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Cohomology-free diffeomorphisms of low-dimension tori

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Abstract. We study cohomology-free (c.f.) diffeomorphisms of the torus T^n. A diffeomorphism is c.f. if every smooth function f on T^n is cohomologous to a constant f_0, i.e. there exists a C^∞ function h so that $h - h \circ \varphi = f - f_0$. We show that the only c.f. diffeomorphisms of T^n, $1 \leq n \leq 3$, are the smooth conjugations of Diophantine translations. For $n = 4$, we prove the same result for c.f. *orientation-preserving* diffeomorphisms.

0. *Introduction*

Let M be a C^∞ manifold and φ a smooth diffeomorphism of M. All the objects we consider here are C^∞. We say that φ is *cohomology-free* (c.f.) if every smooth function on M is cohomologous to a constant, i.e. for every function f on M there exists a smooth function h on M and a constant f_0 so that $h - h \circ \varphi = f - f_0$.

Katok [**13**, Problem 17] proposes the following.

Problem. For a given manifold M what are the cohomology-free diffeomorphisms?

In this generality very little is known. We first observe that c.f. diffeomorphisms are uniquely ergodic and in fact the space of invariant distributions is one-dimensional. We show in Proposition 1.2 that a c.f. diffeomorphism φ of an orientable manifold M leaves invariant a volume form Ω in the sense that

$$\varphi^*\Omega = (\deg \varphi)\Omega,$$

where $\deg \varphi$ is the degree of φ. So φ is minimal.

The aim of this article is to give an answer to Katok's question when the manifold is a low-dimensional torus. We prove the following.

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THEOREM A. *The cohomology-free diffeomorphisms of the torus T^n, $1 \le n \le 3$, are C^∞ conjugate to Diophantine translations.*

THEOREM B. *The only cohomology-free orientation-preserving diffeomorphisms of the torus T^4 are the C^∞ conjugations of Diophantine translations.*

Theorem A is proved in §§2 and 3 and Theorem B is proved in §4. §§5 and 6 are auxiliary results. The cohomology of the foliation given by the fibres of a torus bundle is discussed in §5. In §6 we adapt to our needs the isotopy theorem for volume forms due to Moser [**19**].

In Example 1.9 we give an example of an analytic, minimal and uniquely ergodic diffeomorphism of T^2 that reverses orientation. This shows the crucial role of the cohomological triviality of the diffeomorphism in Theorem A.

In the proof of our results, the analogue for the torus T^2 of the so-called 'last geometric theorem of Poincaré' as conjectured by Arnold in [**2**] and [**3**] and proven by Conley and Zehnder in [**7**] plays an important role. It says that if an area-preserving diffeomorphism of T^2 is homotopic to the identity and has mean translation 0, then the diffeomorphism has at least three fixed points.

1. *On the cohomology-free diffeomorphisms*

1.1. We use c.f. as an abbreviation for cohomology-free. For a closed orientable manifold we have the following.

PROPOSITION 1.2. *Let M be a closed orientable C^∞ manifold and let φ be a cohomology-free diffeomorphism of M. Then there exists a smooth volume form Ω on M so that*

$$\varphi^*\Omega = (\deg\varphi)\Omega, \tag{1.1}$$

where $\deg\varphi$ denotes the degree of φ.

Proof. Fix a volume form Ω_0 on M with total volume 1. Notice that

$$D\varphi^n = \prod_{j=0}^{n-1} D\varphi \circ \varphi^j.$$

From this we get

$$\frac{1}{n}(\log|\det D\varphi^n|) = \frac{1}{n}\sum_{j=0}^{n-1}\log|\det D\varphi| \circ \varphi^j, \tag{1.2}$$

where $\det D\varphi$ is defined by the identity

$$\varphi^*\Omega_0 = (\det D\varphi)\Omega_0.$$

Let μ be the unique φ-invariant normalized probability measure on M.

Now, following an idea of Denjoy [**8**], we see from (1.2) and the unique ergodicity of φ that $\log|\det D\varphi^n|/n$ converges uniformly to $\mu(\log|\det D\varphi|)$ and since $\int_M \det D\varphi^n\Omega_0 = \deg\varphi^n = \pm 1$ we have $\mu(\log|\det D\varphi|) = 0$. As φ is c.f. the functional equation $h - h\circ\varphi = \log|\det D\varphi|$ has a unique C^∞ solution h such that $\int_M e^h\Omega_0 = 1$. It is easy to see that if $\Omega = e^h\Omega_0$, then $\varphi^*\Omega = (\deg\varphi)\Omega$, proving the Proposition. □

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It follows from the above proposition that φ preserves the Lebesgue measure of M and since φ is uniquely ergodic then φ is minimal (i.e. every orbit of φ is dense).

We say that a 1-form w_0 is *almost φ-invariant* if $\varphi^* w_0 = w_0 + df$.

Remark 1.3. Let $\varphi : M \to M$ be a c.f. C^∞ diffeomorphism and let w_0 be an almost φ-invariant 1-form. Then there exists a unique φ-invariant 1-form w cohomologous to w_0 (w_0 need not to be closed).

This holds since φ is c.f. so that the functional equation $h - h \circ \varphi = f$ has a C^∞ solution $h : M \to R$, where f is as above. Now $w = w_0 + dh$ is the φ-invariant 1-form.

We restrict ourselves to the study of the c.f. diffeomorphisms of the torus T^n.

Remark 1.4. Let φ be the diffeomorphism given on the covering R^n by $\varphi = A + F$, where A is an integral matrix with determinant of absolute value 1 and $F : R^n \to R^n$ is *$\mathcal{Z}^n$-periodic*, i.e. $F(x + \ell) = F(x)$ for all $x \in R^n$ and $\ell \in \mathcal{Z}^n$. The Lefschetz number of φ is given by

$$L(\varphi) = \det(I - A). \tag{1.3}$$

We will denote by $\sigma(A)$ the *spectrum* of A. Thus by the Lefschetz fixed point theorem, $1 \in \sigma(A)$ if φ has no fixed point. Choose a vector $v = (m_1, \ldots, m_n)$ in $\mathcal{Z}^n$ so that ${}^tAv = v$ and g.c.d. $(m_1, \ldots, m_n) = 1$. By [**20**, Theorem II.1] v extends to a basis $B = \{v, v_1, \ldots, v_{n-1}\}$ of $\mathcal{Z}^n$ so that

$$P^{-1}\,{}^tAP = \begin{pmatrix} 1 & * \cdots * \\ 0 & \\ \vdots & {}^tA_{n-1} \\ 0 & \end{pmatrix}, \tag{1.4}$$

where P is the matrix whose columns are the vectors of B and $\det P = 1$. Of course, $\det A_{n-1} = \deg \varphi$.

Now if $\sigma(A) = \{1\}$, then B can be chosen so that A_{n-1} is triangular. This can be proved by induction on n.

A translation T_α of T^n is *Diophantine* [**5**, **12**] if $\alpha \in R^n$ satisfies a Diophantine condition

$$\|k \cdot \alpha\| \geq \frac{C}{|k|^{n+\beta}}, \quad C, \beta > 0 \tag{1.5}$$

for all $k = (k_1, \ldots, k_n) \in \mathcal{Z}^n - \{0\}$, where $\|x\| = \inf\{|x - \ell|, \ell \in \mathcal{Z}^n\}$ defines a metric on T^n, $|x| = \sup_j |x_j|$ and $k \cdot \alpha = k_1\alpha_1 + \cdots + k_n\alpha_n$.

The following result is probably known [**14**].

THEOREM 1.5. *The only cohomology-free affine diffeomorphisms of the torus T^n are the Diophantine translations.*

Proof. Let T_α be a Diophantine translation and let f be a smooth function on T^n whose average over T^n is zero. Using Fourier series, one can see that the solution h to the equation $h - h \circ T_\alpha = f$ must satisfy $|\hat{h}(k)| \leq (|k|^{n+\beta}/c)|\hat{f}(k)|$ for all $k \in \mathcal{Z}^n$, which shows that h is smooth. Thus T_α is c.f. Now, if $A + \alpha$ induces a c.f. affine diffeomorphism of T^n, then it is minimal. Thus $\sigma(A) = \{1\}$ [**11**] and $A = I + N$, where N is nilpotent. We show that if $N \neq 0$ there is a countable family of invariant distributions which are

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not measures, and thus the affine diffeomorphism is not c.f. To each $k \in \mathcal{Z}^n$, ${}^tNk \neq 0$, we associate an invariant distribution by

$$\rho_k(f) = \sum_{m \in \mathcal{Z}} e^{2\pi i a^m(0)k} \hat{f}({}^tA^m k) \tag{1.6}$$

for every C^∞ function $f : T^n \to \mathbb{C}$ where $\hat{f}$ denotes, as usual, the Fourier transform of f.

