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Gosset Polytopes in Picard Groups of del Pezzo Surfaces

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Abstract. In this article, we study the correspondence between the geometry of del Pezzo surfaces S_r and the geometry of the r-dimensional Gosset polytopes $(r-4)_{21}$. We construct Gosset polytopes $(r-4)_{21}$ in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ whose vertices are lines, and we identify divisor classes in $\operatorname{Pic} S_r$ corresponding to $(a-1)$-simplexes ($a \leq r$), $(r-1)$-simplexes and $(r-1)$-crosspolytopes of the polytope $(r-4)_{21}$. Then we explain how these classes correspond to skew a-lines($a \leq r$), exceptional systems, and rulings, respectively.

As an application, we work on the monoidal transform for lines to study the local geometry of the polytope $(r-4)_{21}$. And we show that the Gieser transformation and the Bertini transformation induce a symmetry of polytopes 3_{21} and 4_{21}, respectively.

1 Introduction

The celebrated Dynkin diagrams appear as the key ingredients in many areas of mathematical research. In the geometry of polytopes, they represent the dihedral angles between the hyperplanes generating the polytopes, and in the algebraic geometry of surfaces, they are the intersections between the simple roots generating a root space. In fact, the diagrams in each of above areas of research correspond to the relationships presenting symmetry groups which commonly appear in each study on the objects represented by the graphs. In particular, the Dynkin diagrams of the Lie groups E_r, $3 \leq r \leq 8$ correspond to both the Weyl groups $W(S_r)$ of del Pezzo surfaces S_r and the symmetry group of the r-dimensional semiregular E_r-polytopes $(r-4)_{21}$, which are also known as Gosset polytopes. Therefore, there is a natural correspondence between the geometry of the del Pezzo surface and the geometry of the $(r-4)_{21}$ polytope. This article explores the correspondence between del Pezzo surfaces and $(r-4)_{21}$ polytopes.

The del Pezzo surfaces are smooth irreducible surfaces S_r whose anticanonical class $-K_{S_r}$ is ample. We can construct the del Pezzo surfaces by blowing up $r \leq 8$ points from $\mathbb{P}^2$, unless it is $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, it is well known that there are 27 lines on a cubic surface S_6 and the configuration of these lines is acted on by the Weyl group E_6 ([9, 10, 13]). The set of 27 lines in S_6 is bijective with the set of vertices of a Gosset 2_{21} polytope, *i.e.*, an E_6-polytope. Similar correspondences were found for the 28 bitangents in S_7 and the tritangent planes for S_8. The bijection between lines in S_6 and vertices in 2_{21} was applied to study the geometry of 2_{21} by Coxeter ([4]). And the complete list (see [17]) of bijections between the divisor classes containing lines and vertices is well known and applied in many different research fields. In particular, the

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classical application can be found in the study of Du Val [11]. These divisor classes, which are also called *lines*, play key roles in this article.

The study of lines in del Pezzo surfaces has developed in many different directions. Recently, Leung and Zhang related the configurations of the lines to the geometry of the line bundles over del Pezzo surfaces via representation theory [15, 16]. Other interesting research regarding the lines in del Pezzo surfaces and their symmetry groups can be found in [1, 12, 18].

A *line* in Pic S_r is equivalently a divisor class l with $l^2 = -1$ and $K_{S_r} \cdot l = -1$. We observe that the Weyl group $W(S_r)$ acts as an affine reflection group on the affine hyperplane given by $D \cdot K_{S_r} = -1$. Furthermore, $W(S_r)$ acts on the set of lines in Pic S_r. Therefrom, we construct a Gosset polytope $(r-4)_{21}$ in Pic $S_r \otimes \mathbb{Q}$ whose vertices are exactly the lines in Pic S_r. For a Gosset polytope $(r-4)_{21}$, faces are regular simplexes except for the facets which consist of $(r-1)$-simplexes and $(r-1)$-crosspolytopes. Since the faces in $(r-4)_{21}$ are basically configurations of vertices, we obtain natural characterization of faces in $(r-4)_{21}$ as divisor classes in Pic S_r.

Now we want to use the algebraic geometry of del Pezzo surfaces to identify the divisor classes corresponding to the faces in $(r-4)_{21}$. For this purpose, we consider divisor classes which we call *skew a-lines, exceptional systems*, and *rulings* in Pic S_r.

A *skew a-line* is an extension of the definition of lines in S_r. We show that each skew a-line represents an $(a-1)$-simplex in an $(r-4)_{21}$ polytope. In fact, the skew a-lines also satisfy $D^2 = -a$ and $D \cdot K_{S_r} = -a$. Furthermore the divisors with these conditions are equivalently skew a-lines for $a \leq 3$.

An *exceptional system* is a divisor class in Pic S_r whose linear system gives a regular map from S_r to $\mathbb{P}^2$. As this regular map corresponds to a blowing up from $\mathbb{P}^2$ to S_r, naturally we relate exceptional systems to $(r-1)$-simplexes in $(r-4)_{21}$ polytopes, which are one of two types of facets appearing in $(r-4)_{21}$ polytopes. We show that the set of exceptional systems in Pic S_r is bijective to the set of the $(r-1)$-simplexes in $(r-4)_{21}$ polytopes for $3 \leq r \leq 7$.

A *ruling* is a divisor class in Pic S_r that gives a fibration of S_r over $\mathbb{P}^1$. And we show that the set of rulings in Pic S_r is bijective with the set of $(r-1)$-crosspolytopes in the $(r-4)_{21}$ polytope. Furthermore, we explain the relationships between lines and rulings according to the incidence between the vertices and $(r-1)$-crosspolytopes. This leads us to the fact that a pair of proper crosspolytopes in the $(r-4)_{21}$ gives the blowing down maps from S_r to $\mathbb{P}^1 \times \mathbb{P}^1$.

After proper comparison between divisor classes obtained from the geometry of the polytope $(r-4)_{21}$ and those given by the geometry of a del Pezzo surface, we arrive at the following correspondences.

del Pezzo surface S_r	E-semiregular polytopes $(r-4)_{21}$
lines	vertices
skew a-lines $1 \leq a \leq r$	$(a-1)$-simplexes $1 \leq a \leq r$
exceptional systems	$(r-1)$-simplexes $(r < 8)$
rulings	$(r-1)$-crosspolytopes

The nature of these correspondences is macroscopic, but we need a microscopic explanation of the correspondences to decode the local geometry of the $(r-4)_{21}$ polytopes. Thus, we consider the monoidal transform for lines on del Pezzo surfaces and describe the local geometry of the $(r-4)_{21}$ polytopes. This blowing up procedure on lines can be applied to rulings to get a useful recursive description. This will be discussed along with the corresponding geometry on the polytope $(r-4)_{21}$ in [14] and a future article.

As another application, we consider the pairs of lines in $\operatorname{Pic} S_7$ (resp. $\operatorname{Pic} S_8$) with intersection 2 (resp. 3) that are related to the 28 bitangents (resp. tritangent plane). And we define the Gieser transformation (resp. Bertini transformation) on the polytope 3_{21} (resp. 4_{21}) and show that this is a symmetry.

Research on regular and semiregular polytopes along the Coxeter–Dynkin diagrams have a long history which may be well known only as facts. So we begin the next section with preliminaries on the theories of the regular and semiregular polytopes.

2 Regular and Semiregular Polytopes

In this article, we deal with polytopes having highly nontrivial symmetries. Their symmetry groups, along with the corresponding Coxeter–Dynkin diagrams, play key roles. In this section, we revisit the general theory of regular and semiregular polytopes according to their symmetry groups and Coxeter–Dynkin diagrams. Especially, we consider a family of semiregular polytopes known as Gosset figures (k_{21} according to Coxeter). The combinatorial data of Gosset figures along with the group actions will be used everywhere in this article. For further details about the theory, the reader may consult Coxeter's papers [5–8].

Let P_n be a convex n-polytope in an n-dimensional euclidean space. For each vertex O, the midpoints of all the edges emanating from a vertex O in P_n form an $(n-1)$-polytope if they lie in a hyperplane. We call this $(n-1)$-polytope the *vertex figure* of P_n at O.

A polytope P_n $(n > 2)$ is said to be *regular* if its facets are regular and there is a regular vertex figure at each vertex. When $n = 2$, a polygon P_2 is regular if it is equilateral and equiangular. Naturally, the facets of regular P_n are all congruent, and the vertex figures are all the same.

We consider two classes of regular polytopes.

(i) A *regular simplex* α_n is an n-dimensional simplex with equilateral edges. For example, α_1 is a line-segment, α_2 is an equilateral triangle, and α_3 is a tetrahedron. Note α_n is a pyramid based on α_{n-1}. Thus the facets of a regular simplex α_n form a regular simplex α_{n-1}, and the vertex figure of α_n is also α_{n-1}. Furthermore, the symmetry group of α_n is the Coxeter group A_n with order $(n+1)!$.

(ii) A *crosspolytope* β_n is an n-dimensional polytope whose $2n$-vertices are the intersects between an n-dimensional Cartesian coordinate frame and a sphere centered at the origin. For instance, β_1 is a line-segment, β_2 is a square, and β_3 is an octahedron. Note that β_n is a bipyramid based on β_{n-1}, and the n-vertices in β_n form α_{n-1} if a choice is made of one vertex from each Cartesian coordinate line. So the vertex figure of a crosspolytope β_n is also a crosspolytope β_{n-1}, and the facets of β_n are α_{n-1}. The symmetry group of β_n is the Coxeter group B_n (or C_n) with order $2^n n!$.

k	E_{k+4}	order of E_{k+4}	k_{21}-polytopes
-1	$A_1 \times A_2$	12	triangular prism
0	A_4	5!	rectified 5-cell
1	D_5	$2^4 5!$	demipenteract
2	E_6	$72 \times 6!$	E_6-polytope
3	E_7	$8 \times 9!$	E_7-polytope
4	E_8	$192 \times 10!$	E_8-polytope

Table 1: k_{21} Polytopes

Remark 1 Even though the full symmetry group of β_n is the Coxeter group (or C_n), the combinatorics of β_n can be described in terms of the Coxeter group D_n with order $2^{n-1}n!$, and as the Dynkin diagram of D_n is naturally imbedded in that of E-type, we choose D_n as the symmetry group of β_n instead of B_n (or C_n).

A polytope P_n is called *semiregular* if its facets are regular and its vertices are equivalent, namely, the symmetry group of P_n acts transitively on the vertices of P_n.

