}$. Also assuming that all $\tilde{\boldsymbol{y}}_{t+1}$ have a very small common support "around" some given $\boldsymbol{y}^e_{t+1}$, decision theory suggests that G, to the first order, depends on the expectation[5] of the random variable $\tilde{\boldsymbol{y}}_{t+1}(\omega_t)$, which is denoted $\boldsymbol{y}^e_{t+1}(\omega_t)$ (and is close to $\boldsymbol{y}^e_{t+1}$).

Taking into account the previous remark, the heterogeneity of expectations across agents, and assuming again the existence of adequate derivatives, it is reasonable to linearize, around any initially given situation, denoted (0), the above expression as follows[6]:

$$\boldsymbol{y}_t = U(0)\boldsymbol{y}_t + V(0)\boldsymbol{y}_{t-1} + \int W(0, \omega_t)\, \boldsymbol{y}^e_{t+1}(\omega_t)\, d\omega_t,$$

where $\boldsymbol{y}_t$, $\boldsymbol{y}_{t-1}$, and $\boldsymbol{y}^e_{t+1}(\omega_t)$ now denote small deviations from the initial values of $\boldsymbol{y}_t$, $\boldsymbol{y}_{t-1}$, and $\boldsymbol{y}^e_{t+1}$; and $U(0)$, $V(0)$, and $W(0, \omega_t)$ are $n \times n$ square matrices.

Such a linearization is valid everywhere, but we will consider it only around a steady state of the system. Hereafter, $\boldsymbol{y}_t$, $\boldsymbol{y}_{t-1}$, etc. denote deviations from the steady state and $U(0)$, $V(0)$, and $W(0, \omega_t)$ are simply U, V, and $W(\omega_t)$.

Supposing $I - U$ is invertible, we have

$$\boldsymbol{y}_t = [(I - U)^{-1} V]\,\boldsymbol{y}_{t-1} + (I - U)^{-1} \int W(\omega_t)\,\boldsymbol{y}^e_{t+1}(\omega_t)\,d\omega_t.$$

When expectations are homogeneous, $\boldsymbol{y}^e_{t+1}(\omega_t) = \boldsymbol{y}^e_{t+1}$, the system becomes

$$\boldsymbol{y}_t = B\boldsymbol{y}^e_{t+1} + D\boldsymbol{y}_{t-1},$$

with

$$B = (I - U)^{-1} W,$$

where

$$W = \int W(\omega_t)\,d\omega_t,$$

and when y is one-dimensional, using the corresponding small Greek letters, we write the system as $y_t = \beta y^e_{t+1} + \delta y_{t-1}$.

With the new notation, assuming W invertible, we can also write the initial system as

$$\boldsymbol{y}_t = D\boldsymbol{y}_{t-1} + BW^{-1} \int W(\omega_t)\,\boldsymbol{y}^e_{t+1}(\omega_t)\,d\omega_t,$$

or, for the one-dimensional version,

$$y_t = \delta y_{t-1} + \beta \varpi^{-1} \int \varpi(\omega_t)\,y^e_{t+1}(\omega_t)\,d\omega_t. \qquad \textbf{(2)}$$

At this stage, one may also notice that Nash equilibria of the game[7] coincide with perfect-foresight equilibria of the dynamical economy, whatever the dimension of the vector $\boldsymbol{y}$.

2.3. Alternative Formulation

As made explicit in Section 2.2.1, we have so far assumed that agents were able to condition their time t strategies on the equilibrium value of $\boldsymbol{y}_t$. Whether this is an appropriate assumption depends on the economic structure and we therefore consider the alternative assumption, which in some cases will be more natural, that the strategies can only be made contingent on $\boldsymbol{y}_{t-1}$. We outline how the previous argument needs to be modified.

The best response function of agent ω_t is instead $G[\omega_t, \boldsymbol{y}_{t-1}, \tilde{\boldsymbol{y}}_t(\omega_t), \tilde{\boldsymbol{y}}_{t+1}(\omega_t)]$, where $\tilde{\boldsymbol{y}}_t(\omega_t)$ denotes the random expectation by ω_t of $\boldsymbol{y}_t$, and $\tilde{\boldsymbol{y}}_{t+1}(\omega_t)$ denotes the random expectation by ω_t of $\boldsymbol{y}_{t+1}$. The dynamics at time t are governed by

$$\boldsymbol{y}_t = F\left(\Pi_{\omega_t}\{G[\omega_t, \boldsymbol{y}_{t-1}, \tilde{\boldsymbol{y}}_t(\omega_t), \tilde{\boldsymbol{y}}_{t+1}(\omega_t)]\}, \boldsymbol{y}_{t-1}\right).$$

Note that when all agents have the same point expectations $\boldsymbol{y}^*_t, \boldsymbol{y}^*_{t+1}$, the equilibrium equations determine the "temporary equilibrium mapping"

$$\tilde{Q}\left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}^*_t, \boldsymbol{y}^*_{t+1}\right) = \boldsymbol{y}_t - F\big[\Pi_{\omega_t}\big\{G\left(\omega_t, \boldsymbol{y}_{t-1}, \boldsymbol{y}^*_t, \boldsymbol{y}^*_{t+1}\right)\big\}, \boldsymbol{y}_{t-1}\big].$$

For the linearization, we follow the same line of argumentation as above and now obtain

$$\boldsymbol{y}_t = V\boldsymbol{y}_{t-1} + \int Z(\omega_t)\,\boldsymbol{y}_t^*(\omega_t)\,d\omega_t + \int W(\omega_t)\,\boldsymbol{y}_{t+1}^*(\omega_t)\,d\omega_t,$$

and for the one-dimensional system we write this as

$$y_t = \delta y_{t-1} + \beta_0\zeta^{-1}\int \zeta(\omega_t)y_t^*(\omega_t)\,d\omega_t + \beta_1\varpi^{-1}\int \varpi(\omega_t)y_{t+1}^*(\omega_t)\,d\omega_t, \quad \textbf{(3)}$$

where $\zeta = \int \zeta(\omega_t)\,d\omega_t$ and $\varpi = \int \varpi(\omega_t)\,d\omega_t$.

Coming, as above, to the case of homogeneous expectations, we have

$$y_t = \delta y_{t-1} + \beta_0 y_t^* + \beta_1 y_{t+1}^*. \quad \textbf{(4)}$$

2.4. Example

We consider a simple extension of the standard OLG (overlapping generations) model, which may be viewed as somewhat artificial, but which provides a convenient and pedagogical illustration of our framework. This is an OLG model with a fiscal policy feedback rule. Agents live for two periods, working when young and consuming when old. There is no population growth: When the old die they are replaced by an equal number of young agents with identical utility functions. When the economy begins at $t = 1$, there is an initial old generation. One unit of the single (perishable) output is produced for each unit of labor. There is a fixed quantity of money M, held by the initial old generation at $t = 1$. At each time t the young agents decide how much to produce, trading their output, net of (real) lump-sum taxes, for money at price p_t in competitive markets. Government consumption (per young agent), g_t, is given by some known specified function of current and lagged prices and is financed by lump-sum taxes levied on the young. The government and the old purchase output on the competitive market and the equilibrium price is determined by market clearing.

