Relative Equivariant Motives and Modules

Baptiste Calmès, Alexander Neshitov, and Kirill Zainoulline

Abstract. We introduce and study various categories of (equivariant) motives of (versal) flag varieties. We relate these categories with certain categories of parabolic (Demazure) modules. We show that the motivic decomposition type of a versal flag variety depends on the direct sum decomposition type of the parabolic module. To do this we use localization techniques of Kostant and Kumar in the context of generalized oriented cohomology as well as the Rost nilpotence principle for algebraic cobordism and its generic version. As an application, we obtain new proofs and examples of indecomposable Chow motives of versal flag varieties.

1 Introduction

The theory of Chow motives of twisted flag varieties has been a topic of intensive investigation for decades. Inspired by results on motives of quadrics by Rost [25] and Vishik [27] and on motives of Severi–Brauer varieties by Karpenko [11], it has developed into a powerful tool to study quadratic forms, linear algebraic groups, and the associated homogeneous spaces over arbitrary fields (see [6] for applications to the theory of quadratic forms). Several important invariants of algebraic groups, *e.g.*, canonical dimension, can be interpreted using the language of motives (see [24]).

We fix a split reductive linear algebraic group G over a field k of characteristic 0, its split maximal torus T, and a Borel subgroup $B \supset T$. In this paper, we focus on the study of motivic decompositions of versal flag varieties E/P, where E is a versal G-torsor and $P \supset B$ is a standard parabolic subgroup of G. We refer the reader to [10] for a recent discussion concerning Chow groups and K-theory of versal flags and to [21, 23] for recent results about motivic decompositions. It was shown in [21] that direct sum decompositions of the motive of complete versal flag E/B correspond to direct sum decompositions of the D-module $D^\star$, where D is the associated affine Hecke-type algebra for G and $D^\star$ is the T-equivariant cohomology of G/B. In the second author's PhD thesis [20], this result was extended to all versal E/P's, where P is special (all P-torsors are locally trivial in Zariski topology). Our goal is to push these results further by investigating motives of E/P's for arbitrary parabolic subgroups P.

We work in a bit more of a general situation than the theory of Chow motives. Namely, we consider an oriented cohomology theory h in the sense of Levine-Morel [19] and the theory of the associated h-motives (see [8] for definitions and basic

properties of this category). As the first step, we establish several connections between the following pseudoabelian categories: h-motives $\mathfrak{M}_k$ of versal flags, G-equivariant motives $\mathfrak{M}_G$, and relative equivariant motives $\mathfrak{M}_{G|T}$ of split flags. More precisely, we show that there is a chain of 1–1 correspondences between the direct sum decompositions of the respective objects

$$(1.1) \qquad [\text{versal } E/P] \in \mathfrak{M}_k \overset{\text{Thm.4.1}}{\longleftrightarrow} [G/P] \in \mathfrak{M}_G \overset{\text{Cor.4.2}}{\longleftrightarrow} [G/P] \in \mathfrak{M}_{G|T}.$$

To do this we use the generalization (Lemma A.13) from Chow groups to algebraic cobordism of [28, Lemma 6.2] and the Rost nilpotence for free theories proven in [8].

As the second step, we introduce the category of W-equivariant motives $\mathfrak{M}_T^W$, where W is the Weyl group of G with respect to T and show (Corollary 3.5) that there is the natural inclusion of endomorphism rings

$$(1.2) \qquad \operatorname{End}_{\mathfrak{M}_{G|T}}([G/P]) \hookrightarrow \operatorname{End}_D(D_P^\star),$$

where $D_P^\star$ is the so-called parabolic D-module with respect to the $\odot$-action of [18, §3]. (In the context of the usual cohomology, $D_P^\star$ coincides with $H_T(G/P)$ and the $\odot$-action is simply the induced W-action studied by Brion, Knutson, Peterson, Tymoczko, and others.) Informally speaking, (1.1) and (1.2) say that the motivic decomposition type of $[E/P]$ is bounded by the direct sum decomposition type of $D_P^\star$. In particular, if $D_P^\star$ is indecomposable (as a D-module), then so is $[E/P]$ (as a motive).

Next, we investigate the endomorphism ring $E_D = \operatorname{End}_D(D_P^\star)$ using the localization techniques of [3, 4] (these are generalizations of the respective techniques for Chow groups and K-theory of Kostant and Kumar [13, 14]). We describe endomorphisms of E_D with respect to the localization basis (Lemma 5.4) and the Schubert basis for the Chow theory (Corollary 6.2). We relate E_D with the induced (integral/modular) representations of the Weyl group W and its parabolic subgroup W_P. In particular, we show that if $D_P^\star$ is indecomposable, then so is $\operatorname{Ind}_{W_P}^W \mathbf{1}$ (Corollary 6.3).

Finally, we apply all these results to obtain new proofs and examples of indecomposable motives of versal flag varieties. Propositions 7.1, 7.4, 7.7, and 7.10 provide new proofs (by showing the indecomposability of the D-module $D_P^\star$) of the classical results about versal Severi-Brauer varieties and quadrics. In Proposition 7.12, we provide a new example of indecomposable motive of the 6-dimensional involution variety, the form of a split projective quadric twisted by a versal HSpin_8-torsor. Observe that results on quadrics and involution varieties (7.4, 7.7, 7.10, and 7.12) cannot be obtained from [20, 21].

The paper is organized as follows. First, we recall basic properties of equivariant oriented cohomology and of the associated category of equivariant motives; we introduce and study the categories of relative equivariant motives and W-equivariant motives. Then we investigate relations between relative equivariant motives and parabolic modules. In Section 5, we study the endomorphisms of parabolic modules using the localization techniques of [3, 4]. In the next section, we study the same endomorphisms for the Chow theory with respect to the Schubert basis using the results of [18]. In the last section, we discuss applications to Chow motives of versal flags (Sever–Brauer varieties, 4 and 6-dimensional quadrics and involution varieties). In the appendix, we prove the generic nilpotence for algebraic cobordism.

2 Categories of Equivariant Motives

In this section we recall basic properties of equivariant oriented cohomology and of the associated category of equivariant motives. This is essentially the compilation of results of [9,16,17] summarized in [4, §2]. We also introduce and study the categories of relative equivariant motives and W-equivariant motives.

Equivariant cohomology

Given a split reductive algebraic group G over a field k of characteristic 0 and given a graded oriented cohomology theory h of Levine-Morel [19, §1], let h_G be a G-equivariant oriented cohomology theory defined on the category of smooth quasi-projective G-varieties over k that satisfies properties listed in [4, §2]. More precisely, for restrictions (natural transformations res_ϕ of [4, §2]) and for the property [4, (A2)], we only require that, given a split maximal torus T of G, there is the respective T-equivariant oriented cohomology h_T that satisfies the same properties except for restrictions, and there are natural transformations of functors

$$\mathrm{res}_\phi \colon h_T \longrightarrow h \circ \mathrm{Res}_\phi \quad \text{and} \quad \mathrm{res}_{\phi'} \colon h_G \longrightarrow h_T \circ \mathrm{Res}_{\phi'}$$

that commute with push-forwards and smooth pull-backs, where Res_ϕ and $\mathrm{Res}_{\phi'}$ are the forgetful functors of [4, §2] for the inclusions $\phi \colon \{1\} \hookrightarrow T$ and $\phi' \colon T \hookrightarrow G$.

A key property of h (and of h_G) is the Quillen formula [19, Lemma 1.1.3] (see also [16, §2.5])

$$c_1^h(\mathcal{L}_1 \otimes \mathcal{L}_2) = F\big(c_1^h(\mathcal{L}_1), c_1^h(\mathcal{L}_2)\big),$$

where c_1^h is the first (equivariant) characteristic class in the theory h (resp. h_G), $\mathcal{L}_i$ is a (equivariant) line bundle over X and $F(x, y) \in R[[x, y]]$ is the associated formal group law over the coefficient ring $R = h(\mathrm{pt})$ (here $\mathrm{pt} = \mathrm{Spec}\, k$).

Conversely, given a formal group law F over R, one defines the algebraic oriented cohomology theory as

$$h(-) := \Omega(-) \otimes_{\Omega(\mathrm{pt})} R,$$

where Ω is the algebraic cobordism of Levine-Morel (universal cohomology theory) over the Lazard ring $\mathbb{L} = \Omega(\mathrm{pt})$ and the map $\Omega(\mathrm{pt}) \to R$ is determined by F. Such theories h are called *free theories* [8, §2.12]. We refer to [8] for the basic facts and properties of free theories.

By the following process (the Borel construction) one can produce many examples of equivariant oriented theories of [4, §2], such as equivariant Chow theory of [26, §1] and [5], the BG-style equivariant K-theory of [26, §3] (see [12] and [15] for the relation with Thomason's equivariant K-theory), equivariant algebraic cobordism of [9,17].

Consider a system of G-representations V_i and its open subsets $U_i \subseteq V_i$ such that

- G acts freely on U_i and the quotient U_i/G exists as a variety over k;
- $V_{i+1} = V_i \oplus W_i$ for some representation W_i;
- $U_i \subseteq U_i \oplus W_i \subseteq U_{i+1}$, and $U_i \oplus W_i \to U_{i+1}$ is an open inclusion;
- $\mathrm{codim}(V_i \backslash U_i)$ strictly increases as i does and $\lim_i \dim V_i = \infty$.

Such a system is called a *good system of representations of G*.

Now let X be a smooth G-variety (*e.g.*, a projective homogeneous variety). Following [9, §3 and §5] the inverse limit induced by pull-backs

$$\varprojlim_i \mathtt{h}(U_i \times^G X), \quad U_i \times^G X = (U_i \times X)/G,$$

does not depend on the choice of the system (V_i, U_i) and, hence, defines the equivariant oriented cohomology $\mathtt{h}_G(X)$. For the definition and properties of restriction maps we refer to [5, p. 608] and [16, §2.2].

In this paper we only deal with equivariant theories obtained by the Borel construction.

Equivariant motives

Fix a Borel subgroup B of G. Consider the full additive subcategory of the category of G-equivariant $\mathtt{h}$-motives generated by the motives (and their direct summands) of flag varieties G/P for all standard parabolic subgroups P of G containing B. We refer to [8, §2], [23], and [22] for definitions and basic properties of this category. Recall only that morphisms in this pseudo-abelian category are given by cohomology classes in $\mathtt{h}_G(X \times Y)$, where G acts diagonally on the product of smooth projective G-varieties X and Y and the composition is given by the usual correspondence product:

$$(2.1) \qquad \beta \circ \alpha := p_{XZ*}(p_{XY}^*(\alpha) p_{YZ}^*(\beta)), \quad \alpha \in \mathtt{h}_G(X \times Y), \ \beta \in \mathtt{h}_G(Y \times Z)$$

(here p_{XY} is the projection $X \times Y \times Z \to X \times Y$ and p_{XY}^*, p_{XY*} is the induced equivariant pull-back and push-forward respectively).

Taking the respective graded components (*e.g.*, $\dim X$ component of $\mathtt{h}_G(X \times X)$ for irreducible X) we obtain the category of graded equivariant motives which we call simply *G-equivariant motives* and denote by $\mathfrak{M}_G$. We refer the reader to [6, §63-64] for the comparison between graded and non-graded motives in case $\mathtt{h} = \mathrm{CH}$.

To simplify the notation, we will omit the grading degree for the cohomology, *i.e.*, we will write $\mathtt{h}_G(X \times X)$ instead of $\mathtt{h}_G^{\dim X}(X \times X)$. Observe that the actual grading can be easily traced down using the fact that the pull-backs preserve the codimension (degree) m and the push-forwards preserve the dimension $\dim X - m$ (codegree) of cohomology classes.

Relative equivariant motives

Fix a split maximal torus $T \subset B$. Similarly, let $\mathfrak{M}_T$ denote the category generated by T-equivariant motives of G/P's, *i.e.*, morphisms in $\mathfrak{M}_T$ are given by classes in $\mathtt{h}_T(X \times Y)$. Since the forgetful map

$$\mathrm{res}_T^G : \mathtt{h}_G(X \times Y) \longrightarrow \mathtt{h}_T(X \times Y)$$

commutes with smooth pull-backs and push-forwards, it induces the forgetful functor

$$\mathcal{R}es_T^G : \mathfrak{M}_G \longrightarrow \mathfrak{M}_T.$$

Definition 2.1   The categorical image of $\mathcal{R}es_T^G$, *i.e.*, the wide subcategory (subcategory containing all the objects) of $\mathfrak{M}_T$ with morphisms given by classes in $\mathrm{im}(\mathrm{res}_T^G)$, is called the category of *relative equivariant motives* and denoted by $\mathfrak{M}_{G|T}$.

We denote the endomorphism ring of $[X]$ in $\mathfrak{M}_{G|T}$ (resp. in $\mathfrak{M}_G$) by $\mathcal{E}_{G|T}(X)$ (resp. $\mathcal{E}_G(X)$). By definition,

$$\mathcal{E}_{G|T}(X) = \mathrm{im}\left(\mathrm{res}_T^G : \mathrm{h}_G(X \times X) \longrightarrow \mathrm{h}_T(X \times X)\right)$$

is an algebra (with multiplication given by the correspondence product) over the commutative ring

$$\mathcal{E}_{G|T}(\mathrm{pt}) = \mathrm{im}\left(\mathrm{res}_T^G : \mathrm{h}_G(\mathrm{pt}) \longrightarrow \mathrm{h}_T(\mathrm{pt})\right).$$

To simplify the notation, we denote $\mathcal{E}_{G|T}(\mathrm{pt})$ by $\mathcal{E}_{G|T}$ and $\mathcal{E}_G(\mathrm{pt}) = \mathrm{h}_G(\mathrm{pt})$ by $\mathcal{E}_G$.