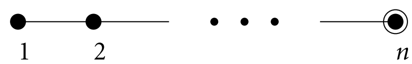
Here, we consider the semiregular k_{21} polytopes discovered by Gosset, which are $(k+4)$-dimensional polytopes whose symmetry groups are the Coxeter group E_{k+4}, $-1 \leq k \leq 4$. Note that the vertex figure of k_{21} is $(k-1)_{21}$ and the facets of k_{21} are regular simplexes α_{k+3} and crosspolytopes β_{k+3}. Table 1 contains the list of k_{21} polytopes.

The *Coxeter groups* are reflection groups generated by the reflections with respect to hyperplanes (called mirrors), and the *Coxeter–Dynkin diagrams* of Coxeter groups are labeled graphs where the nodes are indexed mirrors and the labels on edges present the order n of dihedral angle π/n between two mirrors. If two mirrors are perpendicular, namely $n = 2$, no edge joins two nodes presenting the mirrors because there is no interaction between the mirrors. Since the dihedral angle $\pi/3$ appears very often, we only label the edges when the corresponding order is $n > 3$. Each Coxeter–Dynkin diagram contains at least one ringed node which represents an active mirror, *i.e.,* there is a point off the mirror, and constructing a polytope begins with reflecting the point through the active mirror.

We call the Coxeter–Dynkin diagram of α_n (respectively β_n and k_{21}) with the Coxeter group A_n (respectively D_n and E_n) A_n-type (respectively D_n- and E_n-type), and each Coxeter–Dynkin diagram of A_n, D_n and E_n-type has only one ringed node and no labeled edges. Here, it is important to note that the full symmetry group of β_n is actually the Coxeter group B_n (or C_n), but as in Remark 1, we use the Coxeter group D_n to describe the combinatorics of β_n. For the above cases, the following simple procedure using the Coxeter–Dynkin diagram describes the possible faces and calculates the total number of them (see also [6, 8]).

The Coxeter–Dynkin diagram of each face P' is a connected subgraph Γ containing the ringed node. And the subgraph obtained by taking off all the nodes joined with the subgraph Γ represents the isotropy group $G_{P'}$ of P'. Furthermore, the index between the symmetry group G of the ambient polytope and isotropy group $G_{P'}$ gives the total number of such faces. In particular, by removing the ringed node and transferring to the ring the node adjacent to it, we obtain the subgraph corresponding to the isotropy group of a vertex, and in fact the isotropy group is the symmetry group of the vertex figure.

Regular simplex α_n with symmetry group A_n.

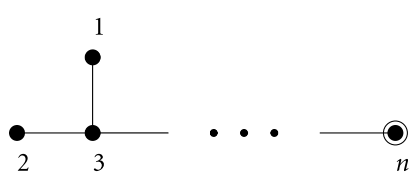


Coxeter–Dynkin diagram of $4\alpha_n$.

The diagram of the vertex figure is A_{n-1}-type because it is represented by the subgraph remaining after removing the ringed node and transferring the ring to the adjacent node, and the facet is only α_{n-1} because the subgraph of A_{n-1}-type is the largest connected subgraph containing the ringed node in the graph of A_n-type. Furthermore, since all the possible subgraphs containing the ringed node are A_k-type, only regular simplexes α_k, $0 \le k \le n-1$ appear as faces. And for each α_k in α_n, the possible total number $N_{\alpha_k}^{\alpha_n}$ is

$$N_{\alpha_k}^{\alpha_n} = [A_n : A_k \times A_{n-k-1}] = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}.$$

Cross polytope β_n with symmetry group D_n. Again, although B_n is the full symmetry group of β_n, we consider D_n as in Remark 1.



Coxeter–Dynkin diagram of β_n.

The diagram of the vertex figure is D_{n-1}-type because the subgraph remaining after removing the ringed node represents D_{n-1}, and the facet is only α_{n-1} since the subgraph of A_{n-1}-type is the biggest subgraph containing the ringed node in D_n-type. Only regular simplexes α_k, $k = 0, \dots, n-1$ appear as faces since the possible subgraphs containing the ringed node are only A_k-type. And for each α_k in β_n, the possible total number $N_{\alpha_k}^{\beta_n}$ is

$$N_{\alpha_k}^{\beta_n} = [D_n : A_k \times D_{n-k-1}] = \frac{2^{n-1} n!}{(k+1)! 2^{n-k-2} (n-k-1)!} = 2^{k+1} \binom{n}{k+1}.$$

In particular, each β_n contains $N_{\alpha_0}^{\beta_n} = 2n$ vertices, and these vertices form n-pairs with the common center.

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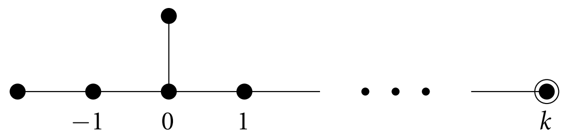
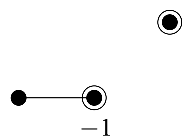
Gosset polytope k_{21}, a $(k+4)$-dimensional polytope with symmetry group E_{k+4}, $-1 \leq k \leq 4$.



Coxeter–Dynkin diagram of k_{21} $k \neq -1$.

For $k \neq -1$, the diagram of the vertex figure is E_{k+3}-type and the facets are the regular simplex α_{k+3} and the crosspolytope β_{k+3}, since the subgraphs of A_{k+3}-type and D_{k+3}-type appear as the biggest subgraph containing the ringed node in E_{k+3}-type. But all the lower dimensional faces are regular simplexes.

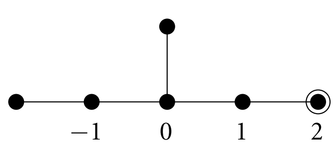
Case $k = -1$ is a bit different from other cases since there are two ringed nodes.



Coxeter–Dynkin diagram of -1_{21}.

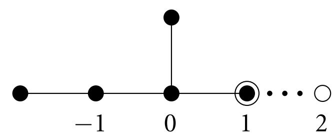
The vertex figure is an isosceles triangle instead of an equilateral triangle because the corresponding diagram is obtained by taking off a ringed node in the A_2-type subgraph. And the facets are the regular triangle α_2 given by the A_2-type subgraph and the square β_2 given by the subgraph taking off the unringed node.

As above, we can calculate the total number of faces in k_{21} by using Coxeter–Dynkin diagrams. For instance, we calculate 2_{21}. After removing the ringed node labelled 2 and transferring the ring to the node labelled 1, we obtain a subgraph of E_5-type, and therefore the vertex figure of 2_{21} is 1_{21}. Since the subgraphs of A_5-type and D_5-type are all the biggest possible subgraphs in the Coxeter–Dynkin diagram of 2_{21}, there are two types of facets in 2_{21}, which are 5-simplexes and 5-crosspolytopes, respectively. And all other faces in 2_{21} are simplexes for the same reason. In the following calculation for 2_{21}, the nodes marked by empty nodes represent deleted nodes.

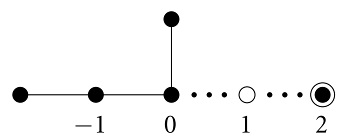


Coxeter–Dynkin diagram of 2_{21}.

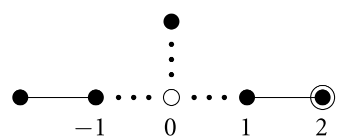
(i) Vertices in 2_{21}: $N^{2_{21}}_{\alpha_0} = [E_6 : E_5] = 27$.



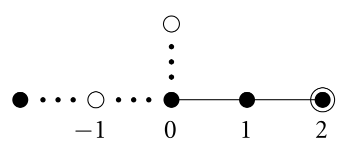
(ii) 1-simplexes(edges) in 2_{21}: $N^{2_{21}}_{\alpha_1} = [E_6 : A_1 \times E_4] = 216$.



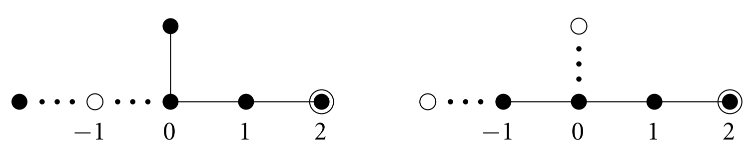
(iii) 2-simplexes(faces) in 2_{21}: $N^{2_{21}}_{\alpha_2} = [E_6 : A_2 \times E_3] = 720$.



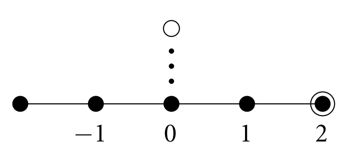
(iv) 3-simplexes(cells) in 2_{21}: $N^{2_{21}}_{\alpha_3} = [E_6 : A_3 \times A_1] = 1080$.



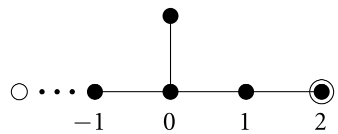
(v) 4-simplexes in 2_{21}: $N^{2_{21}}_{\alpha_4} = [E_6 : A_4 \times A_1] + [E_6 : A_4] = 648$.



(vi) 5-simplexes in 2_{21}: $N^{2_{21}}_{\alpha_5} = [E_6 : A_5] = 72$.



(vii) 5-crosspolytopes in 2_{21}: $N^{2_{21}}_{\beta_5} = [E_6 : D_5] = 27$.



As we apply the same procedure to the other E-polytopes, we get the following table.

E_{k+4}-polytope(k_{21})	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
β_{k+3}	3	5	10	27	126	2160
vertex	6	10	16	27	56	240
α_1	9	30	80	216	756	6720
α_2	2	30	160	720	4032	60480
α_3		5	120	1080	10080	241920
α_4			16	648	12096	483840
α_5				72	6048	483840
α_6					576	207360
α_7						17280

Table 2: Numbers of faces in k_{21}

3 Del Pezzo Surfaces S_r

A del Pezzo surface is a smooth irreducible surface whose anticanonical class $-K_S$ is ample. It is well known that a del Pezzo surface S_r, unless it is $\mathbb{P}^1 \times \mathbb{P}^1$, can be obtained from $\mathbb{P}^2$ by blowing up $r \leq 8$ points in generic positions; namely, no three points are on a line, no six points are on a conic, and for $r = 8$, not all of them are on a plane curve whose singular point is one of them (see [9, 13, 17]).

Notation We do not use different notations for the divisors and the corresponding classes in Picard group unless there is the possibility of confusion.

We denote such a del Pezzo surface by S_r and the corresponding blowup by $\pi_r \colon S_r \to \mathbb{P}^2$. And $K^2_{S_r} = 9 - r$ is called the degree of the del Pezzo surface. Each exceptional curve and the corresponding class given by blowing up is denoted by e_i, and both the class of $\pi_r^*(h)$ in S_r and the class of a line h in $\mathbb{P}^2$ are referred to as h. Then we have

$$h^2 = 1, \quad h \cdot e_i = 0, \quad e_i \cdot e_j = -\delta_{ij} \quad \text{for } 1 \leq i, j \leq r,$$

and the Picard group of S_r is $\operatorname{Pic} S_r \simeq \mathbb{Z}h \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ with the signature $(1, -r)$. And $K_{S_r} = -3h + \sum_{i=1}^{r} e_i$.

