CONSERVATION AS TRANSLATION†

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Abstract. Glivenko's theorem says that classical provability of a propositional formula entails intuitionistic provability of the double negation of that formula. This stood right at the beginning of the success story of negative translations, indeed mainly designed for converting classically derivable formulae into intuitionistically derivable ones. We now generalise this approach: simultaneously from double negation to an arbitrary nucleus; from provability in a calculus to an inductively generated abstract consequence relation; and from propositional logic to any set of objects whatsoever. In particular, we give sharp criteria for the generalisation of classical logic to be a conservative extension of the one of intuitionistic logic with double negation.

§1. Introduction. Glivenko's theorem says that, in propositional logic, classical provability of a formula entails intuitionistic provability of the double negation of that formula [45]. This stood right at the beginning of the success story of negative translations, indeed mainly designed for converting classically derivable formulae into intuitionistically derivable ones. Typically, a negative translation is an operation on formulae in predicate logic such that each formula is classically equivalent to its translation and is classically derivable exactly when the translation is intuitionistically derivable. Negative translations have already been put into the context of nuclei [30, 114] or nucleus [30], and have proved to be useful also in computer science [48], set theory [4], arithmetic and analysis [107], eventually contributing to program extraction [101] and proof mining [63]. Double negation over intuitionistic logic is indeed a typical instance of a nucleus [5, 50, 60, 76, 93, 105, 106, 114].[1]

As compared to recent literature on related topics [4, 15, 31, 36, 40, 48, 49, 57, 68, 77, 78, 83, 85, 94, 107, 111, 119], the main purpose of the present work is to

Received: June 19, 2023.
2020 *Mathematics Subject Classification*: Primary 03F50, Secondary 03B55, 03G27.
Key words and phrases: Glivenko, nucleus, entailment relation, negative translation, double negation.

†This note grew out from a chapter of the first author's doctoral thesis [33], and is a revised and extended version of a conference paper [35]. As compared to the latter, the main amendments include: the study of j-homogeneous functions and j-translations; a more informative statement of the Gödel conservation theorem; the new Kolmogorov conservation theorem and its corollary, the Kuroda conservation theorem; and an application to lax logic.

[1] This and other lists of references below are meant rather indicative than exhaustive.

doi:10.1017/S1755020324000170

generalise Glivenko's theorem and the theory of negative translations: simultaneously from double negation to an arbitrary nucleus; from provability in a calculus to an abstract consequence relation; and from propositional logic to any set of objects whatsoever. As pointed out by van den Berg [114] and Escardó–Oliva [29, 30], a generalised version of the Gentzen negative translation for arbitrary nuclei is already known in logic [4], locale theory [60] and topos theory [62]; and van den Berg himself gives a generalisation of the minimal Kuroda negative translation to nuclei in logic. We aim for a deeper insight into these generalisations, dubbed j-translations, by passing to arbitrary sets endowed with abstract consequence relations.[2]

To this end we move to a nucleus j over a Hertz–Tarski consequence relation in the form of a (single-conclusion) entailment relation $\rhd$ à la Scott [14, 103]. Assuming that $\rhd$ is inductively generated by axioms and rules, we propose two natural extensions (§2.2): $\rhd_j$ generalises the provability of double negation, and $\rhd^j$ is inductively defined by adding the generalisation of double negation elimination to the inductive definition of $\rhd$. By their very definitions, $\rhd^j$ satisfies all the axioms and rules of $\rhd$, and $\rhd_j$ satisfies all the axioms of $\rhd$. But when does $\rhd_j$ also satisfy all the rules of $\rhd$? Glivenko conservation, Corollary 3.7, says that $\rhd^j$ extends $\rhd_j$, and that the two relations coincide precisely when $\rhd_j$ is closed under the non-axiom rules used to inductively generate $\rhd$, which of course is the case whenever there are no such non-axiom rules (Corollary 3.8). While it may happen that this closure condition fails, and hence $\rhd^j$ is not conservative over $\rhd_j$, following Gödel one can ensure conservation by adding to $\rhd$ suitable rules generalising the double negation shift (Theorem 3.5).

We then investigate into a different generalisation by weakening the conditions on the function: instead of a nucleus j, we consider a j-homogeneous function k and the related entailment relation $\rhd_{(k)}$ which generalise the Kolmogorov negative translation and its provability. Similarly to the above, we notice that $\rhd^j$ extends $\rhd_{(k)}$, and that the two relations coincide precisely when $\rhd_{(k)}$ is closed under the rules that are used to inductively generate $\rhd$ (Theorem 3.13). A variant of this last result is shown for $k = jJ$—which is intended to generalise the Kuroda negative translation—where J is j-homogeneous (Corollary 3.14).

In logic, the prime instance of course is Glivenko's theorem (Proposition 5.4) as a syntactical conservation theorem (see also [36]):

$$\Gamma \rhd_c \varphi \iff \Gamma \rhd_i \neg\neg\varphi,$$

where $\rhd_c$ and $\rhd_i$ denote classical and intuitionistic propositional logic. Simultaneously we re-obtain Gödel's theorem [46], otherwise ascribed to Gabbay [39] (Proposition 5.7), which says that, in predicate logic,

$$\Gamma \rhd_c \varphi \iff \Gamma \rhd_* \neg\neg\varphi,$$

where $\rhd_*$ is any extension (by additional axioms) of intuitionistic predicate logic that satisfies the double negation shift:

$$\forall x \neg\neg\varphi \rhd \neg\neg\forall x\varphi.$$

[2] Most recently van den Berg [115] has put forward a theory of nuclei for Miquel's implicative algebras [72].

Next, we regain Kolmogorov's result [64] (Proposition 5.8(i)) that, in predicate logic,

$$\Gamma \rhd_c \varphi \iff k\Gamma \rhd_i k\varphi,$$

where k is the Kolmogorov negative translation inductively defined by

$$\begin{array}{lll} k\top = \neg\neg\top, & k\bot = \neg\neg\bot, & kP = \neg\neg P, \\ k(\varphi \wedge \psi) = \neg\neg(k\varphi \wedge k\psi), & k(\varphi \vee \psi) = \neg\neg(k\varphi \vee k\psi), & k(\varphi \to \psi) = \neg\neg(k\varphi \to k\psi), \\ k(\forall x\, \varphi) = \neg\neg\forall x\, k\varphi, & k(\exists x\, \varphi) = \neg\neg\exists x\, k\varphi. & \end{array}$$

Analogous statements can be made for the Gentzen [41, 42] and for the minimal Kuroda [66, 73, 114] negative translation (Proposition 5.8(ii) and (iii)).

