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REALIZING METRICS OF CURVATURE ≤ −1 ON CLOSED SURFACES IN FUCHSIAN ANTI-DE SITTER MANIFOLDS

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Abstract

We prove that any metric with curvature less than or equal to -1 (in the sense of A. D. Alexandrov) on a closed surface of genus greater than 1 is isometric to the induced intrinsic metric on a space-like convex surface in a Lorentzian manifold of dimension $(2+1)$ with sectional curvature -1. The proof is by approximation, using a result about isometric immersion of smooth metrics by Labourie and Schlenker.

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1. Introduction

In the following, S is a closed, connected, oriented surface. When we speak about a metric with curvature less than or equal to k or greater than or equal to k, this means that S is endowed with a distance d satisfying a curvature bound in the sense of A.D. Alexandrov; see, for example, [6] or Section 5. This metric notion of curvature bound was initially introduced in the 1940s to characterize the induced metric on the boundary of convex bodies of the Euclidean space [2]. (In the present paper, the word *metric* is used for distance, and *induced metric* means the induced intrinsic distance.) While introducing this seminal notion, Alexandrov proved the following statement.

THEOREM 1.1. *Let d be a metric of curvature greater than or equal to* 0 *on the sphere S. Then there exists a convex surface in the Euclidean space whose induced metric is isometric to* (S, d).

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Theorem 1.1 was generalized in many ways. Some of them are contained in the following statement; see the introduction of [9] for details.

THEOREM 1.2. *Let $k \in \mathbb{R}$ and let d be a metric of curvature greater than or equal to k on a closed surface S. Then there exists a Riemannian manifold R homeomorphic to $S \times \mathbb{R}$ of constant sectional curvature k that contains a convex surface whose induced metric is isometric to (S, d).*

In 2017, Fillastre and Slutsky proved an analogous results for metrics with curvature bounded from above [10].

THEOREM 1.3. *Let d be a metric of curvature less than or equal to 0 on a closed surface S of genus greater than 1. Then there exists a flat Lorentzian manifold L homeomorphic to $S \times \mathbb{R}$ that contains a space-like convex surface whose induced metric is isometric to (S, d).*

A natural question is whether an analogue of Theorem 1.2 holds for metrics with curvature bounded from above. The case $k = 0$ is given by Theorem 1.3. In the present paper, we solve the case where k is negative. Up to a homothety, this reduces to the case $k = -1$. So the main result of the present paper is the following theorem.

THEOREM 1.4. *Let d be a metric with curvature less than or equal to -1 on a closed surface S of genus greater than 1. Then there exists a Lorentzian manifold L of sectional curvature -1 homeomorphic to $S \times \mathbb{R}$ that contains a space-like convex surface whose induced metric is isometric to (S, d).*

The proof of Theorem 1.4 is given by a classical approximation procedure, along the lines of [10]. The proof relies on the smooth analogue of Theorem 1.4 proved by Labourie and Schlenker; see Theorem 4.1. We prove Theorem 1.4 showing that the universal cover of (S, d) is isometric to a convex surface in anti-de Sitter space (see Section 2), invariant under the action of a discrete group of isometries leaving invariant a totally geodesic hyperbolic surface. Such groups are usually called *Fuchsian*, and the quotient of a suitable part of anti-de Sitter space by such a group may be called a Fuchsian anti-de Sitter manifold. The main issues in our case, in comparison to [10], is that we lose the vector space structure given by the Minkowski space—it is the Lorentzian analogue of the problem of going from Euclidean space to hyperbolic space. Also, the analogue of an approximation result that is straightforward in the flat case occupies the whole of Section 5 here.

Let us describe more precisely the content of the present paper. In Section 2 we recall some definitions related to anti-de Sitter space, and define the induced metric on convex surfaces in this space. In Section 3 we look at surfaces invariant under the action of Fuchsian groups, and prove several compactness results. In Section 5, we check that any metric with curvature less than or equal to -1 on S can be approximated by a sequence of distances given by Riemannian metrics with sectional curvature less than -1. In Section 4 all the elements are put together to provide a proof of Theorem 1.4.

The case where k is positive is still missing to obtain a Lorentzian analogue of Theorem 1.2. An issue is that it is not clear if the approximation results used in Section 5 can be applied in the less than or equal to 1 curvature case.

2. Convex surfaces in anti-de Sitter space

2.1. Anti-de Sitter space. In the following we describe a geometric model of the anti-de Sitter space (of dimension 3) we are most interested in, and illustrate some of its features. Good references for this material are [3, 4, 12, 14].

Let us consider the symmetric bilinear form

$$b(x, y) = -x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3$$

of signature $(2, 2)$ on $\mathbb{R}^4$.

DEFINITION 2.1. We define $\widehat{AdS^3}$ as

$$\widehat{AdS^3} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid b(x, x) = -1\},$$

endowed with the pseudo-Riemannian metric induced by the restriction of the bilinear form b to its tangent spaces.

Hence $\widehat{AdS^3}$ is a Lorentzian manifold, and it can be checked that its sectional curvature is -1.

A tangent vector v to $\widehat{AdS^3}$ at a point x is called:

$$\begin{cases} \text{space-like} & \text{if } b(v, v) > 0. \\ \text{time-like} & \text{if } b(v, v) < 0. \\ \text{light-like} & \text{if } b(v, v) = 0. \end{cases}$$

Now let $x, y \in \mathbb{R}^4$. We say that $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $x = \lambda y$.

DEFINITION 2.2. We define the anti-de Sitter space of dimension 3 as follows:

$$AdS^3 = \widehat{AdS^3} / \sim$$

endowed with the quotient metric.

It is easy to see that $\widehat{AdS^3}$ is a double cover of AdS^3. The pseudo-Riemannian metric induced on $\widehat{AdS^3}$ goes down to the quotient.

By definition AdS^3 is a subset of the projective space. In order to better visualize it, we look at its intersection with an affine chart and see its image in $\mathbb{R}^3$. Let

$$\varphi_0 : \mathbb{RP}^3 \setminus \{x_0 = 0\} \to \mathbb{R}^3$$

be an affine chart of $\mathbb{RP}^3$ defined by

$$\varphi_0([x_0, x_1, x_2, x_3]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{2-1}$$

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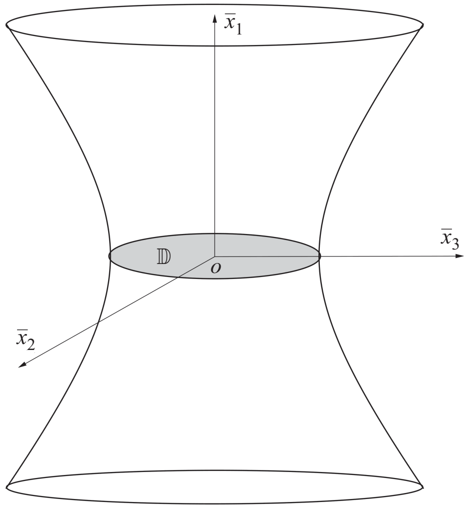

FIGURE 1. Image of AdS^3 in the affine chart φ_0.

Then $\varphi_0(AdS^3 \setminus \{x_0 = 0\})$ gives

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 < 0 \Rightarrow -1 - \left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 + \left(\frac{x_3}{x_0}\right)^2 < 0,$$

so in this affine chart AdS^3 fills the domain

$$-\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 < 1,$$

which is the interior of a one-sheeted hyperboloid. Notice that AdS^3 is not contained in a single affine chart. In the affine chart φ_0 we are missing a totally geodesic plane at infinity, corresponding to $\{x_0 = 0\}$.