For any irreducible curve C on a del Pezzo surface S_r, we have $C \cdot K_{S_r} < 0$ since $-K_{S_r}$ is ample. Furthermore, if the curve C has a negative self-intersection, C must be a smooth rational curve with $C^2 = -1$ by the adjunction formula.

The ample $-K_{S_r}$ on a del Pezzo surface S_r is very useful for dealing with $\operatorname{Pic} S_r$. The inner product given by the intersection on $\operatorname{Pic} S_r$ induces a negative definite metric on $(\mathbb{Z}K_{S_r})^{\perp}$ in $\operatorname{Pic} S_r$ where we can also define natural reflections.

To define reflections on $(\mathbb{Z}K_{S_r})^{\perp}$ in $\operatorname{Pic} S_r$, we consider a root system

$$R_r := \{d \in \operatorname{Pic} S_r \mid d^2 = -2,\ d \cdot K_{S_r} = 0\},$$

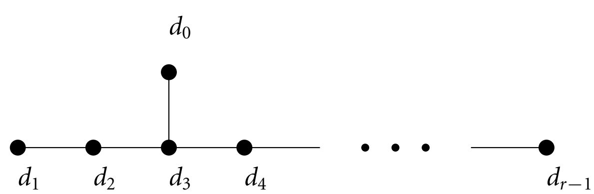
with simple roots

$$d_0 = h - e_1 - e_2 - e_3, d_i = e_i - e_{i+1}, \quad 1 \leq i \leq r-1.$$

Each element d in R_r defines a reflection on $(\mathbb{Z}K_{S_r})^{\perp}$ in $\operatorname{Pic} S_r$

$$\sigma_d(D) := D + (D \cdot d)d \ \text{ for } D \in (\mathbb{Z}K_{S_r})^{\perp}$$

and the corresponding Weyl group $W(S_r)$ is E_r, where $3 \leq r \leq 8$ with the Dynkin diagram



Dynkin diagram of E_r $r \geq 3$.

The definition of the reflection σ_d on $(\mathbb{Z}K_{S_r})^{\perp}$ can be used to obtain a transformation both on $\operatorname{Pic} S_r$ and on $\operatorname{Pic} S_r \otimes \mathbb{Q} \simeq \mathbb{Q}h \oplus \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_r$ via the linear extension of the intersections of divisors in $\operatorname{Pic} S_r$. Here $\operatorname{Pic} S_r \otimes \mathbb{Q}$ is a vector space with the signature $(1, -r)$.

3.1 Affine Hyperplanes and the Reflection Groups

Later on, we deal with divisor classes D satisfying equations $D \cdot K_{S_r} = \alpha$, $D^2 = \beta$ that are preserved by the action of Weyl group $W(S_r)$. Here we know that $W(S_r)$ is generated by the reflections on $(\mathbb{Z}K_{S_r})^{\perp}$ given by simple roots. To extend the action of $W(S_r)$ properly, we want to show that these reflections are defined on $\operatorname{Pic} S_r$ and preserve the above equations. Furthermore, we see that $W(S_r)$ acts as a reflection group on the set of divisor classes with $D \cdot K_{S_r} = \alpha$.

We consider an *affine hyperplane section* in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ defined by

$$\tilde{H}_b := \{D \in \operatorname{Pic} S_r \otimes \mathbb{Q} \mid -D \cdot K_{S_r} = b\},$$

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where b is an arbitrary real number and an affine hyperplane section $H_b := \tilde{H}_b \cap \operatorname{Pic} S_r$ in $\operatorname{Pic} S_r$. Since $-K_{S_r}$ is ample, we are interested in $b \geq 0$.

By the fact that $K_{S_r}^2 = 9 - r > 0$, $3 \leq r \leq 8$, and the Hodge index theorem,

$$0 = (K_{S_r} \cdot (D_1 - D_2))^2 \geq K_{S_r}^2 (D_1 - D_2)^2, \quad D_1, D_2 \in H_b,$$

the inner product on $\operatorname{Pic} S_r$ induces a negative definite metric on H_b. As a matter of fact, the induced metric is defined on $\operatorname{Pic} S_r \otimes \mathbb{Q}$, and we can also consider the induced norm by fixing a center $\frac{-b}{9-r} K_{S_r}$ in the affine hyperplane section $-D \cdot K_{S_r} = b$ in $\operatorname{Pic} S_r \otimes \mathbb{Q}$. This norm is also negative definite.

Lemma 3.1 (i) *Let $\tilde{H}_b$ $(b \geq 0)$ be an affine hyperplane section in* $\operatorname{Pic} S_r \otimes \mathbb{Q}$ *defined above and let $\frac{-b}{9-r} K_{S_r}$ be a center on the affine hyperplane section. The classes D in $H_b = \tilde{H}_b \cap$* $\operatorname{Pic} S_r$ *with a fixed self-intersection are on a sphere with center $\frac{-b}{9-r} K_{S_r}$ in $\tilde{H}_b$.*

(ii) *For each root d in R_r, the corresponding reflection σ_d defined on* $\operatorname{Pic} S_r \otimes \mathbb{Q}$ *is an isometry preserving K_{S_r} and acts as a reflection on each hyperplane section $\tilde{H}_b$ with the center $\frac{-b}{9-r} K_{S_r}$.*

Proof (i) Consider

$$\Big(D - \frac{-b}{9-r} K_{S_r}\Big)^2 = D^2 + \frac{2b}{9-r} D \cdot K_{S_r} + \frac{b^2}{(9-r)^2} K_{S_r}^2 = D^2 - \frac{b^2}{(9-r)} \leq 0$$

and the last inequality is given by the Hodge index theorem,

$$b^2 = (D \cdot K_{S_r})^2 \geq D^2 K_{S_r}^2 = D^2 (9 - r).$$

(ii) Each root d in R_r satisfies $d \cdot K_{S_r} = 0$ and $d^2 = -2$. Therefore, we have

$$\sigma_d(K_{S_r}) = K_{S_r} + (d \cdot K_{S_r}) d = K_{S_r},$$

and for each $D_1, D_2 \in \operatorname{Pic} S_r \otimes \mathbb{Q}$

$$\sigma_d(D_1) \cdot \sigma_d(D_2) = (D_1 + (d \cdot D_1) d) \cdot (D_2 + (d \cdot D_2) d) = D_1 \cdot D_2.$$

Furthermore, for each class D in $\operatorname{Pic} S_r \otimes \mathbb{Q}$, the self-intersection D^2 and $D \cdot K_{S_r}$ are invariant under σ_d. This implies σ_d acts on the hyperplane section $\tilde{H}_b$. Moreover, the hyperplane in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ preserved by the action of σ_d is given by an equation $d \cdot D = 0$ for $D \in \operatorname{Pic} S_r \otimes \mathbb{Q}$, and each center $\frac{-b}{9-r} K_{S_r}$ of $\tilde{H}_b$ is in this hyperplane. And because each class D in $\tilde{H}_b$ can be written as

$$D = D_3 + \frac{-b}{9-r} K_{S_r} \quad \text{for some } D_3 \in \tilde{H}_0,$$

we have

$$\sigma_d(D) = \sigma_d\Big(D_3 + \frac{-b}{9-r} K_{S_r}\Big) = \sigma_d(D_3) + \frac{-b}{9-r} K_{S_r} \in \tilde{H}_0 + \frac{-b}{9-r} K_{S_r} = \tilde{H}_b.$$

Since σ_d is a reflection on $\tilde{H}_0$, we can derive a fact that the isometry σ_d acts as an affine reflection on $\tilde{H}_b$ for the center $\frac{-b}{9-r} K_{S_r}$. ∎

Generically, the hyperplanes in $\mathrm{Pic}\, S_r \otimes \mathbb{Q}$ induce affine hyperplanes in $\tilde{H}_b$ and they may not share a common point. But the reflection hyperplane of each reflection σ_d in $\mathrm{Pic}\, S_r \otimes \mathbb{Q}$ gives a hyperplane in $\tilde{H}_b$ containing the center because it is given by a condition $K_{S_r} \cdot d = 0$. Therefrom, the above lemma gives the following corollary.

Corollary 3.2 *The affine reflections on $\tilde{H}_b$ given by simple roots in R_d generate the Weyl group $W(S_r)$.*

Remark The Weyl group $W(S_r)$, generated by the simple roots in R_d also preserves the self-intersection of each divisor in $\mathrm{Pic}\, S_r$ and acts on each $H_b = \mathrm{Pic}\, S_r \cap \tilde{H}_b$ as an affine reflection group. According to [12], the Weyl group $W(S_r)$ is the isotropy group of K_{S_r} in the automorphism group of $\mathrm{Pic}\, S_r$.

4 Gosset Polytopes $(r-4)_{21}$ in $\mathrm{Pic}\, S_r \otimes \mathbb{Q}$

In this section, we identify a special class in $\mathrm{Pic}\, S_r$, which is known as a line, and construct Gosset polytopes $(r-4)_{21}$ in $\mathrm{Pic}\, S_r \otimes \mathbb{Q}$ as the convex hull of the set of lines. And we study the divisor classes representing faces in $(r-4)_{21}$.

4.1 Lines in del Pezzo surface S_r

The configuration of lines on a del Pezzo surface S_r has attracted much attention because of its high degree of symmetry related to the Weyl group $W(S_r)$ of E_r-type. When $r \leq 6$, the anticanonical class $-K_{S_r}$ on the del Pezzo surface S_r is very ample and its linear system gives an imbedding to $\mathbb{P}^{9-r}$, where $K_{S_r}^2 = 9 - r$. And a smooth rational curve C in S_r is mapped to a line in $\mathbb{P}^{9-r}$ if and only if it is an exceptional curve in S_r. Furthermore, the divisor class D containing the curve C satisfies $D \cdot K_{S_r} = -1 = D^2$ and vice versa [17]. Since the last equation is true for each del Pezzo surface, we also call the divisors with the these conditions in del Pezzo surfaces *lines*. As the symmetry group of lines in the cubic is the Weyl group E_6, the symmetry group of lines in S_r is the Weyl group E_r.