While the double negation nucleus $j\varphi = \neg\neg\varphi$ is an instance of the continuation nucleus, it is tantamount to the same case $j\varphi = \neg\varphi \to \varphi$ of the Peirce nucleus [30]. What does our main result mean for other nuclei in logic? The open nucleus $j\varphi = A \to \varphi$ prompts a form of the deduction theorem (Proposition 5.11(i)), while the closed nucleus $j\varphi = \varphi \vee A$ yields a variant of the reduction from intuitionistic to minimal logic going back to Johansson [59] (Proposition 5.11(ii)). Last but not least, by observing that the modal operator $\bigcirc$ in propositional lax logic is a nucleus, we get a general version of strong conservativity [32] (Proposition 5.13).

1.1. Preliminaries. We proceed in a constructive and predicative way, keeping the concepts elementary and the proofs direct. If a formal system is desired, our work can be placed in a suitable fragment of Aczel's *Constructive Zermelo–Fraenkel Set Theory* (*CZF*) [1–3, 6, 7] based on intuitionistic first-order predicate logic.

By a *finite set* we understand a set that can be written as $\{a_1, \dots, a_n\}$ for some $n \geqslant 0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We thus deviate from the terminology prevalent in constructive mathematics and set theory [6, 7, 11–13, 69, 71]: to reserve the term 'finite' to sets which are in *bijection* with $\{1, \dots, n\}$ for a necessarily unique $n \geq 0$. Those exactly are the sets which are finite in our sense and are *discrete* too, i.e., have decidable equality [71].

§2. Entailment relations. Closely following [35, 89, 90] we briefly recall the basics of entailment relations. Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. Once abstracted from the context of logical formulae, all but one of Tarski's axioms of *consequence* [109][3] can be put as

$$\frac{U \ni a}{U \rhd a} \qquad \frac{\forall b \in U (V \rhd b) \qquad U \rhd a}{V \rhd a} \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}\,,$$

where $U, V \subseteq S$ and $a \in S$. These axioms also characterise a finitary covering or Stone covering in formal topology [95];[4] see further [17, 18, 75, 76, 96, 97]. The notion of consequence has presumably been described first by Hertz [51–53] (see also [9, 67]).

[3] Tarski has further required that S be countable.

[4] This is from where we have taken the symbol $\rhd$, used also [16, 116] to denote a 'consecution' [87].

Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ of a relation $\rhd$ as above where

$$U^{\rhd} = \{a \in S : U \rhd a\}.$$

Rather than with Tarski's notion, we henceforth work with its (tantamount) restriction to finite subsets, i.e., a *(single-conclusion) entailment relation*.[5] This is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ such that

$$\frac{}{U, a \rhd a}\,\mathrm{Refl} \qquad \frac{V \rhd b \qquad V', b \rhd a}{V, V' \rhd a}\,\mathrm{Trans} \qquad \frac{U \rhd a}{U, U' \rhd a}\,\mathrm{Mono}$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V = U \cup V$ and $V, b = V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we reserve the letters $U, V, W, \ldots$; we sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$.

REMARK 2.1. *The rule Refl is equivalent, by Mono, to the axiom* $a \rhd a$.

Redefining

$$T^{\rhd} = \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\}$$

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$. By writing $T \rhd a$ in place of $a \in T^{\rhd}$, the entailment relations thus correspond exactly to the relations satisfying Tarski's axioms above.

Given an entailment relation $\rhd$, by setting $a \leqslant b \equiv a \rhd b$ we get a preorder on S; whence the conjunction $a \approx b$ of $a \leqslant b$ and $b \leqslant a$ is an equivalence relation.

Quite often an entailment relation $\rhd$ is inductively generated from axioms by closing up with respect to the three *structural rules* Refl, Mono and Trans above [92]. In the present note some leeway is required by allowing for rules other than Refl, Mono and Trans in the inductive generation of $\rhd$. These *non-structural rules* have the form[6]

$$\frac{U_1 \rhd b_1 \quad \ldots \quad U_n \rhd b_n}{U \rhd b}\,r \qquad \text{or, more compactly,} \qquad \frac{\{U_i \rhd b_i : i \leqslant n\}}{U \rhd b}\,r.$$

As an axiom is nothing but a rule with no premiss, we explicitly use *non-axiom rule* to indicate a rule that has at least one premiss, whereas in general we do not excluded that a *rule* be an axiom. If the three structural rules are the only non-axiom rules employed for inductively generating an entailment relation $\rhd$, we stress this by saying that $\rhd$ is *generated only by axioms*.

Given an inductively generated entailment relation $\rhd$ and a set of rules $*$, then we call $\rhd$ *plus* $*$ the entailment relation $\rhd_*$ inductively generated by all the rules that either are used for generating $\rhd$ or belong to $*$. We also say that $\rhd_*$ is an *inductive extension* of $\rhd$, and we call $*$ the set of *additional rules*.

5 In the present note there is no need for abstract *multi-conclusion* consequence or entailment à la Scott [102–104], Lorenzen's contributions to which are currently under scrutiny [24, 25]. The relevance of multi-conclusion entailment to constructive algebra, point-free topology, etc. has been pointed out in [14], and has widely been used, e.g., in [19–23, 26, 27, 69, 80, 88–91, 98, 99, 117, 118].

6 Examples are logical, mathematical and modal rules.

A main feature of inductive generation is that if $\rhd$ is an entailment relation generated inductively by certain rules, then $\rhd \subseteq \rhd'$ for every entailment relation $\rhd'$ satisfying those rules. By an *extension* $\rhd'$ of an entailment relation $\rhd$ we mean in general an entailment relation $\rhd'$ such that $\rhd \subseteq \rhd'$. We say that an extension $\rhd'$ of $\rhd$ is *conservative* if also $\rhd \supseteq \rhd'$ and thus $\rhd = \rhd'$ altogether [36, 89, 90].

2.1. Nuclei over entailment relations. In the context of an entailment relation, what do we mean by a nucleus?

DEFINITION 2.2. *Given a set S endowed with an entailment relation $\rhd$, we say that a function $j\colon S \to S$ is a* nucleus (over $\rhd$) *if for all $a, b \in S$ and $U \in \mathrm{Fin}(S)$ the following hold*:

$$\frac{U, a \rhd jb}{U, ja \rhd jb}\,Lj \qquad\qquad \frac{U \rhd b}{U \rhd jb}\,Rj\,.$$

REMARK 2.3.

(i) *By Refl and Trans, the rule Rj can be expressed by an axiom, viz.*

$$b \rhd jb. \tag{1}$$

(ii) *By Lj, from $ja \rhd ja$ we get $j^2a \rhd ja$ and thus $j^2a \approx ja$.*

(iii) *The rule Rj is tantamount to the inverse of Lj. In particular, we have*

$$\frac{U \rhd jb}{\overline{jU \rhd jb}}\,,$$

where $jU = \{ju\colon u \in U\}$ and the double line stands for equivalence. This characterisation is already known in particular cases, e.g., the following (iv).