Throughout the paper, we denote by $\mathbb{D}$ the disc

$$\begin{cases} \bar{x}_2^2 + \bar{x}_3^2 < 1 \\ \bar{x}_1 = 0 \end{cases}$$

in the affine chart φ_0 (see Figure 1).

It is clear from the construction that in the affine chart φ_0, geodesics (respectively, totally geodesic planes) are given by the intersection between affine lines (respectively, affine planes) in $\mathbb{R}^3$ and the interior of the one-sheeted hyperboloid described above. A plane P is space-like if the restriction of the induced metric on P is positive definite. A *convex space-like surface* in anti-de Sitter space is a surface that is convex in an affine chart and has only space-like planes as support planes. The *boundary at infinity* of AdS^3 is given by

$$\{[x] \in \mathbb{RP}^3 : b(x, x) = 0\}/ \sim$$

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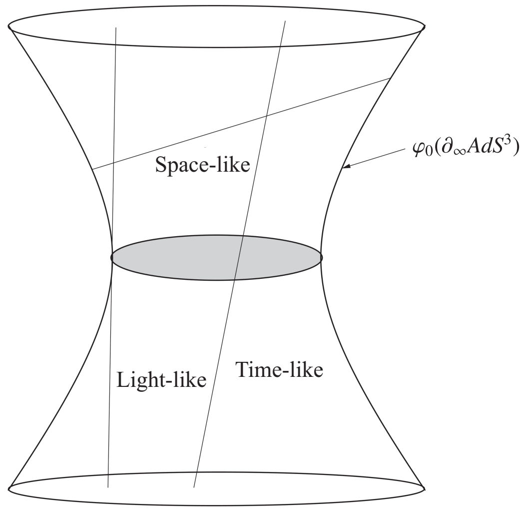

FIGURE 2. Geodesics in an affine model of AdS^3.

and we denote it by $\partial_\infty AdS^3$ (and by $\varphi_0(\partial_\infty AdS^3)$ the boundary in the affine chart φ_0). We can distinguish the type of geodesics in the image of anti-de Sitter space in the affine chart as follows (see Figure 2).

- A geodesic in AdS^3 is space-like if it meets $\partial_\infty AdS^3$ in two different points.
- A geodesic in AdS^3 is light-like if it meets $\partial_\infty AdS^3$ in only one point.
- A geodesic in AdS^3 is time-like if it is strictly contained in the hyperboloid.

Note that $AdS^3 \cap \{x \in \mathbb{R}^4 \mid x_1 = 0\} =: H_0$ is isometric to the hyperbolic plane. We use this fact to define the following map. Let $\tilde{\Psi} : \mathbb{H}^2 \times \mathbb{R} \longrightarrow AdS^3$ be the map defined by $\tilde{\Psi}(x, t) = \exp_x(tV)$, where

- $\tilde{\Psi}(\mathbb{H}^2, 0) = H_0$, and $x \mapsto \tilde{\Psi}(x, 0)$ is an isometry;
- V is a choice of a unit vector field orthogonal to H_0, for the anti-de Sitter metric.

Indeed, we have $\tilde{\Psi}(x, t) = \cos(t)x + \sin(t)V$ with $V = (0, -1, 0, 0)$. For a given x, $t \mapsto \tilde{\Psi}(x, t)$ is a time-like geodesic loop with time-length 2π. We call an *AdS cylinder* the cylinder $\mathbb{H}^2 \times [0, \pi/2[$ endowed with the Lorentzian metric g_{AdS}, which is the pullback of the anti-de Sitter metric by $\tilde{\Psi}$. Let us denote $AdS^3 \cap \{x \in \mathbb{R}^4 \mid x_1 = r\} =: H_r$. The induced metric onto H_r is homothetic to the hyperbolic metric with factor $(1 - r^2)$, and clearly $\tilde{\Psi}(\mathbb{H}^2, t) = H_{\sin(t)}$. In turn,

$$g_{AdS}(x, t) = \cos^2(t) g_{\mathbb{H}^2}(x) - dt^2$$

where $g_{\mathbb{H}^2}$ is the metric on the hyperbolic plane.

It is suitable to work with the image of $\tilde{\Psi}$ in the affine chart considered above. Let us denote $\Psi = \varphi_0 \circ \tilde{\Psi}$. The set $\Psi(\mathbb{H}^2 \times [0, \pi/2[)$ is indeed a Euclidean half-cylinder in $\mathbb{R}^3$

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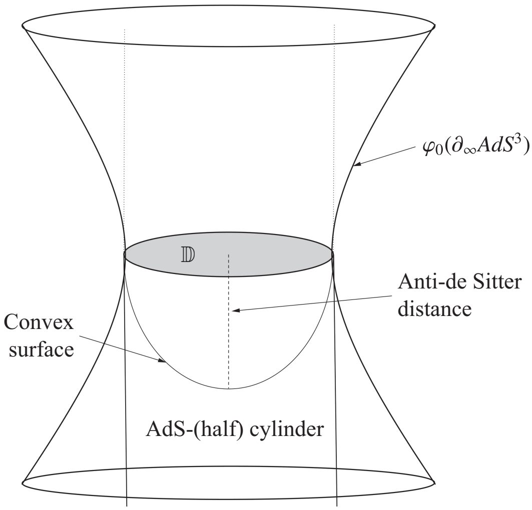

FIGURE 3. The AdS cylinder and a convex surface inside.

(see Figure 3). We have $\Psi(\mathbb{H}^2, 0) = \mathbb{D}$ and, for $x \in \mathbb{H}^2$, $t \mapsto \tilde{\Psi}(x, t)$ is a vertical half line from $\mathbb{D}$. We call an *affine AdS cylinder* the image of $\mathbb{H}^2 \times [0, \pi/2[$ by Ψ. For convexity reasons, we need only to consider a half cylinder.

2.2. Convex functions. For a function $u : \mathbb{H}^2 \to [0, \pi/2[$, we denote

$$S_u = \{(x, u(x)) \mid x \in \mathbb{H}^2\}.$$

For every $x \in \mathbb{H}^2$ we denote by $\bar{x} = \Psi(x, 0)$ the corresponding point on the disc $\mathbb{D}$, where Ψ is the map introduced in the previous section. The image of S_u in the affine AdS cylinder is the graph of a function over $\mathbb{D}$, which we denote by $\bar{u}$. We denote by $S_{\bar{u}}$ the image of S_u. Hence, $\bar{u} : \mathbb{D} \to \mathbb{R}$ and

$$(\bar{x}, \bar{u}(\bar{x})) = \Psi(x, u(x)).$$

For a point $\bar{x} \in \mathbb{D}$ we use the notation $\bar{x} = (\bar{x}_2, \bar{x}_3)$ for its Euclidean coordinates, and its Euclidean norm is $\| \bar{x} \| = \sqrt{\bar{x}_2^2 + \bar{x}_3^2}$. By the considerations of the preceding section, we immediately obtain the following relation.

LEMMA 2.3. *In the above notation,* $\bar{u}(\bar{x}) = -\tan(u(x)) \sqrt{1 - \| \bar{x} \|^2}$.

DEFINITION 2.4. Let $u : \mathbb{H}^2 \to \mathbb{R}$ be a function. We say that u is *C-convex* if

- $u \geq 0$ and there is $R < \pi/2$ such that $u \leq R < \pi/2$;
- the corresponding function $\bar{u}$ is convex.