We define the set of lines on $\mathrm{Pic}\, S_r$ as $L_r := \{l \in Pic(S_r) \mid l^2 = l \cdot K_{S_r} = -1\}$. By the adjunction formula, a divisor in this class represents a rational smooth curve in S_r. By going through a simple calculation, we can obtain the number of lines in L_r, and, moreover, the number of lines in $\mathrm{Pic}(S_r)$ is the same as the number of vertices in Gosset polytopes $(r-4)_{21}$.

del Pezzo Surfaces	S_3	S_4	S_5	S_6	S_7	S_8
number of Lines	6	10	16	27	56	240
Gosset Polytopes $(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
number of Vertices	6	10	16	27	56	240

In fact, this bijection between lines and vertices is a well-known fact ([10, 17]). In this article, this fact induces significant implications after our construction of Gosset polytopes in $\mathrm{Pic}\, S_r$, where each vertex automatically represents a line.

First of all, we need to consider intersections between lines and roots in *Pic* S_r. The possible intersections of the lines in *Pic* S_r can be obtained by the Hodge index theorem, $(K_{S_r} \cdot (l_1 \pm l_2))^2 \geq K_{S_r}^2 (l_1 \pm l_2)^2$. And we have

$$\frac{2}{9-r} + 1 \geq l_1 \cdot l_2 \geq -1.$$

Therefore, two distinct lines l_1 and l_2 in Pic S_r can have intersections such as

$$l_1 \cdot l_2 = \begin{cases} 0, 1 & 3 \leq r \leq 6, \\ 0, 1, 2 & r = 7, \\ 0, 1, 2, 3 & r = 8. \end{cases}$$

Furthermore, $l_1 \cdot l_2 = 2$ for $r = 7$ and $l_1 \cdot l_2 = 3$ for $r = 8$ satisfy the equalities in the Hodge index theorem, and we have equivalences

$$l_1 \cdot l_2 = 2 \Longleftrightarrow l_1 + l_2 = -K_{S_7} \quad \text{for } r = 7,$$
$$l_1 \cdot l_2 = 3 \Longleftrightarrow l_1 + l_2 = -2K_{S_8} \quad \text{for } r = 8.$$

Recall that for a reflection σ_d given by a root d, if l is a line, $\sigma_d(l)$ is also a line by Lemma 3.1. Moreover we have $\sigma_d(l) \cdot l = (l + (l \cdot d)d) \cdot l = -1 + (l \cdot d)^2$.

From the above possible numbers of the intersections of lines, the possible intersections between a line l and a root d are given as

$$l \cdot d = \begin{cases} 0, \pm 1 & 3 \leq r \leq 7, \\ 0, \pm 1, \pm 2 & r = 8. \end{cases}$$

Since $\sigma_d(l) \cdot l + 1$ must be a square of these integers, it is easy to see that any two lines l_1 and l_2 with $l_1 \cdot l_2 = 1$ or 2 cannot be mapped to each other by a reflection σ_d given by a root d.

Lemma 4.1 (i) *For each line l in S_r, the reflection σ_d given by a root d preserves the line l if and only if $d \cdot l = 0$.*

(ii) *Any two distinct lines in* Pic S_r $(r \leq 7)$ *are skew if and only if there is a reflection σ_d given by a root d that reflects these lines to each other. For S_8, this statement is true with an extra condition that a root d is chosen to have intersection* 1 *with one of the lines.*

We say that distinct lines l_1 and l_2 are *skew* if $l_1 \cdot l_2 = 0$.

Proof (i) Trivial from $l = \sigma_d(l) = l + (d \cdot l)d$.

(ii) If $l_1$and l_2 are skew, namely $l_1 \cdot l_2 = 0$, then $l_1 - l_2$ is a root, and the corresponding reflection $\sigma_{l_1 - l_2}$ satisfies $\sigma_{l_1 - l_2}(l_1) = l_2$ and $\sigma_{l_1 - l_2}(l_2) = l_1$. Conversely, when $3 \leq r \leq 7$, if two distinct lines l_1 and l_2 satisfy $l_2 = \sigma_d(l_1) = l_1 + (d \cdot l_1)d$, for a root d with $d \cdot l_1 \neq 0$, then $l_1 \cdot l_2 = -1 + (d \cdot l_1)^2 = 0$, according to above list of possible intersections between a line and a root. The case $r = 8$ is similar to the other once we add the condition that $d \cdot l_1 = \pm 1$. ∎

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By Lemma 3.1, the action of Weyl group $W(S_r)$ preserves the conditions $l^2 = l \cdot K_{S_r} = -1$, and therefore $W(S_r)$ acts on the set of lines L_r on S_r. Furthermore, by the following theorem, there is only one orbit of $W(S_r)$ in the set of lines L_r, and it implies that the bijection between the set of lines and the set of vertices in the above is more than the correspondence between sets.

Theorem 4.2 *For each del Pezzo surface S_r, the set of lines L_r on S_r is the set of vertices of a Gosset polytope $(r-4)_{21}$ in a hyperplane section $\tilde{H}_1$.*

Proof Recall that the center of $\tilde{H}_1$ is $\frac{-K_{S_r}}{9-r}$ and the distance between a line and the center in $\tilde{H}_1$ is $-1-1/(9-r)$. Therefore, the set of lines L_r sits in a sphere in $\tilde{H}_1$ with the center $\frac{-K_{S_r}}{9-r}$. Furthermore, the convex hull of L_r in $\tilde{H}_1$ is a convex polytope. We want to show that this polytope is a Gosset polytope $(r-4)_{21}$. We construct a Gosset polytope $(r-4)_{21}$ in the convex hull of L_r, and we show that the convex hull is same with the polytope $(r-4)_{21}$. By Lemma 3.1, the set L_r is acted on by the Weyl group $W(S_r)$. We choose a line e_r in L_r and consider the generators of $W(S)$ given by simple roots d_i $(0 \leq i \leq r-1)$. Since $d_i \cdot e_r = 0$ except when $i = r-1$, the reflection given by d_{r-1} is only active among the generators. The line e_r and the generators of $W(S_r)$ give the Coxeter–Dynkin diagram of E_r-type with a ringed node at d_{r-1}. Therefore, via the action of $W(S_r)$ on e_r as in Section 2, we obtain a Gosset polytope $(r-4)_{21}$ containing e_r in the convex hull of L_r. Now, as we know, the number of vertices in $(r-4)_{21}$ is the same as the number of lines in L_r, therefore the polytope $(r-4)_{21}$ and the convex hull of L_r are the same. ■

This theorem implies that the Weyl group $W(S_r)$ acts transitively on L_r. We also see that the integral classes representing lines in $\operatorname{Pic}(S_r)$ are honest vertices of $(r-4)_{21}$ in an affine hyperplane H_1. But we also denote a line l in $\operatorname{Pic} S_r$ by V_l if it is considered as a vertex of $(r-4)_{21}$.

Corollary 4.3 *The number of lines in the del Pezzo surface S_r is the same as the number of vertices of the E_r-semiregular polytope. Furthermore, the Weyl group $W(S_r)$ acts transitively on the set of lines L_r on S_r.*

Remark From Section 2, we know that the isotropy group of a vertex of $(r-4)_{21}$ is the same with the symmetry group of the vertex figure which is E_{r-1}-type. We can check this for the Gosset polytope $(r-4)_{21}$ in $\operatorname{Pic} S_r \otimes \mathbb{Q}$. By Corollary 4.3, we can choose an exceptional class e_r without losing generality. The generators of the isotropy group of e_r are the simple roots in the Dynkin diagram of E_r that are perpendicular to e_r. And the relationships of these simple roots are presented as a subdiagram of the Dynkin diagram of E_r by taking off the node d_{r-1}. Therefore, the isotropy group of e_r in $W(S_r)$ is the Weyl group of $W(S_{r-1})$ of E_{r-1}-type, and similarly the isotropy group of each line in S_r is conjugate to $W(S_{r-1})$.

4.1.1 Intersections of Lines and Configuration of Vertices

As the configuration of lines is our main issue, the intersections between lines characterize how the corresponding vertices are related to the polytope $(r-4)_{21}$. Here we discuss the relationship via the properties of the lines with fixed intersections. The complete and uniform description will be given in the last section.

For a fixed line l, the isotropy group of l is E_{r-1}-type, and, moreover, it is the same with the symmetry group of the vertex figure of $(r-4)_{21}$. In fact, the vertex set of vertex figures corresponds to the set of lines intersecting l at zero by the following lemma.

Lemma 4.4 *For distinct lines l_1 and l_2 in* $\operatorname{Pic} S_r$*, the vertices V_{l_1} and V_{l_2} in the Gosset polytope $(r-4)_{21}$ in $\tilde{H}_1$ are joined by an edge if and only if $l_1 \cdot l_2 = 0$.*

Proof The semiregular polytope $(r-4)_{21}$ in $\tilde{H}_1$ is the convex hull of its vertices. Therefore, a fixed vertex and the vertices edged to it have the minimal distance among the distance between vertices. Since the metric on the hyperplane $\tilde{H}_1$ is negative definite (see §3), the distance $(l_1 - l_2)^2$ among lines in L_r is maximal if and only if $l_1 \cdot l_2 = 0$. Therefore we have the lemma. ■

Remark In fact, this lemma is one case of skew 2-lines of Theorem 5.1 in the next subsection, which is proved by a different argument.

For a fixed line l in a del Pezzo surface S_r, $3 \le r \le 8$, we consider the set $N_k(l, S_r) := \{l' \in L_r \mid l' \cdot l = k\}$. From the above lemma, we have $|N_0(l, S_r)| = |L_{r-1}| = [E_{r-1} : E_{r-2}]$, where the last equality is given by Theorem 4.2.

The simple comparison according to the above list of intersections of lines leads us to the following useful lemma.

Lemma 4.5 *For each line l in* $\operatorname{Pic} S_r$ *$(4 \le r \le 8)$, $|N_1(l, S_r)|$ equals the number of $(r-2)$-crosspolytopes in the polytope $(r-5)_{21}$,* i.e., $[E_{r-1} : D_{r-2}]$.

Proof Since Weyl group $W(S_r)$ acts transitively, we can choose an exceptional class e_r, and the results on this line also hold for the other lines. When $4 \le r \le 6$, a line e_r intersects the other lines with 0 or 1. Since $N_0(e_r, S_r)$ is the number of lines in the vertex figure, we have

$$N_1(e_r, S_r) = |L_r| - |N_0(e_r, S_r)| - |\{e_r\}| = |L_r| - |L_{r-1}| - 1.$$

So $N_1(e_4, S_4) = 3$, $N_1(e_5, S_5) = 5$, and $N_1(e_6, S_6) = 10$. These are exactly the numbers of crosspolytopes in -1_{21}, 0_{21}, and 1_{21}, respectively.

For $r = 7$, a line e_7 meets the other lines at 0, 1, or 2. As we saw above, $-K_{S_7} - e_7$ is the only line with intersecting e_7 by 2. So we have

$$\begin{aligned} N_1(e_7, S_7) &= |L_7| - |N_0(e_7, S_7)| - |N_2(e_7, S_7)| - |\{e_7\}| \\ &= |L_7| - |L_6| - 2 = 27 \end{aligned}$$

and this is the number of 5-crosspolytopes in 2_{21}.