(iv) *If we consider a structure with a* relative pseudo-complement *operator, i.e., a binary function $\to$ that satisfies*

$$\frac{U, a \rhd b}{\overline{U \rhd a \to b}}\,RPC\,,$$

then it is easy to show that nuclei are characterised by a single equivalence:

$$a \to jb \approx ja \to jb. \tag{2}$$

This was observed, e.g., in [61, 62, 70] *for nuclei in locale theory. In fact, by means of Trans and RPC, (2) is another way of writing Lj and its inverse, which, as noticed in (ii), is tantamount to Rj. Moreover, the following hold by means of RPC, Lj and Rj*:

$$j(a \to b) \rhd a \to jb, \tag{3}$$

$$j(a \to b) \rhd ja \to jb. \tag{4}$$

Note that (3) and (4) are equivalent in view of (2). As we will see in the applications (§5), the converses of (3) and (4) do not hold in general.

EXAMPLE 2.4. *The above notion of a nucleus includes as a special case the notion of a nucleus over a locale* [5, 58, 60–62, 76, 86, 93, 105, 106], *which is well-known as a* *point-free way to put subspaces. In fact, if S is a locale with partial order* $\leqslant$, *then*

$$U \rhd a \iff \bigwedge U \leqslant a$$

defines an entailment relation [27] *such that any given map* $j : S \to S$ *is a nucleus over* $\rhd$ *precisely when* j *is a nucleus over the locale* S. *The latter means that j satisfies*

$$ja \wedge jb \leqslant j(a \wedge b) \tag{5}$$

on top of the conditions for j being a closure operator on S, which can be put as $a \leqslant ja$ *and*

$$a \leqslant jb \implies ja \leqslant jb\,. \tag{6}$$

In the presence of $a \leqslant ja$, *which corresponds to Rj in the form of* (1), *the conjunction of* (5) *and* (6) *is equivalent to*

$$c \wedge a \leqslant jb \implies c \wedge ja \leqslant jb\,,$$

which in turn subsumes Lj. So the two notions of a nucleus coincide.

EXAMPLE 2.5.

(i) *Every entailment relation* $\rhd$ *has the trivial nucleus* $j = \mathrm{id}$.
(ii) *Let an algebraic structure* $\mathbf{S}$ *come with a unary self-inverse function j (e.g., take a group as* $\mathbf{S}$ *and the inverse as j). The entailment relation* $\rhd$ *of* $\mathbf{S}$*-substructures is inductively defined by*

$$a_1, \ldots, a_n \rhd f(a_1, \ldots, a_n) \tag{7}$$

for every n*-ary function* f *characteristic of* $\mathbf{S}$, *including j. Then j is a nucleus over* $\rhd$. *Axiom* (1) *is just* (7) *for* $f = j$, *therefore the rule Rj holds. In particular,* $j^2 = \mathrm{id}$ *implies* $j(a) \rhd a$, *which, together with Trans, gives the rule Lj.*
(iii) *Double negation* $\neg\neg$ *is a nucleus over intuitionistic logic* $\rhd_i$ *as an entailment relation between propositional or first-order predicate formulae (see* §5.2 *for further details).*

2.2. Extensions induced by nuclei. Every nucleus over $\rhd$ induces two natural extensions of $\rhd$ as follows.

DEFINITION 2.6. *Let j be a nucleus over an entailment relation* $\rhd$ *on a set S. We understand by:*

– *the* weak j-extension *(or* Kleisli extension*) of* $\rhd$ *the relation* $\rhd_j \subseteq \mathrm{Fin}(S) \times S$ *defined by*

$$U \rhd_j a \iff U \rhd ja;$$

– *the* strong j-extension *of* $\rhd$ *the entailment relation* $\rhd^j \subseteq \mathrm{Fin}(S) \times S$ *inductively generated by the rules of* $\rhd$ *together with the* stability axiom *for j:*

$$ja \rhd^j a. \tag{8}$$

In the terminology coined before, $\rhd^j$ *is nothing but* $\rhd$ *plus the stability axiom for j.*

As we will see later on (Corollary 3.7),

$$\rhd \subseteq \rhd_j \subseteq \rhd^j.$$

LEMMA 2.7. *Let S be a set with an entailment relation $\rhd$ and let j be a nucleus over $\rhd$. Then both $\rhd^j$ and $\rhd_j$ are entailment relations that extend $\rhd$.*

Proof. The statement for $\rhd^j$ holds by the very definition of $\rhd^j$. As for $\rhd_j$: By (1) and Remark 2.1, Refl carries over to $\rhd$ from $\rhd_j$. Mono is inherited from $\rhd$, and so is Trans in view of Lj:

$$\dfrac{U \rhd ja \qquad \dfrac{V, a \rhd jb}{V, ja \rhd jb}\,\mathrm{L}j}{U, V \rhd jb}\,\mathrm{Trans}\ .$$

Finally, also $\rhd \subseteq \rhd_j$ is a consequence of (1). □

REMARK 2.8.

(i) *Stability and Rj together give $a \approx^j ja$, where $\approx^j$ is the intersection of $\rhd^j$ and its converse on singletons.*
(ii) *By Refl in the form of $a \rhd a$ (Remark 2.1), stability holds for $\rhd_j$ too, that is, $ja \rhd_j a$.*

Under appropriate circumstances, Remark 2.8 will help to obtain $\rhd^j \subseteq \rhd_j$ (see Corollaries 3.7 and 3.8).

REMARK 2.9. *The nucleus j on $\rhd$ is a nucleus also on $\rhd_j$ and $\rhd^j$. In fact, by Lemma 2.7 both extensions inherit axiom (1) from $\rhd$, and actually satisfy the following strengthening of Lj:*

$$\dfrac{U, a \rhd b}{U, ja \rhd b}\,Lj^+\ .$$

While Lj^+ for $\rhd_j$ is just Lj for $\rhd$, stability $ja \rhd a$ is tantamount to Lj^+ for any entailment relation $\rhd$ whatsoever.

To better understand whether and when $\rhd_j$ coincides with $\rhd^j$, we first study a concrete example.

EXAMPLE 2.10. *Consider deduction in minimal propositional logic $\rhd_m$ with the* closed nucleus $c_\bot : \varphi \mapsto \varphi \vee \bot$ *(see §5.3 for details). This $\rhd_m$ is inductively generated by certain axioms plus the rule*

$$\dfrac{\Gamma, \varphi \rhd_m \psi}{\Gamma \rhd_m \varphi \to \psi}\,R{\to}$$

which cannot be expressed as an axiom. By its very definition, $\rhd_m^{c_\bot}$ too satisfies $R\to$. Does also $\rhd_{mc_\bot}$ satisfy this rule? If this were the case, then by definition of $\rhd_{mc_\bot}$ we would have

$$\dfrac{\Gamma, \varphi \rhd_m \psi \vee \bot}{\Gamma \rhd_m (\varphi \to \psi) \vee \bot}\ .$$

As $\bot \rhd_m \psi \vee \bot$, we would obtain $\rhd_m (\bot \to \psi) \vee \bot$. However, since minimal logic has the disjunction property and neither disjunct is provable in general, this cannot be the case. So $\rhd_{c_\bot}$ does not satisfy the rule R→.