Example 2.2 Consider the variety of complete flags G/B. Since the forgetful map is injective by [21, Lemma 4.5], we can identify $\mathcal{E}_G(G/B)$ with the convolution ring $(\mathrm{h}_G(G/B \times G/B), \circ) \simeq (\mathrm{h}_T(G/B), \circ)$ of [21, §4].

The W-action and motives

Let W be the Weyl group of G with respect to T and let X be a G-variety. There is a natural (left) action of W on $\mathrm{h}_T(X)$. It can be either realized by pull-backs induced by the right action of W on each step of the Borel construction $U \times^T X$ via

$$(u, x)T \cdot \sigma T = (u\sigma, \sigma^{-1}x)T, \quad \sigma \in N_G(T),$$

where U is taken to have a right G-action; or through the natural isomorphism $\mathrm{h}_T(X) \simeq \mathrm{h}_G(G/T \times X)$ and the G-equivariant right action of W on the variety $G/T \times X$ given on points by $(gT, x) \cdot \sigma T = (gT\sigma, x)$.

The forgetful map $\mathrm{res}_T^G : \mathrm{h}_G(X) \to \mathrm{h}_T(X)$ factors through the W-invariants $\mathrm{h}_T(X)^W$ by definition.

Lemma 2.3 *Consider the diagonal action of G on the products $X \times Y$ of G/P's and, hence, the induced action of W on $\mathrm{h}_T(X \times Y)$. For any $\alpha \in \mathrm{h}_T(X \times Y)^W$ and $\beta \in \mathrm{h}_T(Y \times Z)^W$, we have $\beta \circ \alpha \in \mathrm{h}_T(X \times Z)^W$ in $\mathfrak{M}_T$.*

Proof It follows from the fact that the projections p_{XY}, p_{YZ}, and p_{XZ} in the definition of the correspondence product (2.1) are G-equivariant. So the induced pull-backs p_{XY}^*, p_{YZ}^* and the push-forward p_{XZ*} are W-equivariant by definition of the W-action. ∎

Definition 2.4 The subcategory of $\mathfrak{M}_T$ with morphisms given by correspondences from $\mathrm{h}_T(X \times Y)^W$ of the lemma is called the category of *W-equivariant motives* and it is denoted by $\mathfrak{M}_T^W$.

Observe that the forgetful functor factors as

$$\mathcal{R}es_T^G : \mathfrak{M}_G \xrightarrow{\mathcal{R}es_{G|T}} \mathfrak{M}_{G|T} \xrightarrow{\mathcal{R}es_T^W} \mathfrak{M}_T^W \xrightarrow{\mathcal{R}es_T} \mathfrak{M}_T,$$

where $\mathcal{R}es_{G|T}$ is full and $\mathcal{R}es_T^W$, $\mathcal{R}es_T$ are faithful by definition.

3 Equivariant Motives and Parabolic Modules

In this section, we investigate relations between relative equivariant motives and parabolic modules. Our main result here is Corollary 3.5.

Realization functor

Consider the contravariant motivic realization functor

$$\mathcal{R}_T \colon \mathfrak{M}_T \longrightarrow S\text{-mod}$$

to the category of S-modules, where $S = \mathrm{h}_T(\mathrm{pt})$ is the equivariant coefficient ring. It is given on objects by $[X] \mapsto \mathrm{h}_T(X)$ and on morphisms as

$$\alpha \in \mathrm{h}_T(X \times Y) \longmapsto \alpha_* \colon \mathrm{h}_T(Y) \xrightarrow{p_Y^*} \mathrm{h}_T(X \times Y) \xrightarrow{\alpha \cdot} \mathrm{h}_T(X \times Y) \xrightarrow{p_{X*}} \mathrm{h}_T(X).$$

Restricting to W-equivariant correspondences, we obtain the realization functor

$$\mathcal{R}_T^W \colon \mathfrak{M}_T^W \longrightarrow S_W\text{-mod},$$

from $\mathfrak{M}_T^W$ to the category of S_W-modules, where S_W is the twisted group algebra of W. The latter is the free left S-module with basis δ_w, $w \in W$ and multiplication given by the twisted product

$$(s\delta_w) \cdot (s'\delta_{w'}) = sw(s')\delta_{ww'}$$

(here the W-action is given by $\delta_w(a) = w(a)$, $w \in W$, $a \in \mathrm{h}_T(X)$).

Lemma 3.1   *There is a commutative diagram of faithful functors*

$$
\begin{array}{ccc}
\mathfrak{M}_T^W & \xrightarrow{\;\mathcal{R}es_T\;} & \mathfrak{M}_T \\[2pt]
{\scriptstyle \mathcal{R}_T^W}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{R}_T} \\[2pt]
S_W\text{-mod} & \xrightarrow{\;res_T\;} & S\text{-mod,}
\end{array}
$$

where res_T is induced by the ring inclusion $S \hookrightarrow S_W$, $s \mapsto s\delta_1$.

Proof   The functors $\mathcal{R}es_T$ and res_T are faithful by definition.

Since all G/P's are T-equivariant cellular spaces, the realization maps on morphisms $\alpha \mapsto \alpha_*$ are Künneth isomorphisms of [21, Lemma 3.7]. Hence, $\mathcal{R}_T$ is faithful (for a correspondence α in $\mathfrak{M}_T$, $\alpha_* = 0 \implies \alpha = 0$) and so that $\mathcal{R}_T^W$.   ∎

Remark 3.2   Observe that [21, Lemma 3.7] is stated in [21] only for the product of the equivariant cellular space X with itself, however, the same proof works for products of different spaces leading to the Künneth isomorphism of $\mathrm{h}^G(S)$-modules $\mathrm{h}^G(X \times_S Y) \to \mathrm{Hom}_{\mathrm{h}^G(S)}(\mathrm{h}^G(X), \mathrm{h}^G(Y))$ in the notation of [21].

Corollary 3.3   *The composite of functors*

$$\mathcal{R}_T^W \circ \mathcal{R}es_T^W \colon \mathfrak{M}_{G|T} \longrightarrow S_W\text{-mod}, \qquad [X] \longmapsto \mathrm{h}_T(X), \quad \alpha \longmapsto \alpha_*$$

is faithful. In particular, there is the inclusion of rings

$$\mathcal{E}_{G|T}(X) \hookrightarrow \mathrm{End}_{S_W}\big(\mathrm{h}_T(X)\big).$$

Parabolic Demazure modules

We recall the algebraic construction of the cohomology $\mathrm{h}_T(G/P)$ from [4] and the W-action on it.

First, following [4, §3], we identify the equivariant coefficient ring $S = \mathrm{h}_T(\mathrm{pt})$ with the so called formal group algebra

$$R[[x_\lambda]]_{\lambda \in T^*}\big/\big(x_0, x_{\lambda+\mu} - F(x_\lambda, x_\mu)\big)$$

corresponding to the formal group law F of the theory h.

At the next step, assuming that S satisfies [4, Assumption 5.1] (which basically says that all x_α's are regular in S) we localize S at all x_α's (characteristic classes) corresponding to positive roots α of the root system Σ of G. The resulting localized ring is denoted by Q.

Now consider the respective localized twisted group algebra Q_W of W. The subalgebra of Q_W generated by the so-called Demazure elements $X_\alpha = x_\alpha^{-1}(1 - \delta_\alpha)$, $\alpha \in \Sigma$ and elements of S is called the *formal affine Demazure algebra* and is denoted by D.

Example 3.4 By [21] the algebra D can be identified with the convolution algebra $(\mathrm{h}_G(G/B \times G/B), \circ)$. For the Chow theory, D coincides with the nil (affine) Hecke algebra, and for the K-theory it gives the 0-affine Hecke algebra.

Let W^P denote the set of unique minimal left coset representatives of W/W_P. Consider a free Q-module Q_{W/W_P} on the basis $\{\delta_w\}$ indexed by $w \in W^P$. We denote by D_P the image of D under the canonical projection $p: Q_W \to Q_{W/W_P}$, and call it the *parabolic Demazure algebra*.

We will extensively use two key results concerning D_P: The first (see [4, Theorem 8.11]) says that the cohomology ring $\mathrm{h}_T(G/P)$ can be identified with the S-dual $D_P^\star = \mathrm{Hom}_S(D_P, S)$. The second (see [3, Thm. 11.9]) identifies $D_P^\star$ with its image in the cohomology $S_{W/W_P}^\star$ of the T-fixed point locus of G/P under the algebraic moment map.

To see how the algebraic moment map behaves with respect to the W-action, we identify the T-fixed point locus of G/P with a finite constant scheme W/W_P with trivial T-action. Then we have

$$\begin{aligned}
\mathrm{h}_T(W/W_P) &= \mathrm{h}(U \times^T W/W_P) = \oplus_{x W_P \in W/W_P} S \\
&= \mathrm{Hom}(W/W_P, S) = S_{W/W_P}^\star,
\end{aligned}$$

where the class of $(u, x W_P) T$ maps to $x W_P \mapsto [uT]$. Consider the left W-action on $\mathrm{Hom}(W/W_P, S)$ given by

$$(w \cdot f)(x) = w \cdot f(w^{-1}x), \quad x \in W/W_P, \ f \in \mathrm{Hom}(W/W_P, S).$$

We claim that the induced pullback map (the algebraic moment map)

$$D_P^\star = \mathrm{h}_T(G/P) \longrightarrow \mathrm{h}_T(W/W_P) = S_{W/W_P}^\star \text{ is } W\text{-equivariant.}$$

Indeed, the action of W on $U \times^T G/P$ is given by

$$(u, gP)T \cdot \sigma T = (u\sigma, \sigma^{-1}gP)T$$

for any U in the Borel construction. Restricted to $U \times^T W/W_P$, the action of $w = \sigma T$ gives a map

$$U \times^T W/W_P \longrightarrow U \times^T W/W_P, \quad (u, xW_P)T = (u\sigma, \sigma^{-1}xW_P)T.$$

So its pullback defines an endomorphism of $\mathrm{Hom}(W/W_P, S)$ given by $f \mapsto w \cdot f$, where $w \cdot f \colon x \mapsto wf(w^{-1}x)$.

By the very definition, the W-action on $S^\star_{W/W_P} = \mathrm{Hom}(W/W_P, S)$ and, hence, its restriction on $D_P^\star$ coincides with the $\odot$-action of [18, §3]. Together with Corollary 3.3 it gives the following corollary.

Corollary 3.5 *For any parabolic subgroup P of G there is the inclusion of rings*

$$\mathcal{E}_{G|T}(G/P) \hookrightarrow \mathrm{End}_{S_W}(D_P^\star).$$

Moreover, in [18] it was shown that the $\odot$-action of Q_W on the Q-dual Q^*_{W/W_P} restricts to the action of D on $D_P^\star$. Since Q_W is the localization of S_W and Q^*_{W/W_P} is the localization of $D_P^\star$, any S_W-equivariant endomorphism of $D_P^\star$ extends uniquely to the Q_W-equivariant endomorphism of Q^*_{W/W_P}, which then restricts to the unique D-equivariant endomorphism of $D_P^\star$. So we can replace the endomorphism ring $\mathrm{End}_{S_W}(D_P^\star)$ of the corollary by $\mathrm{End}_D(D_P^\star)$.

Definition 3.6 The D-module $D_P^\star$ with respect to the $\odot$-action is called the *parabolic module*.

Direct sum decompositions

Observe that (see [1, Corollary 6.20]) if $\mathcal{E}$ is the endomorphism ring of an object M in some pseudoabelian category $\mathcal{A}$ (*e.g.*, $M = [X]$, $\mathcal{E} = \mathcal{E}_{G|T}(X)$ in $\mathcal{A} = \mathfrak{M}_{G|T}$), then there is a 1–1 correspondence between direct sum decompositions of M and complete finite systems of pairwise orthogonal idempotents $\{p_i\}_{i \in I}$ in $\mathcal{E}$:

$$(3.1) \qquad M = \bigoplus_{i \in I} \mathrm{coker}(p_i) \quad \longleftrightarrow \quad \sum_{i \in I} p_i = \mathrm{id}, \quad p_i \circ p_j = 0, \quad i \neq j.$$

Moreover, two direct summands $\mathrm{coker}(p_i)$ and $\mathrm{coker}(p_j)$ of M are isomorphic in $\mathcal{A}$ (hence, so are the idempotents p_i and p_j of $\mathcal{E}$) if and only if there exist $\theta_{ij} \in p_i \circ \mathcal{E} \circ p_j$ and $\theta_{ji} \in p_j \circ \mathcal{E} \circ p_i$ such that $\theta_{ij} \circ \theta_{ji} = p_i$ and $\theta_{ji} \circ \theta_{ij} = p_j$.

We say that two direct sum decompositions of M (resp. two systems of idempotents in $\mathcal{E}$) are isomorphic if they coincide up to isomorphisms and permutations of summands. Finally, we say that M is indecomposable if there are no non-trivial direct summands of M (resp. if there are no non-trivial idempotents in $\mathcal{E}$).