For $r = 8$, a line e_8 may have intersection 0, 1, 2, or 3 with other lines. Later, we use a transformation between lines in $\operatorname{Pic} S_8$ defined by $-(2K_{S_8} + l)$, for each line l in L_8. Observe that a line l intersects e_8 at 0 if and only if $-(2K_{S_8} + l)$ intersects e_8 at 2.

Therefore $N_0(e_8, S_8) = N_2(e_8, S_8)$. Since $-2K_{S_8} - e_8$ is the only line in L_8 intersecting e_8 at 3, we get

$$\begin{aligned} N_1(e_8, S_8) &= |L_8| - |N_0(e_8, S_8)| - |N_2(e_8, S_8)| - |N_3(e_8, S_8)| - |\{e_8\}| \\ &= |L_8| - 2|L_7| - 2 = 126. \end{aligned}$$

and this is the number of 6-crosspolytopes in 3_{21}. ■

Remark The above results on $N_k(l, S_r)$ appear again when we study the monoidal transforms to lines in del Pezzo surfaces.

4.2 Faces in Gosset Polytope $(r-4)_{21}$

By Theorem 4.2, we construct the Gosset polytope $(r-4)_{21}$ in $\operatorname{Pic} S_r \otimes \mathbb{Q}$, and we want to characterize each face in $(r-4)_{21}$ as a class in $\operatorname{Pic} S_r$. There are k-simplexes $0 \le k \le r-1$ and $(r-1)$-crosspolytopes in $(r-4)_{21}$.

To identify each face in $(r-4)_{21}$, we want to use the barycenter of the face. By Theorem 4.2, each vertex of the polytope $(r-4)_{21}$ represents a line in S_r, and the true centers of simplexes (resp. crosspolytopes) are written as $(l_1 + \cdots + l_k)/k$ (resp. $(l_1' + l_2')/2$) in $\tilde{H}_1$, which may not be elements in $\operatorname{Pic} S_r$. Therefore, alternatively, we choose $(l_1 + \cdots + l_k)$ as the center of a face so that $(l_1 + \cdots + l_k)$ is in $\operatorname{Pic} S_r$.

4.2.1 Simplexes in $(r-4)_{21}$

Each a-simplex in $(r-4)_{21}$ shares its edges with $(r-4)_{21}$ and consists of $a+1$ vertices in $(r-4)_{21}$ joined with edges to each other. According to our understanding, these vertices correspond to lines in L_r, where they are skew to each other, namely disjoint, by Lemma 4.4. Therefrom, one can show that any set of $(a+1)$ mutually disjoint lines in L_r corresponds to an a-simplex in $(r-4)_{21}$. And we consider the set of centers of a-simplexes $0 \le a \le r-1$ in $(r-4)_{21}$ defined by

$$A_r^a := \{D \in \operatorname{Pic}(S_r) \mid D = l_1 + \cdots + l_{a+1}, l_i \text{ disjoint lines in } L_r\}.$$

Thanks to convexity, we expect each center in A_r^a to represent an a-simplex in $(r-4)_{21}$, and we prove this algebraically as follows. Therefrom, we have a bijection between A_r^a and the set of a-simplexes in $(r-4)_{21}$.

Lemma 4.6 *All the centers of a-simplexes in* $(r-4)_{21}$ *are distinct.*

Proof Since $a = 0$ is trivial, we assume that $1 \le a \le r-1$. Let l_i, $1 \le i \le a+1$, be skew lines of an a-simplex P in $(r-4)_{21}$, and assume there is another set of disjoint lines l_i', $1 \le i \le a+1$, of an a-simplex P' whose center is same with the center of P. We observe that

$$(l_1 + \cdots + l_{a+1}) \cdot l_{a+1}' = (l_1' + \cdots + l_{a+1}') \cdot l_{a+1}' = -1.$$

Therefore, l_{a+1}' is one of the lines l_i, $1 \le i \le a+1$. And by induction, the choice of lines l_a and the choice of lines l_a' are the same. Therefore, the a-simplexes P and P' are the same. ■

Remark The referee pointed out that this lemma holds for any convex polytope, namely, the center of a face F of a polytope P determines the face uniquely.

4.2.2 Crosspolytopes in $(r-4)_{21}$

An $(r-1)$-crosspolytope consists of $(r-1)$ pairs of vertices, where all the pairs of vertices share a common center that is also the center of the polytope. Furthermore, a vertex in the polytope is joined to the other vertices by edges in the polytope except for a vertex making the center with the given vertex. According to Theorem 4.2, an $(r-1)$-crosspolytope in $\operatorname{Pic} S_r$ is determined by $(r-1)$ pairs of lines. Since these pairs share a common center, we consider two pairs of lines from the polytopes such as (l_1, l_2) and (l_1', l_2') with $l_1+l_2 = l_1'+l_2'$. Because l_1 is joined to l_1' and l_2', by Lemma 4.4 we have $l_1 \cdot l_2 = l_1 \cdot (l_1'+l_2'-l_2) = 1$. Therefore, each pair of lines in the crosspolytope have intersection 1. In the following theorem, it turns out that a couple of lines with intersection 1 characterize an $(r-1)$-crosspolytope in $(r-4)_{21}$. It is useful to note that one can get $l_i \cdot l_1' = 0 = l_i \cdot l_2'$, $i = 1, 2$, for the above pairs, and it implies that any two pairs of lines with intersection 1 sharing a common center correspond to two diagonal pairs of vertices in a square. Therefore, a center cannot be shared by more than one $(r-1)$-crosspolytope.

Theorem 4.7 *For a del Pezzo surface S_r,*

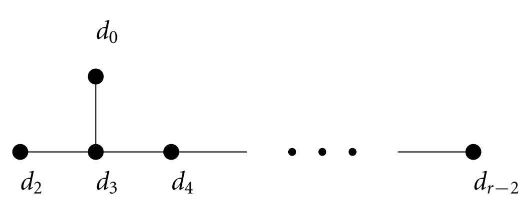
$$B_r := \{D \in \operatorname{Pic}(S_r) \mid D = l_1 + l_2 \text{ where } l_1, l_2 \text{ lines with } l_1 \cdot l_2 = 1\},$$

is the set of centers of $(r-1)$-crosspolytopes in $(r-4)_{21}$.

Proof From the above, we know each center of an $(r-1)$-crosspolytope in $(r-4)_{21}$ gives an element of B_r. Therefore, the cardinality of B_r is at least the total number of $(r-1)$-crosspolytopes in $(r-4)_{21}$, which is $[E_r : D_{r-1}]$ (see § 2).

Now we want to show that $|B_r| = [E_r : D_{r-1}]$ by calculating the number of pairs of lines with intersection 1.

Since $W(S_r)$ acts transitively on L_r, we can focus on a line e_r in $\operatorname{Pic} S_r$. The isotropy subgroup of e_r in $W(S_r)$ is E_{r-1}-type, as we saw in §§4.1. We choose another line $h - e_1 - e_r$ with $(h - e_1 - e_r) \cdot e_r = 1$. Since $W(S_r)$ preserves the intersection, the isotropy subgroup E_{r-1} of e_r has an orbit of $h - e_1 - e_r$ such that each line in the orbit has intersection 1 with e_r. The isotropy subgroup of $h - e_1 - e_r$ in E_{r-1} is generated by the subgraph of Dynkin diagram which is perpendicular to $h - e_1 - e_r$, *i.e.*,



.

Therefore it is of D_{r-2}-type, and moreover, the total number of elements in the orbit is given by $[E_{r-1} : D_{r-2}]$. In fact, all the lines in L_r intersecting e_r by 1 are in this orbit

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by Lemma 4.5. Thus, we obtain that the number of lines intersecting by 1 with a fixed line is $[E_{r-1}:D_{r-2}]$.

Now the number of pairs in L_r with intersection 1 is given by

$$\frac{1}{2}|L_r| \cdot \text{number of lines intersecting by 1 with a fixed line}$$
$$= \frac{1}{2}[E_r : E_{r-1}] \cdot [E_{r-1}:D_{r-2}] = \frac{1}{2}[E_r:D_{r-2}].$$

On the other hand, the total number of pairs of lines with intersection 1 given by the $(r-1)$-crosspolytopes in $(r-4)_{21}$ is obtained from

$$\frac{1}{2}|\{(r-1)\text{-crosspolytope in } (r-4)_{21}\}|$$
$$\times |\{ \text{pairs of lines with intersection 1 in an } (r-1)\text{-crosspolytope}\}|$$
$$= \frac{1}{2}[E_r:D_{r-1}] \cdot [D_{r-1}:D_{r-2}] = \frac{1}{2}[E_r:D_{r-2}].$$

Therefore, all the pairs of lines with intersection 1 in $\operatorname{Pic} S_r$ are obtained from $(r-1)$-crosspolytopes in $(r-4)_{21}$. ■

5 The Geometry of Gosset Polytopes in $\operatorname{Pic} S_r \otimes \mathbb{Q}$

In this section, we explain the correspondences between faces of Gosset polytopes $(r-4)_{21}$ and certain divisor classes in del Pezzo surfaces.

Here we consider the following divisor classes on a del Pezzo surface S_r satisfying one of the following conditions.

(i) $L_r^a := \{D \in \operatorname{Pic} S_r \mid D = l_1 + \cdots + l_a, l_i \text{ disjoint lines in } L_r\}$,
(ii) $\mathcal{E}_r := \{D \in \operatorname{Pic} S_r \mid D^2 = 1, K_{S_r} \cdot D = -3\}$,
(iii) $F_r := \{D \in \operatorname{Pic} S_r \mid D^2 = 0, K_{S_r} \cdot D = -2\}$.

We call the divisor classes in L_r^a *skew a-lines* for each a. Note that a skew a-line D satisfies $D^2 = -a$ and $K_{S_r} \cdot D = -a$. In fact, as we will see, skew a-lines are equivalent to the divisors with $D^2 = -a$ and $K_{S_r} \cdot D = -a$, when $1 \le a \le 3$.

The linear system of a divisor in the class D with $D^2 = 1$, $K_{S_r} \cdot D = -3$ induces a regular map to $\mathbb{P}^2$ ([10]). We call a divisor class in $\operatorname{Pic} S_r$ with these conditions an *exceptional system*. As we will see, each choice of disjoint lines in S_r which produces blowdowns from S_r to $\mathbb{P}^2$ gives one of these divisor classes, and the converse is also true for $r < 8$. Note that for S_6, each effective divisor $\tilde{D}$ with $\tilde{D}^2 = 1$ and $K_{S_r} \cdot \tilde{D} = -3$ corresponds to the twisted cubic surface (see [2]).