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It is worth noting that for $R \geq 0$, if $u = R$, then the graph of the map defined by $\bar{u}(\bar{x}) = -\tan(R)\sqrt{1 - \| \bar{x} \|^2}$ is a half ellipsoid. Also, $|\bar{u}(\bar{x})| \leq \tan(R)\sqrt{1 - \| \bar{x} \|^2}$. It follows that if u is C-convex, then $\bar{u}$ is bounded and satisfies $\bar{u}|_{\partial\mathbb{D}} = 0$.

It is also clear that a bounded convex function $\bar{u} : \mathbb{D} \to \mathbb{R}$ vanishes everywhere if it vanishes in a point of the open disc $\mathbb{D}$. So we have $\bar{u} \leq 0$ by definition, and $\bar{u} < 0$ or $\bar{u} = 0$.

Let us note the following lemma.

LEMMA 2.5. *In the image of* AdS^3 *by* φ_0*, we have the following properties.*

1. *Every time-like line passes through the disc* $\mathbb{D}$.
2. *Every light-like line which does not pass through the boundary* $\partial_\infty\mathbb{D}$ *must pass through the disc* $\mathbb{D}$.
3. *A cone with basis the disc* $\mathbb{D}$ *and with apex in the affine cylinder is a convex space-like surface.*

PROOF. The proofs of the two first points are almost immediate. For the third point, either a support plane of the cone does not meet the closure of $\mathbb{D}$, hence it is space-like, or a support plane of the cone contains a half-line of the cone, then it meets the boundary of the disc, but by assumption this half-line is not vertical, hence not light-like, so the plane is space-like. □

LEMMA 2.6. *Let* $u : \mathbb{H}^2 \to [0, \pi/2[$ *be a C-convex function. Then the surface* S_u *is space-like.*

PROOF. Let p be a point on the image of S_u in the affine half cylinder, and let C_p be the cone with basis the disc $\mathbb{D}$ and apex p. By definition, this cone is contained in the affine half cylinder. By convexity, a support plane to the surface at p is a support plane of the cone, so by Lemma 2.5 it must be space-like. □

We say that a sequence $(u_n)_n$ of C-convex functions is *uniformly bounded* if there is $R < \pi/2$ such that for any n, $u_n < R$.

LEMMA 2.7. *Let* $(u_n)_n$ *be a sequence of uniformly bounded C-convex functions. Up to extracting a subsequence,* $(u_n)_n$ *converges to a C-convex function u, uniformly on compact sets.*

PROOF. This is a classical property of the corresponding convex functions $\bar{u}_n$, [16, Theorem 10.9], in the special case when the surfaces vanish on the boundary of the disc $\mathbb{D}$. □

Let u_n, $n > 1$, be uniformly bounded C-convex functions converging to a C-convex function $u = u_0$. Let $c : I \to \mathbb{H}^2$ be a Lipschitz curve and $\bar{c} : I \to \mathbb{D}$ be its image by Ψ. Then $\bar{u} \circ \bar{c}$, $\bar{u}_n \circ \bar{c}$ are Lipschitz—the Lipschitz nature of $\bar{c}$ is independent of a choice of a Riemannian metric on the disc. By the Rademacher theorem, there exists a set I_0 of Lebesgue measure 0 in I such that for all $n \in \mathbb{N}$, $\bar{u}_n$ is differentiable on $I \setminus I_0$.

LEMMA 2.8. *Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ be uniformly bounded C-convex functions converging to a C-convex function u, and let $c : I \to \mathbb{H}^2$ be a Lipschitz curve. Up to extracting a subsequence, for almost all t,*

$$(u_n \circ c)'(t) \to (u \circ c)'(t).$$

PROOF. The following proof is a straightforward adaptation of [10, Lemma 3.6]. We first prove the lemma for the corresponding functions $\bar{u}_n$ and $\bar{u}$, then we deduce the proof for u_n and u using continuity and Lemma 2.3. We consider that $\bar{c}$ is parameterized by arc-length.

Let $\langle \cdot, \cdot \rangle$ be the Euclidean metric on the affine cylinder, and we use the notation $\binom{a}{b}$, with $a \in \mathbb{D}$ and $b \in \mathbb{R}$. Let t be such that the derivatives exist. Let X be the unit vector $\binom{0}{1}$ and Y the unit vector $\binom{\bar{c}'(t)}{0}$; we have $\langle X, Y \rangle = 0$. The tangent vector to the curve $\binom{\bar{c}}{\bar{u}_n \circ \bar{c}}$ at every point $\binom{\bar{c}(t)}{(\bar{u}_n \circ \bar{c})(t)}$ is given by

$$V_n = (\bar{u}_n \circ \bar{c}(t))' X + Y,$$

and in the plane P spanned by X and Y, the vector

$$N_n = (\bar{u}_n \circ \bar{c}(t))' Y - X$$

is orthogonal to V_n for $\langle \cdot, \cdot \rangle$. Now because $\bar{u}_n$ and $\bar{u}$ are equi-Lipschitz on any compact set of $\mathbb{D}$ (see [16, Theorem 10.6]) there exists k such that $|(\bar{u}_n \circ \bar{c})'(t)| \leq k$ for all $n \in \mathbb{N}$, and

$$\| N_n \| \leq |(\bar{u}_n \circ \bar{c})'(t)| \, \| Y \| + \| X \| \leq |(\bar{u}_n \circ \bar{c})'(t)| + 1 \leq k + 1,$$

so $\| N_n \|$ are uniformly bounded. Hence, up to extracting a subsequence, $(N_n)_n$ converges to a vector N. Note that N is not the zero vector, otherwise $\langle N_n, X \rangle$ would converge to 0, which is impossible because $\langle N_n, X \rangle = -1$.

Let T_n be the intersection of the convex surface $S_{\bar{u}_n}$ defined by $\bar{u}_n$ and the plane P. The set T_n is a convex set in P, and V_n is a tangent vector, hence by convexity for any $\bar{y} \in \mathbb{D} \cap P$,

$$\left\langle N_n, \begin{pmatrix} \bar{c}(t) \\ \bar{u}_n \circ \bar{c}(t) \end{pmatrix} - \begin{pmatrix} \bar{y} \\ \bar{u}_n(\bar{y}) \end{pmatrix} \right\rangle \geq 0,$$

and passing to the limit we get

$$\left\langle N, \begin{pmatrix} \bar{c}(t) \\ \bar{u} \circ \bar{c}(t) \end{pmatrix} - \begin{pmatrix} \bar{y} \\ \bar{u}(\bar{y}) \end{pmatrix} \right\rangle \geq 0,$$

which says that N is a normal vector to T (the intersection of $S_{\bar{u}}$ with P), hence

$$\left\langle N, (\bar{u} \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \bar{c}'(t) \\ 0 \end{pmatrix} \right\rangle = 0.$$

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So there exists λ such that

$$(\bar{u} \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \bar{c}'(t) \\ 0 \end{pmatrix} = \lambda \lim_{n \to \infty} (\bar{u}_n \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \bar{c}'(t) \\ 0 \end{pmatrix}.$$

By identification it follows that $\lambda = 1$, hence $(\bar{u}_n \circ \bar{c})'(t)$ must converge to $(\bar{u} \circ \bar{c})'(t)$.