Translating the inclusion of Corollary 3.5 into the language of idempotents and direct sum decompositions we obtain

Corollary 3.7 *If $D_P^\star$ is indecomposable (as the parabolic D-module), then so is $[G/P]$ in $\mathfrak{M}_{G|T}$ (as the motive).*

4 *G*-equivariant vs. Relative Equivariant Motives

In this section we relate the categories of *G*-equivariant motives and relative equivariant motives of split flags. Our main results are Theorem 4.1 and Corollary 4.2.

Motives of versal flag varieties

Let E be a G-torsor over a field k. Consider the (non-equivariant) category of h-motives $\mathfrak{M}_k$ given by the motives (and their direct summands) of the twisted flag varieties E/P's for all parabolic subgroups P's (the full subcategory of the category of motives considered in [8, §3]). Observe that the morphisms in $\mathfrak{M}_k$ are given by graded correspondences in $\mathrm{h}(X \times Y)$.

Following [21, §4], consider the composite of $P \times P$-equivariant maps

$$E \times E \xrightarrow{\simeq} E \times G \longrightarrow G,$$

where first map is the isomorphism by the definition of a torsor, the second map is the projection on the second factor, the action of $P \times P$ on the $E \times E$ is given by $(e_1, e_2) \cdot (p_1, p_2) = (e_1 p_1, e_2 p_2)$ and on G is given by $g \cdot (p_1, p_2) = p_1^{-1} g p_2$. The induced pullback (see [21, Lemma 4.6] for $H = P$)

$$\gamma \colon \mathrm{h}_G(G/P \times G/P) = \mathrm{h}_{P^2}(G) \longrightarrow \mathrm{h}_{P^2}(E^2) = \mathrm{h}(E/P \times E/P)$$

is the ring homomorphism $\gamma \colon \mathcal{E}_G(G/P) \to \mathcal{E}(E/P)$ from endomorphisms of the G-equivariant motive $[G/P]$ to endomorphisms of the (non-equivariant) motive $[E/P]$ (here we identify $\mathrm{h}_G((G/P)^2)$ with $\mathrm{h}_{P^2}(G)$ as in [21, (7)]).

Consider the following important case of a torsor E and variety E/P. Suppose G is a split reductive group defined over a field k' of characteristic 0. Recall that a G-torsor E is called *versal* (or *generic*) if it is the generic fiber of the quotient map $GL_N \to GL_N/G$ for an embedding of G into GL_N for some $N \geq 1$, *i.e.*, E is a G-torsor over the function field $k = k'(GL_N/G)$ (see *e.g.*, [10]). Given such E, the respective twisted flag variety E/P over k is called the versal flag variety. Informally speaking, the versal torsor E (resp. E/P) can be viewed as the 'most twisted' form of G (resp. G/P).

Lifting of idempotents

We recall several facts concerning the lifting of idempotents following [24, §2].

Suppose $\gamma \colon \mathcal{E}' \to \mathcal{E}$ is a ring homomorphism. We say that γ lifts idempotents and isomorphisms between them if for any complete finite system of pairwise orthogonal idempotents $\{p_i\}_{i \in I}$ in $\mathcal{E}$ there exists a complete system of pairwise idempotents $\{q_i\}_{i \in I}$ in $\mathcal{E}'$ such that $\gamma(q_i) = p_i$ and for any θ_{ij}, θ_{ji} defining an isomorphism between p_i and p_j there exist $\theta'_{ij}, \theta'_{ji}$ which define an isomorphism between q_i and q_j with $\gamma(\theta'_{ij}) = \theta_{ij}, \gamma(\theta'_{ji}) = \theta_{ji}$. Such γ then induces a 1–1 correspondence between isomorphisms classes of systems of idempotents in $\mathcal{E}$ and $\mathcal{E}'$.

We can additionally assume that both rings $\mathcal{E}$ and $\mathcal{E}'$ are graded, *e.g.*, for motives. In this case γ has to be a homomorphism of graded rings, all idempotents have degree 0 and all isomorphisms are homogeneous (see [24, Definition 2.3]). In view of (3.1), such γ will induce a 1–1 correspondence between isomorphism classes of direct sum decompositions of the respective motives.

We are now ready to state our main result.

Theorem 4.1 *Suppose G is a split reductive algebraic group over a field k of charac-
teristic 0. Suppose h is a free theory. Consider the associated G-equivariant theory h_G
and let $\mathfrak{M}_G$ be the respective category of G-equivariant motives. Suppose E is a versal
G-torsor over k.*

*The pull-back γ induces a 1–1 correspondence between isomorphism classes of direct
sum decompositions of the motive $[G/P]$ in $\mathfrak{M}_G$ and direct sum decompositions of the
motive $[E/P]$ in $\mathfrak{M}_k$.*

Proof Consider the flag variety G/P. Let $\{U_i\}_i$ be the sequence of G-representations
from the Borel construction for h_G. Then $\mathrm{h}_G(G/P \times G/P) = \varprojlim_i \mathrm{h}\big((U_i^2 \times G)/P^2\big)$,
where the latter quotient is defined on points $(u_1, u_2) \in U_i^2$, $g \in G$, $(p_1, p_2) \in P^2$ as

$$(U_i^2 \times G)/P^2 = (U_i^2 \times G)/(u_1, u_2, g) \sim (u_1 p_1, u_2 p_2, p_1^{-1} g p_2).$$

Let $V \supset E$ be the ambient G-representation, *i.e.*, E is G-invariant open in V. The
map γ is the limit of surjective maps (*cf.* the proof of [21, Lemma 7.9])

$$\gamma_i \colon \mathrm{h}\big((U_i^2 \times G)/P^2\big) \xrightarrow{\simeq} \mathrm{h}\big((U_i^2 \times V \times G)/P^2\big) \twoheadrightarrow$$
$$\mathrm{h}\big((U_i^2 \times E \times G)/P^2\big) \xrightarrow{\simeq} \mathrm{h}\big((U_i^2 \times E \times E)/P^2\big),$$

where the first map is an isomorphism, because of the homotopy invariance prop-
erty [4, (A4)] for the projection $V \times G \to G$, the second map is surjective by the local-
ization property [4, (A3)], and the last map is induced by the isomorphism $E \times G \simeq
E \times E$ from the definition of a G-torsor.

We now slightly modify the proof of [28, Lemma 3.2]. By the localization sequence
for h each element ϕ in the kernel of γ_i lies in the image of

$$\mathrm{h}\big((U_i^2 \times Z \times G)/P^2\big) \xrightarrow{\iota_*} \mathrm{h}\big((U_i^2 \times V \times G)/P^2\big),$$

where $Z = V \backslash E \xhookrightarrow{\iota} V$ is the closed complement of E in V. Consider the quotient
$(U_i^{d+1} \times G^d)/P^{d+1}$ defined as $U_i^{d+1} \times G^d$ modulo

$$(u_1, \dots, u_{d+1}, g_1, \dots, g_d) \sim (u_1 p_1, \dots, u_{d+1} p_{d+1}, p_1^{-1} g_1 p_2, \dots, p_1^{-1} g_d p_{d+1})$$

for $(u_1, \dots, u_{d+1}) \in U_i^{d+1}$, $(g_1, \dots, g_d) \in G^d$ and $(p_1, \dots, p_{d+1}) \in P^{d+1}$. Consider the
maps

$$\pi_{j,j'} \colon (U_i^{d+1} \times G^d)/P^{d+1} \longrightarrow (U_i^2 \times G)/P^2, \quad 1 \le j < j' \le d+1,$$

which descend from the standard projection $U_i^{d+1} \to U_i^2$ on the j and j' compo-
nents and the map $G^d \to G$ defined on points by $(g_1, \dots, g_d) \mapsto g_{j-1}^{-1} g_{j'-1}$ for $j > 1$
and $(g_1, \dots, g_d) \mapsto g_{j'-1}$ for $j = 1$. Then the d-fold correspondence product on
$\mathrm{h}((U_i^2 \times G)/P^2)$ is given by

$$\phi \circ \cdots \circ \phi = (\pi_{1,d+1})_* \Big(\prod_{j=1}^{d} \pi_{j,j+1}^*(\phi) \Big).$$

Since for each j there is the Cartesian square

$$
\begin{array}{ccc}
(U_i^{d+1} \times Z \times G^d)/P^{d+1} & \xrightarrow{\iota} & (U_i^{d+1} \times V \times G^d)/P^{d+1} \\
\downarrow & & \downarrow{\scriptstyle \pi_{j,j+1}} \\
(U_i^2 \times Z \times G)/P^2 & \longrightarrow & (U_i^2 \times V \times G)/P^2,
\end{array}
$$

for every $\phi \in \mathrm{h}\big((U_i^2 \times V \times G)/P^2\big)$ supported on $(U_i^2 \times Z \times G)/P^2$, the element $\pi_{j,j+1}^*(\phi)$ is supported on $(U_i^{d+1} \times Z \times G^d)/P^{d+1}$. By Lemma A.13 applied to

$$
p_{j,j+1} \colon Y = (U_i^{d+1} \times V \times G^d)/P^{d+1} \longrightarrow X = (U_i^2 \times V \times G)/P^2,
$$

we obtain that for $d > \dim(V)/\operatorname{codim}(Z)$, we have $\prod_{j=1}^d \operatorname{im}(\iota_*) = 0$, and, hence, $\phi^{\circ d} = 0$. Therefore, γ is a limit of surjective maps with nilpotent kernels.

Now by [20, Prop. 6.2.1], each inclusion $U_i \to U_{i+1}$ induces a surjective homomorphism $\ker(\gamma_{i+1}) \to \ker(\gamma_i)$. So by [20, Lemma 4.3.4], which is the limit-generalization of [24, Prop. 2.6], the map γ lifts idempotents and isomorphisms between idempotents in the sense of [24, Def. 2.3]. Hence, it induces the 1–1 correspondence between isomorphism classes of direct sum decompositions of $[G/P]$ in $\mathfrak{M}_G$ and $[E/P]$ in $\mathfrak{M}_k$. ∎

Corollary 4.2 *Under the hypothesis of the theorem, there is a 1–1 correspondence between direct sum decompositions of the motive $[G/P]$ in $\mathfrak{M}_G$ and in $\mathfrak{M}_{G|T}$.*

Proof In the commutative diagram

$$
\begin{array}{ccc}
\mathrm{h}_G(G/P \times_k G/P) & \xrightarrow{\gamma} & \mathrm{h}(E/P \times_k E/P) \\
{\scriptstyle \operatorname{res}_T^G}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathrm{h}_T(G/P \times_k G/P) & \xrightarrow{\operatorname{res}_k^T} & \mathrm{h}(G/P \times_{\bar{k}} G/P),
\end{array}
$$

the map ϕ has nilpotent kernel by the Rost Nilpotence Principle for free theories (see [8, Corollary 4.4]). By the theorem, γ is a limit of surjective maps with nilpotent kernels, hence, so is the restriction res_T^G. The result then follows again by the lifting of idempotents (see *e.g.,* [20, Lemma 4.3.4]). ∎

Summarizing the above arguments, we obtain the following chain of 1–1 correspondences between direct sum decompositions of the respective motives:

$$
[\text{versal } E/P] \in \mathfrak{M}_k \longleftrightarrow [G/P] \in \mathfrak{M}_G \longleftrightarrow [G/P] \in \mathfrak{M}_{G|T}.
$$

Combining this with Corollary 3.7 we obtain the following corollary.

Corollary 4.3 *If the parabolic D-module $D_P^\star$ is indecomposable, then so is the motive (in $\mathfrak{M}_k$) of the versal flag variety E/P.*

5 Localized Endomorphisms

In this section, we study the endomorphism ring $\mathrm{End}_D(D_P^\star)$ using the localization techniques of [3, 4] and the results from [18, §3]. Our main results are Corollary 5.2 and Lemma 5.4.

Endomorphisms of localized modules

Consider the Q-dual Q_{W/W_P}^* with the basis $\{f_w\}$, $w \in W^P$ dual to $\{\delta_w\}$. Following [18, §3], the $\odot$-action of Q_W on Q_{W/W_P}^* is defined by

$$q\delta_w \odot pf_v = qw(p)f_{\overline{wv}}, \quad q, p \in Q, \quad v \in W^P,$$

where $\overline{wv}$ denotes the minimal coset representative of wvW_P. Consider the endomorphism ring $E_Q = \mathrm{End}_{Q_W}(Q_{W/W_P}^*)$ of Q_W-modules with respect to the $\odot$-action.

Since the Q_W-module Q_{W/W_P}^* is generated by f_1, any $\phi \in E_Q$ is uniquely determined by its value at f_1, that is,

$$(5.1) \qquad\qquad \phi(f_1) = \sum_{w \in W^P} c_w f_w, \quad c_w \in Q.$$

Moreover, we have the following lemma.

Lemma 5.1 $\phi \in E_Q \iff v(c_w) = c_{\overline{vw}}, \; \forall v \in W_P, \; \forall w \in W^P.$

Proof $\Rightarrow$: Since $\delta_v \odot \phi(f_1) = \phi(\delta_v \odot f_1) = \phi(f_1)$ for all $v \in W_P$, we obtain

$$\sum_{w \in W^P} v(c_w) f_{\overline{vw}} = \sum_{w \in W^P} c_w f_w \quad \text{for all } v \in W_P.$$

$\Leftarrow$: For each $w' \in W^P$, we set $\phi(f_{w'}) := \delta_{w'} \odot \phi(f_1)$. It is enough to show that $\delta_w \odot \phi(f_{w'}) = \phi(\delta_w \odot f_{w'})$ for all $w, w' \in W^P$ or, equivalently,

$$\delta_{ww'} \odot \phi(f_1) = \phi(f_{\overline{ww'}}) = \delta_{\overline{ww'}} \odot \phi(f_1).$$

But $ww' = \overline{ww'}v$, $v \in W_P$, and $\delta_v \odot \phi(f_1) = \phi(f_1)$ by the assumption.   ∎

Let $^PW^P$ denote the set of minimal double W_P-coset representatives. Since each $w \in W^P$ can be written uniquely as $w = vu$, where $v \in W_P$ and $u \in {}^PW^P$, we can rewrite (5.1) as $\phi(f_1) = \sum_{w \in W^P, \, w=vu} v(c_u) f_w$, and we obtain the following corollary.