The moral of Example 2.10 is that $\rhd$ may already have non-axiom rules, such as R→, which carry over to $\rhd^j$ by its very definition, and thus need to hold in $\rhd_j$ too for the former to be conservative over the latter. To deal with this issue, we introduce the following concept.

DEFINITION 2.11. *We say that a rule r that holds for $\rhd$ is* compatible *with j over $\rhd$ if r also holds for $\rhd_j$. We usually omit "over j" if this is clear from the context.*

REMARK 2.12.

(i) *Refl, Mono, Trans are compatible with every nucleus j, by Lemma 2.7.*
(ii) *Every composition r of compatible rules is compatible. In fact, the derivation that gives r in $\rhd$ can be translated smoothly into $\rhd_j$, as all the applied rules are compatible.*
This is very useful: to check compatibility for all the rules of an entailment relation $\rhd$, it suffices to check compatibility for any set of rules that generate $\rhd$.
(iii) *Each axiom $a_1, \dots, a_n \rhd b$ can be viewed as a rule with no premiss, and as such is compatible with any given nucleus j, simply by Rj. It follows that, if an entailment relation $\rhd$ is generated only by axioms, then every rule that holds for $\rhd$ is compatible with any nucleus j over $\rhd$.*

2.3. Homogeneous functions and translations. Throughout this subsection, let j be a nucleus over an entailment relation $\rhd$ on a set S.

DEFINITION 2.13. *We say that a function $k : S \to S$ is:*

– *j-*homogeneous (from $\rhd^j$ to $\rhd$) *if k satisfies the following two conditions:*

$$kja \rhd ka, \tag{9}$$

$$a \approx^j ka. \tag{10}$$

– *A j-*translation (from $\rhd^j$ to $\rhd$) *if k satisfies (10) and*

$$\frac{U \rhd^j b}{kU \rhd kb}\ . \tag{11}$$

REMARK 2.14.

(i) *Condition (10) implies the converse of (11), that is,*

$$\frac{kU \rhd kb}{U \rhd^j b}\ .$$

In fact, since $\rhd^j$ extends $\rhd$, we have that $kU \rhd kb$ implies $kU \rhd^j kb$, which is equivalent to $U \rhd^j b$ by means of $a \approx^j ka$.
(ii) *The nucleus j is j-homogeneous: it follows immediately from $j^2a \rhd ja$ (Remark 2.3(ii)) and $a \approx^j ja$ (Remark 2.8).*
(iii) *Every j-translation is j-homogeneous: by applying (11) to (8) we get (9).*

(iv) *The nucleus j is a j translation precisely when*

$$\frac{U \rhd^j b}{jU \rhd jb}\ .$$

(v) *We have seen that j-translations are j-homogeneous. We want to stress that the two notions are distinct via a counterexample. The closed nucleus* $c_\bot : \varphi \mapsto \varphi \vee \bot$ *over minimal propositional logic* $\rhd_m$ *is* $c_\bot$*-homogeneous by (ii) but it is not a* $c_\bot$*-translation: this is a direct consequence of Corollary 3.7 and the fact that* $R{\to}$ *is not compatible with* $c_\bot$ *(see Example 2.10).*

(vi) *Every j-translation k that satisfies Rk also satisfies Lk. In fact:*

$$\dfrac{\dfrac{\dfrac{\dfrac{a, U \rhd kb}{a, U \rhd^j kb}\,\textit{extension} \qquad \dfrac{}{kb \rhd^j b}\,(11)}{a, U \rhd^j b}\,\textit{Trans}}{ka, kU \rhd kb}\,(10)}{ka, U \rhd kb}\,*\qquad .$$

Where $(*)$ *is subsequent applications of Trans with*

$$\dfrac{\dfrac{}{u \rhd u}\,\textit{Refl}}{u \rhd ku}\,Rk$$

for each $u \in U$*. It follows that a j-translation k is a nucleus if and only if it satisfies Rk.*

§3. Conservation as translation.

We next present a few *conservation theorems for nuclei*, typically giving necessary and sufficient conditions. Throughout this section, let S be a set with an entailment relation $\rhd$ inductively generated by rules, and let j be a nucleus over $\rhd$.

3.1. Glivenko and Gödel conservation.

DEFINITION 3.1. *Given a rule*

$$\frac{\{U_i \rhd b_i : i \leqslant n\}}{U \rhd b}\,r\ ,$$

to obtain its j-version r_j *we put* j *in front of every consequent:*

$$\frac{\{U_i \rhd jb_i : i \leqslant n\}}{U \rhd jb}\,r_j\ .$$

REMARK 3.2.

(i) *By definition of* $\rhd_j$*, the j-version* r_j *holds for* $\rhd$ *if and only if the original rule r holds for* $\rhd_j$*. This means that if r holds for* $\rhd$*, then r is compatible with j over* $\rhd$ *if and only if* r_j *holds for* $\rhd$*.*

(ii) *By* $j^2 a \approx ja$ *(Remark 2.3(iii)),* r_{jj} *holds for* $\rhd$ *if and only if* r_j *holds for* $\rhd$*. In particular,* r_j *is compatible with j over* $\rhd$ *if and only if* r_j *holds for* $\rhd$*.*

(iii) *By* $ja \approx^j a$ *(Remark 2.8), if r holds for* $\rhd$*, then* r_j *holds for* $\rhd^j$*.*

Definition 3.3. *Given an entailment relation $\rhd$ generated by rules, let its* intermediate j-extension $\rhd_{\langle j \rangle}$ *be $\rhd$ plus the j-version r_j of all the rules r in the inductive definition of $\rhd$.*

Remark 3.4.

(i) *We have $\rhd \subseteq \rhd_{\langle j \rangle} \subseteq \rhd^j$ by Remark 3.2(iii).*
(ii) *By Remarks 2.12 and 3.2(i), to obtain $\rhd_{\langle j \rangle}$ it suffices to add the j-version of all the non-axiom rules in the inductive definition of $\rhd$ that are not already compatible with j.*

Theorem 3.5 (Gödel conservation). *Let $\rhd_*$ be $\rhd$ plus additional rules that hold for $\rhd^j$ and such that j is a nucleus also over $\rhd_*$. Then $\rhd_*^j = \rhd^j$; in particular, $\rhd^j$ extends $\rhd_{*j}$, that is, $\rhd_{*j} \subseteq \rhd^j$. Moreover, the following are equivalent:*

(a) *$\rhd^j$ is conservative over $\rhd_{*j}$, that is, $\rhd^j \subseteq \rhd_{*j}$;*
(b) *the nucleus j is a j-translation from $\rhd^j$ to $\rhd_*$;*
(c) *$\rhd_{*j}$ is an inductive extension of $\rhd_*$;*
(d) *all the non-axiom rules that generate $\rhd_*$ are compatible with j over $\rhd_*$, that is, they hold for $\rhd_{*j}$;*
(e) *$\rhd_*$ is an extension of $\rhd_{\langle j \rangle}$, that is, $\rhd_{\langle j \rangle} \subseteq \rhd_*$.*

Proof. First, note that $\rhd \subseteq \rhd_* \subseteq \rhd^j$. Since $\rhd \subseteq \rhd_*$, we have $\rhd^j \subseteq \rhd_*^j$. On the other hand, as stability and all the rules of $\rhd_*$ hold for $\rhd^j$, we also have $\rhd_*^j \subseteq \rhd^j$. Therefore $\rhd_*^j = \rhd^j$.