The functions $\bar{u}_n \circ \bar{c}$ and $u_n \circ c$ are defined from $I \subset \mathbb{R}$ to $\mathbb{R}$. By Lemma 2.3.

$$u_n \circ c(t) = \arctan\left(\frac{\bar{u}_n \circ \bar{c}(t)}{h(t)}\right),$$

where $h(t) = -\sqrt{1 - \| \bar{c}(t) \|^2}$, hence $u_n \circ c$ is clearly differentiable almost everywhere for all $n \in \mathbb{N}$ and

$$(u_n \circ c)'(t) = \frac{(\bar{u}_n \circ \bar{c})'(t)h(t) - (\bar{u}_n \circ \bar{c})(t)h'(t)}{h^2(t) + (\bar{u}_n \circ \bar{c})^2(t)}. \tag{2-2}$$

Also we have (by hypothesis), for almost all t, that

$$(u_n \circ c)(t) \underset{n \to \infty}{\longrightarrow} (u \circ c)(t);$$

hence by continuity (in the relation given by Lemma 2.3) it is clear that

$$(\bar{u}_n \circ \bar{c})(t) \underset{n \to \infty}{\longrightarrow} (\bar{u} \circ \bar{c})(t).$$

Then by the preceding arguments and by continuity again in (2-2) and passing to the limit, it follows that $(u_n \circ c)'(t)$ converges to $(u \circ c)'(t)$. □

Let $u : \mathbb{H}^2 \to \mathbb{R}$ be a C-convex function. For $c : [0, 1] \to \mathbb{H}^2$ a Lipschitz curve, $(c, u \circ c)$ is a curve on S_u, and its length for the anti-de Sitter metric is

$$\mathcal{L}_u(c) = \int_0^1 \sqrt{\cos^2(u \circ c(t))\|c'(t)\|^2_{\mathbb{H}^2} - (u \circ c)'^2(t)}dt. \tag{2-3}$$

By Lemma 2.8 above and using the dominated convergence theorem, we get the following proposition.

Proposition 2.9. *Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ be uniformly bounded C-convex functions converging to a C-convex function u, and let $c : I \to \mathbb{H}^2$ be a Lipschitz curve. Up to extracting a subsequence, $\mathcal{L}_{u_n}(c) \to \mathcal{L}_u(c)$.*

The induced (intrinsic) metric d_{S_u} on S_u is the pseudo-distance induced by $\mathcal{L}_u$: for $x, y \in S_u$, $d_{S_u}(x, y)$ is the infimum of the lengths of Lipschitz curves between x and y contained in S_u. Note that as the AdS cylinder has a Lorentzian metric, the induced distance between two distinct points on S_u may be equal to 0, which is a major difference with respect to the case of induced metrics on surfaces in a Riemannian space.

DEFINITION 2.10. We denote by d_u the pullback of d_{S_u} on $\mathbb{H}^2$, so that for every point $x, y \in \mathbb{H}^2$,

$$d_u(x, y) = d_{S_u}((x, u(x)), (y, u(y))).$$

From (2-3), as $\cos \leq 1$, we clearly have the following result.

LEMMA 2.11. *In the above notation, for $x, y \in \mathbb{H}^2$, $d_u(x, y) \leq d_{\mathbb{H}^2}(x, y)$.*

3. Fuchsian invariance

3.1. Convergence of surfaces implies convergence of metrics. The aim of this section is to state Proposition 3.9. The arguments are quite general and close to those of [10]. The main point is Lemma 3.4 below, which is the AdS analogue of [10, Corollary 3.11].

Recall that a Fuchsian group is a discrete group of orientation-preserving isometries acting on the hyperbolic plane. In the present paper we restrict this definition to the groups also acting freely and cocompactly.

DEFINITION 3.1. A *Fuchsian C-convex function* is a couple (u, Γ), where u is a C-convex function and Γ is a Fuchsian group such that for all $\sigma \in \Gamma$ we have $u \circ \sigma = u$.

We often abuse terminology, using the term 'Fuchsian' for a single function u, so that the Fuchsian group remains implicit.

DEFINITION 3.2. Let $(\Gamma_n)_n$ be a sequence of discrete groups. $(\Gamma_n)_n$ converges to a group Γ if there exist isomorphisms $\tau_n : \Gamma \to \Gamma_n$ such that for all $\sigma \in \Gamma$, the $\tau_n(\sigma)$ converge to σ.

DEFINITION 3.3. We say that a sequence of Fuchsian C-convex functions $(u_n, \Gamma_n)_n$ converges to a pair (u, Γ), if u is a C-convex function, Γ is a Fuchsian group such that $(u_n)_n$ converges to u and $(\Gamma_n)_n$ converges to Γ.

It is easy to see that if (u_n, Γ_n) is a sequence of Fuchsian C-convex functions that converges to a pair (u, Γ), then (u, Γ) is a Fuchsian C-convex function; see, for example, [10, Lemma 3.17]. Recall the definition of the distance d_u from Definition 2.10. Recall also that a C-convex function is differentiable almost everywhere. At a point where u is differentiable, we denote by $\|\cdot\|_u$ the norm induced by the ambient anti-de Sitter metric on the tangent of S_u at this point.

LEMMA 3.4. *Let u be a C-convex function. Let $K := \inf(\|v\|_u / \|v\|_{\mathbb{H}^2})$, and let $d_{\mathbb{H}^2}$ be the distance given by the hyperbolic metric (e.g. $d_{\mathbb{H}^2} = d_u$ for $u = 0$). Then $d_u(x, y) \geq K d_{\mathbb{H}^2}(x, y)$.*

Moreover, if u is Fuchsian, then $K > 0$.

PROOF. Let c be a Lipschitz curve between two points $x, y \in \mathbb{H}^2$. Let v be the tangent vector field of $(c, u \circ c)$ whenever it exists. We have

$$\mathcal{L}_u(c) = \int_a^b \|v\|_u \geq K \int_a^b \|v\|_{\mathbb{H}^2} \geq K d_{\mathbb{H}^2}(x, y)$$

and the first result follows as by definition $d_u(x, y)$ is an infimum of lengths.

Now let us suppose that u is Fuchsian. Let us suppose that $K = 0$; that is, there is a sequence $(x_n)_n$ such that u is differentiable at each x_n, and $v_n \neq 0$ in $T_{x_n}\mathbb{H}^2$ such that $\|v_n\|_u \,/\, \|v_n\|_{\mathbb{H}^2} \to 0$. Without loss of generality, let us suppose that $\|v_n\|_{\mathbb{H}^2} = 1$. Let σ_n be isometries of $\mathbb{H}^2$ that send (x_n, v_n) to a given pair (x, v), and let $u_n := u \circ \sigma_n$. As u is Fuchsian, there exists $\beta < \pi/2$ such that $u \leq \beta$, and in turn $u_n \leq \beta$. By Lemma 2.7, up to considering a subsequence, $(u_n)_n$ converges to a C-convex function u_0. As we supposed that $\|v_n\|_u \to 0$, S_{u_0} must have a light-like support plane, which contradicts Lemma 2.6. □

Note that Lemma 3.4 indicates that in the Fuchsian case, d_u is a distance and not just a pseudo-distance.

Let us recall the following classical result; see, for example, [10, Lemma 3.14]. The homeomorphisms in the statement below could also be constructed by hand, for example using canonical polygons as fundamental domains for the Fuchsian groups; see [8, Section 6.7].