To show that ρ_k is a-invariant observe that

$$\rho_k(f \circ a) = \sum_{m \in \mathcal{Z}} e^{2\pi i [a^m(0) + A^m a^{-1}(0)]k} \hat{f}({}^tA^{m-1} k)$$

and $a^m(0) + A^m a^{-1}(0) = a^{m-1}(0)$. Thus a is a Diophantine translation by [**12**] and [**5**], finishing the proof. □

A *closed ordered coframe* of T^n is an ordered n-tuple $\{w_1, \ldots, w_n\}$ of closed 1-forms which are linearly independent at every point of T^n.

LEMMA 1.6. *Let φ be a C^∞ diffeomorphism of T^n and let $w = \{w_1, \ldots, w_n\}$ be a closed coframe cohomologous to the canonical coframe $dx = \{dx_1, \ldots, dx_n\}$, i.e. $w_j = dx_j + dh_j$, $1 \leq j \leq n$. Then φ is conjugate to an affine diffeomorphism a of T^n, induced by $A + \alpha$, and the conjugating diffeomorphism ψ is homotopic to the identity if and only if $\varphi^* w = {}^tAw$ (in matrix notation).*

Proof. Clearly, if $\psi \circ \varphi = a \circ \psi$ then $\varphi^* w = {}^tAw$, where $\psi^* dx_j = w_j$, $1 \leq j \leq n$. Now since w and dx are closed cohomologous coframes then by [**6**] there exists a C^∞ diffeomorphism ψ homotopic to the identity such that $\psi^* dx = w$. Thus $a = \psi \circ \varphi \circ \psi^{-1}$ is an affine diffeomorphism, since $a^* dx = {}^tAdx$.

COROLLARY 1.7. *Let φ be a c.f. diffeomorphism of T^n such that the only eigenvalue of $\varphi_* : H_1(T^n, R) \hookleftarrow$ is 1. Then φ is conjugate to a Diophantine translation by a C^∞ diffeomorphism.*

Proof. By Remark 1.4 we may assume that φ is given on the covering R^n by a diffeomorphism of the form

$$\varphi = A + F + \alpha, \tag{1.7}$$

where A is triangular, i.e. $A_{ij} = 0$ if $i < j$, F has Lebesgue measure zero in T^n and $\alpha \in R^n$. By Lemma 1.6 and Theorem 1.5 it suffices to construct a closed coframe $w = \{w_1, \ldots, w_n\}$ of T^n cohomologous to the canonical coframe dx and such that $\varphi^* w = {}^tAw$ in matrix notation. We construct w inductively using the cohomological triviality of φ. Let $w_1 = dx_1 + dh_1$, where $h_1 - h_1 \circ \varphi = F_1$,

$$w_2 = dx_2 + dh_2 \quad \text{where } h_2 - h_2 \circ \varphi = F_2 - A_{21} h_1 \tag{1.8}$$

and $w_j = dx_j + dh_j$, where $h_j - h_j \circ \varphi = F_j - \sum_{i=1}^{j-1} A_{ji} h_i$, $1 \leq j \leq n$. It follows that $\varphi^* w = {}^tAw$ and by the minimality of φ we see that w is a coframe, proving the corollary. □

We now show that a c.f. diffeomorphism φ of T^n is quasi-unipotent on homology.

PROPOSITION 1.8. *Let φ be a c.f. diffeomorphism of the torus T^n. Then all the eigenvalues of the induced mapping $\varphi_* : H_1(T^n, R) \hookleftarrow$ are roots of unity and 1 is an eigenvalue.*

Proof. By a theorem of Manning in [**18**] the spectral radius $\mathrm{sp}(\varphi_*)$ of $\varphi_* : H_1(T^n; R) \hookleftarrow$ is bounded above by the exponential of the topological entropy $h(\varphi)$ of φ, i.e.

$$h(\varphi) \geq \log \mathrm{sp}(\varphi_*). \tag{1.9}$$

Now, by a theorem communicated to us by A. Katok, after reading a preliminary version of this paper, the topological entropy of a c.f. diffeomorphism of an orientable manifold is zero. Thus

$$\mathrm{sp}(\varphi_*) = 1. \tag{1.10}$$

Since the characteristic polynomial of φ_* is a monic over the integers, then by a well-known result of algebra all the eigenvalues of φ_* are roots of unity. Of course, by the Lefschetz fixed point theorem, 1 is an eigenvalue of φ_*. □

The above proposition is not sharp as is shown by the following.

Example 1.9. We construct an analytic, minimal and uniquely ergodic orientation-reversing diffeomorphism of the torus T^2. On the covering R^2 it is of the form

$$\varphi(x, y) = (x + \alpha, -y + f(x)), \tag{1.11}$$

where α is a special Liouville number and f is a $\mathcal{Z}$-periodic function. It is sufficient to choose α and f so that

$$\varphi^2(x, y) = (x + 2\alpha, y - f(x) + f(x + \alpha)) \tag{1.12}$$

is minimal and uniquely ergodic.

By a well-known result of Furstenberg [**9**] if the functional equation

$$\chi(x) - \chi(x + 2\alpha) = -f(x) + f(x + \alpha) \tag{1.13}$$

has no measurable solution, then φ^2 is minimal and uniquely ergodic. By a theorem of Krygin [**16**] if the number $\beta = 2\alpha$ has a sequence of convergents p_n/q_n such that

$$\beta - \frac{p_n}{q_n} = \frac{\theta_n}{2^{q_n} q_n^2}, \tag{1.14}$$

where $|\theta_n|$ is a bounded sequence, then there exists an analytic $\mathcal{Z}$-periodic function such that the diffeomorphism induced by

$$\psi(x, y) = (x + \beta, y + g(x)) \tag{1.15}$$

is uniquely ergodic. The function g is constructed by a uniformly convergent series

$$g(x) = \sum_{k=1}^{\infty} c_k (e^{2\pi i b_k x} + e^{-2\pi i b_k x}) \tag{1.16}$$

with positive coefficients c_k. The b_k are selected as some subsequence of $\{q_n\}$.

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We construct the number α inductively by a continued fraction $[0, a_1, a_2, \ldots]$ as follows. Observe first that the partial quotients a_n and the convergents to α must satisfy

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_0 = 0, \quad p_1 = 1$$

and

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad q_1 = a_1, \quad \text{for } n \geq 2.$$

So define inductively the odd numbers

$$a_{k+1} = 3^{q_k}, \tag{1.17}$$

where $p_k/q_k = \langle 0, a_1, \ldots, a_k \rangle$ is the kth convergent of α. As the a_k are odd integers, then $p_{3n+2} = 2r_n - 1$ are odd integers and

$$q_{3n+2} = 2l_n \quad \text{are even.} \tag{1.18}$$

We claim that $(2r_n - 1)/l_n$ are convergents of the number $\beta = 2\alpha$.

This holds since from (1.18) and a well-known property of the convergents [**23**] we have

$$\left| \alpha - \frac{2r_n - 1}{2l_n} \right| \leq \frac{1}{q_{3n+3} q_{3n+2}} \leq \frac{1}{q_{3n+3} l_n}, \tag{1.19}$$

and from (1.17) and (1.18) we get

$$q_{3n+3} > 3^{q_{3n+2}} q_{3n+2} > 2^{l_n} l_n. \tag{1.20}$$

Now from (1.19) and (1.20) we get

$$\left| \beta - \frac{2r_n - 1}{l_n} \right| < \frac{1}{2^{l_n} l_n^2}, \tag{1.21}$$

where $\beta = 2\alpha$. Thus by a well-known result of Legendre [**23**] $(r_n - 1)/l_n$ are convergents of β proving our claim. Hence condition (1.14) is satisfied for the above sequence and (1.15) and (1.16) hold.