Since $a \approx_*^j ja$ by Remark 2.8(i), (b) is tantamount to

$$\frac{U \rhd^j b}{jU \rhd_* jb}$$

which by the inverse of Lj (see Remark 2.3(iii)) is equivalent to (a).

As $\rhd^j$ is an inductive extension of $\rhd_*$, (c) follows from (a); and (c) trivially entails (d).

To deduce (a) from (d), assume (d) and consider one by one the rules that generate $\rhd^j$: Refl, Mono, Trans hold for $\rhd_j$ since $\rhd_j$ is an entailment relation by Lemma 2.7; stability (8) holds for $\rhd_j$ by Remark 2.8(i); all the rules that generate $\rhd_*$ hold for $\rhd_j$ since they are either compatible with j by assumption (d), or axioms and thus compatible with j by Remark 2.12(iii). As $\rhd^j$ is the smallest extension of $\rhd_*$ satisfying all these rules, we obtain (a).

By Remark 3.2(ii) with $\rhd_{\langle j \rangle}$ in place of $\rhd$, (d) and thus (a) hold with $\langle j \rangle$ in place of $*$. So, if (e), then

$$\rhd^j = \rhd_{\langle j \rangle j} \subseteq \rhd_{*j} \subseteq \rhd^j$$

(see also Figure 1); whence (a).

To obtain (e) from (d), suppose (d) and let r be a rule in the inductive definition of $\rhd$. By (d), r holds for $\rhd_{*j}$ which, by Remark 3.2(i) with $\rhd_*$ in place of $\rhd$ means that r_j holds for $\rhd_*$. Hence all the rules of $\rhd_{\langle j \rangle}$ hold for $\rhd_*$, and thus (e). □

Remark 3.6. *Let $\rhd_*$ be an inductive extension of $\rhd$.*

(i) *For j to be a nucleus over $\rhd_*$ too, it is sufficient to check Lj. In fact, Rj is tantamount to (1), which as an axiom is inherited by $\rhd_*$ from $\rhd$ by the very definition of extension.*

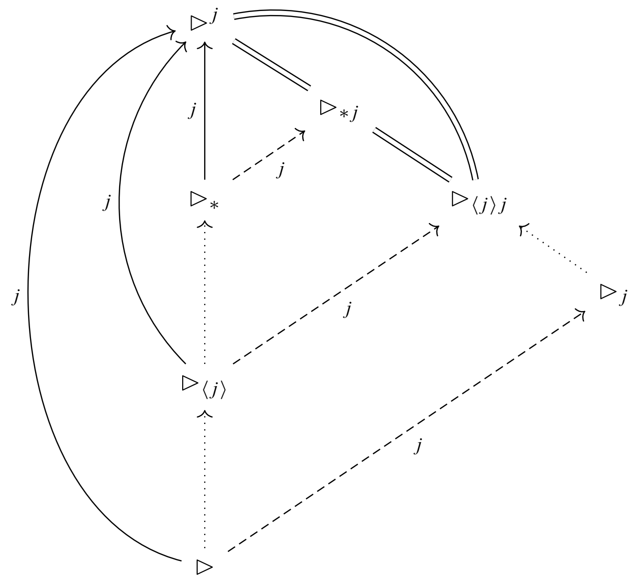

Figure 1. Diagram of the entailment relations involved in the situation of Theorem 3.5. A solid arrow denotes a strong j-extension, a dashed arrow denotes a weak j-extension, a dotted arrow denotes a generic extension, and a double line denotes a conservative extension. The intuition is that, if the outer triangle does not satisfy the desired properties, then we can move to an inner triangle that works.

(ii) *Lj is trivially satisfied whenever* $* = \emptyset$.
(iii) *If Lj can be proved admissible for* $\rhd$ *by exclusively composing the generating rules of* $\rhd$, *then the same composition equally shows that Lj is admissible for* $\rhd_*$.

COROLLARY 3.7 (Glivenko conservation). $\rhd^j$ *extends* $\rhd_j$, *that is,* $\rhd_j \subseteq \rhd^j$; *and the following are equivalent*:

(a) $\rhd^j$ *is conservative over* $\rhd_j$, *that is,* $\rhd^j \subseteq \rhd_j$;
(b) *the nucleus j is a j-translation from* $\rhd^j$ *to* $\rhd$;
(c) $\rhd_j$ *is an inductive extension of* $\rhd$;
(d) *all the non-axiom rules that generate* $\rhd$ *are compatible with j, that is, they hold for* $\rhd_j$;
(e) $\rhd_{\langle j \rangle}$ *is conservative over* $\rhd$, *that is,* $\rhd_{\langle j \rangle} \subseteq \rhd$.

Proof. This is Theorem 3.5 with $\rhd_* = \rhd$. □

COROLLARY 3.8. *If* $\rhd$ *is inductively generated only by axioms, then j is a j-translation and* $\rhd^j$ *is a conservative extension of* $\rhd_j$, *that is,* $\rhd^j = \rhd_j$.

3.2. Kolmogorov and Kuroda conservation. We now generalise Corollary 3.7 from the nucleus j to a j-homogeneous function.

DEFINITION 3.9. *For the given nucleus j over an entailment relation* $\rhd$, *and any function* $k : S \to S$ *whatsoever, we define the relation* $\rhd_{(k)} \subseteq \mathrm{Fin}(S) \times S$ *by*

$$U \rhd_{(k)} a \iff kU \rhd ka.$$

DEFINITION 3.10. *Given a rule r that holds for* $\rhd$, *we say that*:

- r *is* Kolmogorov compatible *with* k (*over* $\rhd$) *if it also holds for* $\rhd_{(k)}$,
- r *is* j-Kuroda compatible *with* k (*over* $\rhd$) *if it also holds for* $\rhd_{j(k)}$.

We thus have an analogue of Remark 2.12:

REMARK 3.11.

(i) *Refl, Mono, Trans are trivially Kolmogorov compatible with* k. *This means that* $\rhd_{(k)}$ *is an entailment relation; however in general it is not an extension of the original* $\rhd$.
(ii) *Compatibility entails j-Kuroda compatibility, hence Refl, Mono, Trans are Kuroda compatible with k by Remark 2.12.*
(iii) *Every composition r of Kolmogorov (resp. j-Kuroda) compatible rules is Kolmogorov (resp. j-Kuroda) compatible. In fact, the derivation that gives r in* $\rhd$ *can be translated smoothly into* $\rhd_{(k)}$ (*resp.* $\rhd_{j(k)}$), *as all the applied rules are Kolmogorov (resp. j-Kuroda) compatible.*
This means that if we want to check Kolmogorov (resp. j-Kuroda) compatibility for all the rules of an entailment relation $\rhd$, *it suffices to check it for any set of rules that generate* $\rhd$.