To set up the model formally, we assume that the utility of agent ω_t is given by

$$\Theta_{\omega_t}[c_{t+1}(\omega_t)] - \Lambda_{\omega_t}[n_t(\omega_t)],$$

where Θ_{ω_t} is increasing and concave, Λ_{ω_t} is increasing and convex and both functions are assumed to be smooth. The quantity of labor supplied when young (which is equal to the quantity of output produced by ω_t) is denoted by $n_t(\omega_t)$, and $c_{t+1}(\omega_t)$ is the quantity of consumption when old. The agent is subject to the budget constraint $c_{t+1}(\omega_t) = [n_t(\omega_t) - g_t](p_t/p_{t+1})$, where $g_t = g(p_t, p_{t-1})$. Although g need not depend on p_t, for the treatment in this paper we assume that g depends nontrivially on p_{t-1}.

When young the agent's problem is to maximize utility subject to the budget constraint. The details now depend on the information assumptions of the model,

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and we start with those of Section 2.2.1. Thus, we assume that p_t is observable at t when agent ω_t is deciding on $n_t(\omega_t)$, or at least that $n_t(\omega_t)$ can be made conditional on p_t. Assuming an interior solution and point-expectations $p^e_{t+1}(\omega_t)$ of p_{t+1}, the necessary first-order condition for the choice of agent ω_t is

$$\left[p_t/p^e_{t+1}(\omega_t)\right]\Theta'_{\omega_t}\left\{[n_t(\omega_t)-g(p_t,p_{t-1})]\left[p_t/p^e_{t+1}(\omega_t)\right]\right\}=\Lambda'_{\omega_t}[n_t(\omega_t)],$$

yielding an optimum choice of the form $n_t(\omega_t)=G[\omega_t,p_t,p_{t-1},p^e_{t+1}(\omega_t)]$.

The decisions of an old agent do not depend on expectations and in fact are trivial since their optimal strategy is simply to exchange all money holdings for goods. The model is completed by the definition $n_t=\int n_t(\omega_t)\,d\omega_t$ and the market-clearing equation $p_t[n_t-g(p_t,p_{t-1})]=M$, which can be solved locally as $p_t=\phi(n_t,p_{t-1})$. Combining this latter equation with

$$n_t=\int G\left[\omega_t,p_t,p_{t-1},p^e_{t+1}(\omega_t)\right]d\omega_t,$$

we obtain an equation of the form (1) with $y_t=p_t$. Linearization is of the form (2).

Suppose we now change the information assumptions so that when deciding on $n_t(\omega_t)$ the agent is unable to observe p_t or n_t (or to be able to condition their choices on these values). The government spending rule continues to depend at least on p_{t-1} and possibly on p_t [in which case we assume that $g_t=g(p_t,p_{t-1})$ is made conditionally on p_t]. Thus the timing we are assuming is that, first, each agent ω_t must commit to $n_t(\omega_t)$. Then, g_t and p_t are jointly determined, given n_t, by $g_t=g(p_t,p_{t-1})$ and $p_t(n_t-g_t)=M$.

The model is essentially unchanged except that, when choosing $n_t(\omega_t)$, the young agent does not know p_t or g_t. In the optimization problem, these are therefore replaced by their expectations. It is assumed that the function $g(p_t,p_{t-1})$ is known. For this alternative formulation the first-order condition for the young agent thus becomes

$$\begin{aligned}&\left[p^*_t(\omega_t)/p^*_{t+1}(\omega_t)\right]\Theta'_{\omega_t}\left(\left\{n_t(\omega_t)-g\left[p^*_t(\omega_t),p_{t-1}\right]\right\}\left[p^*_t(\omega_t)/p^*_{t+1}(\omega_t)\right]\right)\\&\quad=\Lambda'_{\omega_t}[n_t(\omega_t)],\end{aligned}$$

where $p^*_t(\omega_t)$ and $p^*_{t+1}(\omega_t)$ denote the expectations of agent ω_t, based on the more restrictive information set. The rest of the argument proceeds analogously, following now Section 2.3, with the linearization in $y_t=p_t$ taking the form (3).

2.5. A Coordination Criterion: Strong Rationality

Consider a perfect-foresight equilibrium of our economy. Why would agents coordinate expectations on such a path? As argued in the introduction, we rely on a criterion, called "strong rationality" by Guesnerie (1992, 1993), which is the following:

Assume that all agents in the system have beliefs that restrict the strategies possibly played by the others and consequently that induce restrictions on the possible paths of the system. Then, assume that these restrictions, either on strategies or on the associated paths, are common knowledge among the agents, that is, that everyone knows them, that everyone knows that everyone knows them, ... If agents are assumed to be Bayesian rational, this assumption triggers a mental process[8] that starts as follows:

Given the common-knowledge restrictions, each agent deletes strategies from his strategy set and two cases occur:

- Either the initial restrictions are further reinforced in such a way that the set of possible paths of the system is tightened,
- or the initial restrictions are neither reinforced nor even confirmed, and in the latter case the hypothetical CK assumption that was stated is self-defeating.
- In the first case, if we assume that Bayesian rationality is known, that it is known that it is known, ..., and ultimately that it is CK, the process may go on, along further steps, mimicking the process of determination of rationalizable strategies. When it converges toward a unique "equilibrium path," we say that such an equilibrium is strongly rational.

With the methodology just sketched, we shall here start from the following assumption

(Hypothetical) CK Assumption [(H) CKA]: It is CK that (the strategies are restricted so that)[9] the actual path E of the economy lies in some well-defined neighborhood $V(E^*)$ of some given perfect-foresight trajectory E^*.

The perfect-foresight trajectory that we shall consider here is a trajectory that lies in the stable manifold of the dynamical system, that is, in the one-dimensional case, on a constant-growth-rate path that converges to the steady state. The choice of neighborhood is discussed below.

The trajectory under consideration will be said to be *locally strongly rational* (LSR) whenever assertion A triggers conclusion B:

(A) (H) CKA holds, with $V(E^*)$, a local neighborhood of the candidate perfect-foresight equilibrium E^*.

(B) It is CK that the actual trajectory E *is* the candidate perfect-foresight trajectory E^*.

3. ONE-DIMENSIONAL LINEAR MODEL

From now on, we focus on the linear or linearized system, but keeping in mind the strategic structure just sketched. Since the linearization will be valid only in a neighborhood of the stationary solution $y_t = 0$, we will not consider solutions that diverge from the stationary solution. We also focus, in the present paper, on the univariate case. In this section, we consider the first formulation, in which agents have knowledge of y_t when making decisions:

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$$y_t = \delta y_{t-1} + \beta \varpi^{-1} \int \varpi(\omega_t) y^e_{t+1}(\omega_t)\, d\omega_t. \tag{5}$$

The alternative formulation is taken up in Section 5.