We claim that

$$-f(x) + f(x + \alpha) = g(x), \tag{1.22}$$

where f is an analytic $\mathcal{Z}$-periodic function. This holds since

$$f(x) = \sum_{k=1}^{\infty} d_k e^{2\pi i b_k x} + \sum_{k=1}^{\infty} \overline{d}_k e^{-2\pi i b_k x},$$

where

$$d_k = \frac{c_k}{e^{2\pi i b_k \beta} - 1} \tag{1.23}$$

and $b_k = l_{n_k}$ is a subsequence of l_n. So f defines an analytic function since $\lim_{k\to\infty} \|l_{n_k}\beta\| = \frac{1}{2}$. Thus from (1.12), (1.17) and (1.22), $\psi = \varphi^2$ is minimal and uniquely ergodic.

Theorem A follows immediately from Proposition 1.8 and Corollary 1.7 if the manifold is the circle.

Let M be a compact metric space, $\varphi : M \hookleftarrow$ and $f : M \to R$ be continuous mappings. Consider the series

$$\sum_{j=0}^{\infty} f \circ \varphi^j \tag{1.24}$$

the partial sums

$$S_k(f, \varphi) = \sum_{j=0}^{k-1} f \circ \varphi^j$$

and the Cesàro sums

$$\sigma_n(f, \varphi) = \frac{1}{n} \sum_{k=1}^{n} S_k(f, \varphi).$$

We say that the series (1.24) is C^0-*Cesàro convergent* to s if the sequence $\sigma_n(f, \varphi)$ is uniformly convergent to s.

PROPOSITION 1.10. *Let M be a compact metric space, $\varphi : M \hookleftarrow$ be a continuous uniquely ergodic mapping, μ be the normalized φ-invariant probability measure and $f : M \to R$ be a continuous function. Then the functional equation*

$$\chi - \chi \circ \varphi = f, \quad \mu(f) = 0 \tag{1.25}$$

has a continuous solution $s : M \to R$, $\mu(s) = 0 \Leftrightarrow \sum_{j=0}^{\infty} f \circ \varphi^j$ is C^0-Cesàro convergent to s.

Proof. Let $s : M \to R$, $\mu(s) = 0$ be a continuous solution to (1.25). An easy computation shows that

$$\sigma_n(f, \varphi) = s - \frac{1}{n} \sum_{k=1}^{n} s \circ \varphi^k \tag{1.26}$$

and as φ is uniquely ergodic, then $\sigma_n(f, \varphi)$ converges uniformly to s.

Consider the identity

$$\sigma_n(f, \varphi) - \sigma_n(f, \varphi) \circ \varphi = f - \frac{1}{n} \sum_{j=1}^{n} f \circ \varphi^j. \tag{1.27}$$

Now if $\sigma_n(f, \varphi)$ converges uniformly to $s : M \to R$, then s is a continuous solution to equation (1.25) because $\frac{1}{n} \sum_{j=1}^{n} f \circ \varphi^j$ converges uniformly to $\mu(f) = 0$ since φ is uniquely ergodic. Moreover, $\mu(s) = 0$. □

COROLLARY 1.11. *Let $F \to M \xrightarrow{\pi} N$ be a fibre bundle and let $\varphi : M \hookleftarrow$ be a c.f. fibre preserving diffeomorphism. Then the induced diffeomorphism $\psi : N \hookleftarrow$ is also c.f.*

Proof. Let $f : N \to R$ be a C^∞ function, $\mu(f) = 0$. As φ is c.f. then there exists a C^∞ solution $h : M \to R$ to the equation

$$h - h \circ \varphi = f \circ \pi. \tag{1.28}$$

Now observe that the Cesàro sums of $\sum_{j=0}^{\infty} f \circ \pi \circ \varphi^j$ satisfy

$$\sigma_n(f \circ \pi, \varphi) = \sigma_n(f, \overline{\varphi}) \circ \pi \tag{1.29}$$

since, by assumption, $\pi \circ \varphi = \overline{\varphi} \circ \pi$. Thus $\sigma_n(f, \overline{\varphi}) \circ \pi$ converges uniformly to h. Therefore $h = \overline{h} \circ \pi$ where $\overline{h} : N \hookleftarrow$ is a C^∞ solution to the equation $\overline{h} - \overline{h} \circ \overline{\varphi} = f$ and $\overline{\varphi}$ is c.f. □

2. *The cohomology free diffeomorphisms of T^2*

We show that the *only c.f. diffeomorphisms of T^2 are the differentiable conjugations of Diophantine translations*. Let φ be a C^∞ c.f. diffeomorphisms of T^2. Then φ is minimal and by Proposition 1.8, $\sigma(\varphi_*) \subset \{-1, 1\}$. Now by Corollary 1.7 to show that φ is conjugate to a Diophantine translation of T^2 we have to show that -1 is not an eigenvalue. Suppose $-1 \in \sigma(\varphi_*)$. Then by Remarks 1.3 and 1.4, φ is, on the covering space R^2, of the form

$$\varphi(x, y) = (x + \alpha, nx - y + f(x, y)), \tag{2.1}$$

where n is an integer.

Now by a theorem of Moser [**19**], see Corollary 6.2, we may assume that $\varphi^*(dx \wedge dy) = -dx \wedge dy$ which implies that $f_y = 0$ and f does not depend on y.

We claim that $T_\alpha(x) = x + \alpha$ induces a Diophantine rotation on S^1. By Theorem 1.5 it is sufficient to show that T_α is c.f. To show this, let g be a C^∞ $\mathcal{Z}$-periodic function on R. Since φ is c.f., the functional equation

$$h - h \circ \varphi = g \circ \pi \tag{2.2}$$

has a C^∞ solution, where π is the projection on the first factor. Differentiating (2.2) we get

$$h_y + h_y \circ \varphi = 0.$$

So $h_y = 0$ since φ is minimal. Thus h is a solution to the equation $h(x) - h(x + \alpha) = g(x)$, showing that T_α is c.f. We note that since α is a Diophantine number, then we also have a C^∞ solution k to the equation

$$k(x) + k(x + \alpha) = -f(x). \tag{2.3}$$

Now $\varphi^* w = n\,\mathrm{d}x - w$, where $w = dy + d(k \circ \pi)$, and as $\varphi^* dx = dx$, then by Lemma 1.6 φ is differentiably conjugate to the affine mapping

$$a(x, y) = (x + \alpha, -y + nx)$$

which by Theorem 1.5 shows that a is not c.f. Thus φ is not c.f., giving a contradiction.

3. *The cohomology free diffeomorphisms of the torus T^3*

We show that *the only c.f. diffeomorphisms of T^3 are smooth conjugations of Diophantine translations*.

We see from 1.4 that a minimal diffeomorphism φ of T^3 is given, on the covering R^3, by

$$\varphi = A + F,$$

where the integral matrix A is of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ n_1 & & \\ n_2 & B & \end{pmatrix} \tag{3.1}$$

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with $\det B = \deg\varphi$ and $F : R^n \to R^n$ is $\mathcal{Z}^n$-periodic. By Proposition 1.8 all the eigenvalues of B are roots of unity. Now since $\varphi^* dx = dx + dF_1$ then, by Remark 1.3, if φ is c.f. so we have a φ-invariant 1-form $w = dx + dh$. Hence, by the isotopy theorem [**17**] there exists a C^∞ diffeomorphism ψ isotopic to the identity such that $\psi^* w = dx$. So $\psi \circ \varphi \circ \psi^{-1}$ leaves dx invariant and we may, in addition, assume that φ leaves dx invariant. Now by Proposition 1.2 there exists a volume form Ω such that $\varphi^*\Omega = (\deg\varphi)\Omega$. Thus by an adapted version of a theorem of Moser, see Corollary 6.2, we may assume that $\Omega = dx \wedge dy \wedge dz$.

We need the following lemma.