REMARK 3.12. *Remark 2.3(iii) yields* $\rhd_{(jk)} = \rhd_{j(k)}$ *and* $\rhd_{(j)} = \rhd_j$.

THEOREM 3.13 (Kolmogorov conservation). *Let* $k : S \to S$ *be j-homogeneous. Then* $\rhd^j$ *extends* $\rhd_{(k)}$, *that is,* $\rhd_{(k)} \subseteq \rhd^j$; *and the following are equivalent*:

(a) $\rhd^j$ *is conservative over* $\rhd_{(k)}$, *that is,* $\rhd^j \subseteq \rhd_{(k)}$;
(b) *the j-homogeneous function k is a j-translation*;
(c) $\rhd_{(k)}$ *is an inductive extension of* $\rhd$;
(d) *all the rules in the inductive definition of* $\rhd$ *are Kolmogorov compatible with* k, *that is, they hold for* $\rhd_{(k)}$.

Proof. The fact that $\rhd_{(k)} \subseteq \rhd^j$ is just Remark 2.14(i). The implications from (a) to (c) and from (c) to (d) hold just as in Theorem 3.5. Suppose that all the non-structural rules in the inductive definition of $\rhd$ are Kolmogorov compatible with k, and that $U \rhd^j b$. We show that $kU \rhd kb$ by induction on the derivation of $U \rhd^j b$: the cases involving structural rules are trivial; the case of the stability axiom $ja \rhd^j a$ is tantamount to $kja \rhd ka$, which holds since k is j-homogeneous; consider the case in which $U \rhd^j b$ is derived from a non-structural rule r in the inductive definition of $\rhd$, i.e.,

$$\frac{\{U_i \rhd^j b_i : i \leqslant n\}}{U \rhd^j b}\, r \quad ,$$

then

$$\cfrac{\cfrac{U_i \rhd^j b_i}{\{kU_i \rhd kb_i : i \leqslant n\}}\text{ induction hypothesis}}{kU \rhd kb}\, r \qquad ,$$

where r can be applied because of the Kolmogorov compatibility. This proves that (d) implies (b). Finally, the fact that (b) implies (a) is just (11). □

By setting $k = j$ in Theorem 3.13, by Remark 3.12 we have $\rhd_{(j)} = \rhd_j$, which in particular means that compatibility and Kolmogorov compatibility are equivalent. Therefore Corollary 3.7 except for condition (e) is a special case of Theorem 3.13.

COROLLARY 3.14 (Kuroda conservation). *Let S be a set with an entailment relation $\rhd$ inductively generated by rules, and let j be a nucleus over $\rhd$. Let $J : S \to S$ be j-homogeneous. Then $\rhd^j$ is an extension of $\rhd_{j(J)}$, that is, $\rhd_{j(J)} \subseteq \rhd^j$; and the following are equivalent*:

(a) *$\rhd^j$ is conservative over $\rhd_{j(J)}$, that is, $\rhd^j \subseteq \rhd_{j(J)}$;*
(b) *the function jJ is a j-translation;*
(c) *$\rhd_{j(J)}$ is an inductive extension of $\rhd$;*
(d) *all the non-structural rules in the inductive definition of $\rhd$ are Kuroda j-compatible with J, that is, they hold for $\rhd_{j(J)}$.*

Proof. First, note that if J is j-homogeneous, then so is $k = jJ$. Then the claim follows immediately from Theorem 3.13 and Remark 3.12. □

§4. Logic as entailment. We now want to see how the conservation results that we have just proved apply to logic.

Throughout this section and the following one, if not stated otherwise, the overall assumption is that S is a set of propositional formulae containing $\top, \bot$ and closed under the connectives $\vee, \wedge, \to$.[7] By *minimal (propositional) logic* $\rhd_m$ we mean the fragment of propositional intuitionistic logic without the principle of *ex falso sequitur quodlibet*. More precisely, we define $\rhd_m$ as the least entailment relation $\rhd$ that satisfies the *deduction theorem*

$$\frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \to \psi}\,\mathrm{R}{\to}$$

and the following axioms:

$$\begin{array}{lll} \varphi, \psi \rhd \varphi \wedge \psi, & \varphi \wedge \psi \rhd \varphi, & \varphi \wedge \psi \rhd \psi, \\ \varphi \rhd \varphi \vee \psi, & \psi \rhd \varphi \vee \psi, & \varphi \vee \psi, \varphi \to \delta, \psi \to \delta \rhd \delta, \\ \varphi, \varphi \to \psi \rhd \psi, & & \\ \rhd \top. & & \end{array}$$

Of course, we understand this as an inductive definition. In this setting, negation $\neg$ is not given as a primitive operator, but it is rather defined by

$$\neg\varphi = \varphi \to \bot.$$

The above system for minimal logic is equivalent to the **G3**-style calculus in Table 1; meaning that they inductively generate the same entailment relation.

[7] It is worth noting that, while we explicitly talk about logic, everything in this section can easily be transferred into any setting with logic-like operators, such as lattice theory, locale theory [60], topos theory [62] and such.

Table 1. *Sequent calculus-like rules for minimal propositional logic.*

$$\frac{\Gamma, \varphi, \psi \rhd \delta}{\Gamma, \varphi \wedge \psi \rhd \delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}\,\mathrm{R}\wedge$$

$$\frac{\Gamma, \varphi \rhd \delta \qquad \Gamma, \psi \rhd \delta}{\Gamma, \varphi \vee \psi \rhd \delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi}\,\mathrm{R}\vee_1 \qquad \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}\,\mathrm{R}\vee_2$$

$$\frac{\Gamma \rhd \varphi \qquad \Gamma, \psi \rhd \delta}{\Gamma, \varphi \to \psi \rhd \delta}\,\mathrm{L}{\to} \qquad \frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \to \psi}\,\mathrm{R}{\to}$$

$$\frac{}{\Gamma \rhd \top}\,\mathrm{R}\top$$

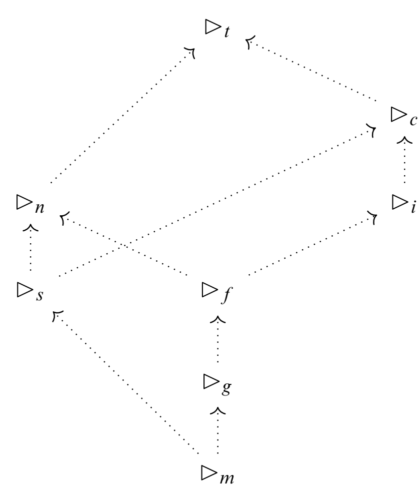

Figure 2. Diagram of the logics introduced in §4, mostly based on [81, figure 4.1].