The divisor in class D in F_r with $D^2 = 0$, $K_{S_r} D = -2$ corresponds to the fiber class of a *ruling* on S_r. Here a ruling is a fibration of S_r over $\mathbb{P}^1$ whose generic fiber is a smooth rational curve. The rulings on the del Pezzo surfaces S_r play an important role in research on line bundles on del Pezzo surfaces according to representation theory (see [15, 16]). We also call the divisor classes in F_r *rulings*.

5.0.1 Dual lattices and Theta Functions

The cardinalities of L_r, L_r^a, $\mathcal{E}_r$, and F_r are finite because each set is a compact and integral subset of an affine hyperplane section in Pic S_r. Furthermore, one can obtain them by solving the corresponding Diophantine equations. But when $r \geq 6$, the cardinalities are very big, so that we need another way to guarantee the validity of our results of the simple calculation. The following argument deals with somewhat different objects than we are aiming at. Thus we sketch the key ideas only and discuss the details in another article. In the argument below, we follow the notation in [3].

First of all, we observe that the hyperplane section H_0 in Pic S_r is the root lattice Γ_r of the root system of E_r, $r \geq 6$. Here the set of minimal vectors in Γ_r is the set of roots. We consider the translation of $H_b = \tilde{H}_b \cap \operatorname{Pic} S_r$ along the vector K_{S_r} in Pic $S_r \otimes \mathbb{Q}$. If $\frac{b}{9-r}K_{S_r}$ is an integral vector, then $H_b + \frac{b}{9-r}K_{S_r}$ in $\tilde{H}_0$ is Γ_r. When $r = 7$ and $b \in \mathbb{Z}{-}2\mathbb{Z}$ (resp. $r = 6$ and $b \in \mathbb{Z}{-}3\mathbb{Z}$), there is a glue vector v in $\tilde{H}_0$ with norm $3/2$ (resp. $4/3$) such that $v + \Gamma_r$ is $H_b + \frac{b}{9-r}K_{S_r}$. Therefore, for each $b \in \mathbb{Z}$, H_b is transformed into the union of $v + \Gamma_r$ in $\tilde{H}_0$, where v is a glue vector which can be the origin. In fact, this union is the dual lattice of Γ_r. Now in order to figure out the total number of integral solutions $D \in \operatorname{Pic} S_r$ satisfying $D^2 = a$ and $D \cdot K_{S_r} = -b$, we send D to $\tilde{H}_0$ by $D + \frac{b}{9-r}K_{S_r}$ and compute

$$-\left(D + \frac{b}{9-r}K_{S_r}\right)^2 = -a + \frac{b^2}{9-r}.$$

Then we look up the coefficient of degree $-a + \frac{b^2}{9-r}$ in the theta series of dual lattices of Γ_r. This gives us the number of solutions for the equations.

For example, to identify the number of elements in L_r via the theta series, we observe

$$K_{S_r} l = -1,\ l^2 = -1$$
$$\iff \quad \left(l + \frac{K_{S_r}}{9-r}\right) \cdot K_{S_r} = 0, \left(l + \frac{K_{S_r}}{9-r}\right)^2 = -1 - \frac{1}{9-r},$$

and the number of lines in Pic S_r is the same with the the coefficient of degree $(1+\frac{1}{9-r})$ in the theta series of the dual lattices of the root lattice of E_r. If we consider L_7, the divisor classes in L_7 are transformed into the dual lattice of Γ_7 by $D + 1/2K_{S_7}$ with norm $3/2$. And the coefficient of the degree $3/2$ in the theta series of the dual lattice of E_7 is 56 ([3]).

Note that the root lattice Γ_8 of E_8 is self dual, and the theta series of self dual lattice Γ_8 is given as

$$\Theta_{\Gamma_8} = \sum_{m=0}^{\infty} N_m q^m,\ N_m = 240\sigma_3\left(\frac{m}{2}\right),$$

where $\sigma_r(m) = \sum_{d|m} d^r$. As we will see, lines and rulings in S_8 correspond to lattice points with norms 2 and 4, respectively. And the coefficients of q^2 and q^4 are 240 and $240(1 + 2^3) = 2160$.

5.1 Special Divisor Classes of del Pezzo Surface S_r and Faces of Gosset Polytope $(r-4)_{21}$

In this subsection, we show L_r^a, $\mathcal{E}_r$, and F_r are bijective to faces in $(r-4)_{21}$. By Lemma 3.1, the conditions in L_r^a, $\mathcal{E}_r$, and F_r are preserved by the action of $W(S_r)$, the Weyl group of the root spaces in $\operatorname{Pic} S_r$. Therefore, the correspondences between these divisor classes on del Pezzo surfaces and the faces in $(r-4)_{21}$ will be more than numerical coincidences. Furthermore, since the geometry of faces in E_r-semiregular polytopes is basically configuration of vertices, it is natural to study the divisor classes in $\operatorname{Pic} S_r$ with respect to the configurations of lines.

There are two types of faces in $(r-4)_{21}$, which are a-simplexes$(0 \leq a \leq r-1)$ and $(r-1)$-crosspolytopes. In particular, the facets of $(r-4)_{21}$ consist of $(r-1)$-simplexes and $(r-1)$-crosspolytopes. In this subsection, these top degree faces correspond to the divisor classes representing rational maps from S_r to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively.

5.1.1 Skew a-Lines and $(a-1)$-Simplexes

In Subsection 4.2, we showed that A_r^a, the set of centers of a-simplexes in $(r-4)_{21}$, is bijective to the set of a-simplexes in $(r-4)_{21}$. In fact the set A_r^a is the same as the set of divisor classes so-called *skew* $(a+1)$*-lines*,

$$L_r^{a+1} := \{D \in \operatorname{Pic}(S_r) \mid D = l_1 + \cdots + l_{a+1}, l_i \text{ disjoint lines in } L_r\}.$$

Therefore, we have the following theorem.

Theorem 5.1 *The set of skew a-lines in* $\operatorname{Pic} S_r$, $1 \leq a \leq r$ *is bijective to the set of* $(a-1)$*-simplexes,* $\alpha_{(a-1)}$*, in the Gosset polytope* $(r-4)_{21}$.

Remark • In particular, a skew 1-line is just a *line* in L_r that corresponds to the vertices of E_r-semiregular polytopes, and skew 2-lines represent edges in $(r-4)_{21}$.

- Since the vertices of an $(a-1)$-simplex correspond to the lines consisting of a skew a-line in S_r, each $(a-1)$-simplex represents a rational map from S_r to S_{r-a} obtained by blowing down the disjoint a-lines in S_r. In particular, each $(r-1)$-simplex in $(r-4)_{21}$ gives a rational map from S_r to $\mathbb{P}^2$.
- As the Weyl group $W(S_r)$ act transitively on the set of $(a-1)$-simplexes in the Gosset polytope $(r-4)_{21}$, $W(S_r)$ acts transitively on the set of skew a-lines.

It is easy to see that skew a-lines in $\operatorname{Pic} S_r$ satisfy $D^2 = -a$ and $K_{S_r}D = -a$, which are called *a-divisors*. The number of a-divisors in S_r, $2 \leq a \leq r$, can be obtained from the theta series of dual lattices of the root lattice of E_r. First we observe

$$K_{S_r}D = -a,\ D^2 = -a$$
$$\iff \Big(D + a\frac{K_{S_r}}{9-r}\Big) \cdot K_{S_r} = 0, \Big(D + a\frac{K_{S_r}}{9-r}\Big)^2 = -a\Big(1 + \frac{a}{9-r}\Big),$$

and according to [3], in Table 3 we have the table of the number of 2-divisors and 3-divisors.

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a	S_3	S_4	S_5	S_6	S_7	S_8
2	6	30	80	216	756	6720
3	2	30	160	720	4032	60480

Table 3: Number of 2-and 3-divisors in S_r

In fact, the table gives the number of 1-simplexes and 2-simplexes in Gosset polytopes $(r-4)_{21}$ (see §2). Therefore, by Lemma 4.6 and Theorem 5.1, the divisor classes D in $\operatorname{Pic} S_r$ with $D^2 = -a$ and $K_{S_r}D = -a$, $1 \le a \le 3$ are skew a-lines. This gives the following theorem.

Theorem 5.2 *For $1 \le a \le 3$, each divisor class D in $\operatorname{Pic} S_r$ with $D^2 = -a$ and $K_{S_r}D = -a$ can be written as the sum of skew lines $l_1, \dots, l_a$, where the choice is unique up to the permutation.*

5.1.2 Exceptional Systems and $(r-1)$-Simplexes in $(r-4)_{21}$

Recall that $(r-1)$-simplexes in $(r-4)_{21}$ are one of two types of facets in $(r-4)_{21}$, and each $(r-1)$-simplex in $(r-4)_{21}$ corresponds to disjoint r-lines in $\operatorname{Pic} S_r$ giving a rational map from S_r to $\mathbb{P}^2$. But the above correspondence between $(r-1)$-simplex in $(r-4)_{21}$ and skew r-lines in $\operatorname{Pic} S_r$ is somewhat coarse and we want to have another approach. Here we consider classes which we called exceptional systems.

An *exceptional system* is a divisor class D on a del Pezzo surface S_r with $D^2 = 1$ and $D \cdot K_{S_r} = -3$, and the linear system of D gives a regular map from S_r to $\mathbb{P}^2$ ([10]). We denote the set of exceptional systems in $\operatorname{Pic} S_r$ as

$$\mathcal{E}_r := \{D \in \operatorname{Pic}(S_r) \mid D^2 = 1, K_{S_r} \cdot D = -3\}.$$

When $r = 6$, each linear system with the above conditions contains a twisted cubic curve and it is also known that there are 72 of such classes. In fact, according to the correspondence between skew a-lines in S_6 and $(a-1)$-simplexes in 2_{21}, we can see the number of exceptional systems in S_6 equals the number of 5-simplexes. As we saw in the remarks following Theorem 5.1, each $(r-1)$-simplex in $(r-4)_{21}$ corresponds to the rational map from S_r to $\mathbb{P}^2$. Therefore, it is natural to compare the set of $(r-1)$-simplex in $(r-4)_{21}$ and the set of exceptional systems. Here we explain that each $(r-1)$-simplex in $(r-4)_{21}$ is related to an exceptional system, and moreover, two sets of these are bijective for $3 \le r \le 7$.