Corollary 5.2 $\phi \in E_Q$ is uniquely determined by the coefficients $c_u \in Q^{W_P \cap u W_P u^{-1}}$, $u \in {}^PW^P$.

Example 5.3   Let G be of Dynkin type A_n and let P be of type A_{n-1}, i.e., $G/P = \mathbb{P}^n$. We have $W = \langle s_1, \ldots, s_n \rangle$, $W_P = \langle s_2, \ldots, s_n \rangle$, and $W^P = \{1, v_1, v_2, \ldots, v_n\}$, where s_i denotes the i-th simple reflection and $v_i = s_i s_{i-1} \ldots s_1$.

For any $\phi \in \mathrm{End}_{Q_W}(Q_{W/W_P}^*)$, presentation (5.1) can be written as

$$\phi(f_1) = c_0 f_1 + c_1 f_{v_1} + \cdots + c_n f_{v_n}, \text{ where } c_i \in Q.$$

Since there are only two double cosets W_P and $W_P s_1 W_P$, ϕ is determined by two coefficients $c_0 \in Q^{\langle s_2, \ldots, s_n \rangle}$ and $c_1 \in Q^{\langle s_3, \ldots, s_n \rangle}$ with $c_j = \frac{v_j}{v_1}(c_1), 1 \le j \le n$.

Localized endomorphisms

The localization $D \subset Q_W$ induces the inclusion

$$E_D = \mathrm{End}_D(D_P^\star) \longhookrightarrow E_Q = \mathrm{End}_{Q_W}(Q_{W/W_P}^\star).$$

We investigate its image.

According to [18, §3] $D_P^\star$ is a D-module generated by the class of a point

$$[\mathrm{pt}] = x_{\Pi/P} f_1 \in S_{W/W_P}^\star,$$

where $x_{\Pi/P} = x_\Pi/x_P$, $x_P = \prod_{\alpha \in \Sigma_P^-} x_\alpha$ and $x_\Pi = \prod_{\alpha \in \Sigma^-} x_\alpha$, Σ_P^- is the set of all negative roots of the root subsystem for P.

Therefore, any $\phi \in E_D$ is uniquely determined by its value on $[\mathrm{pt}]$. On the other hand, $\phi([\mathrm{pt}])$ belongs to $D_P^\star$ as an element of $S_{W/W_P}^\star \subset Q_{W/W_P}^\star$ if and only if it satisfies the criteria of [3, Thm. 11.9]. Combining these, we obtain the following lemma.

Lemma 5.4 *An endomorphism $\phi \in E_Q$ comes from E_D if and only if its coefficients $c_w \in Q$ satisfy*

$$x_{\Pi/P} c_w \in S \text{ and } x_{w(\alpha)} \mid x_{\Pi/P}(c_w - c_{\overline{s_{w(\alpha)}w}}) \text{ for all } \alpha \notin \Sigma_P.$$

The localization $S_W \subset Q_W$ also induces the inclusion

$$E_S = \mathrm{End}_{S_W}(S_{W/W_P}^\star) \longhookrightarrow E_Q = \mathrm{End}_{Q_W}(Q_{W/W_P}^\star),$$

where $\phi \in E_Q$ comes from E_S if and only if all the coefficients c_w of ϕ are in S. From now on we identify E_D and E_S with their images in E_Q. We have the following lemma.

Lemma 5.5 $E_S \subset E_D$ *in* E_Q.

Proof Consider an endomorphism $\phi \in E_S$ with $\phi(f_1) = \sum c_w f_w$, $c_w \in S$. By Lemma 5.4, it is enough to show that

$$x_{w(\alpha)} \mid x_{\Pi/P}(c_w - c_{\overline{s_{w(\alpha)}w}}) \text{ for all } \alpha \notin \Sigma_P.$$

If $w(\alpha) \notin \Sigma_P$, then $x_{w(\alpha)} \mid x_{\Pi/P}$, hence we may assume $w(\alpha) \in \Sigma_P$. By Lemma 5.1, $c_{\overline{s_{w(\alpha)}w}} = s_{w(\alpha)}(c_w)$. Then $x_{w(\alpha)} \mid (c_w - s_{w(\alpha)}(c_w))$ by [2, Corollary 3.4]. ∎

Finally, observe that the endomorphism ϕ, which maps $[\mathrm{pt}]$ to $\sum_{w \in W^P} f_w \in D_P^\star$, clearly belongs to E_D but not to E_S in general.

6 Endomorphisms in Terms of the Schubert Basis

In this section we concentrate on the study of the Chow theory $\mathrm{h}(-) = \mathrm{CH}(-) \otimes R$ (here, $R = \mathbb{Z}$ or $R = \mathbb{Z}/p\mathbb{Z}$, where p is a prime). In this case, S is the polynomial ring in the basis of characters T^*, $X_i = X_{\alpha_i} = (1 - \delta_{s_i})/\alpha_i$ corresponds to the classical divided difference operator Δ_i for the simple root α_i and D is the nil (affine) Hecke algebra.

Endomorphisms of D-modules

Recall that there is the free S-basis of $h_T(G/P)$ given by the so called Schubert classes ξ_w, $w \in W^P$. The latter are the classes of the Schubert varieties of dimensions $l(w)$ (the length of w), the closures of the orbits BwP/P in G/P. We have the following well-known formula for the $\odot$-action by divided difference operators on the Schubert classes (see *e.g.,* [18, Example 4.8]):

$$X_j \odot \xi_v = \begin{cases} \xi_{s_j v} & \text{if } l(s_j v) \geq l(v) \text{ and } s_j v \in W^P, \\ 0 & \text{otherwise.} \end{cases}$$

We now describe endomorphisms of $D_P^\star$ with respect to the Schubert basis ξ_w.

Let $\phi \in E_D = \text{End}_D(D_P^\star)$. As an endomorphism of free S-modules ϕ is uniquely determined by its values on the classes ξ_w, $w \in W^P$, that is,

$$\phi(\xi_w) = \sum_v a_{v,w} \xi_v,$$

where $(a_{w,v})$ is the corresponding generalized matrix of coefficients in S (here the coefficients are indexed by the product of left Bruhat posets in v and w).

Since D is generated by X_j's and elements of S, $\phi \in E_D$ if and only if

$$X_j \odot \phi(\xi_w) = \phi(X_j \odot \xi_w) \text{ for all } j \text{ and } w \in W^P.$$

Using the properties of the $\odot$-action of [18, §3], we compute the left-hand side as follows (for $v, w \in W^P$)

$$
\begin{aligned}
X_j \odot \sum_v a_{v,w} \xi_v &= \sum_v (s_j(a_{v,w}) X_j + \Delta_j(a_{v,w})) \odot \xi_v \\
&= \sum_{s_j v \in W^P,\ l(s_j v) \geq l(v)} s_j(a_{v,w}) \xi_{s_j v} + \sum_v \Delta_j(a_{v,w}) \xi_v \\
&= \sum_{v' \in W^P,\ l(s_j v') \leq l(v')} s_j(a_{s_j v', w}) \xi_{v'} + \sum_v \Delta_j(a_{v,w}) \xi_v.
\end{aligned}
$$

As for the right-hand side, we obtain

$$\phi(X_j \odot \xi_w) = \begin{cases} \sum_v a_{v,s_j w} \xi_v & \text{if } l(s_j w) \geq l(w) \text{ and } s_j w \in W^P, \\ 0 & \text{otherwise.} \end{cases}$$

Combining, we obtain the following recursive formulas for the coefficients $a_{v,w}$.

Lemma 6.1 *If $l(s_j w) \geq l(w)$ and $s_j w \in W^P$, then*

$$a_{v,s_j w} = \begin{cases} s_j(a_{s_j v, w}) + \Delta_j(a_{v,w}) & \text{if } l(s_j v) \leq l(v), \\ \Delta_j(a_{v,w}) & \text{otherwise.} \end{cases}$$

If $l(s_j v) \leq l(v)$, then (observe that $s_j \Delta_j = \Delta_j$ in the Chow theory)

$$a_{s_j v, w} = \begin{cases} s_j(a_{v,s_j w}) - \Delta_j(a_{v,w}) & \text{if } l(s_j w) \geq l(w) \text{ and } s_j w \in W^P, \\ -\Delta_j(a_{v,w}) & \text{otherwise.} \end{cases}$$

We then obtain the following analogues of Lemma 5.1 and Corollary 5.2.

Corollary 6.2 *An endomorphism $\phi \in E_D$ is uniquely determined by the W_P-invariant coefficients $a_v = a_{v,1} \in S^{W_P}$, $v \in W^P$ such that $l(s_j v) \geq l(v)$ for all $s_j \in W_P$.*

Proof It follows immediately by the recurrent formulas that ϕ is uniquely determined by its value at $[\mathrm{pt}] = \xi_1$, *i.e.*, by the coefficients a_v. Comparing $X_j \odot \phi(\xi_1)$ and $\phi(X_j \odot \xi_1)$, we obtain that for all $s_j \in W_P$,

$$\Delta_j(a_v) = \begin{cases} -a_{s_j v} & \text{if } l(s_j v) \leq l(v), \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Homogeneous endomorphisms

Recall that the endomorphism ring of the relative equivariant motive consists of graded correspondences $\alpha \in \mathrm{CH}^n_T(G/P \times G/P)$, where $n = \dim G/P$. The respective realization map $\phi = \alpha_*$ then preserves the dimensions of cycles and, hence, the dimensions $l(w)$ of the respective Schubert classes ξ_w. In other words, the inclusion of Corollary 3.5

$$\mathcal{E}_{G|T}\big([G/P]\big) \longrightarrow E_D = \mathrm{End}_D(D_P^\star), \qquad \alpha \longmapsto \alpha_*,$$

factors through the subring $E_D^{(0)}$ of endomorphisms ϕ where each coefficient $a_{v,w}$ is a homogeneous polynomial of degree $\dim \xi_v - \dim \xi_w = l(v) - l(w)$. We call such ϕ a homogeneous endomorphism.

If $\phi \in E_D^{(0)}$, then $\phi([\mathrm{pt}]) = \sum_{v \in W^P} a_v \xi_v$, where $a_v = a_{v,1} \in S$ is homogeneous of degree $l(v)$. Since $\xi_v = \sum_{w \in W^P} c_{w,v} f_w$ where all $c_{w,v}$s are homogeneous of degree $n - l(v)$, we get $\phi(f_1) = \sum_{v \in W^P} c_w f_w$ where each coefficient $c_w = \frac{1}{x_{\Pi/P}} \sum_v a_v c_{w,v}$ is the quotient of homogeneous polynomials of degree n.

Now ϕ comes from E_S of Lemma 5.5 if and only if each $c_w \in S$ is a polynomial of degree 0, *i.e.*, each $c_w \in R$. Since $R_W = R[W]$, $R^\star_{W/W_P} = R[W/W_P]$, and the $\odot$-action restricted to $S = R$ coincides with the left action of W on W/W_P, we obtain the following corollary.

Corollary 6.3 *We have $E_S \cap E_D^{(0)} = \mathrm{End}_{R[W]}(R[W/W_P])$. In particular, if $D_P^\star$ is indecomposable (as a D-module with $R = \mathbb{Z}$), then so is the induced integral representation $\mathrm{Ind}_{W_P}^W \mathbf{1}$ of the Weyl group W.*

Idempotent matrices

Observe that $a_{v,w} = 0$ if $l(v) - l(w) < 0$. So the generalized matrix of coefficients $(a_{v,w})$ of a homogeneous endomorphism ϕ is always a lower triangular matrix.

Lemma 6.4 *The generalized matrix of coefficients $(a_{v,w})$ of an idempotent ϕ is a lower triangular idempotent matrix. In particular, its coefficients satisfy*

$$\sum_{u,\, l(u) \leq l(v)} a_u a_{v,u} = a_v, \quad v, u \in W^P,$$

and its diagonal is the direct sum of idempotent matrices

$$\bigoplus_{d=0}^{n} (a_{v,w})_{\{(v,w)|l(v)=l(w)=d\}}, \quad n = \dim G/P.$$

Proof Any idempotent ϕ is homogeneous, so $(a_{v,w})$ is triangular and idempotent. By definition, $\phi(\phi([\mathrm{pt}])) = \phi([\mathrm{pt}])$, which leads to

$$\sum_v a_v \xi_v = \phi\Big(\sum_u a_u \xi_u\Big) = \sum_u a_u \phi(\xi_u) = \sum_{u,v} a_u a_{v,u} \xi_v$$

and, hence, to the formula for the coefficients. The latter statement follows from the properties of triangular idempotent matrices.   ∎

7 Applications to Chow Motives

The purpose of this section is to demonstrate how the recursive formulas of Lemma 6.1 can be applied to show indecomposability of certain D-modules $D_P^\star$ and, hence, of Chow motives of versal flag varieties E/P. Depending on the case, we work either over $R = \mathbb{Z}$ or over $R = \mathbb{Z}/p\mathbb{Z}$, where p is the respective modular (torsion) prime. Note that the indecomposability over $\mathbb{Z}/p\mathbb{Z}$ obviously implies the indecomposability over $\mathbb{Z}$.