LEMMA 3.5. *Let $(\Gamma_n)_n$ be a sequence of Fuchsian groups converging to a group Γ and τ_n the isomorphisms given in Definition 3.2. There exist homeomorphisms $\phi_n : \mathbb{H}^2/\Gamma \longrightarrow \mathbb{H}^2/\Gamma_n$ whose lifts $\tilde{\phi}_n$ satisfy, for any $\sigma \in \Gamma$,*

$$\tilde{\phi}_n \circ \sigma = \tau_n(\sigma) \circ \tilde{\phi}_n$$

and such that $(\tilde{\phi}_n)_n$ converges to the identity map uniformly on compact sets, that is,

$$\forall x \in \mathbb{H}^2, \quad \tilde{\phi}_n(x) \underset{n\to\infty}{\longrightarrow} x.$$

Now, let u be a C-convex function and S_u the surface described by u. The length structure $\mathcal{L}_u$ given by (2-3) induces a (pseudo-)distance d_{S_u}. In turn, d_{S_u} induces a length structure denoted by $L_{d_{S_u}}$ defined in the following way: the length of a curve $(c, u \circ c) : [0, 1] \to S_u$ is defined as

$$\begin{aligned} L_{d_{S_u}}(c, u \circ c) &= \sup_\delta \sum_{i=1}^n d_{S_u}((c(t_i), u \circ c(t_i)), (c(t_{i+1}), u \circ c(t_{i+1}))), \\ &= \sup_\delta \sum_{i=1}^n d_u(c(t_i), c(t_{i+1})) = L_{d_u}(c) \quad \text{(see Definition 2.10)} \end{aligned}$$

where the sup is taken over all the decompositions

$$\delta = \{(t_1, \ldots, t_n) \mid t_1 = 0 \leq t_2 \leq \cdots \leq t_n = 1\},$$

We have the following proposition.

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PROPOSITION 3.6. *Let $(u_n)_n$ be a sequence of convex functions such that:*

- *d_{u_n} is a complete distance with Lipschitz shortest paths;*
- *$\mathcal{L}_{u_n} = L_{d_{u_n}}$ on the set of Lipschitz curves;*
- *there exists $R < \pi/2$ with $0 \leq u_n < R$.*

Then, up to extracting a subsequence, $(u_n)_n$ converges to a convex function u and $(d_{u_n})_n$ converges to d_u uniformly on compact sets.

PROOF. The proof of this proposition is similar to that in [10, Proposition 3.12]. It was made using Proposition 2.9, the only difference is to use Lemma 2.11 and 3.4 instead of [10, Corollary 3.11]. □

We recall that in this paper we are using approximation by smooth surfaces. We note also that by Lemmas 2.11 and 3.4, d_{u_n} are complete distances on $\mathbb{H}^2$, and we have $\mathcal{L}_{u_n} = L_{d_{u_n}}$ (because of smoothness; see [7] for more details). We deduce the following lemma.

LEMMA 3.7. *Let (u_n, Γ_n) be Fuchsian C-convex functions such that:*

- *$(u_n, \Gamma_n)_n$ converges to a pair (u, Γ);*
- *there exists $R < \pi/2$ with $0 \leq u_n < R$;*
- *d_{u_n} are distances with Lipschitz shortest paths;*
- *d_{u_n} converge to d_u, uniformly on compact sets.*

Then on any compact set of $\mathbb{H}^2$, $d_{u_n}(\tilde{\phi}_n(.), \tilde{\phi}_n(.))$ uniformly converges to d_u, where $\tilde{\phi}_n$ is given by Lemma 3.5.

PROOF. By Lemmas 2.11 and 3.4, the topology induced by d_u on $\mathbb{H}^2$ is the topology for the hyperbolic metric. It follows that for the maps $\tilde{\phi}_n$ of Lemma 3.5, we have that on compact sets, the maps $x \mapsto d_{u_n}(\tilde{\phi}_n(x), x)$ uniformly converge to 0. By the triangle inequality, we have

$$d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_u(x, y) \leq d_{u_n}(\tilde{\phi}_n(x), x) + d_{u_n}(\tilde{\phi}_n(y), y) + d_{u_n}(x, y) - d_u(x, y).$$

By the preceding arguments and Proposition 3.6, for n sufficiently large the right-hand side is uniformly less than any $\epsilon > 0$. On the other hand, by the triangle inequality again, we have

$$\begin{aligned} d_u(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) &= d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) \\ &\leq d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, \tilde{\phi}_n(x)) + d_{u_n}(y, \tilde{\phi}_n(y)) \\ &\quad + d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)), \end{aligned}$$

which is uniformly less than any $\epsilon > 0$ for n sufficiently large (by the same arguments). □

By definition, if (u, Γ) is a Fuchsian C-convex function, then Γ acts by isometries on d_u. In turn, d_u defines a distance on the compact surface $\mathbb{H}^2/\Gamma$.

Definition 3.8. For a Fuchsian C-convex function (u, Γ), we denote by $\bar{d}_u$ the distance defined by d_u on $\mathbb{H}^2/\Gamma$.

The reason for introducing the maps $\tilde{\phi}_n$ from Lemma 3.5 is the following corollary of Lemma 3.7. Its proof is formally the same as that of [10, Proposition 3.19]. (The definition of uniform convergence of metric spaces is recalled in Definition 5.1.)

Proposition 3.9. *Let (u_n, Γ_n) be Fuchsian C-convex functions converging to a pair (u, Γ). Up to extracting a subsequence, $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ uniformly converges to $(\mathbb{H}^2/\Gamma, \bar{d}_u)$.*

3.2. Convergence of metrics implies convergence of groups. The aim of this section is to prove Proposition 3.10, which may be seen as a kind of converse of Proposition 3.9. The distance $\bar{d}_u$ was defined in Definition 3.8.

Proposition 3.10. *Let (S, d) be a metric of curvature less than or equal to -1 and let (u_n, Γ_n) be smooth Fuchsian C-convex functions, such that the sequence $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ uniformly converges to (S, d). Up to extracting a subsequence,*

- *$(\Gamma_n)_n$ converges to a Fuchsian group Γ;*
- *there exists $\beta < \pi/2$ such that $0 \le u_n < \beta$.*

Under the hypothesis of Proposition 3.10, let us first prove the convergence of groups. We first have a consequence of simple hyperbolic geometry; see [10, Corollary 4.2].

Lemma 3.11. *There exist $G > 0$ and $N > 0$ such that for any $n > N$, for any $x \in \mathbb{H}^2$, for every element $\sigma_n \in \Gamma_n \setminus \{0\}$,*

$$d_{u_n}(x, \sigma_n(x)) \ge G.$$

Proposition 3.12. *Under the hypothesis of Proposition 3.10, up to extracting a subsequence, the sequence $(\Gamma_n)_n$ converges to a Fuchsian group Γ.*

Proof. First, by Lemma 2.11 we have that, for all $x, y \in \mathbb{H}^2$,

$$d_{u_n}(x, y) \le d_{\mathbb{H}^2}(x, y),$$

and by Lemma 3.11 we have that there exist $G > 0$ and $N > 0$ such that, for any $n > N$ and for any $x \in \mathbb{H}^2$,

$$G \le d_{u_n}(x, \sigma_n(x)) \le d_{\mathbb{H}^2}(x, \sigma_n(x));$$

in particular, if

$$L_{\sigma_n} = \min_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(x, \sigma_n(x)),$$

we have

$$G \le L_{\sigma_n}.$$

The length is uniformly bounded from below, hence by a classical result of Mumford [13] we can deduce that, up to extracting a subsequence, the sequence of groups converges. □

LEMMA 3.13. *Under the assumptions of Proposition 3.10, there exists $M < \pi/2$ such that for all n, there is $x_n \in \mathbb{H}^2$ such that $u_n(x_n) < M$.*

PROOF. Suppose that the result is false: for a sequence $M_k \to \pi/2$, there is n_k such that $u_{n_k} \geq M_k$. By the definition of the length structure (2-3), it follows that $d_{u_{n_k}} \leq \cos(M_k) d_{\mathbb{H}^2}$. In turn, $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ has a subsequence converging to 0, which is a contradiction. □

PROPOSITION 3.14. *Under the hypothesis of Proposition 3.10, there exists $\beta < \pi/2$ such that, for any $n \in \mathbb{N}$, for any $x \in \mathbb{H}^2$,*

$$u_n(x) < \beta.$$

PROOF. Let us consider the affine model of anti-de Sitter space. As the sequence of groups converges, there exists a compact set $C \subset \mathbb{D}$ that contains a fundamental domain for Γ_n for all n. Hence, the points x_n given by Lemma 3.13 can be chosen to all belong to C. The result follows because the convex maps $\bar{u}_n$ on the disc are zero on the boundary, so for any compact set C in the interior of the disc, the difference between the minimum and the maximum of $\bar{u}_n$ on C cannot be arbitrarily large. □

Proposition 3.10 is now proved.