First, we observe

$$\begin{aligned} &D \cdot K_{S_r} = -3, D^2 = 1 \\ &\qquad\qquad \Longleftrightarrow \Big(D + \frac{3K_{S_r}}{9-r}\Big) \cdot K_{S_r} = 0, \Big(D + \frac{3K_{S_r}}{9-r}\Big)^2 = 1 - \frac{9}{9-r} \end{aligned}$$

and from the theta series of dual lattice of root lattice of E_r, the number of exceptional systems can be listed as below. We observe that the numbers of exceptional systems

del Pezzo Surfaces S_r	S_3	S_4	S_5	S_6	S_7	S_8
number of exceptional systems	2	5	16	72	576	17520
$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
number of $(r-1)$-simplexes	2	5	16	72	576	17280

Table 4:

in del Pezzo surfaces and the numbers of top degree subsimplexes in $(r-4)_{21}$ are the same except $r=8$.

To explain the correspondence, we define a transformation Φ from $\mathcal{E}_r$ to L_r^r by $\Phi(D_t) =: K_{S_r} + 3D_t$ for $D_t \in \mathcal{E}_r$. It is well defined because

$$\Phi(D_t)\cdot K_{S_r} = (K_{S_r} + 3D_t)\cdot K_{S_r} = -r, \quad \Phi(D_t)^2 = -r.$$

In the following theorem, the transformation Φ leads to a correspondence between $\mathcal{E}_r$ and L_r^r.

Theorem 5.3 *When $3 \le r \le 8$, each $(r-1)$-simplex in $(r-4)_{21}$ corresponds to an exceptional system in the del Pezzo surfaces S_r. Moreover, for $3 \le r \le 7$ the Weyl group $W(S_r)$ acts transitively on $\mathcal{E}_r$, the set of exceptional systems in the del Pezzo surface S_r, and $\mathcal{E}_r$ is bijective to L_r^r, the set of skew r-lines in* $\operatorname{Pic} S_r$.

Proof First we observe that the above transformation Φ is injective and equivariant for the action of the Weyl group $W(S_r)$, by Lemma3.1. We consider a class h in $\operatorname{Pic}(S_r)$ which is an exceptional system. Here Φ sends h to the sum of disjoint exceptional classes $e_1 + \cdots + e_r$, which is a skew r-line representing one of $(r-1)$-simplexes by Theorem 5.1. And the orbit containing h in $\mathcal{E}_r$ corresponds to the orbit containing $e_1 + \cdots + e_r$ in L_r^r. Since the action of $W(S_r)$ on L_r^r is transitive, each skew r-line representing an $(r-1)$-simplex corresponds to an exceptional system. According to Table 4, $\mathcal{E}_r$ also acts transitively on $W(S_r)$ for $3 \le r \le 7$. ■

Remark When $r=8$, the set of exceptional systems has two orbits. One orbit, with 17280 elements, corresponds to the set of skew 8-lines in S_8, and the other orbit, with 240 elements, corresponds to the set of E_8-roots, since for each E_8-root d, $-3K_{S_8} + d$ is an exceptional system.

5.1.3 Rulings and Crosspolytopes

Now the crosspolytopes that are the other type of facets in $(r-4)_{21}$ are the only remaining faces in $(r-4)_{21}$. In Subsection 4.2, the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$ is bijective to the set of their centers, defined by B_r. Observe that each element in B_r, $l_1 + l_2$ with $l_1 \cdot l_2 = 1$ satisfies $(l_1+l_2)^2 = 0$, $K_{S_r}\cdot(l_1+l_2) = -2$, and we consider the divisor classes with these conditions. Furthermore, the divisor classes correspond to the rational maps from S_r to $\mathbb{P}^1 \times \mathbb{P}^1$.

The divisor class f on del Pezzo surface $\operatorname{Pic} S_r$ with $f^2 = 0$, $K_{S_r}\cdot f = -2$ is called a *ruling* since the divisor in it corresponds to a fibration of S_r over $\mathbb{P}^1$ whose generic fiber is a smooth rational curve. The set of rulings in Pic S_r is denoted as

$$F_r := \{f \in \mathrm{Pic}(S_r) \mid f^2 = 0, K_{S_r} \cdot f = -2\}.$$

The number of rulings in Pic S_r can be obtained from the theta series of the dual lattice of root lattice of E_r as follows:

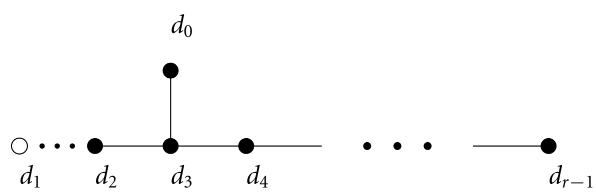
$$K_{S_r} f = -2,\ f^2 = 0$$
$$\iff \left(f + \frac{2K_{S_r}}{9-r}\right) \cdot K_{S_r} = 0,\ \left(f + \frac{2K_{S_r}}{9-r}\right)^2 = -\frac{4}{9-r}.$$

Furthermore, we get the following parallel list of the numbers of rulings in del Pezzo surface S_r and the numbers of crosspolytopes in $(r-4)_{21}$.

del Pezzo Surfaces S_r	S_3	S_4	S_5	S_6	S_7	S_8
number of rulings	3	5	10	27	126	2160
$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
number of $(r-1)$-crosspolytopes	3	5	10	27	126	2160

Table 5:

To get a rough idea about these parallel lists of numbers, we consider the following. By Lemma 3.1, the set of rulings in S_r is acted on by the Weyl group $W(S_r)$. We consider a ruling $h - e_1$. The isotropy group of $h - e_1$ is generated by the simple roots perpendicular to $h - e_1$ with the relationships represented as



Thus it is D_{r-1}-type, and the number of elements in the orbit containing $h - e_1$ is $[E_r : D_{r-1}]$. This procedure is exactly the same as counting the number of $(r-1)$-crosspolytopes in an $(r-4)_{21}$. As a matter of fact, the action of the Weyl group $W(S_r)$ on rulings is transitive from the following theorem. A part of following theorem appears in [1, 15].

Theorem 5.4 *For each ruling f in a del Pezzo surface S_r, there is a pair of lines l_1 and l_2 with $l_1 \cdot l_2 = 1$ such that f is equivalent to the sum of $l_1 + l_2$. Furthermore, the set of rulings in S_r is bijective to the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$ and is acted transitively upon by the Weyl group $W(S_r)$.*

Proof From Subsection 4.2, B_r (the set of centers of $(r-1)$-crosspolytopes in $(r-4)_{21}$) is already a subset of rulings F_r. Since B_r is bijective to the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$, $|B_r| = |F_r|$. Therefore $B_r = F_r$ and each ruling can be written as the two lines with intersection 1. This also shows that F_r is bijective to the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$. Since this correspondence is naturally equivariant for the Weyl group $W(S_r)$ action and the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$ acted transitively on by $W(S_r)$, F_r is also transitively acted on by $W(S_r)$. ∎

According to Theorem 4.2, the vertices of an $(r-1)$-crosspolytope must correspond to lines in the linear system of a ruling. Here we consider the pairs of antipodal vertices in the $(r-1)$-crosspolytope and their correspondences in a ruling. The antipodal vertices in the pair are the two bipolar points in the crosspolytope, which is a bipyramid. The number of the pairs of antipodal vertices in an $(r-1)$-crosspolytope is $(r-1)$. Hence, we get the following corollary.

Corollary 5.5 *For each ruling f in a del Pezzo surface S_r there are $(r-1)$-pairs of lines with 1-intersection whose sum is f. Furthermore, each of these pairs corresponds to antipodal vertices of the $(r-1)$-crosspolytope corresponding to f in $(r-4)_{21}$.*

From the above corollary, we obtain the following useful lemma.

Lemma 5.6 *For a ruling f and a line l in* $\operatorname{Pic} S_r$, *the following are equivalent.*

(i) $f \cdot l = 0$.
(ii) *$f - l$ is a line.*
(iii) *The vertex represented by l in $(r-4)_{21}$ is one of the vertices of the $(r-1)$-crosspolytope corresponding to f.*

Remark As the $(r-1)$-simplex faces in $(r-4)_{21}$ are related to the blowing down from S_r to $\mathbb{P}^2$, the crosspolytope which is the other type of facets in $(r-4)_{21}$ gives the blowing down from S_r to $\mathbb{P}^1 \times \mathbb{P}^1$. Here we give an example of this blowing down. This will be discussed further in another article.

For cubic surfaces S_6, it is well known that there are two disjoint lines l_a and l_b, and five parallel lines l_i, $1 \le i \le 5$ that meet both l_a and l_b. Furthermore, for each pair l_a (resp. l_b) and l_i, there is a line l_{ai} (resp. l_{bi}) intersecting l_a (resp. l_b) and l_i. Blowing down of five disjoint lines l_i gives a rational map from S_6 to $\mathbb{P}^1 \times \mathbb{P}^1$. From the facts in this article, for each line l_i, $l_{ai} + l_i$ produces the same ruling, namely, l_i and l_{ai} for $1 \le i \le 5$ correspond to ten vertices of a 5-crosspolytope. And the same fact is true for lines l_i and l_{bi}, $1 \le i \le 5$. Therefore proper choice of 5-crosspolytopes in a Gosset polytope 2_{21} gives a blowing down from S_6 to $\mathbb{P}^1 \times \mathbb{P}^1$.

For example, we consider two disjoint lines e_6 and $2h - \sum_{k=1}^{6} e_k + e_6$ in S_6. For the line e_6, there are five pairs of lines e_i and $h - e_i - e_6$, $1 \le i \le 5$, where e_6, e_i, and $h - e_i - e_6$ have 1-intersections to each other. Similarly, the line $2h - \sum_{k=1}^{6} e_k + e_i$ has five pairs of lines $2h - \sum_{k=1}^{6} e_k + e_i$ and $h - e_i - e_6$, $1 \le i \le 5$. And we find five disjoint lines $h - e_i - e_6$, $1 \le i \le 5$, that give a blowing down to $\mathbb{P}^1 \times \mathbb{P}^1$. Here we observe that the five pair of lines for e_6 are from a ruling $(h - e_i - e_6) + e_i = h - e_6$ and correspond to the antipodal vertices in the crosspolytope in 2_{21} corresponding to $h - e_6$. Similarly the five pairs of lines for $h - e_i - e_6$ are from a ruling

$$(h - e_i - e_6) + \left(2h - \sum_{k=1}^{6} e_k + e_i\right) = 3h - \sum_{k=1}^{6} e_k - e_6.$$

This observation induces interesting relationships between the geometry of cubic surfaces and the combinatorial data on a Gosset polytope 2_{21}. Furthermore, as each del Pezzo surface blows down to $\mathbb{P}^1 \times \mathbb{P}^1$, we can search the similar works on the del Pezzo surfaces of the other degree and the corresponding Gosset polytopes.