LEMMA 3.1. *Let B be a unipotent $n \times n$ real matrix, i.e. $B^m = I$ for some positive integer m, and let T_α be a Diophantine translation of the torus T^p. Let $r : T^p \to R^n$ be any C^∞ function with Haar measure zero. Then there exists a unique C^∞ solution $h : T^p \to R^n$ to the functional equation*

$$h - Bh \circ T_\alpha = r.$$

Proof. The formal solution in Fourier transform of the above equation is

$$(I - e^{2\pi i k\cdot\alpha} B)\hat{h}(k) = \hat{r}(k)$$

and as the eigenvalues of B are roots of unity and $k \cdot \alpha \notin \mathcal{Z}$ for all $k \in \mathcal{Z}^p - \{0\}$, then we can solve for h and get

$$\hat{h}(k) = (I - e^{2\pi i k\cdot\alpha} B)^{-1}\hat{r}(k).$$

Since B is unipotent, an easy computation gives

$$|\hat{h}(k)| \leq \frac{L}{|1 - e^{2\pi i k\cdot m\alpha}|}|\hat{r}(k)|, \quad \forall k \in \mathcal{Z}^p - \{0\}$$

for some constant $L > 0$ and all $k \neq 0$. As α is Diophantine, then

$$|\hat{h}(k)| < CL|k|^{p+\beta}|\hat{r}(k)|$$

showing that h is a C^∞ function. □

We now distinguish two cases.

3.2. The diffeomorphism φ preserves orientation. By Corollary 1.7 we have to show $\sigma(B) = \{1\}$. Suppose $1 \notin \sigma(B)$. The idea of the proof is to show the existence of a free action ψ of S^1 on T^3 which is invariant under φ, i.e. $\varphi \circ \psi_t = \psi_t \circ \varphi$ for all $t \in S^1$. Thus φ gives a morphism $(\varphi, \overline{\varphi})$ of a circle bundle $S^1 \to T^3 \xrightarrow{\tau} T^2$. Now by Corollary 1.11 the diffeomorphism $\overline{\varphi}$ of T^2 is c.f. and by §2 we conclude that $\overline{\varphi}$ is conjugate to a Diophantine translation of T^2. Thus 1 is an eigenvalue of B, giving a contradiction.

We may, as before, assume that φ leaves invariant the 1-form dx and the canonical volume element $\Omega = dx \wedge dy \wedge dz$. Thus φ has, on the covering, the form

$$\varphi(x, y, z) = (x + \alpha, G) \quad \text{where } G(x, y, z) = B\begin{pmatrix} y \\ z \end{pmatrix} + F \tag{3.2}$$

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and $F : R^3 \to R^2$ is a $\mathcal{Z}^3$-periodic function.

The main difficulty lies in proving the existence of a φ-invariant free action of S^1 as above. Instead of determining the action ψ directly we first look for a φ-invariant closed 2-form Θ so that

$$dx \wedge \Theta = \Omega. \tag{3.3}$$

Now the φ-invariant volume form Ω will convert Θ into a divergence-free φ-invariant vector field X, i.e.

$$\Theta = i_X \Omega \quad \text{and} \quad \varphi_* X = X. \tag{3.4}$$

We show using a fixed point theorem due to Conley and Zehnder [**7**] that X generates the desired action.

To find Θ we first start with the integral linear closed 2-form

$$\theta_0 = dy \wedge dz + m_1\, dx \wedge dy + m_2\, dx \wedge dz \tag{3.5}$$

which is φ^*-invariant in $H^2(T^3, R)$, i.e.

$$dx \wedge \theta_0 = \Omega \quad \text{and} \quad \varphi^*\theta_0 = \theta_0 + dx \wedge \eta, \tag{3.6}$$

where $dx \wedge \eta = d\xi$ is an exact 2-form.

Let $\mathcal{F}$ be the foliation given by the submersion $p : T^3 \to S^1$, $p(x, y, z) = x$. Observing that the 1-form η is d_f-closed and $dx \wedge \eta$ is exact we may, by Proposition 5.4, choose η to be of the form

$$\eta = r_1'(x)\, dy + r_2'(x)\, dz + df. \tag{3.7}$$

Now, since φ preserves orientation, the matrix B in (3.1) is either unipotent or is the matrix

$$\begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}, \tag{3.8}$$

m integer.

Let h be given as in Lemma 3.1 if B is unipotent. If B is the matrix in (3.8), it is easy to see that we also have a smooth solution to the equation

$$h - Bh \circ T_\alpha = r, \quad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{3.9}$$

Now if $\theta = \theta_0 + dh_1 \wedge dy + dh_2 \wedge dz$, then from (3.9) one can see that

$$\varphi^*\theta = \theta + dx \wedge dg,$$

where

$$g = f + (h_1' \circ T_\alpha)F_1 + (h_2' \circ T_\alpha)F_2$$

and F is the mapping given in (3.2). Since φ is c.f. there exists a smooth solution k to the equation $k - k \circ \varphi = g$. An easy computation shows that the closed 2-form

$$\Theta = \theta_0 + dh_1 \wedge dy + dh_2 \wedge dz + dx \wedge dk \tag{3.10}$$

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is invariant under φ. Since Ω and Θ are φ-invariant, then the vector field X in (3.4) is also invariant, i.e. $\varphi_* X = X$ since $i_{\varphi_* X}\Omega = i_X\Omega$. Actually, one can see from (3.10) that

$$X = \frac{\partial}{\partial x} - (m_2 + h_2' + k_z)\frac{\partial}{\partial y} + (m_1 + h_1' + k_y)\frac{\partial}{\partial z}. \tag{3.11}$$

The flow ψ_t of X is given, on the covering R^3, by

$$\psi_t(x, y, z) = (x + t, y + u_t, z + v_t). \tag{3.12}$$

We notice that since $L_X\Theta = 0$, then $\psi_t^*\Theta = \Theta$ for all t since φ and ψ_1 commute then (3.2) and (3.12) give rise to the cocycle

$$BU - U \circ \varphi = F - F \circ \psi_1, \quad \text{where } U = (u_1, v_1). \tag{3.13}$$

Now, integrating along the fibres of the torus bundle $p : T^3 \to S^1$ [**10**] one obtains

$$B \fint U\, dy\, dz - T_\alpha^* \fint U\, dy\, dz = 0$$

and since α is irrational we see that $\fint U\, dy\, dz$ is constant and as 1 is not an eigenvalue of B this constant must be zero.

Thus, by a theorem of Conley and Zehnder in [**7**], the restriction of ψ_1 to each fibre has at least three fixed points and as φ and ψ_1 commute and φ is minimal then ψ_1 is the identity mapping. Hence X generates a φ-invariant free action of S^1, finishing the proof. □

3.3. The Diffeomorphism φ reverses orientation. We may assume, by Proposition 1.8 and Remark 1.4, that the diffeomorphism φ is, on the covering R^3, homotopic to the integral matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ n_1 & 1 & 0 \\ n_2 & n_3 & -1 \end{pmatrix}.$$

By Remark 1.3 and the isotopy theorem [**17**] we may assume that dx is invariant under φ. Thus φ is, on the covering, of the form

$$\varphi(x, y, z) = (x + \alpha_1, n_1 x + y + f, n_2 x + n_3 y - z + g). \tag{3.14}$$

The cohomological triviality of φ gives rise to a smooth function h on T^3 and a constant α_2 such that $h - h \circ \varphi = f - \alpha_2$. If $w = dy + dh$ then $\varphi^* w = n_1\, dx + w$ and the closed 2-form $W = dx \wedge w$ is φ-invariant. We want to construct a diffeomorphism ψ such that $\psi^*\, dx = dx$ and $\psi^*\, dy = w$ and modify φ by conjugation by ψ in order to simplify φ. For this, we show that the submersion $p : T^3 \to T^2$, $p(x, y, z) = (x, y + h)$ is a φ-invariant principal circle bundle. This holds since by Proposition 1.2 there exists a volume form $\lambda\Omega$, Ω being the standard volume form of T^3, so that $\varphi^*\lambda\Omega = -\lambda\Omega$.

Let X be the vector field corresponding to the 2-form W, i.e. $W = i_X\lambda\Omega$. Since W is φ-invariant and $\varphi^*\lambda\Omega = -\lambda\Omega$, then one can see that $\varphi_* X = -X$. Observe that the orbits of X are the fibres of p which are circles. Thus all of them are closed. We have to show they have the same period. This can be seen by observing that the union of all

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orbits of a given period is a closed φ-invariant set and, as φ is minimal, this set is either the empty set or T^3.