7.1 Projective Spaces

We fix a simple split group G of type A_n and its standard parabolic subgroup P of type A_{n-1}. The respective G/P is isomorphic to the projective space $\mathbb{P}^n$. We consider the adjoint form $G = PGL_{n+1}$ so that T^* is the root lattice with the basis given by simple roots $\{\alpha_1, \dots, \alpha_n\}$. Set $R = \mathbb{Z}$. The algebra D is then the usual nil-Hecke algebra over the polynomial ring $S = \mathbb{Z}[T^*] = \mathbb{Z}[\alpha_1, \dots, \alpha_n]$. The Weyl group is given by $W = \langle s_1, \dots, s_n \rangle$, where s_i is the simple reflection corresponding to α_i, $W_P = \langle s_2, s_3, \dots, s_n \rangle$ and $W^P = \{1, v_1, v_2, \dots, v_n\}$, where $v_i = s_i s_{i-1} \dots s_1$.

Consider an idempotent $\phi \in E_D^{(0)}$ and the associated generalized matrix of coefficients $(a_{v,w})$. For simplicity of notation, set

$$c_{i,j} := a_{v_i, v_j}, \quad c_{i,0} := a_{v_i, 1}, \quad c_{0,j} = a_{1, v_j} \text{ for } i, j \geq 1, \quad \text{and } c_{0,0} = a_{1,1}.$$

So $(c_{i,j}) = (a_{v,w})$ is a lower-triangular idempotent $(n+1) \times (n+1)$-matrix in homogeneous polynomials $c_{i,j} \in \mathbb{Z}[\alpha_1, \dots, \alpha_n]$ of degrees $\deg c_{i,j} = i - j$.

The recursive formulas of Lemma 6.1 turn into:

$$\begin{aligned}
c_{i,i} &= s_i(c_{i-1,i-1}) + \Delta_i(c_{i,i-1}) && \text{if } i \geq 1, \\
c_{i,j} &= \Delta_j(c_{i,j-1}) && \text{if } i \neq j, \, j \geq 1, \\
c_{i,0} &= (-1)^{n-i} \Delta_{i+1,\dots,n}(c_{n,0}) && \text{for } n > i \geq 1.
\end{aligned}$$

which can be expressed by the diagram of operators in Figure 1. These formulas imply (over $\mathbb{Z}$)

$$c_{i,j} = \Delta_{j,\dots,1}(c_{i,0}) = (-1)^{n-i} \Delta_{j,\dots,1,i+1,\dots,n}(c_{n,0}) \text{ for } i > j \geq 1 \quad \text{(under the diagonal)},$$

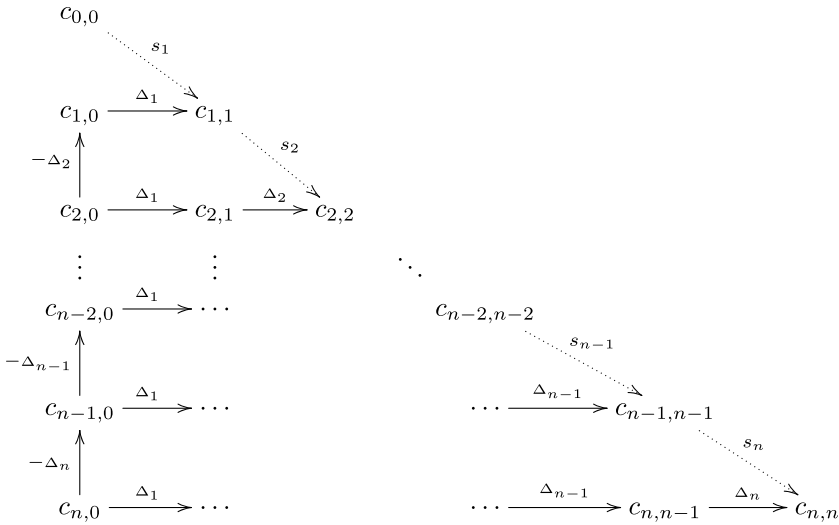$$c_{i,i} = s_i(c_{i-1,i-1}) + (-1)^{n-i} \Delta_{i,\dots,1,i+1,\dots,n}(c_{n,0}) \qquad \text{(on the diagonal)}.$$

$$c_{0,0}$$

$$c_{1,0} \xrightarrow{\Delta_1} c_{1,1}$$

$$-\Delta_2 \Bigg\uparrow$$

$$c_{2,0} \xrightarrow{\Delta_1} c_{2,1} \xrightarrow{\Delta_2} c_{2,2}$$

$$\vdots \qquad \vdots \qquad \ddots$$

$$c_{n-2,0} \xrightarrow{\Delta_1} \cdots \qquad\qquad c_{n-2,n-2}$$

$$-\Delta_{n-1} \Bigg\uparrow \qquad\qquad s_{n-1}$$

$$c_{n-1,0} \xrightarrow{\Delta_1} \cdots \qquad \cdots \xrightarrow{\Delta_{n-1}} c_{n-1,n-1}$$

$$-\Delta_n \Bigg\uparrow \qquad\qquad\qquad s_n$$

$$c_{n,0} \xrightarrow{\Delta_1} \cdots \qquad \cdots \xrightarrow{\Delta_{n-1}} c_{n,n-1} \xrightarrow{\Delta_n} c_{n,n}$$

Figure 1: The case of a projective space.

In particular, the matrix $(c_{i,j})$ and, hence, the idempotent ϕ is uniquely determined by the homogeneous polynomial $c_{n,0}$ of degree n.

Using relations in the nil Hecke algebra D for the divided difference operators Δ_j and the fact that $\Delta_j(c_{n,0}) = 0$ for all $j \neq 1, n$ (Corollary 6.2) we get

$$\Delta_k \circ \Delta_{i-1,\dots,1,i+1,\dots,n}(c_{n,0}) = 0, \quad \text{for all } k \neq i.$$

So $c_{i,i-1} \in S^{W_i}$, where W_i is generated by all simple reflections except the i-th one. Since $c_{i,i-1}$ has degree 1, we can express it as $c_{i,i-1} = b_1\alpha_1 + \cdots + b_n\alpha_n$, $b_i \in \mathbb{Z}$. Then $c_{i,i-1} \in S^{W_i}$ is equivalent to

$$b_2 = 2b_1, \quad b_3 = 3b_1, \dots, b_i = ib_1 = (n+1-i)b_n, \dots, b_{n-2} = 3b_n, \quad b_{n-1} = 2b_n.$$

Assume $n + 1 = p^r$ for some prime p and $r \geq 1$. Then

$$p \mid \Delta_i(c_{i,i-1}) = 2b_i - b_{i-1} - b_{i+1} = b_1 + b_n.$$

Since ϕ is an idempotent, all diagonal elements $c_{i,i}$ are idempotent as well by Lemma 6.4, *i.e.*, each $c_{i,i}$ is either 0 or 1. The recursive formulas and the fact that $p \mid \Delta_i(c_{i,i-1})$ then imply that $c_{i,i} = c_{i-1,i-1}$ for all i, *i.e.*, that there are no nontrivial idempotents in E_D. In other words, we obtain the following proposition.

Proposition 7.1 *Let $G = PGL_{n+1}$, where $n = p^r - 1$ for some $r \geq 1$ and a prime p. Let P be the standard parabolic subgroup of type A_{n-1}.*

Then the D-module $D_P^\star$ is indecomposable over $\mathbb{Z}$.

Remark 7.2 In view of Corollary 4.3, this fact implies the celebrated result by Karpenko [11] on indecomposability of the integral motive of the Severi–Brauer variety of a generic algebra. On the other side, according to Corollary 6.3, it also

implies that the induced integral representation $\mathrm{Ind}_{S_n}^{S_{n+1}}\mathbf{1}$ of the symmetric group is indecomposable.

7.2 Klein Quadrics

Let G be a simple split group of type A_3 with the character lattice $T^* = \langle \alpha_1, \alpha_2, \alpha_3, \omega_2 \rangle$ (here ω_2 is the respective fundamental weight), *i.e.*, $G = SL_4/\mu_2$. Let P be the standard parabolic subgroup of type $A_1 \times A_1$, *i.e.*, $G/P = Gr(2,4)$ is a split 4-dimensional smooth projective quadric. Again, we set $R = \mathbb{Z}$. By definition, the Weyl group $W = \langle s_1, s_2, s_3 \rangle$, $W_P = \langle s_1, s_3 \rangle$, and the set of minimal coset representatives W^P is given by the Hasse diagram
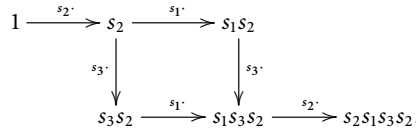
Consider an idempotent $\phi \in E_D^{(0)}$ and the associated generalized matrix $(a_{v,w})$. For simplicity, set $a_{ijk\dots}^{lmn\dots} = a_{s_i s_j s_k \dots, s_l s_m s_n \dots}$ and $a_\varnothing = a_1$.

By the recursive formulas of Lemma 6.1 and of Corollary 6.2, we obtain over $\mathbb{Z}$:

$$
\begin{aligned}
(7.1) \qquad a_2^2 &= a_\varnothing + \Delta_{2,1,3}(a_{132}), \\
a_{12}^2 &= a_2^2 - \Delta_{1,2,3}(a_{132}) \quad \text{and} \quad a_{32}^2 = a_2^2 - \Delta_{3,2,1}(a_{132}), \\
a_{32}^{12} &= -\Delta_{1,2,1}(a_{132}) \quad \text{and} \quad a_{12}^{32} = -\Delta_{3,2,3}(a_{132}), \\
a_{132}^{132} &= a_{12}^{12} + \Delta_3(\Delta_{1,2} - \Delta_{2,1})(a_{132}) = a_{32}^{32} + \Delta_1(\Delta_{3,2} - \Delta_{2,3})(a_{132}), \\
a_{2132}^{2132} &= a_{132}^{132} + \Delta_{2,1,3,2}(a_{2132}) + \Delta_{2,1,3}s_2(a_{132}),
\end{aligned}
$$

which can be also expressed as the diagram of operators in Figure 2.

Lemma 7.3 *For any homogeneous polynomial g of degree* 3, *we have the following chain of congruences modulo* 2, *i.e., over* $\mathbb{Z}/2\mathbb{Z}$:

$$
\Delta_{3,2,1}(g) \equiv \Delta_{1,2,3}(g) \equiv \Delta_{3,2,3}(g) \equiv \Delta_{1,2,1}(g),
$$
$$
\Delta_{2,1,3}(g) \equiv \Delta_{3,1,2}(g) \equiv \Delta_{1,3,2}(g) \equiv 0.
$$

Proof   As for the first chain of congruences, observe that $\Delta_1(g) \equiv \Delta_3(g) \equiv 0$ for any polynomial g that does not contain α_2, and the computations are symmetric with respect to α_1 and α_3. So it is enough to check it only on monomials $\alpha_2^2 \alpha_1$ and $\alpha_2 \omega_2 \alpha_1$. Direct computations mod 2 then give

$$
\begin{aligned}
\Delta_{3,2,1}(\alpha_2^2 \alpha_1) &\equiv \Delta_{3,2}(\alpha_1^2) \equiv \Delta_3(\alpha_2) \equiv 1, \text{ and} \\
\Delta_{1,2,3}(\alpha_2^2 \alpha_1) &\equiv \Delta_{1,2}(\alpha_1 \alpha_3) \equiv \Delta_1(\alpha_1 + \alpha_2 + \alpha_3) \equiv 1; \\
\Delta_{3,2,1}(\alpha_2 \omega_2 \alpha_1) &\equiv \Delta_{3,2}(\omega_2 \alpha_1) \equiv \Delta_3(\alpha_1 + \alpha_2 - \omega_2) \equiv 1, \text{ and} \\
\Delta_{1,2,3}(\alpha_2 \omega_2 \alpha_1) &\equiv \Delta_{1,2}(\omega_2 \alpha_1) \equiv \Delta_1(\alpha_1 + \alpha_2 - \omega_2) \equiv 1.
\end{aligned}
$$

Similarly, we get $\Delta_{3,2,3}(g) \equiv \Delta_{1,2,1}(g)$ and $\Delta_{3,2,3}(g) \equiv \Delta_{1,2,3}(g)$.
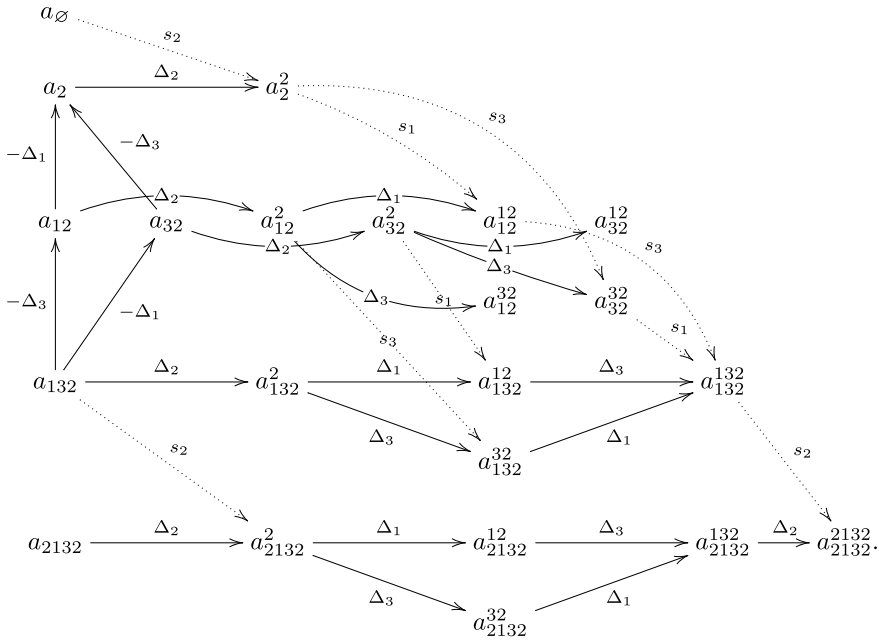
Figure 2: The case of Klein's quadric.