3.1. Perfect-Foresight Solutions

Perfect-foresight solutions satisfy $y_t = \beta y_{t+1} + \delta y_{t-1}$.

The *roots* λ_1, λ_2 of the associated quadratic

$$\beta\lambda^2 - \lambda + \delta = 0$$

are *real*, if and only if $\beta\delta \leq 1/4$. From now, *we restrict attention to this case.*

Then, the solutions can be written as

$$y_t = k_1\lambda_1^t + k_2\lambda_2^t, \tag{6}$$

where k_1, k_2 are real and $k_1 + k_2 = \hat{y}_0$, where $\hat{y}_0$ is the value given at $t = 0$.

Two particular solutions of interest are

$$y_t = \hat{y}_0\lambda_1^t \quad \text{and} \quad y_t = \hat{y}_0\lambda_2^t.$$

Letting $P(\lambda) = \beta\lambda^2 - \lambda + \delta$ and noting that $P(-1) = \beta + \delta + 1$, $P(1) = \beta + \delta - 1$, and $P'(1) = 2\beta - 1$, one can easily establish the following:

(i) Both roots are *explosive*, that is, have absolute value larger than 1, if either $\beta + \delta > 1$ and $\beta < 1/2$ or $\beta + \delta < -1$ and $\beta > -1/2$;
(ii) both roots are less than 1 in absolute value if either $\beta + \delta > 1$ and $\beta > 1/2$ or $\beta + \delta < -1$ and $\beta < -1/2$, often called the *indeterminate* case; and
(iii) one root is larger than 1 in absolute value and the other root is smaller than 1 in absolute value if $|\beta + \delta| < 1$. This is often called the *saddle-point stable* case. As indicated earlier, we will not consider explosive solutions since they diverge from the stationary solution. Both the saddle-point stable and indeterminate cases are of interest. These arise in the regions shown in Figure 1: EX denotes the regions of explosive roots, IN the regions of indterminacy, and SP the saddle-point stable region.

We note that, in the saddle-point stable case, for every initial y_0, there is a unique nonexplosive perfect-foresight solution and it converges to $y = 0$. In contrast, in the indeterminacy case, for every initial y_0, there is a continuum of paths converging to $y = 0$ and in the explosive case every path is explosive unless initially $y_0 = 0$.

3.2. Common-Knowledge Initial Restrictions

Fix the perfect-foresight solution $y_t(\lambda) = \hat{y}_0\lambda^t$, where $\lambda = \lambda_1$ or $\lambda = \lambda_2$. Here, $\hat{y}_0$ is the given initial condition. We are thus focusing on one of the perfect-foresight solutions: $y(\lambda) = \{y_t(\lambda)\}_{t=0}^{\infty}$.

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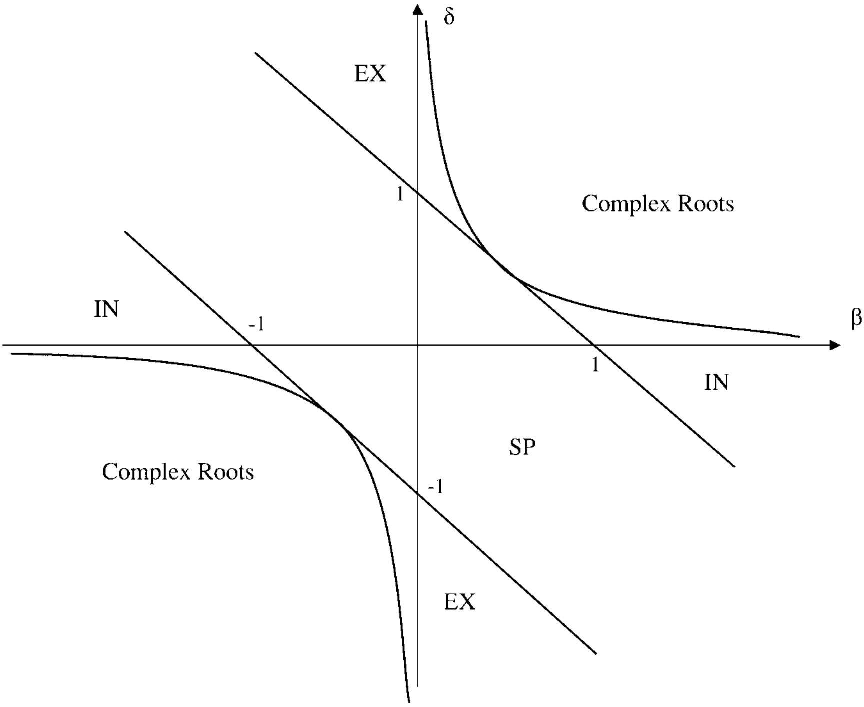

Figure 1. Basic case.

A (hypothetical) CK assumption, from which we start, might state, in the spirit of what has just been announced, that the actual trajectory $\{y_s\}_0^\infty$ is close to the equilibrium perfect-foresight trajectory $y(\lambda)$, a fact formalized in Condition $C0$:

Condition $C0$: $\forall s = 1, \ldots, \infty$, y_s lies between $(\lambda - \epsilon)^s \hat{y}_0$ and $(\lambda + \epsilon)^s \hat{y}_0$ for some specified $\epsilon > 0$.

Note, however, that for high s, such a condition says that y_s is close to zero, but not in a very precise way, since the ratio of the upper bound to the lower bound tends to infinity, when s itself tends to infinity. As a consequence, growth rates between two periods are not bounded from the assumption, but such growth rates play a key role in fixing expectations (as will become still clearer later). To say things in another way, the idea of proximity of trajectories that we have introduced in the tentative definition reflects proximity in a sense reminiscent of a C^0 topology on a space of functions, when we need to assume proximity in the sense of a C^1 topology.

This is done with Condition $C1$.

Condition $C1$: $\forall s = 1, \ldots, \infty$, y_s lies between $(\lambda - \epsilon) y_{s-1}$ and $(\lambda + \epsilon) y_{s-1}$ for some specified $\epsilon > 0$.

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Note now that $C1$ implies $C0$: Both the path and the slope of y are close to the path and slopes of $y(\lambda)$.

Note also that the assumption is formulated in terms of one-period growth rates. This formulation of the CK assumption is natural in the case, considered in this section, in which agents can condition their actions on y_t. To justify this, remember that the equilibrium equations at time t can be written

$$y_t/y_{t-1} = \delta + \beta(y_t/y_{t-1})\varpi^{-1}\int \varpi(\omega_t)\left[y^e_{t+1}(\omega_t)/y_t\right] d\omega_t \quad \text{or}$$

$$y_t/y_{t-1} = \delta\left\{1 - \beta\varpi^{-1}\int \varpi(\omega_t)\left[y^e_{t+1}(\omega_t)/y_t\right] d\omega_t\right\}^{-1}.$$

In other words, the equilibrium choices of the strategies of all agents—those who react to expectations and those who do not—amounts to a choice of a growth rate between yesterday and today that is influenced, through the choices of agents who react to expectations, only by the distribution, across those agents, of expected growth rates between today and tomorrow. A restriction on the growth rates between today and tomorrow is a special restriction on the strategies chosen by the agents tomorrow, which, however, includes their equilibrium strategies. Since equilibrium tomorrow is determined from the same strategic considerations as those intervening today, this can be viewed as a restriction of the beliefs of tomorrow's agents, who in our story are supposed to be different agents, (but this is not crucial), that is, the belief that the growth rate between tomorrow and the day after tomorrow is restricted.[10]

3.3. Conditions for Local Strong Rationality

3.3.1. Key insight. The natural question to consider is whether the (Hypothetical) CK Assumption C1, from now CKAC1, is internally coherent in the following sense: The assumption being known and believed is compatible with a set of rational reactions of agents that generate a set of possible trajectories that may or may not be compatible with the initial restriction.