Thus $p : T^3 \to T^2$ is isotopic to the trivial circle bundle $S^1 \times T^2 \xrightarrow{\pi} T^2$, i.e. there exists a diffeomorphism ψ of T^3 isotopic to the identity such that $\pi \circ \psi = p$. Hence ψ leaves dx invariant and $\psi^* dy = w$. So, up to a conjugation by ψ, we may assume that

$$\varphi^* dx = dx \quad \text{and} \quad \varphi^* dy = n_1 dx + dy \tag{3.15}$$

and $\pi \circ \varphi = a \circ \pi$, where $a(x, y) = (x + \alpha_1, y + \alpha_2)$ is by Corollary 1.11 and Theorem 1.5 a Diophantine translation. Now apply Corollary 6.2 to modify φ by conjugation to obtain a new diffeomorphism satisfying $\varphi^* \Omega = -\Omega$ and

$$\varphi(x, y, z) = (x + \alpha_1, y + \alpha_2, n_2 x + n_3 y - z + g). \tag{3.16}$$

It follows from (3.16) that $g_z = 0$. Thus Lemma 3.1 gives a smooth solution χ to the equation $\chi + \chi \circ T_\alpha = g$. Consider now the closed coframe

$$\rho = \{dx, dy, \rho_3\}, \tag{3.17}$$

where $\rho_3 = dz - d\chi$. Since, in matrix notation, $\varphi^* \rho = {}^t A \rho$, then by Lemma 1.6 φ is differentiably conjugate to the affine mapping induced by $A + \alpha$, where $\alpha = (\alpha_1, \alpha_2, 0)$, contradicting, by Theoerm 1.5, the cohomological triviality of φ. §§3.2 and 3.3 show that the c.f. diffeomorphisms of the torus T^3 are the smooth conjugations of Diophantine translations.

This finishes the proof of Theorem A.

4. *The orientation-preserving cohomology-free diffeomorphisms of the torus* T^4

Let φ be a c.f. diffeomorphism of T^4. Then φ is minimal and by Proposition 1.8 all the eigenvalues of $\varphi_* : H_1(T^4, R) \hookleftarrow$ are roots of unity and 1 is an eigenvalue. So, on the covering R^4, φ is given by

$$\varphi = A + F, \tag{4.1}$$

where F is a $\mathcal{Z}^4$-periodic mapping from R^4 to R^4 and the characteristic polynomial of the integral matrix A factors over $\mathcal{Z}$ as

$$p(x) = (x - 1)^j q(x), \quad 1 \le j \le 4 \tag{4.2}$$

and $q(1) \neq 0$.

Notice that since φ *preserves orientation*, if $3 \le j$ then 1 is the only eigenvalue of A and by Corollary 1.7 φ is differentiably conjugate to a Diophantine translation. Thus, in this case, we may assume $1 \le j \le 2$. By Remark 1.4, we may, in addition, assume that the matrix A is of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n & 1 & 0 & 0 \\ & L & & B \end{pmatrix}, \tag{4.3}$$

where $\det B = 1$.

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Now, since φ is c.f., there exist constants α_j and smooth $\mathcal{Z}^4$-periodic zero measure functions g_j such that

$$g_j - g_j \circ \varphi = F^j - \alpha_j, \quad 1 \le j \le 2, \tag{4.4}$$

where F^j is the jth coordinate function of F. We also have a smooth $\mathcal{Z}^4$-periodic solution to the functional equation

$$h - h \circ \varphi = g_1.$$

Let $w_1 = dx_1 + dg_1$ and $w_2 = dx_2 + dg_2 - ndh$. An easy computation shows that

$$\varphi^* w_1 = w_1 \quad \text{and} \quad \varphi^* w_2 = n w_1 + w_2. \tag{4.5}$$

So $W = w_1 \wedge w_2$ is φ-invariant and as φ is minimal then W is non-singular, i.e. w_1 and w_2 are linearly independent at each point of T^4. Thus, the submersion $p : T^4 \to T^2$ given by $p(x) = (x_1 + g_1, x_2 + g_2 - nh)$ is a φ-invariant torus bundle and φ induces on the base space T^2 an affine mapping

$$a = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} + \alpha, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{4.6}$$

By Corollary 1.11 a is c.f. and by Theorem 1.5 it must be a Diophantine translation T_α.

Thus $n = 0$ and we do have two φ-invariant 1-forms

$$w_j = dx_j + dg_j, \quad 1 \le j \le 2. \tag{4.7}$$

4.1. A difficulty now arises: we need to linearize simultaneously these two 1-forms, i.e. to show that *there is a diffeomorphism ψ of T^4 homotopic to the identity and such that*

$$\psi^* w_j = dx_j, \quad 1 \le j \le 2. \tag{4.8}$$

But in T^4, even the isotopy theorem is not known to be true. We by-pass this difficulty by making use of a result on the cohomology of fibrations, Proposition 5.4. By [**6**], w_1 and w_2 are simultaneously linearizable if and only if w_1 and w_2 can be extended to a closed coframe $\{w_1, w_2, \rho_1, \rho_2\}$ of T^4.

We claim the following.

4.2. The closed invariant 1*-forms w_1 and w_2 can be extended to a closed coframe of T^4.*

By a Theorem of Saldanha in [**21**] and Proposition 5.4, w_1 and w_2 extend to a d_f-closed coframe, i.e. there exist d_f-closed 1-forms η_1 and η_2 such that

$$\{w_1, w_2, \eta_1, \eta_2\} \tag{4.9}$$

is a coframe of T^4.

Here the foliation $\mathcal{F}$ is given by the fibres of the torus bundle $p : T^4 \to T^2$. Now as the canonical 1-forms dy_j, $1 \le j \le 2$, are closed, they give a d_f-closed ordered pair of tangent 1-forms $dy^f = \{dy_1^f, dy_2^f\}$, of the foliated manifold $(T^4, \mathcal{F})$.

By Proposition 5.4 we have, in matrix notation,

$$dy^f = (P \circ p)\eta + d_f k, \tag{4.10}$$

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where P is a smooth mapping from T^2 to the vector space of all real 2×2 matrices. To prove Claim 4.2 it is sufficient to show that $\det P \neq 0$ for all $x \in T^2$. If $\rho = dy - dk$, then

$$W \wedge \Delta\rho = \det(P \circ p) W \wedge \Delta\eta, \tag{4.11}$$

where $\Delta\eta = \eta_1 \wedge \eta_2$ and $\Delta\rho = \rho_1 \wedge \rho_2$. Notice that $W \wedge \Delta\eta$ is a volume form of T^4. Thus $\det P(x) \neq 0$ for all $x \in T^2$ if and only if $\{w_1, w_2, \rho_1, \rho_2\}$ is a closed coframe of T^4. We show now that $\det P(x) \neq 0$ for all $x \in T^2$. In fact, since φ is a morphism of the torus bundle $p : T^4 \to T^2$, then by Proposition 5.4, (5.12), one has

$$\varphi^* \eta = (M \circ p)\eta + d_f h. \tag{4.12}$$

From (4.10) and (4.12) we get

$$\varphi^* \, dy^f = [(P \circ T_\alpha \circ p)(M \circ p)]\eta + d_f[k \circ \varphi + ((P \circ T_\alpha) \circ p)h],$$

on the other hand, from (4.1), (4.3) and (4.10) we also have

$$\varphi^* \, dy^f = {}^t B(P \circ p)\eta + d_f[{}^t Bk + \pi_2 \circ F],$$

where π_j are the projections of $R^4 = R^2 \times R^2$ onto R^2, $1 \leq j \leq 2$. Thus by Remark 5.5 one has

$$P(x + \alpha)M(x) = {}^t B P(x) \tag{4.13}$$

for all $x \in T^2$.

Iteration of (4.12) and (4.10) leads to

$$P(x + m\alpha)M_m(x) = {}^t B^m P(x), \tag{4.14}$$

where $M_m(x) = \prod_{j=1}^m M(x + (m - j)\alpha)$.

We claim that $\det M(x) \neq 0$ for all $x \in T^2$. From (4.12) we get

$$\varphi^*(W \wedge \Delta\eta) = (\det M \circ p)(W \wedge \Delta\eta) + d\tau$$

and integrating along the fibres of the torus bundle p one has

$$g(x + \alpha) = g(x) \det M(x), \tag{4.15}$$

where $g(x) = \int \Delta\eta(x) \neq 0$ for all $x \in T^2$ since $\Delta\eta$ gives a volume form on each fibre of p.