As for the second chain of congruences, it is enough to verify that $\Delta_{1,3}(h) \equiv 0$ for any quadratic h and $\Delta_{2,1,3}(\alpha_2^3) \equiv 0$. Indeed, for quadratic h, it reduces to $\Delta_{1,3}(\alpha_2^2) \equiv \Delta_1(\alpha_3) \equiv 0$ and

$$\Delta_{2,1,3}(\alpha_2^3) \equiv \Delta_{2,1}(\alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2) \equiv \Delta_2(\alpha_1 + \alpha_3) \equiv 0. \qquad \blacksquare$$

Now by Lemma 6.4, since ϕ is an idempotent, all diagonal entries of the generalized matrix $(a_{v,w})$ are idempotents as well. So the coefficients $a_\varnothing$, a_2^2, a_{132}^{132}, a_{2132}^{2132} can only take values 0 or 1, and the matrix

$$M = \begin{pmatrix} a_{12}^{12} & a_{32}^{12} \\ a_{12}^{32} & a_{32}^{32} \end{pmatrix}$$

is a 2×2-idempotent matrix over $\mathbb{Z}$.

By the recursive formulas (7.1) and Lemma 7.3, we obtain $a_{12}^{32} \equiv a_{32}^{12}$ and $a_{12}^{12} \equiv a_{32}^{32}$, which implies that $M \equiv 0$ (the zero matrix) or $M \equiv I$ (the identity matrix) over $\mathbb{Z}/2\mathbb{Z}$. Therefore, $a_{12}^{32} \equiv a_{32}^{12} \equiv 0$, and we get the following chain of congruences modulo 2 (here we use (7.1) and Lemma 7.3 again)

$$a_\varnothing \equiv a_2^2 \equiv a_{12}^{12} \equiv a_{32}^{32} \equiv a_{132}^{132} \equiv a_{2132}^{2132}.$$

This shows that ϕ is trivial over $\mathbb{Z}/2\mathbb{Z}$.

We claim that ϕ is trivial over $\mathbb{Z}$. As the coefficients $a_\varnothing$, a_2^2, a_{132}^{132}, a_{2132}^{2132} are either 0's or 1's over $\mathbb{Z}$, it remains to show that $M = 0$ or $M = I$ over $\mathbb{Z}$.

Indeed, suppose $M \neq 0$ over $\mathbb{Z}$. Since M is idempotent, the g.c.d. of all non-zero coefficients of M has to be 1, so $M \equiv I$. Suppose $M \neq I$ over $\mathbb{Z}$. Let m be the g.c.d. of
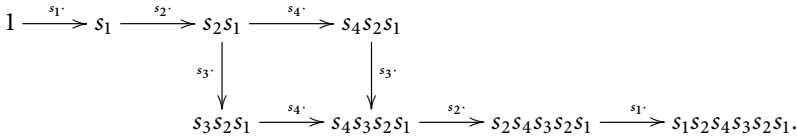
all non-zero coefficients of the difference $M - I$. Since $M - I \equiv 0$, we have $2 \mid m$. On the other side, since $(M - I)^2 = I - M$, we obtain $m = 1$, a contradiction.

Proposition 7.4   *Let G be a split simple group of type A_3 with the character lattice $T^* = \langle \alpha_1, \alpha_2, \alpha_3, \omega_2 \rangle$, i.e., $G = SL_4/\mu_2$. Let P be a parabolic subgroup of type $A_1 \times A_1$. Then the D-module $D_P^\star$ is indecomposable over $\mathbb{Z}$.*

Remark 7.5   Again in view of Corollary 4.3 this fact implies indecomposability of the integral motive of a generic 4-dimensional quadric.

7.3   Involution Varieties

Let G be a split simple group of type D_4 and let P be of type A_3, *i.e.*, G/P is a split 6-dimensional smooth projective quadric. The Weyl group $W = \langle s_1, s_2, s_3, s_4 \rangle$, $W_P = \langle s_2, s_3, s_4 \rangle$ and the set of minimal coset representatives W^P is given by the Hasse diagram

$$1 \xrightarrow{\ s_1 \cdot\ } s_1 \xrightarrow{\ s_2 \cdot\ } s_2 s_1 \xrightarrow{\ s_4 \cdot\ } s_4 s_2 s_1$$

with $s_3\cdot$ and $s_3\cdot$ vertical arrows down to

$$s_3 s_2 s_1 \xrightarrow{\ s_4 \cdot\ } s_4 s_3 s_2 s_1 \xrightarrow{\ s_2 \cdot\ } s_2 s_4 s_3 s_2 s_1 \xrightarrow{\ s_1 \cdot\ } s_1 s_2 s_4 s_3 s_2 s_1.$$

By the recursive formulas of Lemma 6.1 and of Corollary 6.2, we obtain (as before we set $a_{ijk\ldots}^{klm\cdots} = a_{s_i s_j s_k \ldots, s_k s_l s_m \ldots}$ and $a_\varnothing = a_1$):

$$a_1^1 = a_\varnothing + \Delta_{1,2,3,4,2}(a_{24321}),$$
$$a_{21}^{21} = a_1^1 - \Delta_{2,1,3,4,2}(a_{24321}),$$
$$a_{321}^{321} = a_{21}^{21} + \Delta_{3,2,1,4,2}(a_{24321}) \text{ and } a_{421}^{421} = a_{21}^{21} + \Delta_{4,2,1,3,2}(a_{24321}),$$
$$a_{321}^{421} = \Delta_{4,2,1,4,2}(a_{24321}) \text{ and } a_{421}^{321} = \Delta_{3,2,1,3,2}(a_{24321}),$$
$$a_{4321}^{4321} = a_{421}^{421} + \Delta_{3,2,1,4,2}(a_{24321}) - \Delta_{3,4,2,1,2}(a_{24321})$$
$$\qquad = a_{321}^{321} + \Delta_{4,2,1,3,2}(a_{24321}) - \Delta_{4,3,2,1,2}(a_{24321}),$$
$$a_{24321}^{24321} = a_{4321}^{4321} + (\Delta_{2,4,3,2,1} - \Delta_{2,4,3}s_2\Delta_{1,2})(a_{24321}),$$
$$a_{124321}^{124321} = a_{24321}^{24321} + \Delta_{1,2,4,3,2}\big(s_1(a_{24321}) + \Delta_1(a_{124321})\big),$$

which can be also expressed as the diagram of operators in Figure 3.

From this point on, we set $R = \mathbb{Z}/2\mathbb{Z}$. Let $\Delta_{i_1, i_2, \ldots}^d$ denote the image $\Delta_{i_1, i_2, \ldots}$ $(\mathrm{Sym}^d(T^*))$ modulo 2.

PGO_8-**case**

Assume T^* corresponds to the root lattice, *i.e.*, $G = PGO_8$. We have the following analogue of Lemma 7.3 for the type D_4.

Lemma 7.6   *We have (over $\mathbb{Z}/2\mathbb{Z}$) $\Delta_{3,4}^2 = 0$, $\Delta_{3,4}^3 = \langle \alpha_3 + \alpha_4 \rangle$,*

$$\Delta_{3,4}^4 = \big\langle (\alpha_3 + \alpha_4)\alpha_1, (\alpha_3 + \alpha_4)\alpha_3, (\alpha_3 + \alpha_4)\alpha_4 \big\rangle, \quad \Delta_{2,3,4}^4 = \langle \alpha_3 + \alpha_4 \rangle.$$

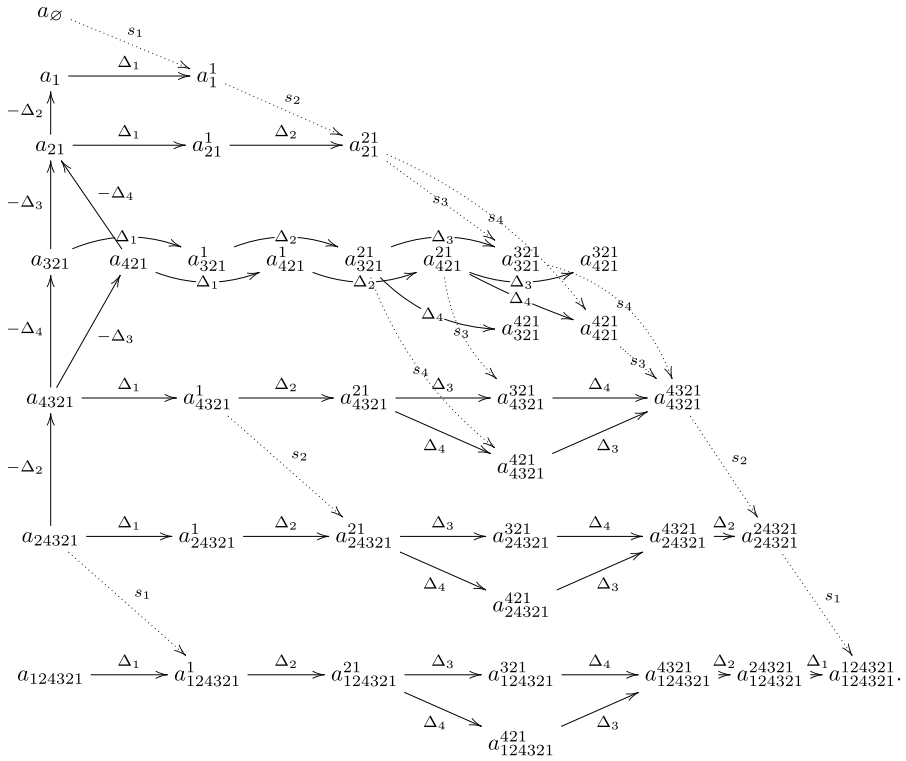*Moreover, it holds for any permutation of the set of subscripts $\{1, 3, 4\}$.*

Figure 3: The PGO-case.

Proof The proof follows from the fact that Δ_3 and Δ_4 are trivial mod 2 on all simple roots except α_2 and that $\Delta_{3,4}(\alpha_2^2) \equiv \Delta_{3,4}(\alpha_2^4) \equiv 0$, $\Delta_{3,4}(\alpha_2^3) \equiv \alpha_3 + \alpha_4$. ∎

From the lemma, we immediately obtain

$$a_{124321}^{124321}, a_1^1 : \Delta_{1,2,3,4}^4 = \Delta_1(\Delta_{2,3,4}^4) = 0,$$
$$a_{24321}^{24321}, a_{21}^{21} : \Delta_{2,3,4}^3 = 0, \; \Delta_{2,1}(\Delta_{3,4}^4) = 0,$$
$$a_{321}^{321}, a_{421}^{421} : \Delta_{3,2,1,4}^4 = 0, \; \Delta_{4,2,1,3}^4 = 0,$$
$$a_{4321}^{4321} : \Delta_{4,2,1,3}^4 = 0, \; \Delta_{3,4}^2 = 0.$$

If $\phi \in E_D$ is an idempotent over $\mathbb{Z}/2\mathbb{Z}$, it gives

$$a_\varnothing \equiv a_1^1 \equiv a_{21}^{21} \equiv a_{321}^{321} \equiv a_{421}^{421} \equiv a_{4321}^{4321} \equiv a_{24321}^{24321} \equiv a_{124321}^{124321}.$$

So there are no non-trivial idempotents over $\mathbb{Z}/2\mathbb{Z}$, and we obtain the following proposition.

Proposition 7.7 *Let $G = PGO_8$ and let P be the maximal parabolic subgroup generated by all simple reflections except the first one.*
Then the D-module $D_P^\star$ is indecomposable over $\mathbb{Z}/2\mathbb{Z}$.

Remark 7.8   In view of Corollary 4.3, this fact implies indecomposability of the $\mathbb{Z}/2\mathbb{Z}$-motive of a generic twisted form (*e.g.*, of involution variety) of a 6-dimensional split quadric.

SO_8-case

Assume that $\omega_1 \in T^*$, that is $G = SO_8$. Then Lemma 7.6 turns into the following lemma.