4. Proof of Theorem 1.4

The proof relies on the two following results.

THEOREM 4.1 [11]. *Let (S, d) be a metric induced by a Riemannian metric of sectional curvature less than -1. Then there exists a C^∞ Fuchsian C-convex $u : \mathbb{H}^2 \to [0, \pi/2[$ such that $\bar{d}_u$ is isometric to d.*

THEOREM 4.2. *Let (S, d) be a metric of curvature ≤ -1. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n are homeomorphic to S and d_n are induced by Riemannian metrics with sectional curvature less than -1.*

Although Theorem 4.2 may seem well known, we found no reference for it, so we prove it in Section 5. Note that we are not aware if the analogue of Theorem 4.2 holds for metrics of curvature less than or equal to 1.

Let d be a metric of curvature less than or equal to -1 on S. From Theorem 4.2, there exists a sequence $(d_n)_n$ of metrics induced by Riemannian metrics with sectional curvature less than -1 on S that converges uniformly to d. By Theorem 4.1, for each $n \in \mathbb{N}$ there exists a Fuchsian C-convex pair (u_n, Γ_n) such that $\bar{d}_{u_n}$ is isometric to d_n and u_n is smooth. By Proposition 3.10 there is a subsequence of $(\Gamma_n)_n$ converging to a Fuchsian group Γ, and $\beta < \pi/2$ such that $0 \leq u_n < \beta$.

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So Lemma 2.7 and Proposition 3.9 apply: up to extracting a subsequence, there is a function u such that the induced distance on $\bar{d}_u$ (the quotient of d_u by Γ) is the uniform limit of $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})$, that is, the uniform limit of (S, d_n). The limit for uniform convergence is unique, up to isometries [6], so $\bar{d}_u$ is isometric to d. Theorem 1.4 is proved, with L the quotient of the AdS cylinder of Section 2 by Γ.

5. Approximation by smooth metrics

In the following we use the uniform convergence, so let us recall its definition.

Definition 5.1. We say that a sequence of metric spaces $(S_n, d_n)_n$ *converges uniformly* to the metric space (S, d) if there exist homeomorphisms $f_n : S \longrightarrow S_n$ such that

$$\sup_{x,y\in S} |d_n(f_n(x), f_n(y)) - d(x, y)| \xrightarrow[n\to\infty]{} 0.$$

If $S_n = S$ and $f_n = \mathrm{id}$, then this is the usual definition of uniform convergence of distance functions.

We want to check that a metric of curvature less than or equal to -1 on the closed surface S can be approximated (in the sense of uniform convergence) by distances induced by Riemannian metrics with sectional curvature less than -1. We first approximate by hyperbolic metrics with conical singularities of negative curvature. Then we 'smooth' those cone metrics.

5.1. Approximation of metrics by polyhedral metrics. Let (X, d_0) be a metric space such that every pair of points can be joined by a shortest path. A *(geodesic) triangle* Δ of X consists of three points $x, y, z \in X$ and shortest paths $[x, y], [y, z]$ and $[z, x]$. A hyperbolic comparison triangle for Δ is a geodesic triangle $\tilde{\Delta}$ in the hyperbolic space with vertices $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, such that $d_0(x, y) = d_{\mathbb{H}^2}(\tilde{x}, \tilde{y})$, $d_0(y, z) = d_{\mathbb{H}^2}(\tilde{y}, \tilde{z})$, $d_0(x, z) = d_{\mathbb{H}^2}(\tilde{x}, \tilde{z})$. The interior angle of $\tilde{\Delta}$ at $\tilde{x}$ is called the *comparison angle* at x of the triangle Δ.

Definition 5.2. Let γ, γ' be two nontrivial shortest paths issued from the same point x. Let $\tilde{\angle}(\gamma(t)x\gamma'(t'))$ be the angle at $\tilde{x}$ of the comparison triangle $\tilde{\Delta}$ with vertices $\tilde{\gamma}(t), \tilde{x}$ and $\tilde{\gamma}'(t')$ in the hyperbolic plane corresponding to the triangle $\Delta(\gamma(t)x\gamma(t'))$ in X. Then the *upper angle* at x of γ and γ' is defined by

$$\limsup_{t,t'\longrightarrow 0} \tilde{\angle}(\gamma(t)x\gamma'(t')). \tag{5-1}$$

Definition 5.3. We say that an intrinsic metric space (X, d_0) is CAT(-1) if the upper angle between any couple of sides of every geodesic triangle with distinct vertices is not greater than the angle between the corresponding sides of its comparison triangle in the hyperbolic plane.

Let $B_{d_0}(x, r)$ be the ball of center x and radius r in (X, d_0).

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DEFINITION 5.4. An intrinsic metric space (X, d_0) has curvature less than or equal to -1 (in the Alexandrov sense), if for any x there exists r such that $B_{d_0}(x, r)$ endowed with the induced (intrinsic) distance is CAT(-1).

Let us recall the notion of bounded integral curvature [1, Ch. I, page 6]. A *simple triangle* is a triangle bounding an open set homeomorphic to a disc, consisting of three distinct points (the vertices of the triangle) and three shortest paths joining these points, and which is convex relative to the boundary, that is, no two points of the boundary of the triangle can be joined by a curve outside the triangle, which is shorter than a suitable part of the boundary joining the points (see [1] for more details).

DEFINITION 5.5. An intrinsic distance d_0 on a surface S is said to be of bounded integral curvature (BIC), if (S, d_0) satisfies the following property. For every $x \in S$ and every neighborhood N_x of x homeomorphic to an open disc, for any finite system $\mathcal{T}$ of pairwise nonoverlapping simple triangles T belonging to N_x, the sum of the excesses

$$\delta_0(T) = \bar{\alpha}_T + \bar{\beta}_T + \bar{\gamma}_T - \pi$$

of the triangles $T \in \mathcal{T}$ with upper angles $(\bar{\alpha}_T, \bar{\beta}_T, \bar{\gamma}_T)$ is bounded from above by a number C depending only on the neighborhood N_x, that is,

$$\sum_{T \in \mathcal{T}} \delta_0(T) \leq C.$$

The main tool for our approximation result is the following theorem.

THEOREM 5.6 [1, Theorem 2, page 59]. *Let $\epsilon > 0$. A compact BIC surface admits a triangulation by a finite number of arbitrary nonoverlapping simple triangles of diameter less than ϵ.*

To prove that the sum of the angles in a cone point is not less than 2π, we also need the following result, which corresponds to [1, Ch. II, Theorem 11, page 47].

LEMMA 5.7. *Let p be a point on a BIC surface such that there is at least one shortest arc containing p in its interior. Then for any decomposition of a neighborhood of p into sectors convex relative to the boundary formed by geodesic rays issued from p such that the upper angles between the sides of these sectors exist and do not exceed π, the total sum of those angles is not less than 2π.*

To get a triangulation of our surface, we use some properties of BIC surfaces, so let us consider the following lemma.