Here, the following lemma is straightforward, under the condition that all the numbers $\varpi(\omega_t)$ have the same sign, for example, positive.

LEMMA 1. *Let either $\lambda = \lambda_1$ or $\lambda = \lambda_2$. If at period t, all agents conjecture that the growth rate between today and tomorrow is between $\lambda + \epsilon$ and $\lambda - \epsilon$, then the actual growth rate is between $\lambda +\ [\delta\beta/(1-\beta\lambda)^2]\epsilon + o(\epsilon^2)$ and $\lambda - [\delta\beta/(1-\beta\lambda)^2]\epsilon + o'(\epsilon^2)$,[11] where o, o' tends to zero with ϵ^2.*

Proof. From $(y_t/y_{t-1}) = \delta\{1 - \beta\varpi^{-1}\int \varpi(\omega_t)[y^e_{t+1}(\omega_t)/y_t]\, d\omega_t\}^{-1}$ and $\lambda - \epsilon \leq [y^e_{t+1}(\omega_t)/y_t] \leq \lambda + \epsilon$, it follows that $\lambda - \epsilon \leq \varpi^{-1}\int \varpi(\omega_t)[y^e_{t+1}(\omega_t)/y_t]\, d\omega_t \leq \lambda + \epsilon$. Since the derivative of $\delta(1-\beta x)^{-1}$ is $\delta\beta/(1-\beta x)^2$, the conclusion obtains immediately. ■

The result allows us to identify two cases:

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If $|\delta\beta/(1-\beta\lambda)^2| > 1$, the actual trajectory generated by the CKAC1 does not necessarily fit the CK conjecture in the following sense: There are trajectories associated with beliefs compatible with CKAC1 that generate paths that are in contradiction with CKAC1; in some sense, the assumption is not fully consistent, it is self defeating . . . .[12]

If $|\delta\beta/(1-\beta\lambda)^2| < 1$, then the growth rate of any trajectory "consistent" with CKAC1, in the sense that at each period the equilibrium is determined by the interaction of rational agents who have beliefs compatible with CKAC1, is between $\lambda - \rho\epsilon$ and $\lambda + \rho\epsilon$, with $0 < \rho < 1$, when ϵ is small enough.

Then, if the initial beliefs on growth rates are maintained through time, it will be the case that actual growth rates will always be between $\lambda - \rho\epsilon$ and $\lambda + \rho\epsilon$. Now, here is the trigger to our CK argument, which will be used repeatedly in the next sections: The conditions on actual growth rates (induced by the initial beliefs) are known by the agents of period 1, who anticipate[13] that it will be known by their successors at any period. *Then, the above process, relating actual growth rates to beliefs, can be iterated once again, and using the full power of the CK assumption, iterated indefinitely*. At stage n of the mental process, it is CK that the growth rate will be between $\lambda - \rho^n\epsilon$ and $\lambda + \rho^n\epsilon$.

3.3.2. Main result. Recall the characteristic quadratic $P(\lambda) = \beta\lambda^2 - \lambda + \delta$. Assuming $P(\lambda)$ has distinct real roots, we denote the roots

$$\lambda_1 = (2\beta)^{-1}(1 - \sqrt{1-4\beta\delta}) \quad \text{and} \quad \lambda_2 = (2\beta)^{-1}(1 + \sqrt{1-4\beta\delta}).$$

Note that λ_1 is closer to 0 than λ_2; that is, $|\lambda_1| < |\lambda_2|$.

We then have the following proposition.

PROPOSITION 1. *Assume all $\varpi(\omega_t) \geq 0$. Under CKAC1, the solution path $y(\lambda_1)$ is LSR and the solution path $y(\lambda_2)$ is not LSR.*

Proof. First note that since $\lambda = \delta(1-\beta\lambda)^{-1}$ for $\lambda = \lambda_1, \lambda_2$, the condition $|\delta\beta/(1-\beta\lambda)^2| < 1$ is equivalent to $|\beta\lambda/(1-\beta\lambda)| < 1$. There are two cases (we ignore the case $\beta\delta = 0$). If $\beta\delta > 0$, then for both $\lambda = \lambda_1, \lambda_2$ we have $0 < \beta\lambda < 1$ so that $\beta\lambda/(1-\beta\lambda) > 0$. Then, $\beta\lambda/(1-\beta\lambda) < 1$ if and only if $2\beta\lambda < 1$. This holds for λ_1 but not for λ_2, and so, $y(\lambda_1)$ is LSR and $y(\lambda_2)$ is not LSR. If $\beta\delta < 0$ then $\beta\lambda_1 < 0$ and $-1 < \beta\lambda_1/(1-\beta\lambda_1) < 0$. However, $\beta\lambda_2 > 1$ and $\beta\lambda_2/(1-\beta\lambda_2) < -1$. Thus again, $y(\lambda_1)$ is LSR and $y(\lambda_2)$ is not LSR. ■

Note that, somewhat surprisingly, the above result applies even for $|\lambda_1| \geq 1$. However, we do not stress this result because, here, we view linear models as satisfactory approximations of nonlinear phenomena, a property that certainly does not hold along an explosive path. The most relevant results, in the just-evoked perspective, concerns (i) the saddle-path case and (ii) the so-called indeterminate case:

(i) In the saddle-point stable case, it is immediate that the unique nonexplosive solution is LSR.

(ii) However, and surprisingly, LSR (local strong rationality) also holds in the case of indeterminacy for $y(\lambda_1)$, the perfect-foresight path with smaller $|\lambda|$. This is surprising because, in the earlier literature, determinacy of the perfect-foresight path has appeared to be a necessary condition for LSR. This is discussed in Section 4.2.

We stress that the above result requires the assumption that all the numbers $\varpi(\omega_t)$ have the same sign, placing bounds on the structural heterogeneity we permit. Evans and Guesnerie (1993) discuss the implications of relaxing this assumption for static models. Here, we emphasize that Proposition 1 is a special case of the more general Proposition 2.