Thus $\det M(x) \neq 0$ for all $x \in T^2$ as claimed.

Now from (4.14), if $\det P(x_0) = 0$ at some point $x_0 \in T^2$ then $\det P(x) = 0$ for all $x \in T^2$ since the translation by α is minimal. This contradicts (4.11) since $W \wedge \Delta\rho$ is cohomologous to the canonical volume form $dx \wedge dy$ of T^4, proving 4.2 and 4.1. We have thus proved the following.

4.3. The *canonical form* of a c.f. orientation-preserving diffeomorphism of T^4 is, on the covering R^4, given by

$$\varphi = A + H,$$

where A is given in (4.3) and the first two coordinate functions of H are constant, i.e. $H = (\alpha_1, \alpha_2, F)$, $F : R^4 \to R^2$. By Corollary 6.2 we may assume that φ preserves the standard volume form $\Omega_0 = dx \wedge dy$ of T^4.

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We now prove the main result of this section.

4.4. The only c.f. orientation-preserving diffeomorphisms of T^4 are the smooth conjugations of Diophantine translations.

By Corollary 1.7 we have to show $\sigma(B) = \{1\}$. Suppose $1 \notin \sigma(B)$. The idea of the proof is to construct a φ-invariant free action ψ of T^2 on T^4, obtaining a principal torus bundle $T^2 \to T^4 \xrightarrow{\tau} T^2$. Now, φ induces a diffeomorphism $\overline{\varphi}$ on the base space T^2. As before, Corollary 1.11 shows that $\overline{\varphi}$ is c.f. Thus, by §2, $\overline{\varphi}$ is a smooth conjugation of a Diophantine translation. So $\overline{\varphi}$ is homotopic to the identity and as $\overline{\varphi}$ is, on the covering R^2, homotopic to B then B is the identity matrix, giving a contradiction.

Instead of constructing ψ directly we first look for a φ-invariant symplectic form Θ which will convert the φ-invariant 1-forms dx_1 and dx_2 into φ-invariant commuting vector fields X_1 and X_2. Then we use a fixed point theorem for measure preserving diffeomorphism of T^2 due to Conley and Zehnder, [**7**] to show that X_1 and X_2 generate a free action ψ of T^2 on T^4, as desired.

To construct Θ we first observe that a routine computation shows the existence of a rational 2-form

$$\theta_0 = dy_1 \wedge dy_2 + r_1\, dx_1 \wedge dy_1 + r_2\, dx_1 \wedge dy_2 + r_3\, dx_2 \wedge dy_1 + r_4\, dx_2 \wedge dy_2, \quad (4.16)$$

$r_j \in Q$, $1 \le j \le 4$, such that

$$\varphi^*\theta_0 = \theta_0 + d\xi \quad \text{where } d\xi = dx_1 \wedge \eta_1 + dx_2 \wedge \eta_2$$

since $\varphi^*\, dx_j = dx_j$, $1 \le j \le 2$. Moreover, η_1 and η_2 are smooth.

Now as $dx \wedge \theta_0 = dx \wedge dy$ is the standard volume form on T^4 then $w_0 = \frac{1}{2}(dx + \theta_0)$ is a symplectic form and as dx is φ-invariant, then w_0 is φ-invariant in cohomology. Since ξ is a d_f-closed 1-form, Proposition 5.4 gives

$$\xi = (r_1 \circ \pi)\, dy_1 + (r_2 \circ \pi)\, dy_2 + \xi_0 + d_f k, \quad \pi(x, y) = x, \quad (4.17)$$

where r_1 and r_2 are smooth real functions on T^2, k is smooth real function on T^4 and $\xi_0 = g_1\, dx_1 + g_2\, dx_2$ is a smooth 1-form on T^4. Since φ is c.f., we have smooth solutions k_j to the equations

$$k_j - k_j \circ \varphi = g_j - c_j, \quad 1 \le j \le 2,$$

where c_j is the integral of g_j over T^4. Let h be the smooth solution to the equation $h - Bh \circ T_\alpha = r$ given by Lemma 3.1 where the coordinate functions of $r : T^2 \to R^2$ are as in (4.17). Now, if

$$\theta_1 = \theta_0 + dh_1 \wedge dy_1 + dh_2 \wedge dy_2 + d(k_1\, dx_1 + k_2\, dx_2) + d\, d_f \chi,$$

where

$$\chi - \chi \circ \varphi = k$$

then, by a straightforward computation, one gets

$$\varphi^*\theta_1 = \theta_1 + d\eta, \quad (4.18)$$

where $\eta = F^1\, d(h_1 \circ T_\alpha) + F^2\, d(h_2 \circ T_\alpha)$. So η is of the form $\eta = \ell_1\, dx_1 + \ell_2\, dx_2$, where ℓ_j are smooth real functions on T^4. Since φ is c.f. and the 1-forms dx_1 and

dx_2 are φ-invariant, we have a smooth solution γ to the equation $\gamma - \varphi^*\gamma = \eta$. Thus $\theta = \theta_1 + d\gamma$ is a φ-invariant smooth 2-form and $dx \wedge \theta = \Omega_0$. Therefore

$$\Theta = \tfrac{1}{2}(dx + \theta) \tag{4.19}$$

is the desired φ-invariant sympletic form. Let X_j be the smooth divergence-free vector fields given by $i_{X_j}\Theta = dx_j$, $1 \le j \le 2$. Since Θ and dx_j are φ-invariant, then

$$\varphi_* X_j = X_j, \quad 1 \le j \le 2. \tag{4.20}$$

Now as $i_{X_j}\,dx_j$ are φ-invariant smooth functions and φ is minimal, they are constant and we may, in addition, assume that $i_{X_j}\,dx_j = 1$ and $i_{X_i}\,dx_2 = 0$. Actually, X_1 and X_2 commute since the Poisson bracket $(dx_1, dx_2) = -d\Theta(X_1, X_2)$ vanishes. So these vector fields generate a locally free action ψ of R^2 on T^4 which is transversal to the fibres of the trivial bundle $\pi(x, y) = x$.

Since φ and $\psi_j = \psi_{e_j}$ commute, $1 \le j \le 2$, and $\psi_j(x, y) = (x + e_j, y + U_j)$, where $U_j : T^4 \to R^2$ are smooth functions, then one has the cocycle equations

$$BU_j - U_j \circ \varphi = F - F \circ \psi_j, \quad 1 \le j \le 2. \tag{4.21}$$

Integration along the fibres of π gives

$$B \fint U_j\,dy - T_\alpha \fint U_j\,dy = 0, \quad 1 \le j \le 2,$$

and as T_α is minimal and B is unipotent, then $\fint U_j\,dy$ is constant and as, by assumption, 1 is not an eigenvalue of B, these constants must be 0. Thus the restriction of ψ_j to each fibre of π has mean translation 0 and by a theorem of Conley and Zehnder [**7**] ψ_j has at least three fixed points in each fibre. As φ and ψ_j commute and φ is minimal, ψ_j must be the identity mapping. Hence the action ψ generated by X_1 and X_2 is free, finishing the proof of 4.4. This proves Theorem B.

5. *On the cohomology of torus bundles*

We discuss briefly some results on the cohomology of the foliation $\mathcal{F}$ given by the fibres of a torus bundle $T^q \to T^n \xrightarrow{\tau} T^p$.

Let $\mathcal{F}$ be a foliation of codimension q of a closed, orientable, connected m-manifold M and let $\Lambda(M)$ be the graded algebra of all C^∞ differential forms on M. If $I(\mathcal{F})$ is the annihilating ideal of $\mathcal{F}$ then $I(\mathcal{F})^{q+1} = 0$. We call $\Lambda(\mathcal{F}) = \Lambda(M)/I(\mathcal{F})$ the graded algebra of the *differential forms tangent to* $\mathcal{F}$. Since, by Frobenius' theorem, $dI(\mathcal{F}) \subset I(\mathcal{F})$, the differential $d : \Lambda(M) \to \Lambda(M)$ induces the *foliated differential* $d_f : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F})$. The cohomology $H^*(\mathcal{F})$ of the differential complex $(\Lambda(\mathcal{F}), d_f)$ is referred to as the *cohomology of the foliated manifold* $(M, \mathcal{F})$. This is a natural generalization of the de Rham cohomology. For more details see [**4**] and [**22**].