6 Applications

6.1 Monoidal Transforms for Lines

In this subsection, we consider lines along the blowup procedure producing del Pezzo surfaces to study the geometry of Gosset polytopes $(r-4)_{21}$ according to the above correspondences.

This blowup procedure can be applied to rulings and other special divisor classes in this article so as to obtain recursive relationships corresponding to faces in Gosset polytopes $(r-4)_{21}$. Furthermore, we can also use this procedure by virtue of interesting combinatorial relationships on faces in Gosset polytopes $(r-4)_{21}$, which are also related to the Cox ring of del Pezzo surfaces [1]. This is explained in [14].

For a fixed vertex P in a Gosset polytope $(r-4)_{21}$, the set of vertices with the shortest distance from P is characterized as the vertex figure of P by the action of an isotropy group of E_{r-1}-type. But the description of the set of vertices with greater distance from P requires a rather indirect procedure. Here, we consider divisor classes producing lines by blowup and apply these to study the set of vertices with greater distance from P according to the correspondences between vertices and lines.

For a fixed line l in a del Pezzo surface S_r, $3 \le r \le 8$, we consider a set

$$N_k(l, S_r) = \{l' \in L_r \mid l' \cdot l = k\}$$

that also presents the set of lines in S_r with the same distance from l. In Section 4, we described the local geometry of the polytope via case study on these sets of lines. Here, by using blowup procedure on lines, we complete the description in a uniform manner.

Since a del Pezzo surface S_r is obtained by blowing up one point on S_{r-1} to a line l in S_r, we can describe divisor classes in S_{r-1} producing lines in S_r after blowing up. In fact, the choice of the above line l can be replaced by the exceptional class e_r in S_r which is the exceptional divisor in S_r given by a blowup of a point from S_{r-1}. The proper transform of a divisor D in $\mathrm{Pic}(S_{r-1})$ producing a line in S_r satisfies

$$(D - me_r)^2 = -1, \quad (D - me_r) \cdot (K_{S_{r-1}} + e_r) = -1$$

for a nonnegative integer m. Therefore, we consider a divisor D in $\mathrm{Pic}(S_{r-1})$ with

$$D^2 = m^2 - 1, \quad D \cdot K_{S_{r-1}} = -m - 1.$$

By the Hodge index theorem we have

$$(m^2-1)K_{S_{r-1}}^2 = D^2K_{S_{r-1}}^2 \le (D\cdot K_{S_{r-1}})^2 = (-m-1)^2,$$

which implies $-1 \le m \le 1+2/(9-r)$. Thus the list of possible m is

$$m = \begin{cases} 0,1 & 4\le r\le 6,\\ 0,1,2 & r=7,\\ 0,1,2,3 & r=8.\end{cases}$$

Definition 6.1 A line l in the Picard group of del Pezzo surface S_r obtained by blowup from a divisor class D in S_{r-1} is called an *m-degree line* if $l = D - me_r$, where e_r is the exceptional class produced by the blowup.

(i) 0-**degree line in** S_r, $4 \le r \le 8$: Each 0-degree line in S_r corresponds to a line in S_{r-1} and the number of 0-degree lines equals the number of lines in S_{r-1}.

(ii) 1-**degree line in** S_r, $4 \le r \le 8$: A divisor D in S_{r-1} with $D^2 = 0$ and $D\cdot K_{S_{r-1}} = -2$ corresponds to a 1-degree line in S_r. Therefore, F_{r-1} the set of rulings in S_{r-1} is bijective to the set of 1-degree lines in S_r.

(iii) 2-**degree line in** S_r, $r = 7, 8$: When $r = 7$ (resp. $r = 8$), there are 2-degree lines given by divisors in $\operatorname{Pic} S_6$ (resp. $\operatorname{Pic} S_7$) with $D^2 = 3$ and $DK_{S_6} = -3$. For $r = 7$, by the Hodge index theorem, $-K_{S_6}$ is the only divisor in S_6 with these equations. For $r = 8$, we can transform the equations to $(D + K_{S_7})^2 = -1$, and $(D + K_{S_7})\cdot K_{S_7} = -1$, which represent lines in S_7. Therefore, the number of 2-degree lines in S_8 is the same as the number of lines in S_7.

(iv) 3-**degree line in** S_8: For $r = 8$, there are 3-degree lines obtained from divisors in $\operatorname{Pic}(S_7)$ with $D^2 = 8$ and $DK_{S_7} = -4$. These equations are equivalent to $(D + 2K_{S_7})^2 = 0$ and $(D + 2K_{S_7})\cdot K_{S_7} = 0$. Therefore, $-2K_{S_7}$ is the only divisor in S_7 producing a 3-degree line in S_8.

Now we obtain the following theorem.

Theorem 6.2 *Let l be a fixed line in a del Pezzo surface S_r, $4 \le r \le 8$, and V_l the vertex corresponding to the line l in the polytope $(r-4)_{21}$.*

(i) *$N_0(l, S_r)$, $4 \le r \le 8$, is bijective to the set of lines L_{r-1} in S_{r-1}, and equivalently, it is also bijective to the set of vertices in the polytope $(r-5)_{21}$.*

(ii) *$N_1(l, S_r)$, $4 \le r \le 8$, is bijective to the set of rulings containing l in S_r which is also bijective to the set of rulings F_{r-1} in S_{r-1}. Equivalently, it is also bijective to the set of $(r-1)$-crosspolytopes in the polytope $(r-4)_{21}$ containing V_l and the set of $(r-2)$-crosspolytopes in the polytope $(r-5)_{21}$.*

(iii) *$N_2(l, S_8)$ is bijective to $N_0(-2K_{S_8} - l, S_8)$, the set of skew lines in S_7 for a line $-2K_{S_8} - l$, and equivalently it is also bijective to the set of lines in S_7.*

(iv) *$N_2(l, S_7) = \{-K_{S_7} - l\}$ and $N_3(l, S_8) = \{-2K_{S_8} - l\}$.*

Proof Since $N_k(l, S_r)$ is equivalent to the k-degree lines in S_r, the above description of blowups for lines and Theorem 5.4 give the theorem. ■

Remarks For $r = 3$, $|N_0(l, S_3)| = 3$ and $|N_1(l, S_3)| = 2$.

For S_7, each element l' in $N_1(l, S_7)$ satisfies $(-K_{S_7} - l) \cdot l' = 0$. Therefore, $N_1(l, S_7)$ is bijective to $N_0(-K_{S_7} - l, S_7)$, and the theorem explains $|F_6| = |N_1(l, S_7)| = |N_0(-K_{S_7} - l, S_7)| = |L_6|$, namely, the number of lines and rulings in S_6 are the same.

For S_8, each line l' in S_8 satisfies $(-2K_{S_7} - l) \cdot l' = 2 - l \cdot l'$. Thus we have $|N_0(l, S_8)| = |N_2(-2K_{S_8} - l, S_8)|$ and $|N_1(l, S_8)| = |N_1(-2K_{S_8} - l, S_8)|$.

6.2 Gieser Transform and Bertini Transform for Lines

Recall that for two lines l_1 and l_2 in S_7, $l_1 \cdot l_2 = 2$ is equivalent to $l_1 + l_2 = -K_{S_7}$. In fact, each pair of lines in S_7 with a 2-intersection represents a bitangent of degree 2 covering from S_7 to $\mathbb{P}^2$ given by $|-K_{S_7}|$. Thus according to this relation, each line in $\operatorname{Pic} S_7$ determines another line in $\operatorname{Pic} S_7$. Similarly, any two lines l_1' and l_2' in S_8 with $l_1' \cdot l_2' = 3$ equivalently hold $l_1 + l_2 = -2K_{S_8}$ and these two lines in S_8 with 3-intersection are related to a tritangent plane of degree 2 covering from S_8 to $\mathbb{P}^2$ given by $|-2K_{S_8}|$. Thus again each line in $\operatorname{Pic} S_8$ determines another line in $\operatorname{Pic} S_8$ with respect to the covering. Here the deck transformation σ of the covering is a biregular automorphism of S_7 (resp. S_8). For a blowdown $\pi\colon S_7 \to \mathbb{P}^2$, $\pi\sigma$ is another blowdown, and the corresponding birational transformation $(\pi\sigma)\pi^{-1}\colon \mathbb{P}^2 \to \mathbb{P}^2$ is called a *Gieser transform* (resp. *Bertini transform*) (see [10, Chapter 8]). Therefore, the Gieser transform corresponds to a transformation G on lines in S_7 defined as

$$G(l) := -(K_{S_7} + l),$$

and we also call G the *Gieser transform on lines* or simply the Gieser transform. Similarly a transformation B on lines in S_8 defined as

$$B(l) := -(2K_{S_8} + l)$$

is referred as the *Bertini transform on lines* or simply the Bertini transform.

Since both the Gieser transform and the Bertini transform are defined on the set of lines, we can extend the definition to any divisor written as a linear sum of lines. Namely, for a divisor class D given as $a_1 l_1 + \cdots + a_m l_m$ in S_7

$$G(D) := a_1 G(l_1) + \cdots + a_m G(l_m),$$

and similarly in S_8,

$$B(D) := a_1 B(l_1) + \cdots + a_m B(l_m).$$

And for lines l_1, l_2 in $\operatorname{Pic} S_7$ and lines l_1', l_2' in $\operatorname{Pic} S_8$, we have

$$\begin{aligned} G(l_1) \cdot G(l_2) &= (K_{S_7} + l_1) \cdot (K_{S_7} + l_2) \\ &= K_{S_7}^2 + (l_1 + l_2) \cdot K_{S_7} + l_1 \cdot l_2 = l_1 \cdot l_2, \\ B(l_1') \cdot B(l_2') &= (2K_{S_8} + l_1') \cdot (2K_{S_8} + l_2') \\ &= 4K_{S_8}^2 + 2(l_1' + l_2') \cdot K_{S_8} + l_1' \cdot l_2' = l_1' \cdot l_2'. \end{aligned}$$

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Therefore, G and B preserve the intersections between lines; G and B are symmetries on 3_{21} and 4_{21}, respectively. Moreover, since all the regular faces we discussed in 3_{21} and 4_{21} are written as linear sums of lines, G and B act on the set of these faces. In summary, we have the following theorem.

Theorem 6.3 *The Gieser transform G on the set lines in* Pic S_7 *and the Bertini transform B on the set of lines in* Pic S_8 *can be extended to a symmetry of* 3_{21} *and* 4_{21}*, respectively.*

Naturally, further studies on Gieser transform G and Bertini transform B are performed along the degree 2 coverings from S_7 to $\mathbb{P}^2$ and from S_8 to $\mathbb{P}^2$. This is continued in [14] and another article.

Remark The above pairs of lines are special cases of Steiner blocks related to the inscribed simplexes in $(r-4)_{21}$. See [14].

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