PROPOSITION 2. *Assume that the sign of $\varpi(\omega_t)$ varies across the set of agents and let $\varpi^+ = \int_{\varpi(\omega_t)\geq 0} \varpi(\omega_t)\, d\omega_t$ and $\varpi^- = \int_{\varpi(\omega_t)<0} \varpi(\omega_t)\, d\omega_t$. Under $CKAC1$ the solution path $y(\lambda_2)$ is not LSR. The solution path $y(\lambda_1)$ is (locally) LSR whenever*

$$-1/[2(\Omega-1)] < \beta\delta < 1/[2(\Omega+1)],$$

where $\Omega = (\varpi^+ - \varpi^-)/\varpi$.

We leave the proof to the reader.[14]

Note that the right-hand side (resp., left-hand side) equals $1/4$ (resp., $-\infty$) when all the coefficients have the same sign, and decreases (resp., increases) with the heterogeneity of individual reactions, as (reasonably) measured by $\Omega = (\varpi^+ - \varpi^-)/\varpi$. Because $\beta\delta < 1/4$ whenever the roots are real, the statement is indeed a generalization of Proposition 1. Also, it precisely establishes the role of heterogeneity in LSR coordination.

4. LESS-RESTRICTIVE CK ASSUMPTIONS

4.1. Central Results

Assuming, as we now do, that $|\lambda_1| < 1$, the CK (common-knowledge) assumption $C1$ requires both that $|y_t|$ be small for large t and that the proportional growth rate of y_t be close to λ. (The latter implies the former under the assumption that we made). As stated earlier, this corresponds to the idea of proximity in C^1 topology. We consider now the effect on the conditions for strong rationality of relaxing the CK assumptions, in the following way:

Condition $C0'$: $\forall s = 1, \ldots, \infty$, $y_s = k_s + h_s y_{s-1}$, where $-\eta \leq k_s \leq \eta$ and $\lambda - \epsilon \leq h_s \leq \lambda + \epsilon$, with $\epsilon > 0$ and $\eta > 0$.

Obviously, $C0'$ is less restrictive than $C1$ since the latter is obtained from $C0'$ by setting $\eta = 0$. It can also be seen from the following result that $C0'$ involves proximity in the C^0 topology.

LEMMA 2. *Under $CKAC0'$, $\forall s = 1, \ldots, \infty$, $y_s = a_s + (g_s)^s \hat{y}_0$ where g_s lies between $\lambda - \epsilon$ and $\lambda + \epsilon$ and where $|a_s| \leq (1 - |\lambda| - \epsilon)^{-1}\eta$. Hence $\sup_s |y_s - y_s(\lambda)|$ is finite and can be made arbitrarily small by choosing ϵ and η sufficiently small.*

Proof. Substituting in recursively,

$$y_s = (k_s + h_s k_{s-1} + h_s h_{s-1} k_{s-2} + \cdots + h_s h_{s-1} \cdots h_2 k_1) + (h_s h_{s-1} \cdots h_1)\hat{y}_0.$$

Clearly, $h_s h_{s-1} \cdots h_1$ lies between $(\lambda - \epsilon)^s$ and $(\lambda + \epsilon)^s$ and

$$|k_s + h_s k_{s-1} + h_s h_{s-1} k_{s-2} + \cdots + h_s h_{s-1} \cdots h_2 k_1| \leq \eta \sum_{i=0}^{\infty} (|\lambda| + \epsilon)^i$$

$$= (1 - |\lambda| - \epsilon)^{-1} \eta.$$

This establishes the first claim, and $|y_s - y_s(\lambda)| = |a_s + (g_s)^s \hat{y}_0 - \lambda^s \hat{y}_0|$. Since $(g_s)^s = (\lambda + \epsilon_s)^s$ for some ϵ_s with $|\epsilon_s| \leq \epsilon$ and using $(\lambda + \epsilon_s)^s = \lambda^s + s\lambda^{s-1}\epsilon_s + o(\epsilon_s^2)$, we have $\max_s |y_s - y_s(\lambda)| \leq |a_s| + (s|\lambda|^{s-1}\epsilon + |o(\epsilon^2)|)|\hat{y}_0|$. Finally, for $|\lambda| < 1$ the quantity $s|\lambda|^{s-1}$ has a maximum for some positive s depending on λ. The result follows. ■

For the model (5), with CK assumptions $C0'$, we obtain the following results, again for the case in which $\varpi(\omega_t)$ all have the same sign, which we take to be positive:

LEMMA 3. *Let either $\lambda = \lambda_1$ or $\lambda = \lambda_2$. If at period t all agents conjecture that $y_{t+1} = k(\omega_t) + h(\omega_t) y_t$ where $-\eta \leq k(\omega_t) \leq \eta$ and $\lambda - \epsilon \leq h(\omega_t) \leq \lambda + \epsilon$, then the actual value of y_t is given by $y_t = \tilde{k} + \tilde{h} y_{t-1}$, where $\tilde{k}$ lies between $\beta/(1 - \beta\lambda)^{-1}\eta + o(\eta^2, \epsilon^2)$ and $-\beta/(1 - \beta\lambda)^{-1}\eta + o(\eta^2, \epsilon^2)$ and $\tilde{h}$ lies between $\lambda + [\delta\beta/(1 - \beta\lambda)^2]\epsilon + o(\epsilon^2)$ and $\lambda - [\delta\beta/(1 - \beta\lambda)^2]\epsilon + o(\epsilon^2)$.*

Proof.

$$\varpi^{-1} \int \varpi(\omega_t) y_{t+1}^e(\omega_t) = k + h y_t,$$

where $|k| \leq \eta$, $|h - \lambda| \leq \epsilon$. Hence, from (5), $y_t = \beta k(1 - \beta h)^{-1} + \delta(1 - \beta h)^{-1} y_{t-1}$. The result now follows from the derivative of the map $(x_1, x_2) \to (\beta x_1(1 - \beta x_2)^{-1}, \delta(1 - \beta x_2)^{-1})$. ■

PROPOSITION 3. *Under CK AC0$'$, the solution path $y(\lambda_1)$ is LSR if and only if the model is saddle-point stable. The solution path $y(\lambda_2)$ is not LSR.*

Proof. First, note that if $|\beta/(1 - \beta\lambda)^{-1}| < 1$ and $|\delta\beta/(1 - \beta\lambda)^2| < 1$, then, under CKAC0', Lemma 3 implies that at each period the equilibrium y_s satisfies $y_s = k_s + h_s y_{s-1}$ where $-\rho\eta \leq k_s \leq \rho\eta$ for some $0 < \rho < 1$ and $\lambda - \varsigma\epsilon \leq h_s \leq \lambda + \varsigma\epsilon$ for some $0 < \varsigma < 1$. Iterating the argument, along the same lines as in the preceding section, we have that, for all n, $-\rho^n \eta \leq k_s \leq \rho^n \eta$ and $\lambda - \varsigma^n \epsilon \leq h_s \leq \lambda + \varsigma^n \epsilon$. Using Lemma 2, this implies that the iteration picks out $y(\lambda)$, which is hence LSR. Similarly, when the above conditions do not hold, $y(\lambda)$ is not LSR.

Finally, we consider when the required conditions hold.