We define the following inductive extensions of minimal logic $\rhd_m$:

extension	additional axiom	
intuitionistic logic $\rhd_i$	$\bot \rhd \varphi$	(12)
Glivenko logic $\rhd_g$	$\rhd \neg\neg(\bot \to \varphi)$	(14)
Clavius logic $\rhd_s$	$\neg\varphi \to \varphi \rhd \varphi$	(16)
trivial logic $\rhd_t$	$\rhd \varphi$	(18)

extension	additional axiom	
classical logic $\rhd_c$	$\neg\neg\varphi \rhd \varphi$	(13)
Frobenius logic $\rhd_f$	$\rhd \bot \vee (\bot \to \varphi)$	(15)
negative logic $\rhd_n$	$\rhd \bot$	(17)

For a discussion of these and other logics, see [81, 82], from whom we took the names for $\rhd_g$ and $\rhd_n$. See also Figure 2 to see how they are related.

4.1. Predicate and infinitary extensions. Let $\rhd_*$ be an inductive extension of $\rhd_m$. When we say that we work in predicate logic, we mean that we add quantifiers $\forall$ and $\exists$

Table 2. *Sequent calculus-like rules for quantifiers. Rules R∀ and L∃ come with the condition that y has to be fresh.*

$$\frac{\Gamma, \varphi[t/x] \rhd \delta}{\Gamma, \forall x\varphi \rhd \delta}\,\mathrm{L}\forall \qquad \frac{\Gamma \rhd \varphi[y/x]}{\Gamma \rhd \forall x\varphi}\,\mathrm{R}\forall$$

$$\frac{\Gamma, \varphi[y/x] \rhd \delta}{\Gamma, \exists x\varphi \rhd \delta}\,\mathrm{L}\exists \qquad \frac{\Gamma \rhd \varphi[t/x]}{\Gamma \rhd \exists x\varphi}\,\mathrm{R}\exists$$

to the language, and hence require that the formulae in S are closed under $\forall, \exists$ as well. Moreover, we extend the inductive definition of $\rhd_*$ by adding the rule

$$\frac{\Gamma \rhd \varphi[y/x]}{\Gamma \rhd \forall x\varphi}\,\mathrm{R}\forall$$

with the condition that y has to be fresh, and the following axioms:

$$\forall x\varphi \rhd \varphi[t/x],$$
$$\varphi[t/x] \rhd \exists x\varphi,$$
$$\exists x\varphi, \forall x(\varphi \to \delta) \rhd \delta,$$

where the latter comes with the condition that x cannot be free in δ. Over $\rhd_*$, these axioms and rule are equivalent to the **G3**-style rules in Table 2.

The definition of a nucleus j given in [114] requires j to be compatible with substitution, that is,

$$j(\varphi[t/x]) = (j\varphi)[t/x].$$

We prefer not to have this as a general assumption, but to make explicit whenever we need it.

When we say that we work in infinitary logic, we mean that we add infinitary connectives $\bigwedge_{i\in\mathbb{N}}$ and $\bigvee_{i\in\mathbb{N}}$ to the language, and hence require that the formulae in S are closed under $\bigwedge_{i\in\mathbb{N}}, \bigvee_{i\in\mathbb{N}}$ as well. Moreover, we extend the inductive definition of $\rhd_*$ by adding the rule

$$\frac{\{\Gamma \rhd \varphi_n : n \in \mathbb{N}\}}{\Gamma \rhd \bigwedge_{i\in\mathbb{N}} \varphi_i}\,\mathrm{R}\bigwedge$$

and the following axioms:

$$\bigwedge_{i\in\mathbb{N}} \varphi_i \rhd \varphi_n \qquad \text{for every } n \in \mathbb{N},$$
$$\varphi_n \rhd \bigvee_{i\in\mathbb{N}} \varphi_i \qquad \text{for every } n \in \mathbb{N},$$
$$\bigvee_{i\in\mathbb{N}} \varphi_i, \bigwedge_{i\in\mathbb{N}} (\varphi_i \to \delta) \rhd \delta.$$

Over $\rhd_*$, these axioms and rule are equivalent to the **G3**-style rules in Table 3.

Table 3. *Sequent calculus-like rules for quantifiers. For each $n \in \mathbb{N}$ there is one rule $L\bigwedge_n$ and one rule $R\bigvee_n$.*

$$\frac{\Gamma, \varphi_n \rhd \delta}{\Gamma, \bigwedge_{i\in\mathbb{N}} \varphi_i \rhd \delta}\,\mathrm{L}{\textstyle\bigwedge_n} \qquad \frac{\{\Gamma \rhd \varphi_n : n \in \mathbb{N}\}}{\Gamma \rhd \bigwedge_{i\in\mathbb{N}} \varphi_i}\,\mathrm{R}{\textstyle\bigwedge}$$

$$\frac{\{\Gamma, \varphi_n \rhd \delta : n \in \mathbb{N}\}}{\Gamma, \bigvee_{i\in\mathbb{N}} \varphi_i \rhd \delta}\,\mathrm{L}{\textstyle\bigvee} \qquad \frac{\Gamma \rhd \varphi_n}{\Gamma \rhd \bigvee_{i\in\mathbb{N}} \varphi_i}\,\mathrm{R}{\textstyle\bigvee_n}$$

§5. Nuclei in logic. Among the logical rules seen in §4, R$\to$, R$\forall$ and R$\bigwedge$ are the only ones that cannot be expressed as axioms. Therefore, when checking compatibility of rules with j, if we do not add other rules that cannot be expressed as axioms, then the only rules we have to deal with are these three.

LEMMA 5.1 (Compatibility criterion). *Let $\rhd$ be an inductive extension of $\rhd_m$, and let j be a nucleus over $\rhd$.*

(i) *In either propositional, predicate or infinitary logic, R$\to$ is compatible with j if and only if*

$$\varphi \to j\psi \rhd j(\varphi \to \psi).$$

(ii) *In predicate logic, if j is compatible with substitution, that is,*

$$j(\varphi[t/x]) = (j\varphi)[t/x],$$

then R$\forall$ is compatible with j if and only if

$$\forall x j\varphi \rhd j\forall x\varphi.$$

(iii) *In infinitary logic, R$\bigwedge$ is compatible with j if and only if*

$$\bigwedge_{i\in\mathbb{N}} j\varphi_i \rhd j \bigwedge_{i\in\mathbb{N}} \varphi_i.$$

Proof. We only prove (i), as the proof of (ii) and (iii) is similar. Suppose that R$\to$ is compatible with j. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to j\psi, \varphi \rhd j\psi}\,\text{axiom}}{\varphi \to j\psi, \varphi \rhd_j \psi}}{\varphi \to j\psi \rhd_j \varphi \to \psi}\,\mathrm{R}{\to}}{\varphi \to j\psi \rhd j(\varphi \to \psi)}\quad .$$