Lemma 7.9   *We have* $\Delta_{3,4}^2 = 0$, $\Delta_{3,4}^3 = \langle \alpha_3 + \alpha_4 \rangle$,

$$\Delta_{3,4}^4 = \big\langle (\alpha_3 + \alpha_4)\alpha_1, (\alpha_3 + \alpha_4)\alpha_3, (\alpha_3 + \alpha_4)\alpha_4, (\alpha_3 + \alpha_4)\omega_1 \big\rangle, \ \Delta_{2,3,4}^4 = \langle \alpha_3 + \alpha_4 \rangle.$$

So we obtain

$$a_{124321}^{124321}, a_1^1 : \Delta_{1,2,3,4}^4 = \Delta_1(\Delta_{2,3,4}^4) = 0$$
$$a_{24321}^{24321}, a_{21}^{21} : \Delta_{2,3,4}^3 = 0, \ \Delta_{2,1}(\Delta_{3,4}^4) = 0,$$

which gives only that

$$a_\varnothing \equiv a_1^1 \equiv a_{21}^{21} \quad \text{and} \quad a_{4321}^{4321} \equiv a_{24321}^{24321} \equiv a_{124321}^{124321}.$$

Since $\Delta_3(f) \equiv \Delta_4(f)$ for any linear f, we have

$$\Delta_{4,2,1,3,2} \equiv \Delta_{3,2,1,3,2}.$$

Moreover, direct computations show that

$$\Delta_{3,2,3}(\alpha_2^2 \alpha_3) \equiv \Delta_{3,2,3}(\alpha_2^2 \alpha_4) \equiv 1,$$

so that $\Delta_{3,2,3}(g) \equiv \Delta_{4,2,4}(g)$. Combining, we obtain

$$\Delta_{4,2,1,3} = \Delta_{4,2,3,1} \equiv \Delta_{3,2,3,1} \equiv \Delta_{4,2,4,1} \equiv \Delta_{3,2,4,1} = \Delta_{3,2,1,4},$$

so $a_{21}^{21} \equiv a_{321}^{321} \equiv a_{421}^{421} \equiv a_{4321}^{4321}$. Hence, there are no non-trivial idempotents as well, and we get the following proposition.

Proposition 7.10   *Let $G = SO_8$ and let P be the maximal parabolic subgroup generated by all simple reflections except the first one.*
   Then the D-module $D_P^\star$ is indecomposable over $\mathbb{Z}/2\mathbb{Z}$.

Remark 7.11   This fact implies indecomposability of the $\mathbb{Z}/2\mathbb{Z}$-motive of a generic 6-dimensional quadric.

$HSpin_8$-case

Assume $\omega_4 \in T^*$, that is, $G = HSpin_8$. Then $T^* = \langle \alpha_2, \alpha_3, \alpha_4, \omega_4 \rangle$.
   We claim that $\Delta_{1,2,3,4,2}(a_{24321}) \equiv 0$. Indeed, let $f = \Delta_{2,3,4,2}(a_{24321}) \in \mathrm{Sym}^1(T^*)$. Then

$$\Delta_3(f) = \Delta_{3,2,3,4,2}(a_{24321}) = \Delta_{2,3,2,4,2}(a_{24321}) = \Delta_{2,3,4,2}(\Delta_4(a_{24321})) = 0.$$

Now for $f = a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + b\omega_4$, we get $\Delta_1(f) \equiv a_2$, but $\Delta_3(f) \equiv a_2$ as well.

Similarly, $\Delta_{3,2,1,4,2}(a_{24321}) \equiv 0$. In this case, denote $f = \Delta_{2,1,4,2}(a_{24321})$. Then $\Delta_1(f) = \Delta_{1,2,1,4,2}(a_{24321}) = \Delta_{2,1,2,4,2}(a_{24321}) = \Delta_{2,1,4,2}(\Delta_4(a_{24321})) = 0$, and $\Delta_3(f) \equiv \Delta_1(f) \equiv 0$.

By the same arguments, $\Delta_{3,2,1,3,2}(a_{24321}) \equiv \Delta_{3,4,2,1,2} \equiv 0$.

Now consider $\Delta_{2,1,3,4,2}(a_{24321})$. Let $g = \Delta_{3,4,2}(a_{24321})$. We have $\Delta_{3,2}(g) = 0$. Let $g = \sum_{2 \le i \le j} c_{ij} \alpha_i \alpha_j + \sum_{2 \le i} b_i \omega_4 \alpha_i + d\omega_4^2$. Then

$$\Delta_2(g) \equiv c_{22}(\alpha_2 + \alpha_3 + \alpha_4) + \alpha_2(c_{23} + c_{24}) + \omega_4(b_3 + b_4).$$

The fact that $\Delta_{3,2}(g) \equiv 0$ implies that $c_{22} + c_{23} + c_{24} \equiv 0$. But

$$\Delta_1(g) \equiv c_{22}\alpha_1 + c_{23}\alpha_3 + c_{24}\alpha_4 + b_2\omega_4,$$

so that $\Delta_{2,1}(g) \equiv (c_{22} + c_{23} + c_{24}) \equiv 0$. Combining, we obtain that

$$a_\varnothing \equiv a_1^1 \equiv a_{21}^{21} \equiv a_{321}^{321} \equiv a_{421}^{321},$$
$$a_{421}^{421} \equiv a_{321}^{421} \equiv a_{4321}^{4321} \equiv a_{24321}^{24321} \equiv a_{124321}^{124321}.$$

So the D-module $D_P^\star$ is either indecomposable over $\mathbb{Z}/2\mathbb{Z}$ or it splits into two indecomposable direct summands with a generating function $1 + t + t^2 + t^3$ (over S) each.

Remark 7.12 This fact implies that the motive of a HSpin_8-versal involution variety E/P is either indecomposable or splits as a direct sum $[E/P] = N \oplus N(3)$, where N is indecomposable with a generating function $1 + t + t^2 + t^3$. Using known results on motives of quadratic forms (*e.g.,* that after splitting the algebra, the motive of a Spin_8-versal quadric splits into 2-fold Rost motives [6, XVII]), it follows that the second decomposition is impossible, *i.e.,* $[E/P]$ is indecomposable.

A Appendix: Generic Nilpotence for Cobordism

This appendix is included to prove Lemma A.13, the obvious generalization from Chow groups to algebraic cobordism of [28, Lemma 6.2]. We therefore closely follow arguments of [28], while making the following adjustments to cobordism: Lemma A.9 is the analogue of [28, Lemma 6.4], and Lemma A.12 is the analogue of [7, Proposition 3.3] that was used in the proof of [28, Lemma 6.2].

Let Sch_k denote the category of reduced schemes of finite type over k and let Sm_k denote its subcategory of smooth schemes over k. Let $\Omega_*(-)$ denote the algebraic cobordism functor of Levine–Morel [19].

Definition A.1 Let $X \in \mathrm{Sch}_k$ and let $i \colon Z \hookrightarrow X$ be a closed subset. We will denote the image of the map $i_* \colon \Omega_*(Z) \to \Omega_*(X)$ by $\Omega_Z(X)$ and say that elements of $\Omega_Z(X)$ are supported on Z.

Definition A.2 We will call a projective map $f \colon Z' \to Z$ in Sch_k a *good cover* if there are filtrations by closed subsets $\varnothing \subseteq Z'_0 \subseteq Z'_1 \subseteq \cdots \subseteq Z'_n = Z'$ and $\varnothing \subseteq Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = Z$ such that

- $\dim Z_i' = \dim Z_i$;
- $f^{-1}(Z_i - Z_{i-1}) \to Z_i - Z_{i-1}$ is an isomorphism;
- $f(Z_i') \subseteq Z_i$.

for any $0 \leqslant i \leqslant n$.

Example A.3   Here is a typical good cover. If $f: X' \to X$ is a projective map that induces an isomorphism of non-empty open subsets $f^{-1}(X - Z) \cong X - Z$ for some closed subset Z of X with $\dim Z < \dim X$, then the map $X' \sqcup Z \to X$ is a good cover; the respective filtrations can be chosen as $\varnothing \subseteq Z \subseteq X' \sqcup Z$ and $\varnothing \subseteq Z \subseteq X$.

Remark A.4   We could have defined good covers inductively by declaring that isomorphisms are good covers and that the map $f: X' \to X$ in Sch_k is a good cover provided there are closed subsets $Z \subseteq X$ and $Z' \subseteq X'$ such that

- $\dim X' = \dim X$;
- $f^{-1}(X - Z) \to X - Z$ is an isomorphism;
- $f(Z') \subseteq Z$ and $f_{|Z'}: Z' \to Z$ is a good cover.

Lemma A.5   *If $f: Z' \to Z$ is a good cover, then $f_*: \Omega_*(Z') \to \Omega_*(Z)$ is surjective.*

Proof   Let $(Z_i)_{i=0}^n$ and $(Z_i')_{i=0}^n$ be filtrations as in Definition A.2. We prove that $\Omega_*(Z_i') \to \Omega_*(Z_i)$ is surjective by induction on i. The case $i = 0$ is obvious, since $Z_0' \cong Z_0$. For the induction step consider the map of localization sequences induced by f_*:

$$
\begin{array}{ccccccc}
\Omega_*\big(f^{-1}(Z_{i-1})\big) & \longrightarrow & \Omega_*(Z_i') & \longrightarrow & \Omega_*\big(f^{-1}(Z_i - Z_{i-1})\big) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega_*(Z_{i-1}) & \longrightarrow & \Omega_*(Z_i) & \longrightarrow & \Omega_*(Z_i - Z_{i-1}) & \longrightarrow & 0
\end{array}
$$

The right vertical map is an isomorphism by Definition A.2. The left vertical map is surjective, because it fits into the composition

$$
\Omega_*(Z_{i-1}') \longrightarrow \Omega_*(f^{-1}(Z_{i-1})) \longrightarrow \Omega_*(Z_{i-1}),
$$

where the composite map is surjective by the induction hypothesis. Hence the central arrow is surjective as well. ∎

Lemma A.6   *Good covers are preserved by flat base-change: if $f: Z' \to Z$ is a good cover and $g: Y \to Z$ is flat, then $f': Y \times_Z Z' \to Y$, the base-change of f to Y, is a good cover.*

Proof   Let $(Z_i')_{i=1}^n$ and $(Z_i)_{i=1}^n$ be filtrations on Z' and Z as in Definition A.2, and let $Y_i = g^{-1}(Z_i)$ and $Y_i' = (g')^{-1}(Z_i')$. These are filtrations on Y and Y' that satisfy Definition A.2. Indeed, we have $\dim Y_i' = \dim Y_i$ by flatness of g (and g'); the map $(f')^{-1}(Y_i - Y_{i-1}) \to Y_i - Y_{i-1}$ is the base change of the isomorphism $f^{-1}(Z_i - Z_{i-1}) \to Z_i - Z_{i-1}$, so it is an isomorphism, and the inclusion $f'(Y_i') \subseteq Y_i$ is obvious. ∎

Lemma A.7 *Let $X \in \mathrm{Sch}_k$ and let E be a vector bundle over X. Suppose that Z is a closed subset of X. Then there exists a projective map $p\colon X' \to X$ with $\dim X' = \dim X$ and a closed subset Z' of X' such that*

(i) *$p^* E$ has a filtration by subbundles whose successive quotients are line bundles;*
(ii) *$p(Z') \subseteq Z$, and the map $p\colon Z' \to Z$ is a good cover.*

Proof Consider the projection $p\colon \mathrm{Fl}(E) \to X$ where $\mathrm{Fl}(E)$ is the variety of complete flags of the vector bundle E over X. Then $p^* E$ has a filtration by tautological subbundles. Any point $x \in X$ is contained in an open neighbourhood U over which the bundle E is trivial, so $\mathrm{Fl}(E)|_U \simeq \mathrm{Fl} \times U$, and there is a section $s\colon U \to p^{-1}(U)$. Let X'_U be the closure of $s(U)$ in $\mathrm{Fl}(E)$. Note that the projection $(p_{|X'_U})^{-1}(U) \to U$ is an isomorphism.

We will construct X' as a disjoint union of schemes of the form X'_U by induction on $d = \dim Z$. When $d = 0$, the set Z is a finite union of closed points $z_1, \dots, z_n$. Each z_i is contained in some neighbourhood U_i as above, and we can take $X' = X'_{U_1} \sqcup \cdots \sqcup X'_{U_n}$. Each X'_{U_i} contains a copy of the point z_i, and we can take Z' to be the disjoint union of those copies. Then $Z' = Z$ is a good cover.

Consider the case of a general d. Let $Z_1, \dots, Z_n$ be irreducible components of Z. For any i, there is a neighbourhood U_i of the generic point of Z_i such that $Z_i \cap U_i$ is disjoint with Z_j for $j \neq i$, and E is trivial over U_i. Then let Z'_i be the closure of $Z_i \cap U_i$ in X'_{U_i}. Observe that the projection $p_{|Z'_i}\colon Z'_i \to Z_i$ induces an isomorphism $(p_{|Z'_i})^{-1}(Z_i \cap U_i) \to Z_i \cap U_i$.