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In the proof of Proposition 1, it was shown that the solution $y(\lambda_2)$ always violates and $y(\lambda_1)$ always satisfies the condition $|\delta\beta/(1-\beta\lambda)^2| < 1$. Using $\beta\lambda^2 - \lambda + \delta = 0$, the condition $|\beta/(1-\beta\lambda)^{-1}| < 1$ is equivalent to $|\beta\lambda/\delta| < 1$. It is straightforward to show that $|\beta\lambda_1/\delta| < 1$ is satisfied when $\delta > 0$ if $\beta + \delta > -1$ and either $\delta > 1/2$ or $\beta + \delta < 1$. Similarly $|\beta\lambda_1/\delta| < 1$ is satisfied when $\delta < 0$ if $\beta + \delta < 1$ and either $\delta < -1/2$ or $\beta + \delta > -1$. It can be verified from Section 3.1 and from Figure 1 that these conditions are compatible with the saddle-point region and exclude the indeterminacy region. ■

4.2. Discussion

From the previous literature, one might suspect that solutions that are strongly rational are necessarily determinate, that is, locally unique. (That the converse need not hold is evident from analysis of the Muth cobweb model [Guesnerie (1992) and Evans and Guesnerie (1993)]: Even globally unique solutions are not always strongly rational.) Intuitively, if a solution is not locally unique, then, since each nearby perfect-foresight solution is necessarily rationalizable, the solution cannot be strongly rational.

The results of Section 3 might appear inconsistent with this view since for certain regions of the parameter space, for example, $\beta + \delta > 1$ and $\beta > 1/2$, the solution $y(\lambda_1)$ is indeterminate but LSR. The results we have just obtained indicate how to reconcile Section 3 with the intuition of the preceding paragraph by careful attention to the definition of "closeness."

In the C^0 topology, there is a determinate solution, namely $y(\lambda_1)$, if and only if $|\lambda_1| < 1 < |\lambda_2|$. This is evident from the set of perfect-foresight solutions (6) using the sup norm. Closeness in C^1 topology requires closeness in C^0 topology and also closeness in growth rates. For the solution $y(\lambda)$ to be determinate in C^1 topology we require that there do not exist perfect-foresight paths $y \neq y(\lambda)$ with $\sup_t |y_t/y_{t-1} - \lambda|$ arbitrarily small. This stricter definition of closeness implies that determinacy in C^1 is more permissive than determinacy in C^0. Considering again the set of perfect-foresight solutions (6) in the real case, we see that for $|\lambda_1| < |\lambda_2|$, all perfect-foresight solutions except $y(\lambda_1)$ have asymptotic growth rates equal to λ_2: If $k_2 \neq 0$, then

$$\lim_{t\to\infty} \frac{y_t}{y_{t-1}} = \lim_{t\to\infty} \frac{k_1\lambda_1^t + k_2\lambda_2^t}{k_1\lambda_1^{t-1} + k_2\lambda_2^{t-1}} = \lim_{t\to\infty} \frac{k_1 k_2^{-1}\lambda_1(\lambda_1/\lambda_2)^{t-1} + \lambda_2}{k_1 k_2^{-1}(\lambda_1/\lambda_2)^{t-1} + 1} = \lambda_2.$$

Thus in the C^1 topology the solution $y(\lambda_1)$ is always determinate, even if the absolute value of λ_2 is less than one, while $y(\lambda_2)$ is never determinate.

To summarize, the stability conditions for local strong rationality generally will depend on the common-knowledge assumptions, with less restrictive CK assumptions yielding correspondingly more restrictive stability conditions for LSR. The results of Section 3, together with the current section, illustrate, despite initial appearances to the contrary, that determinacy is indeed a necessary condition for

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local strong rationality, provided determinacy is defined using a topology that corresponds to the CK assumptions.

5. ALTERNATIVE FORMULATION

A second theme of this paper is that the information structure of the model also plays a role in the LSR stability conditions. We therefore now take up the alternative formulation of the one-dimensional model, which, when linearized, leads to the reduced form (3), reproduced here for convenience:

$$y_t = \delta y_{t-1} + \beta_0 \zeta^{-1} \int \zeta(\omega_t) y_t^*(\omega_t)\, d\omega_t + \beta_1 \varpi^{-1} \int \varpi(\omega_t) y_{t+1}^*(\omega_t)\, d\omega_t.$$

This reduced form is appropriate when agents are unable to observe or condition their strategies on y_t when deciding on their actions. We again assume that all the numbers $\zeta(\omega_t)$ have the same sign and that all the numbers $\varpi(\omega_t)$ have the same sign (for convenience we take them all to be positive).

5.1. Perfect-Foresight Solutions

We start by briefly discussing the perfect-foresight trajectories for the alternative formulation. Under perfect foresight, $y_t = \delta y_{t-1} + \beta_0 y_t + \beta_1 y_{t+1}$. The associated quadratic is now

$$\beta_1 \lambda^2 - (1 - \beta_0)\lambda + \delta = 0.$$

Roots are real, provided $\beta_1 \delta \le (1-\beta_0)^2/4$. Assuming real roots, there are solutions of the form $y_t = k_1 \lambda_1^t + k_2 \lambda_2^t$, where k_1, k_2 are real and $k_1 + k_2 = \hat{y}_0$, and we focus on the solutions

$$y_t = \hat{y}_0 \lambda_1^t \quad \text{and} \quad y_t = \hat{y}_0 \lambda_2^t,$$

where

$$\lambda_1 = \frac{(1-\beta_0) - \sqrt{(1-\beta_0)^2 - 4\beta_1\delta}}{2\beta_1} \quad \text{and}$$

$$\lambda_2 = \frac{(1-\beta_0) + \sqrt{(1-\beta_0)^2 - 4\beta_1\delta}}{2\beta_1}.$$

It can be shown that the saddle-point stable case (one root larger than 1 in absolute value and one root less than 1 in absolute value) arises when $|(\beta_1 + \delta)/(1-\beta_0)| < 1$. The indeterminate case, in which both roots have absolute value less than 1, arises if either (i) $(\beta_1 + \delta)/(1-\beta_0) > 1$ and $\beta_1/(1-\beta_0) > \frac{1}{2}$ or (ii) $(\beta_1 + \delta)/(1-\beta_0) < -1$ and $\beta_1/(1-\beta_0) < -\frac{1}{2}$.