LEMMA 5.8. *A metric of curvature less than or equal to -1 is a BIC surface.*

PROOF. We have that a CAT(0) surface is a BIC surface (proved in [10, Lemma 2.18]), and that a CAT(-1) surface is also a CAT(0) surface (see [5, Ch. II, page 165]). The lemma follows because of the local nature of the definition of BIC and curvature less than or equal to -1. □

For a BIC surface, the angle exists, which means that in (5-1) the limit exists in place of the limsup [1]. So in the following we speak about angles rather than upper angles.

THEOREM 5.9. *Let (S, d) be a metric of curvature less than or equal to -1 on the closed surface S. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n is homeomorphic to S and d_n is the metric induced by a hyperbolic metric with conical singularities of negative curvature on S_n.*

The remainder of this section is devoted to the proof of Theorem 5.9.

Applying Theorem 5.6 and Lemma 5.8, we obtain a triangulation $\mathcal{T}_\epsilon$ of our surface in which every simple triangle has diameter less than ϵ. Replace the interiors of the triangles of $\mathcal{T}_\epsilon$ by the interiors of the hyperbolic comparison triangles. We obtain $(\bar{S}_\epsilon, \bar{d}_\epsilon)$, which is a hyperbolic metric with conical singularities, corresponding to the vertices of the triangles. By construction, $\bar{S}_\epsilon$ is endowed with a triangulation $\bar{\mathcal{T}}_\epsilon$.

LEMMA 5.10. *The total angles around the conical singularities of $\bar{d}_\epsilon$ are not less than 2π.*

PROOF. By a property of the CAT(-1) spaces, we have that every vertex of $\mathcal{T}_\epsilon$ lies in the interior of some geodesic in (S, d) [5, II.5.12]. Applying Lemma 5.7 we immediately get that the sum of the sector angles α_i at any vertex V of the triangulation $\mathcal{T}_\epsilon$ in (S, d) is not less than 2π. By definition of the CAT(-1) spaces, we have that the angles $\alpha_{-1,i}$ of the comparison triangles in the hyperbolic space are not less than the corresponding angles at every vertex V in the triangulation $\mathcal{T}_\epsilon$ in (S, d). It follows that

$$2\pi \leq \sum_i \alpha_i \leq \sum_i \alpha_{-1,i}. \qquad \square$$

We want to prove that the finer the triangulation is, the closer $\bar{d}_\epsilon$ is to d (for the uniform convergence between metric spaces). This relies on a series of lemmas.

LEMMA 5.11. *Let α be the angle at a vertex of a triangle T in a surface of curvature less than or equal to -1, and let α_{-1} be the corresponding angle in a comparison triangle T_{-1} in the hyperbolic space. Then*

$$\alpha_{-1} - \alpha \leq -\text{area}(T_{-1}) - \delta_0(T).$$

PROOF. If β and λ are the other angles of T and β_{-1}, λ_{-1} the corresponding angles in T_{-1}, then we have

$$\alpha_{-1} - \alpha \leq \alpha_{-1} - \alpha + \beta_{-1} - \beta + \lambda_{-1} - \lambda = \delta_0(T_{-1}) - \delta_0(T) = -\text{area}(T_{-1}) - \delta_0(T). \qquad \square$$

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LEMMA 5.12. *If $\mathcal{T}$ is a triangulation of a compact surface (S, d) with curvature less than or equal to -1 by nonoverlapping simple triangles, then*

$$\sum_{T\in\mathcal{T}} \delta_0(T) \geq 2\pi\mathcal{X}(S),$$

with $\mathcal{X}(S)$ the Euler characteristic of S.

PROOF. Let $|T|$ be the number of triangles, $|E|$ the number of edges and $|N|$ the number of vertices in our geodesic triangulation. We have $|E| = \frac{3}{2}|T|$, so that the Euler formula

$$|T| - |E| + |N| = \mathcal{X}(S)$$

implies

$$2|N| - |T| = 2\mathcal{X}(S). \tag{5-2}$$

If we denote by θ_i the sum of the angles of the triangles around a vertex, then, using (5-2), it follows that

$$\sum_{T\in\mathcal{T}} \delta_0(T) = \sum_{i=1}^{N} \theta_i - |T|\pi = \sum_{i=1}^{N} (\theta_i - 2\pi) + 2\pi\mathcal{X}(S)$$

The proof follows because $\theta_i - 2\pi \geq 0$ for all i. □

LEMMA 5.13. *Let T be an isosceles triangle in the hyperbolic space with diameter less than a given ϵ, with edges of length x, x, l and with θ the angle opposite to the edge of length l. Then*

$$l \leq \sinh(\epsilon)\theta.$$

PROOF. By the hyperbolic cosine law,

$$\cosh(l) = \cosh^2(x) - \sinh^2(x)\cos(\theta),$$

which is equivalent to

$$1 + 2\sinh^2\left(\frac{l}{2}\right) = \cosh^2(x) - \sinh^2(x)\left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right),$$

so

$$\frac{l}{2} \leq \sinh\left(\frac{l}{2}\right) = \sinh(x)\sin\left(\frac{\theta}{2}\right) \leq \sinh(\epsilon)\frac{\theta}{2}.$$ □

LEMMA 5.14. *Let $\epsilon > 0$. Let T be a simple triangle in (S, d) of diameter less than ϵ with vertices OXY. Let A (respectively, B) be on the edge OX (respectively, OY) and at distance a (respectively, b) from O. Let T^1_{-1} be a comparison triangle for T in the hyperbolic space, with vertices $O'X'Y'$. Let A' (respectively, B') be the corresponding point of A (respectively, B) (that is, on the edge $O'X'$ (respectively, $O'Y'$) and at distance a (respectively, b) from O'). Then*

$$0 \leq d_{\mathbb{H}^2}(A', B') - d(A, B) \leq -\delta_0(T)\sinh(\epsilon).$$

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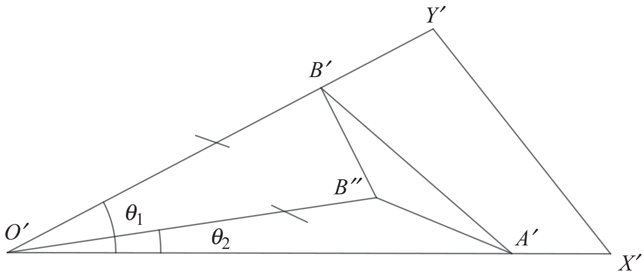

FIGURE 4. Notation for the proof of Lemma 5.14.

PROOF. The first inequality comes from the fact that we are in a CAT(−1) neighborhood ([5, page 158]). Let T^2_{-1} be the comparison triangle for OAB in the hyperbolic space drawn such that the edge of length a is identified with $O'A'$ (see Figure 4). Let B'' be the corresponding comparison point for B in T^2_{-1}; that is, B'' satisfies $d(A,B) = d_{\mathbb{H}^2}(A',B'')$ and $d(O,B) = d_{\mathbb{H}^2}(O',B'')$. By the triangle inequality, we have

$$d_{\mathbb{H}^2}(A',B') - d(A,B) = d_{\mathbb{H}^2}(A',B') - d_{\mathbb{H}^2}(A',B'') \leq d_{\mathbb{H}^2}(B',B''). \tag{5-3}$$

Let θ_1 be the angle at O' of T^1_{-1} (that is, the angle at O' of $O'A'B'$), and let θ_2 be the angle at O' of T^2_{-1} (that is, $O'A'B''$).