To give a torus bundle $T^q \to T^n \xrightarrow{\tau} T^p$ is equivalent to giving an *integral closed p-coframe* $w = \{w_1, \ldots, w_p\}$ on T^n, i.e. the 1-forms w_j, $1 \le j \le p$, are closed and linearly independent at every point of T^n and $w_j = \tau^*\,dx_j$, $1 \le j \le p$.

We now compute the cohomology group $H^q(\mathcal{F})$. Let $\Lambda(M) \xrightarrow{f} \Lambda(\mathcal{F})$, $\theta \mapsto \theta^f$, be the quotient mapping and let $W = w_1 \wedge \cdots \wedge w_p$ be the closed p-form which defines $\mathcal{F}$.

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We have a vector space isomorphism

$$\Lambda^q(\mathcal{F}) \xrightarrow{W\wedge} \Lambda^n(T^n), \quad \theta^f \mapsto W \wedge \theta.$$

Integration along the fibre of τ [**10**] gives an epimorphism

$$\Lambda^q(\mathcal{F}) \xrightarrow{\mathcal{P}} \Lambda^p(T^p) \simeq C^\infty(T^p), \tag{5.1}$$

where

$$\mathcal{P}(\theta^f) = \fint W \wedge \theta.$$

PROPOSITION 5.1. *Let $T^q \to T^n \xrightarrow{\tau} T^p$ be a torus bundle and let $\mathcal{F}$ be the foliation given by τ. Then:*

(i) *the kernel of $\mathcal{P}$ is the subspace $B^q(\mathcal{F})$ of all d_f-exact tangent forms in $\Lambda^q(\mathcal{F})$;*

(ii) *$H^q(\mathcal{F})$ is isomorphic to the space $C^\infty(T^p)$ of all smooth real functions on T^p.*

Proof. Let Ω be a volume form on T^n and let f be any smooth real function on T^n. To prove (i) we have to show

$$\fint f\Omega = 0 \Leftrightarrow f\Omega = W \wedge d\eta \tag{5.2}$$

for some $\eta \in \Lambda^{q-1}(T^n)$ and $W = p^* dx$, where $dx = dx_1 \wedge \cdots \wedge dx_p$ is the standard volume form of T^p.

Clearly, $\fint W \wedge d\eta = dx \fint d\eta = 0$, by Stokes' theorem. Now, let $\fint f\Omega = 0$ and let $\psi : V \times T^q \to \tau^{-1}(V)$ be a local trivialization of τ, i.e. $\tau \circ \psi = \pi$ where $\pi : V \times T^q \to V$ is the projection. We observe that

$$\psi^*(W \wedge \theta) = \pi^* dx \wedge \overline{\theta}, \quad \overline{\theta} = \psi^*\theta = \overline{f}\, dy,$$

where $dy = dy_1 \wedge dy_2$ and, by assumption

$$dx \fint \overline{\theta} = \fint_V \psi^*(W \wedge \theta) = \fint_V W \wedge \theta = 0,$$

where $\fint_V$ means integration along the fibre of the torus bundle $\tau^{-1}(V) \to V$. Thus $\fint \overline{\theta} = 0$, which implies

$$\overline{\theta} = d_f \overline{\eta} \tag{5.3}$$

for some 1-form $\overline{\eta}$ on V.

Now cover T^p by open sets V_λ as above and let $\psi_\lambda : V_\lambda \times T^q \to \tau^{-1}(V_\lambda)$ be the corresponding trivializations. Let $\{\rho_\lambda\}$ be a smooth partition of unity subordinate to the cover $\{V_\lambda\}$. Let $g_\lambda\, dV = \psi_\lambda^*(f\Omega)$. As $\fint f\Omega = 0$ then $\fint_{V_\lambda} f\Omega = \fint g_\lambda\, dV = 0$, where $dV = dx \wedge dy$ is the standard volume form of T^n. Thus from (5.3) one has

$$g_\lambda\, dV = dx \wedge d\overline{\eta}_\lambda$$

$$f\Omega = \sum_\lambda (\rho \circ \tau) W \wedge d\eta_\lambda = W \wedge d\eta,$$

where $\psi_\lambda^* \eta_\lambda = \overline{\eta}_\lambda$, proving (i). Now, (ii) follows immediately from (i) and (5.1). □

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COROLLARY 5.2. *Let φ be a c.f. diffeomorphism of T^n and let $T^q \to T^n \xrightarrow{\tau} T^p$ be a φ-invariant torus bundle, i.e. $\varphi^* W = W$, $W = \tau^* dx$. Let θ_0 be a closed q-form on T^n such that:*

(i) $\varphi^*\theta_0 = \theta_0 + d\eta_0$;

(ii) $\pi^* dx \wedge \theta_0 = \Omega$ *is the φ-invariant volume form on T^n, where π is the projection of $T^n = T^p \times T^q$ on the first factor.*

Then there exists a $(q-1)$-form η on T^n such that $\Omega = W \wedge \theta$, where $\theta = \theta_0 + d\eta$.

Proof. Observe that $W \wedge (\varphi^*\theta_0 - \theta_0) = W \wedge d\eta_0 - f\Omega$ and $\fint f\Omega = 0$. Since φ is c.f. we have a smooth solution g to the equation $g - g \circ \phi = f$. Integration along the fibre of τ leads to the equation

$$\fint g\Omega - \overline{\varphi}^* \fint g\Omega = 0, \tag{5.4}$$

where $\tau \circ \varphi = \overline{\varphi} \circ \tau$, and since $\overline{\varphi}$ is minimal then $\fint g\Omega = 0$. Now, Proposition 5.1(i) gives a smooth 1-form η so that

$$g\Omega = W \wedge d\eta. \tag{5.5}$$

It is easy to see that $\Omega = W \wedge \theta$, where $\theta = \theta_0 + d\eta$, because $W \wedge (d\eta - \varphi^* d\eta) = W \wedge d\xi_0$, finishing the proof. □

In what follows, it would be interesting to have an answer to the following.

Question 5.3. Let $T^q \to M \xrightarrow{\tau} T^p$ be a smooth torus bundle. Does there exist a locally free smooth action of R^q on T^n whose orbits are the fibres of τ?

For $q = 1$ the answer is trivially yes. For $q = 2$ the answer is yes, by a theorem of Saldanha [**21**].

Let $(M, \mathcal{F})$ be a foliated manifold and let q be the dimension of $\mathcal{F}$. By [**6**] and [**22**], to give a locally free action A of R^q *tangent* to $\mathcal{F}$, i.e. the orbits of A are the leaves of $\mathcal{F}$, is equivalent to giving a *d_f-closed coframe ξ^f* tangent to $\mathcal{F}$. A *d_f-closed coframe* is an ordered set $\xi^f = \{\xi_1^f, \dots, \xi_q^f\}$ of d_f-closed tangent forms of $\mathcal{F}$ such that ξ^f is a basis of the tangent space $T_x\mathcal{F}$ at every point x of M.

We have the following.

PROPOSITION 5.4. *Let $T^q \to T^n \xrightarrow{\tau} T^p$ be a torus bundle. Suppose there exists a locally free action A of R^q tangent to the fibres of τ. Then the cohomology of the foliation defined by the fibres of τ is given by*

$$H^j(\mathcal{F}) \simeq H^j(T^q) \otimes C^\infty(T^p).$$

Proof. For the sake of clarity we give a proof for $q = 2$. Let $Y = \{Y_1, Y_2\}$ be the frame of the generators of A and let $\eta = \{\eta_1, \eta_2\}$ be a coframe adapted to A, i.e. $\eta_i(Y_j) = \delta_{ij}$. Thus, as Y_1 and Y_2 commute, η is a d_f-closed coframe. So, $\eta = \{w_1, \dots, w_p, \eta_1, \eta_2\}$ is a d_f-closed coframe of T^n, i.e. $W \wedge d\eta_j = 0$, $1 \leq j \leq 2$, where, as before, $W = w_1 \wedge \cdots \wedge w_p$ and $w_j = \tau^* dx_j$, $1 \leq j \leq p$. Now let $\psi : V \times T^2 \to \tau^{-1}(V)$ be a local trivialization of τ and let Z_j be the vector field tangent to the fibres of the trivial

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bundle $\pi : V \times T^2 \to V$ given by $\psi_* Z_j = Y_j$, $1 \le j \le 2$. We may assume that

$$Z_j = f_{j^1}(x)\frac{\partial}{\partial y_1} + f_{j^2}(x)\frac{\partial}{\partial y_2}, \quad x \in V, 1 \le j \le 2$$

or, in matrix notation,

$$Z = F(x)\frac{\partial}{\partial y}, \tag{5.6}$$

where F is a smooth mapping from V to $G\ell(2, R)$.