5.2. Local Strong Rationality

We again have a key preliminary result:

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LEMMA 4. *Let either $\lambda = \lambda_1$ or $\lambda = \lambda_2$. If at period t, all agents conjecture that the growth rate between yesterday and today is between $\lambda + \epsilon$ and $\lambda - \epsilon$ and also that the growth rate between today and tomorrow lies between these bounds, then the actual growth rate lies between $\lambda + \rho\epsilon + o(\epsilon^2)$ and $\lambda - \rho\epsilon + o'(\epsilon^2)$, where*

$$\rho = max(|\beta_0|, |\beta_0 + 2\lambda\beta_1|).$$

Proof. We have

$$y_t/y_{t-1} = \delta + \beta_0\zeta^{-1}\int \zeta(\omega_t)\left[y_t^*(\omega_t)/y_{t-1}\right] d\omega_t$$

$$+\beta_1\varpi^{-1}\int \varpi(\omega_t)\left[y_t^*(\omega_t)/y_{t-1}\right]\left[y_{t+1}^*(\omega_t)/y_t^*(\omega_t)\right] d\omega_t$$

$$= \delta + \int \left[y_t^*(\omega_t)/y_{t-1}\right]\left\{\beta_0\zeta^{-1}\zeta(\omega_t) + \beta_1\varpi^{-1}\varpi(\omega_t)\left[y_{t+1}^*(\omega_t)/y_t^*(\omega_t)\right]\right\} d\omega_t.$$

Then, $\zeta^{-1}\int \zeta(\omega_t)[y_t^*(\omega_t)/y_{t-1}]\, d\omega_t$ lies between $\lambda - \epsilon$ and $\lambda + \epsilon$ since, by assumption, $y_t^*(\omega_t)/y_{t-1}$ lies between these bounds. Since also $y_{t+1}^*(\omega_t)/y_t^*(\omega_t)$ lies between $\lambda - \epsilon$ and $\lambda + \epsilon$, the expression in curly braces must lie between $\beta_0\zeta^{-1}\zeta(\omega_t) + \beta_1\varpi^{-1}\varpi(\omega_t)(\lambda - \epsilon)$ and $\beta_0\zeta^{-1}\zeta(\omega_t) + \beta_1\varpi^{-1}\varpi(\omega_t)(\lambda + \epsilon)$. The upper and lower bounds for y_t/y_{t-1} are thus the largest and smallest values of the four quantities: $\delta + \beta_0(\lambda - \epsilon) + \beta_1(\lambda - \epsilon)^2$, $\delta + \beta_0(\lambda - \epsilon) + \beta_1(\lambda - \epsilon)(\lambda + \epsilon)$, $\delta + \beta_0(\lambda + \epsilon) + \beta_1(\lambda - \epsilon)(\lambda + \epsilon)$, and $\delta + \beta_0(\lambda + \epsilon) + \beta_1(\lambda + \epsilon)^2$. Using $(\lambda + \epsilon)^2 = \lambda^2 + 2\lambda + o(\epsilon^2)$, $(\lambda - \epsilon)^2 = \lambda^2 - 2\lambda + o(\epsilon^2)$, and $(\lambda - \epsilon)(\lambda + \epsilon) = \lambda^2 + o(\epsilon^2)$, and also that $\delta + \beta_0\lambda + \beta_1\lambda^2 = \lambda$, we obtain that the upper and lower bounds for y_t/y_{t-1} are thus the largest and smallest values of the four quantities $\lambda \pm (\beta_0 + 2\lambda\beta_1)\epsilon + o(\epsilon^2)$ and $\lambda \pm \beta_0\epsilon + o(\epsilon^2)$. The lemma follows. ∎

Arguing as in Section 3, it follows that a perfect-foresight solution $y_t(\lambda) = \hat{y}_0\lambda^t$ is LSR when $\rho < 1$. We can now state the key result for the alternative formulation of our model.

PROPOSITION 4. *Under CKAC1, and assuming real roots, the solution path $y(\lambda_1)$ is (locally) SR if and only if*

$$\beta_1\delta > -1 + (1 - \beta_0)^2/4 \quad and \quad |\beta_0| < 1,$$

while the solution path $y(\lambda_2)$ is not (locally) SR.

Proof. First, consider $y(\lambda_2)$. Since $\beta_0 + 2\lambda_2\beta_1 = 1 + \sqrt{(1 - \beta_0)^2 - 4\beta_1\delta} > 1$, it follows immediately from the preceding lemma that $y(\lambda_2)$ is not SR. Next, consider $y(\lambda_1)$. Again using the Lemma 4, if $|\beta_0| \geq 1$, then $y(\lambda_1)$ is not LSR, whereas if $|\beta_0| < 1$, then $y(\lambda_1)$ is LSR if $\beta_0 + 2\lambda_1\beta_1 = 1 - \sqrt{(1 - \beta_0)^2 - 4\beta_1\delta} > -1$. This last condition is equivalent to the condition $\beta_1\delta > -1 + (1 - \beta_0)^2/4$. ∎

The results of this proposition are illustrated in Figure 2. The shaded area shows the region of strong rationality.

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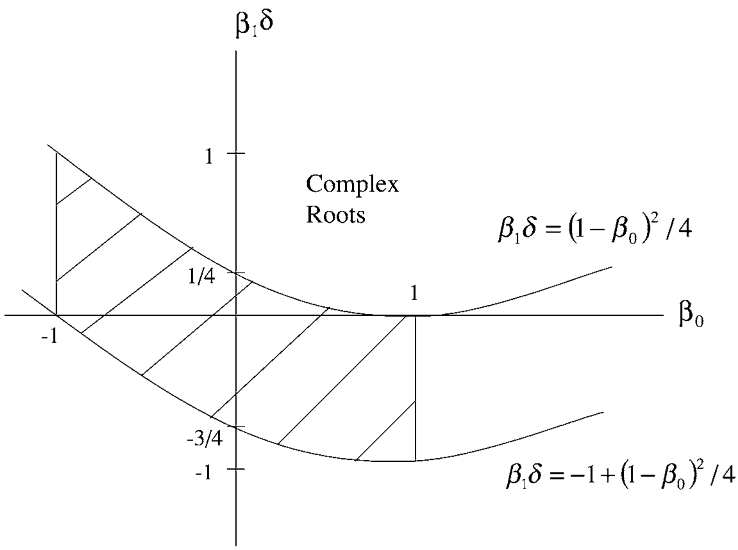

FIGURE 2. Alternative formulation.

We remark that it is also possible to derive the stability conditions for local strong rationality under the less-restrictive CK assumptions $C0'$. This is omitted for reasons of space.

5.3. Discussion

It is revealing to compare the conditions for LSR obtained in this section with those from Section 3. We do so under the same common-knowledge restrictions $CKAC1$. For both cases the solution $y(\lambda_2)$ is never SR. However, for the model of Section 3, the solution $y(\lambda_1)$ is always LSR, whereas in this section additional requirements must be met for $y(\lambda_1)$ to be LSR. This point is particularly transparent for the case $\beta_0 = 0$ under which we have here the additional LSR condition

$$\beta_1\delta > -3/4.$$

For this case the models differ only in that, under the alternative formulation, agents are unable to condition their actions on y_t. Thus, this makes clear the importance of a detailed specification of the information sets for the possibility of achieving the coordination of expectations.