Let $D_i = Z_i - (Z_i \cap U_i)$ and $D = \cup_{i=1}^n D_i$. Then $\dim D < d$ and by the induction hypothesis, there is a map $p_D\colon X'_D \to X$ and a closed subset $Z'_D \subseteq X'_D$ such that $p_D\colon Z'_D \to D$ is a good cover, and $p_D^*(E)$ satisfies (i). Take

$$X' = X'_D \sqcup X'_{U_1} \cdots \sqcup X'_{U_n}, Z' = Z'_D \sqcup Z'_1 \sqcup \cdots \sqcup Z'_n.$$

Consider the map $p_{|Z'}\colon Z' \to Z$. We have $Z - D = \bigsqcup_i (Z_i \cap U_i)$, hence $(p_{|Z'})^{-1}(Z - D) \to Z - D$ is an isomorphism. By Remark A.4, the map $p_{|Z'}$ is a good cover, so X' and Z' satisfy conditions (i) and (ii). ∎

Lemma A.8 *Let $X \in \mathrm{Sch}_k$ and let E be a vector bundle over X that has a filtration by subbundles with line bundle quotients $E_1, \dots, E_d$. Assume that $D_1, \dots, D_d$ are divisors such that E_i is trivial over $X - D_i$ for each i. Let $D = D_1 \cap \cdots \cap D_d$. Then for any $\alpha \in \Omega_Z(X)$ the element $\widetilde{c}_d(E) \cap \alpha$ lies in $\Omega_{Z \cap D}(X)$.*

Proof By the Whitney formula, we have $\widetilde{c}_d(E) = \widetilde{c}_1(E_1) \circ \cdots \circ \widetilde{c}_1(E_d)$. For any $\alpha \in \Omega_Z(X)$, the element $\widetilde{c}_1(E_d) \cap \alpha$ vanishes in $\Omega_{Z - D_d}(X - D_d)$, hence it lies in $\Omega_{Z \cap D_d}(X)$. The statement thus follows by induction on d. ∎

Lemma A.9 *Let E be a vector bundle of rank d on a quasi-projective variety X and let Z be a closed subset of X. Then there is a closed subset $\widetilde{Z}$ of Z such that $\dim \widetilde{Z} \leqslant \dim Z - d$ and for any smooth morphism $f\colon Y \to X$ of schemes in Sch_k and any $\alpha \in \Omega_{f^{-1}(Z)}(Y)$, the element $\widetilde{c}_d(f^* E) \cap \alpha$ is supported on $f^{-1}(\widetilde{Z})$.*

Remark A.10   If $\dim Z - d < 0$, then $\dim \widetilde{Z} < 0$ so $\widetilde{Z} = \varnothing$, and the lemma concludes that $\widetilde{c}_d(f^*E) \cap \alpha = 0$ as already known; see [19, 2.3.13].

Proof   By Lemma A.7, there is a projective map $p: X' \to X$ and a closed subset $Z' \subseteq X$ such that $p_{|Z'}: Z' \to Z$ is a good cover, while p^*E has a filtration with line bundle quotients $E_1, \ldots, E_d$. Using quasi-projectivity of X, let $D_1, \ldots, D_d$ denote divisors such that each E_i is trivial over $X' - D_i$. Set $D = D_1 \cap \cdots \cap D_d$. One can furthermore ensure that $Z' \cap D$ and thus $\widetilde{Z} = p(Z' \cap D)$ has dimension at most $\dim Z' - d = \dim Z - d$. Let $Y' = X' \times_X Y$ and let $g: Y' \to X'$ be the canonical projection. Then $g^{-1}(Z') \to f^{-1}(Z)$ is a good cover by Lemma A.6. By Lemma A.5, the projection $q: Y' \to Y$ induces the surjective push-forward $\Omega_*(g^{-1}(Z')) \to \Omega_*(f^{-1}(Z))$, and thus $\Omega_{g^{-1}(Z')}(Y') \to \Omega_{f^{-1}(Z)}(Y)$ is also surjective. Given $\alpha \in \Omega_{f^{-1}(Z)}(Y)$, choose a preimage $\alpha' \in \Omega_{g^{-1}(Z')}(Y')$. Then $\widetilde{c}_d(f^*E) \cap \alpha = q_*(\widetilde{c}_d(q^*f^*E) \cap \alpha')$, and $\widetilde{c}_d(q^*f^*E) \cap \alpha'$ is supported on $g^{-1}(Z' \cap D)$ by Lemma A.8; thus, $\widetilde{c}_d(f^*E) \cap \alpha$ is supported on $q(g^{-1}(Z' \cap D))$, which is contained in $f^{-1}(\widetilde{Z})$. ∎

Lemma A.11 (*cf.* [28, Lemma 6.3])   *Let $f: V \hookrightarrow B$ and $g: T \hookrightarrow B$ be closed embeddings with regular f and smooth quasi-projective B. Then there exists a closed embedding $h: Z \hookrightarrow V$ such that $\operatorname{codim} h \geqslant \operatorname{codim} g$, and for any smooth morphism $\varepsilon: W \to B$, we have $\operatorname{im}\big(f_W^* \circ (g_W)_*\big) \subseteq \operatorname{im}\big((h_W)_*\big)$ inside $\Omega_*(V_W)$, where f_W, g_W, and h_W are the base changes of f, g, and h to W along ε.*

Proof   Consider the intersection $\widetilde{T}$ of T and Z in B and base-change it along ε to obtain the Cartesian squares

$$
\begin{array}{ccc}
T & \xrightarrow{\ g\ } & B \\
\tilde{f}\big\uparrow & & \big\uparrow f \\
\widetilde{T} & \xrightarrow{\ \tilde{g}\ } & V
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
T_W & \xrightarrow{\ g_W\ } & W \\
\widetilde{f}_W\big\uparrow & & \big\uparrow f_W \\
\widetilde{T}_W & \xrightarrow{\ \widetilde{g}_W\ } & V_W.
\end{array}
$$

By [19, Proposition 6.6.3] $f_W^* \circ (g_W)_* = (\widetilde{g}_W)_* \circ f_W^!$ where the refined pullback $f_W^!$ is given by the composition [19, (6.17), §6.6.2]:

$$
\Omega_*(T_W) \longrightarrow \Omega_*(C_W) \longrightarrow \Omega_*(N_W) \longrightarrow \Omega_{*-d}(\widetilde{T}_W),
$$

where d is the codimension of f, while C_W is the normal cone of $\widetilde{f}_W$ and $N_W = \widetilde{g}_W^*(N_{f_W})$ is the the normal bundle of f_W pulled back to $\widetilde{T}_W$.

Let $N = \widetilde{g}^*(N_f)$ be the pullback of the normal bundle of f and C be the normal cone of the map $\widetilde{f}$, both being bundles over $\widetilde{T}$. Let $q: \mathbb{P}(N \oplus 1) \to \widetilde{T}$ be the structural map. Note that by flatness of $\varepsilon: W \to B$, base changing C (resp. N) to W does yield C_W (resp. N_W). Applying Lemma A.9 to the vector bundle $E = q^*N \otimes \mathcal{O}(1)$ and to the closed subset $\mathbb{P}(C \oplus 1)$ inside $\mathbb{P}(N \oplus 1)$, we get a closed subset $\widetilde{Z}$ of codimension at least d in $\mathbb{P}(C \oplus 1)$ such that for every smooth morphism $\epsilon: W \to B$ and every cobordism class x supported on $\mathbb{P}(C_W \oplus 1)$ the class $\widetilde{c}_d(\epsilon^*E) \cap x$ is supported on $\epsilon^{-1}(\widetilde{Z})$. Set $Z = q(\widetilde{Z})$. By Lemma A.12, the right-hand side of the diagram

$$\begin{array}{ccc}
\Omega_*\big(\mathbb{P}(C_W \oplus 1)\big) \longrightarrow \Omega_*\big(\mathbb{P}(N_W \oplus 1)\big) \xrightarrow{\widetilde{c}_d(\varepsilon^* E) \cap -} \Omega_*\big(\mathbb{P}(N_W \oplus 1)\big) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
\Omega_*(C_W) \longrightarrow \Omega_*(N_W) \longrightarrow \Omega_*(\widetilde{T}_W)
\end{array}$$

is commutative (and so is the left-hand side), while the left vertical map is surjective, since C_W is open in $\mathbb{P}(C_W \oplus 1)$. Therefore, the image of the composition $\Omega_*(C_W) \to \Omega_*(N_W) \to \Omega_{*-d}(\widetilde{T}_W)$ is supported on $q(\varepsilon^{-1}(\widetilde{Z})) \subseteq \varepsilon^{-1}(Z) = Z_W$, *i.e.*, is in $\Omega_{Z_W}(\widetilde{T}_W)$, which maps to $\Omega_{Z_W}(V_W) = \mathrm{im}((h_W)_*)$. Thus, $h\colon Z \hookrightarrow V$ satisfies the requirements. ∎

Lemma A.12 *Let $X \in \mathrm{Sch}_k$ and let $E \to X$ be a rank d vector bundle with a zero section $z\colon X \to E$. Let $q\colon \mathbb{P}(E \oplus 1) \to X$ be the structural map and let $a\colon E \to \mathbb{P}(E \oplus 1)$ be the natural open affine chart sending e to $[e : 1]$. Then the following diagram commutes.*

$$\begin{array}{ccc}
\Omega_*(\mathbb{P}(E \oplus 1)) & \xrightarrow{\widetilde{c}_d(q^* E \otimes \mathcal{O}(1)) \cap -} & \Omega_{*-d}(\mathbb{P}(E \oplus 1)) \\
{\scriptstyle a^*}\downarrow & & \downarrow{\scriptstyle q_*} \\
\Omega_*(E) & \xrightarrow{\quad z^* \quad} & \Omega_{*-d}(X)
\end{array}$$

Proof The composition $\mathcal{O}(-1) \to q^*(E \oplus 1) \to q^* E$ of the natural embedding and projection defines a global section s of the sheaf $q^* E \otimes \mathcal{O}(1) = \underline{\mathrm{Hom}}(\mathcal{O}(-1), q^* E)$. The zero set of s is $X \simeq \mathbb{P}(1)$ (regularly) embedded in $\mathbb{P}(E \oplus 1)$, and this embedding coincides with $\bar{s} = a \circ z$. By [19, Lemma 6.6.7], the operator $\widetilde{c}_d(q^* E \otimes \mathcal{O}(1)) \cap -$ on $\Omega_*(\mathbb{P}(E \oplus 1))$ is given by $\bar{s}_* \bar{s}^*$. The right-down composition is thus $q_* \bar{s}_* \bar{s}^* = \bar{s}^* = z^* a^*$. ∎

Let h be an oriented cohomology theory that is generically constant (*i.e.*, the pullback $\mathrm{h}(k) \to \mathrm{h}(L)$ is an isomorphism for any field extensions L/k), and the canonical map from algebraic cobordism $\Omega(X) \otimes_{\mathbb{L}} \mathrm{h}(k) \to \mathrm{h}(X)$ is surjective for every $X \in \mathrm{Sm}_k$. An example of such theory is given by the free theory of [8, §2.12].

Lemma A.13 *Let X be a smooth quasi-projective variety. Let $i_1\colon Z_1 \hookrightarrow X$ and $i_2\colon Z_2 \hookrightarrow X$ be closed embeddings. Then there exists a closed embedding $i_3\colon Z_3 \hookrightarrow X$ such that for any smooth morphism $\pi\colon Y \to X$,*

$$\mathrm{codim}\, Z_3 \geqslant \mathrm{codim}\, Z_1 + \mathrm{codim}\, Z_2 \quad and \quad \mathrm{im}(i_1')_* \cdot \mathrm{im}(i_2')_* \subseteq \mathrm{im}(i_3')_* \ in\ \mathrm{h}(Y),$$

where $i_j'\colon Z_j \times_X Y \to Y$, $j = 1, 2, 3$ are the base change along π of the respective closed embeddings.

Proof We first consider the case $\mathrm{h} = \Omega$. Applying Lemma A.11 to $B = X \times X$, $f = \Delta_X\colon X \to X \times X$, $g = i_1 \times i_2\colon Z_1 \times Z_2 \to X \times X$, one obtains a closed embedding $h\colon Z \hookrightarrow X$ such that $\mathrm{codim}\, Z \geqslant \mathrm{codim}\, Z_1 + \mathrm{codim}\, Z_2$. Choosing $W = Y \times Y$ and $\varepsilon = \pi \times \pi\colon Y \times Y \to X \times X$, it follows that f_W is the natural inclusion $Y \times_X Y \to Y \times Y$, $h_W\colon Z_{Y \times Y} \hookrightarrow Y \times_X Y$ is the obvious map and

$$\mathrm{im}\big(f_W^* \circ (i_1' \times i_2')_*\big) \subseteq \mathrm{im}\big((h_W)_*\big).$$

The diagonal embedding $Y \to Y \times Y$ factors as $Y \xrightarrow{\phi} Y \times_X Y \xrightarrow{f_W} Y \times Y$. Using the Cartesian square

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & Y \times_X Y \\
{\scriptstyle h_Y}\big\uparrow & & {\scriptstyle h_W}\big\uparrow \\
Z_Y & \xrightarrow{\phi_Z} & Z_{Y \times Y}
\end{array}
$$

and [19, Proposition 6.6.3], we obtain $\phi^* \circ (h_W)_* = (h_Y)_* \circ \phi_Z^!$,

$$
\mathrm{im}(i_1')_* \cdot \mathrm{im}(i_2')_* = \mathrm{im}\left(\Delta_Y^* \circ (i_1' \times i_s')_*\right) \subseteq \mathrm{im}\left((h_Y)_*\right).
$$

Thus, $i_3 = h$ satisfies the requirements. Finally, the natural map $\Omega_*(-) \otimes_{\mathbb{L}} \mathrm{h}(k) \to \mathrm{h}(-)$ is surjective and compatible with push-forwards and the intersection product, so we can replace Ω by h. ∎

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Faculté des Sciences Jean Perrin, Université d'Artois, Rue Jean Souvraz SP 18, 62307 Lens Cedex, France
e-mail : baptiste.calmes@math.cnrs.fr

Department of Mathematics, Western University, Middlesex College, London ON N6A 5B7
e-mail : aneshito@uwo.ca

Department of Mathematics and Statistics, University of Ottawa, 150 Louis-Pasteur, Ottawa ON K1N 6N5
e-mail : kirill@uottawa.ca