If ξ is a closed 1-form, then

$$\xi = g_1\eta_1 + g_2\eta_2 \quad \text{where } Y_1 g_2 = Y_2 g_1. \tag{5.7}$$

Notice that $\xi_0 = \psi^*\xi$ is also d_f-closed, i.e. $dx \wedge d\xi_0 = 0$. Here $\mathcal{F}_0$ is the foliation given by the fibres of π. We claim that

$$H^1(\mathcal{F}_0) = H^1(T^2) \otimes C^\infty(V). \tag{5.8}$$

$\xi_0 = f\,dy_1 + g\,dy_2$ is d_f-closed iff $g_{y_1} = f_{y_2}$ and then, using the Fourier transform, we get $f = r_1(x) + h_{y_2}$ and $g = r_2(x) + h_{y_1}$, which gives

$$\xi_0 = r_1(x)\,dy_1 + r_2(x)\,dy_2 + d_f h, \tag{5.9}$$

where r_1 and r_2 are smooth real functions on T_2 and h is a smooth function on T^4, proving (5.8). If $\rho = \{\rho_1, \rho_2\}$ is the coframe adapted to $Z = \{Z_1, Z_2\}$ given by $\psi^*\rho_j = \eta_j$, $1 \le j \le 2$, then from (5.6) one has

$$dy = {}^tF^{-1}(x)\rho \tag{5.10}$$

in matrix notation. Now from (5.9) and (5.10) we see that ξ_0 can be written as

$$\xi_0 = \bar{s}_1(x)\rho_1 + \bar{s}_2(x)\rho_2 + d_f k, \tag{5.11}$$

where $\bar{s}_j$ are smooth functions on T^2.

Cover T^2 by an open set V_λ as above and let $\{\rho_\lambda\}$ be a smooth partition of unity subordinate to this cover. A routine computation gives

$$\xi^f = (s_1 \circ \tau)\eta_1 + (s_2 \circ \tau)\eta_2 + d_f h, \tag{5.12}$$

where s_j, $1 \le j \le 2$, are smooth functions on T^2 and h is a smooth function on T^4, finishing the proof. □

Remark 5.5. The expression of ξ^f in terms of a given d_f-closed coframe η, as in (5.12) above, is *unique*. If $(s_1 \circ \tau)\eta_1 + (s_2 \circ \tau)\eta_2 = d_f h$ then integration along the fibre of the bundle τ gives

$$s_j(x)\,dx \fint \Delta\eta = 0, \quad 1 \le j \le 2,$$

where $\Delta\eta = \eta_1 \wedge \eta_2$ is the corresponding volume form on the fibres of τ. Thus $s_j = 0$, $1 \le j \le 2$.

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6. *An adapted version of the isotopy theorem of Moser*

In this paper, we make use of an adapted version of a theorem due to Moser [**19**].

Let $T^n = T^p \times T^q$ and denote by (x, y), $x \in T^p$, $y \in T^q$, the points of the torus T^n. As before, we denote by $dx = dx_1 \wedge \cdots \wedge dx_p$ the standard p-form of T^p.

Let $\mathcal{F}$ be the foliation of T^n defined by dx. The *foliated divergence* of a smooth function $F : T^n \to R^q$ is the real function $\mathrm{Div}_f\, F$ on T^n given by

$$\mathrm{Div}_f\, F = F^1_{y_1} + \cdots + F^q_{y_q}, \tag{6.1}$$

F^j, $1 \leq j \leq q$, being the coordinate functions of F.

Let $\Omega_0 = \pm dx \wedge dy$, where $dx \wedge dy$ is the standard volume form of T^n.

We have the following.

THEOREM 6.1. *Suppose $\Omega = \Omega_0 + d\sigma$ is a volume form on the torus T^n such that $\Omega = (1 + \mathrm{Div}_f\, F)\Omega_0$ for some smooth function $F : T^n \to R^q$. Then there exists a C^∞ diffeomorphism ψ of T^n, isotopic to the identity such that:*

(i) $\psi^* dx_j = dx_j$, $1 \leq j \leq p$;

(ii) $\psi^* \Omega = \Omega_0$.

Proof. Consider the path $\Omega_t = \Omega_0 + t\, d\sigma$, $0 \leq t \leq 1$, of cohomologous volume forms on T^n. We want to find a path ψ_t of smooth diffeomorphisms of T^n such that $\psi_0 = \mathrm{id}$, $\psi_1 = \psi$ and:

(i) $\psi_t^* dx_j = dx_j$, $1 \leq j \leq p$;

(ii) $\psi_t^* \Omega_t = \Omega_0$.

where (i) and (ii) hold for all t, $0 \leq t \leq 1$.

We now follow the ideal of Moser [**19**]. Instead of determining the path ψ_t directly we look for a vector field $V : R \to \mathcal{X}(T^n)$, from which we obtain ψ_t by solving the differential equation

$$\frac{d}{dt}\psi_t = V_t \circ \psi_t, \tag{6.2}$$

where $\mathcal{X}(T^n)$ is the Fréchet space of all C^∞ vector fields on the torus T^n. To obtain a more convenient expression for (6.2) we use the formula [**15**]

$$\frac{d}{dt}(\psi_t^* \alpha) = -\psi_t^*(L_{V_t}\alpha), \tag{6.3}$$

where $L_{V_t}\alpha$ is the Lie derivative of a differential form $\alpha \in \Lambda(T^n)$ with respect to V_t. This has the advantage of linearizing the problem. Taking

$$V_t = a_t^1 \frac{\partial}{\partial y_1} + \cdots + a_t^q \frac{\partial}{\partial y_q} \tag{6.4}$$

where

$$a_t^j = \frac{F^j}{1 + t\,\mathrm{Div}_f\, F}, \quad 1 \leq j \leq q,$$

for the vector field, we see from (6.1)–(6.3) that

$$\frac{d}{dt}(\psi_t^*\, dx_j) = 0 \quad \text{and} \quad \frac{d}{dt}(\psi_t^* \Omega_t) = 0,$$

for $q \leq j \leq p$, proving the theorem. □

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COROLLARY 6.2. *Let φ be a minimal C^∞ diffeomorphism of T^n such that:*
(i) $\varphi_t^* \, dx_j = dx_j$, $1 \le j \le p$;
(ii) *φ preserves a smooth volume form Ω on T^n, i.e. $\varphi^*\Omega = (\deg\varphi)\Omega$ and Ω is cohomologous to the standard volume form Ω_0.*
Then there exists a C^∞ diffeomorphism ψ of T^n, isotopic to the identity, such that

$$\psi^*\Omega = \Omega_0 \quad \textit{and} \quad \psi^* \, dx_j = dx_j, \quad 1 \le j \le p.$$

Proof. Notice that

$$\Omega = \Omega_0 + d\tau = (1 + f)\Omega_0. \tag{6.6}$$

In view of Theorem 6.1, we have to show that there exists a smooth mapping $F : T^n \to R^q$ such that

$$f = \mathrm{Div}_f \, F. \tag{6.7}$$

By (i), $\pi \circ \varphi = T_\alpha \circ \pi$, where π is the projection of $T^n = T^p \times T^q$ on the first factor and T_α is a minimal translation of the torus T^p. Integrating Ω along the fibres of π and using (ii) we get $h(x + \alpha) = \pm h(x)$, where h is the smooth real function of T^p given by $\int \Omega = h \, dx$, where dx is the standard volume form of T^p. Now, since T_α is minimal, h is a constant function and we may, in addition, assume that $h = 1$. So, from (6.6) we get $\int f \Omega_0 = 0$ which, by a standard argument using Fourier series, implies (6.7), proving the Corollary. □

The crucial problem on c.f. diffeomorphisms is the following.

Problem. Let $\varphi : M \to M$ be a c.f. diffeomorphism of a compact smooth manifold M and $n \in \mathbb{Z} - \{0\}$. Is φ^n also c.f.?

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