It is also convenient to comment here on the relationship between (local) strong rationality and iterative expectational stability. With homogeneous expectations the alternative formulation of this section can be written

$$y_t = \delta y_{t-1} + \beta_0 y_t^* + \beta_1 y_{t+1}^*.$$

Iterative E-stability of a perfect-foresight solution $y_t = \lambda y_{t-1}$ (where $\lambda = \lambda_1$ or λ_2) is defined in terms of the mapping from the perceived law of motion $y_t = g y_{t-1}$

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to the implied actual law of motion. From the perceived law of motion, we have $y_t^* = g y_{t-1}$ and $y_{t+1}^* = g^2 y_{t-1}$ so that the implied actual law of motion is $y_t = T(g) y_{t-1}$, where $T(g) = \delta + \beta_0 g + \beta_1 g^2$. The solution $y_t = \lambda y_{t-1}$ is said to be iteratively E-stable [e.g., Evans (1985)] if $\lim_{n\to\infty} T^n(g) = \lambda$ for g near λ. Clearly, iterative E-stability holds here if $|T'(\lambda)| = |\beta_0 + 2\beta_1\lambda| < 1$.

From the results of this section, it follows that iterative E-stability is necessary but not sufficient for LSR.[15] This is in line with the earlier results of Evans and Guesnerie (1993) but for a different reason. In our earlier paper, which considered a static expectations model, the LSR conditions were stricter than iterative E-stability in certain cases in which there was sufficient structural heterogeneity. In such cases the possibility of heterogeneous expectations could prevent coordination even when iterative E-stability holds. This possibility does not arise in the current section because we have made the assumption that all numbers $\zeta(\omega_t)$, and all numbers $\varpi(\omega_t)$, have the same sign. However, a new phenomenon arises here from the dynamic structure. Our CK assumptions do not impose any assumption that the deviations from the perfect-foresight path of y_t^* be "consistent" with the deviations from y_{t+1}^*. This leads to a further possibility (which also does not arise in the version of Section 3) of agents failing to coordinate in a way not "tested" by iterative E-stability, and hence to stronger LSR conditions.

6. CONCLUSION

In the univariate linear dynamic expectations model, which we have developed at length, we have studied the conditions under which coordination of expectations on a perfect-foresight path might be expected to arise. To make it possible to "trigger" this coordination, we have assumed common knowledge of the agents that the actual path of the economy lies close to the perfect-foresight path under consideration. If these HCK (hypothetical common knowledge) assumptions are sufficient to imply common knowledge of the perfect-foresight path itself, then we say that it is locally strongly rational. This paper has worked out the LSR stability conditions under several alternative assumptions.

As we have shown, the possibility of coordination of expectations on a perfect-foresight solution, along the lines just recalled, depends to some degree on three points.

First, the stability conditions depend on the nature of the initial beliefs of the agents, a result that technically refers to the topology used to define proximity for the local HCK assumptions. The examination of this question has enabled us to illustrate, in a rather striking way, the fact that local determinacy of the equilibrium path is a necessary condition for LSR. Also, the analysis makes clear that "eductive stability" can be triggered only by quite demanding assumptions.

Second, agents' reactions to expectations should not be too heterogenous. Proposition 2 makes clear the extent to which heterogeneity, in the sense of a violation of the sign condition, makes LSR more demanding.

Third, the LSR stability conditions depend on the precise information structure adopted. This point was illustrated by considering two versions of the model, one in which agents were assumed to observe time t sufficient statistics y_t when formulating their strategies at t and a version in which they could not condition their time t strategies on y_t. In the second version, agents must allow for the possibility of time-varying deviations from the perfect-foresight path and this also leads to stricter LSR stability conditions. Hence, the justification of the saddle-path stable solution as the right solution is not necessarily obtained, even with appropriate CK restrictions and even when the agents are sufficiently homogeneous.

The specific results of this paper suggest that it would be possible and valuable to extend our techniques to obtain LSR conditions for the more general multivariate model developed in the first part of the paper. A key point, in view of this generalization, is the fact that conditions for LSR of an equilibrium, as analyzed here, are closely tied to the "determinacy" of the perfect-foresight dynamics of *growth rates*, a point emphasized by Gauthier (1998). The analysis is left to a companion paper.

NOTES

1. See, for example, Marcet and Sargent (1989) and Evans and Honkapohja (1999) for conditions under which adaptive learning converges to rational expectations in this type of model.

2. This could be called the "local unique rationalizability" viewpoint in the terminology of Bernheim (1984) or Pearce (1984), or the "local dominance solvability" viewpoint in the terminology of Farquharson (1969) and Moulin (1979).

3. This means either that each agent is "physically" different or that the agents have strategies that are independent from period to period. In an OLG interpretation of the model, each agent lives for two periods but only reacts to expectations in the first period of his life.

4. For a more complete discussion, see Evans and Guesnerie (1993, p. 637).

5. This could be formalized along lines similar to those taken by Chiappori and Guesnerie (1991), who also argue that the property is general in economic models that adopt the Bayesian view of uncertainty.

6. This can be viewed as an "axiom," whose field of validity is very large.

7. This applies to Nash equilibria that are perfect, whatever definition of perfection is adopted.

8. We can view the mental process, triggered by the CK assumption, as taking place at the beginning of time, when all agents think simultaneously about present and future decisions. We can also think of it as taking place in the minds of those who are concerned with the initial decisions (born in an OLG interpretation), but who anticipate the mental processes that will be triggered later by the assumption (which is then viewed as a continuing CK assumption).

9. Restrictions on strategies can equivalently be seen, as usual, as restrictions on beliefs or on beliefs on beliefs . . . etc.

10. Equivalently, it can be viewed as a belief, of people at t, on the beliefs, of people at $t+1$, over the beliefs, of people at $t+2$, or on a still higher-order belief.

11. Which number is the upper (or the lower) bound depends on the signs of the parameters, but the two numbers unambiguously define an interval.

12. Naturally, the restriction on growth rates might be a credible policy restriction, that is, might be implemented by a credible government. In this case, the assumption is not "hypothetical" but factual. Under the condition that we are stressing, the restriction has no further power on coordination and may actually have to be implemented in order to remain credible.

13. One can think of the mental process as taking place in the minds of people of period 1, anticipating the mental processes of their successors, endowed initially with the HCKA belief, or

as involving at the beginning of time all the future actors, assuming that they are born mentally not physically, and thinking about the system.

14. A sketch of the proof obtains as follows:

(i) In the proof of Lemma 1, we instead have

$$\lambda - \Omega\epsilon \leq \varpi^{-1} \int \varpi(\omega_t) \left[y^e_{t+1}(\omega_t) \big/ y_t \right] d\omega_t \leq \lambda + \Omega\epsilon.$$

(ii) The actual growth rates in Lemma 1 are therefore instead between $\lambda + [\delta\beta/(1-\beta\lambda)^2]\,\Omega\epsilon + o(\epsilon^2)$ and $\lambda - [\delta\beta/(1-\beta\lambda)^2]\,\Omega\epsilon + o'(\epsilon^2)$.

(iii) The LSR condition is then $|[\delta\beta/(1-\beta\lambda)^2]|\,\Omega < 1$.

(iv) Inserting the value of λ_1, found in Section 3.3.2, we obtain the result by use of simple but tedious algebra.

15. For the model of Section 3, it can be shown that iterative E-stability and the LSR conditions are identical.

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