We have that $\theta_1 - \theta_2$ is the angle at O' of $O'B'B''$, which is isosceles, so by inequality (5-3) and Lemma 5.13 it follows that

$$d_{\mathbb{H}^2}(A',B') - d(A,B) \leq \sinh(\epsilon)(\theta_1 - \theta_2).$$

If β is the angle of T at O, then both θ_1 and θ_2 are angles corresponding to β in the different comparison triangles, so by Lemma 5.11,

$$\theta_1 - \theta_2 = \theta_1 - \beta + \beta - \theta_2 \leq \theta_1 - \beta \leq -\text{area}(T^1_{-1}) - \delta_0(T),$$

which leads to the result. □

Now, let us describe a homeomorphism between (S,d) and $(\bar{S}_\epsilon, \bar{d}_\epsilon)$ in the following way. The triangle $\bar{T}_i$ does not degenerate into segments, since the sum of every pair of sides is greater than the third. Therefore, the triangles T_i can be mapped homeomorphically onto the corresponding triangles $\bar{T}_i$, such that the vertices are sent to vertices and the homeomorphism restricts to an isometry along the edges. We consider any homeomorphism from the interior of the triangles that extends the homeomorphism on the boundary. As the surfaces are triangulated by such triangles, this gives a homeomorphism from S to $\bar{S}_\epsilon$.

For two points H and J on S, we denote by H', J' the corresponding points on $\bar{S}_\epsilon$.

FACT 5.15. In the above notation,

$$-2\epsilon \leq \bar{d}_\epsilon(H',J') - d(H,J) \leq 2\epsilon - 2\pi\mathcal{X}(S)\sinh(\epsilon).$$

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PROOF. The idea of this proof is the same as that of [2, Lemma 2, page 263]. Let us prove the first inequality. Let $H', J' \in \bar{S}_\epsilon$, let γ' be a shortest path joining H' and J', and let γ be a path joining H and J such that the intersection with every triangle T is a shortest path (that is, each connected piece of γ' meeting a triangle T' from a point A' to a point B' on the boundary of T' is associated in T with the shortest path joining the corresponding (in the sense of Lemma 5.14) points A and B).

Let us denote by γ'_i, $i = 0, \ldots, m+1$, the decomposition of γ' given by the triangles it crosses, and by $l(\gamma'_i)$ their lengths.

As (S, d) is CAT(-1), the length of a connected component of the intersection of γ' with T' joining two points of the boundary is greater than the length of the corresponding component of γ in T ([5], page 158). Now, because the diameters are not greater than ϵ, we have $l(\gamma_0) + l(\gamma_{m+1}) \leq 2\epsilon$ and $l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon$. It follows that

$$d(H, J) \leq \sum_{i=1}^{m} l(\gamma_i) + 2\epsilon \leq \sum_{i=1}^{m} l(\gamma'_i) + 2\epsilon \leq \bar{d}_\epsilon(H', J') + 2\epsilon$$

and in turn that

$$-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J).$$

The first inequality is now proved.

Let us now prove the second inequality. Consider a shortest path γ joining H and J in S and let γ' be a path in $\bar{S}_\epsilon$ joining H' and J' such that the intersection with every triangle T' is a shortest path (that is, each connected piece of γ meeting a triangle T from a point A to a point B on the boundary of T is associated in T' with the shortest path joining the corresponding (in the sense of Lemma 5.14) points A' and B').

Let us denote by γ_i, $i = 0, \ldots, m+1$, the decomposition of γ given by the triangles it crosses, and by $l(\gamma_i)$ their lengths. We find that

$$\bar{d}_\epsilon(H', J') - d(H, J) \leq l(\gamma'_0) + l(\gamma'_{m+1}) + \sum_{i=1}^{m} l(\gamma'_i) - l(\gamma_i).$$

Since $l(\gamma'_0)$ and $l(\gamma'_{m+1})$ are not greater than ϵ, we have that $l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon$. By Lemma 5.14 it follows that

$$\bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - \sum_{i=1}^{m} \delta_0(T_i) \sinh(\epsilon),$$

But $\delta_0(T_i)$ are nonpositive and, moreover, the triangles T are relative convex, so γ meets each triangle at most once (because, if the shortest path γ meets the (geodesic) triangle more than once, then there will be two points on the boundary of the triangle joined by a shortest path lying outside of the triangle, which contradicts the fact that the triangles are convex relative to the boundary), so $-\sum_{i=1}^{m} \delta_0(T_i)$ is less than $-\sum_T \delta_0(T)$ for all the triangles of the triangulation of S, which is less than $-2\mathcal{X}(S)$ by Lemma 5.12. The second inequality is now proved, and we are done. □

The lemmas above imply the uniform convergence. Theorem 5.9 is now proved.

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5.2. Approximation of polyhedral metrics by smooth metrics.

PROPOSITION 5.16. *Let d be the metric induced by a hyperbolic metric with conical singularities of negative curvature on the closed surface S. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n is homeomorphic to S, and d_n is a metric induced by a Riemannian metric of sectional curvature less than -1.*

We use the same method as in [15, Lemma 3.9], but we choose the cone in anti-de Sitter space (Figure 5), rather than the hyperbolic space $\mathbb{H}^3$.

PROOF. Let $p \in S$ be a singular point of the polyhedral hyperbolic metric d. Consider a neighborhood U_p of p in S that does not contain any other singular point of d. As the curvature is supposed to be negative, the neighborhood U_p equipped with the restriction of the metric d will be isometric to the neighborhood of a space-like circular cone C_p in the affine model of the anti-de Sitter space, such that the singularity p corresponds to the apex of C_p. Consider a sequence of smooth convex functions, whose graphs coincide with the cone C_p outside a neighborhood of the apex, and converging to C_p (this is very classical; see, for example, [15, Lemma 3.9]).

Using Gauss's formula, one can easily check that the sectional curvature for the induced metric on the smooth approximating surfaces is less than or equal to -1. We can multiply those metrics by any constant $\lambda > 1$ to get the sectional curvature less than -1. As the surfaces differ only on a compact set, and as the approximating sequence is smooth, it follows from (2-3) that the induced distances are uniformly bi-Lipschitz to the hyperbolic metric. From this and Proposition 2.9, it is classical to deduce that the induced distances converge locally uniformly (hence uniformly in this case); see, for example, the proof of [10, Proposition 3.12].

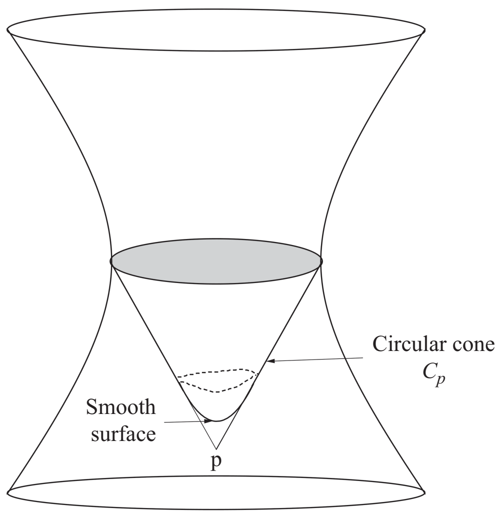

FIGURE 5. Smooth surface and circular cone in anti-de Sitter space.

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The proposition follows by applying this procedure simultaneously to all singular points of the metric d. □

Let d be any metric of curvature less than or equal to -1 on a compact surface S. We obtain a sequence $(d_n)_n$ from Theorem 5.9, and for each d_n, a sequence $(d_{n_k})_k$ from Proposition 5.16. Theorem 4.2 follows from a diagonal argument.

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