Now suppose that $\varphi \to j\psi \rhd j(\varphi \to \psi)$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \varphi \rhd_j \psi}{\Gamma, \varphi \rhd j\psi}}{\Gamma \rhd \varphi \to j\psi}\,\mathrm{R}{\to} \qquad \dfrac{}{\varphi \to j\psi \rhd j(\varphi \to \psi)}\,\text{assumption}}{\Gamma \rhd j(\varphi \to \psi)}\,\text{Trans}}{\Gamma \rhd_j \varphi \to \psi}.$$

□

5.1. *Some notable translations in logic.* We now consider predicate logic. We introduce some classes of functions that generalise some well-known negative translations. Let $\rhd$ be either $\rhd_m$ or $\rhd_i$. Given a nucleus j on $\rhd$, we inductively define $k, g, t, J : S \to S$ as follows:

$$\begin{array}{llll}
kP = jP, & gP = jP, & tP = jP, & JP = P, \\
k\top = j\top, & g\top = j\top, & t\top = j\top, & J\top = \top, \\
k\bot = j\bot, & g\bot = j\bot, & t\bot = j\bot, & J\bot = \bot, \\
k(\varphi \wedge \psi) = j(k\varphi \wedge k\psi), & g(\varphi \wedge \psi) = g\varphi \wedge g\psi, & t(\varphi \wedge \psi) = t\varphi \wedge t\psi, & J(\varphi \wedge \psi) = J\varphi \wedge J\psi \\
k(\varphi \vee \psi) = j(k\varphi \vee k\psi), & g(\varphi \vee \psi) = j(g\varphi \vee g\psi), & t(\varphi \vee \psi) = t\varphi \vee t\psi, & J(\varphi \vee \psi) = J\varphi \vee J\psi \\
k(\varphi \to \psi) = j(k\varphi \to k\psi), & g(\varphi \to \psi) = g\varphi \to g\psi, & t(\varphi \to \psi) = t\varphi \to t\psi, & J(\varphi \to \psi) = J\varphi \to jJ\psi, \\
k(\forall x\, \varphi) = j\forall x\, k\varphi, & g(\forall x\, \varphi) = \forall x\, g\varphi, & t(\forall x\, \varphi) = \forall x\, t\varphi, & J(\forall x\, \varphi) = \forall x\, jJ\varphi, \\
k(\exists x\, \varphi) = j\exists x\, k\varphi, & g(\exists x\, \varphi) = j(\exists x\, g\varphi), & t(\exists x\, \varphi) = \exists x\, t\varphi, & J(\exists x\, \varphi) = \exists x\, J\varphi.
\end{array}$$

The *Kolmogorov j-function* k is named after the *Kolmogorov negative translation*, which is k obtained for $j = \neg\neg$, as seen in Proposition 5.8(i). A refined version of the Kolmogorov j-function is the *Gentzen j-function* g, named after the *Gentzen negative translation*, which is g obtained for $j = \neg\neg$, as seen in Proposition 5.8(ii). An even more refined version is the *prime j-function* t which, however, does not provide a j-translation for $j = \neg\neg$ as the Gentzen negative translation is known to be minimal in this sense [37]. Finally, we call J the *Kuroda j-function*. Its definition follows van den Berg [114], and is based on the *minimal Kuroda negative translation*, which is jJ for $j = \neg\neg$, as seen in Proposition 5.8(iii).

REMARK 5.2.

(i) *The following hold for all* $\varphi \in S$: $\varphi \approx^j t\varphi$, $\varphi \approx^j k\varphi$, $\varphi \approx^j g\varphi$, $\varphi \approx^j J\varphi$. *It follows that* t, k, g, J *are j-homogenous exactly when they satisfy* (9).

(ii) *Every rule in the inductive definition of* $\rhd$ *is Kolmogorov compatible with* t, k, g. *This, together with* (*i*) *and Theorem* 3.13, *implies that* t, k, g *are j-translations exactly when they satisfy* (9).

(iii) *Every rule in the inductive definition of* $\rhd$ *is Kuroda j-compatible with* J. *This, together with* (*i*) *and Corollary* 3.14, *implies that* jJ *is a j-translation exactly when* J *satisfies* (9).

Proof. The properties in (i) are readily seen by induction on φ. We refer to [33, appendix A] for (ii) and (iii).[8] □

[8] As both proofs have several cases, they are somewhat lengthy but straightforward otherwise.

5.2. The continuation nucleus and the Glivenko nucleus. Let $\rhd_*$ be an inductive extension of $\rhd_m$, and fix a formula α. The *continuation nucleus*

$$g_\alpha : \varphi \mapsto (\varphi \to \alpha) \to \alpha$$

has as special case the *Glivenko nucleus* [93, 114]

$$g_\bot : \varphi \mapsto \neg\neg\varphi\,.$$

While g_α is well-known to be a nucleus whenever $\rhd_*$ is $\rhd_m$, $\rhd_i$ or $\rhd_c$ (see, e.g., [114]), to be sure (Remark 3.6) that this is the case for *every* inductive extension $\rhd_*$ of $\rhd_m$ we give a proof in which we only use generating rules of $\rhd_m$; in fact, Trans and R$\to$ suffice together with *modus ponens*.

LEMMA 5.3. *g_α is a nucleus over $\rhd_*$ for every formula α.*

Proof. Rg_α:

$$\dfrac{\dfrac{\Gamma \rhd_* \varphi \qquad \dfrac{}{\varphi, \varphi \to \alpha \rhd_* \alpha}\,\textrm{axiom}}{\Gamma, \varphi \to \alpha \rhd_* \alpha}\,\textrm{Trans}}{\Gamma \rhd_* (\varphi \to \alpha) \to \alpha}\,\textrm{R}\!\to$$

Lg_α:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \varphi \rhd_* (\psi \to \alpha) \to \alpha \qquad \dfrac{}{\psi \to \alpha, (\psi \to \alpha) \to \alpha \rhd_* \alpha}\,\textrm{axiom}}{\Gamma, \psi \to \alpha, \varphi \rhd_* \alpha}\,\textrm{Trans}}{\Gamma, \psi \to \alpha \rhd_* \varphi \to \alpha}\,\textrm{R}\!\to \qquad \dfrac{}{\varphi \to \alpha, (\varphi \to \alpha) \to \alpha \rhd_* \alpha}\,\textrm{axiom}}{\Gamma, (\varphi \to \alpha) \to \alpha, \psi \to \alpha \rhd_* \alpha}\,\textrm{Trans}}{\Gamma, (\varphi \to \alpha) \to \alpha \rhd_* (\psi \to \alpha) \to \alpha}\,\textrm{R}\!\to$$

□

5.2.1. The intuitionistic case: Glivenko's theorem. Take propositional intuitionistic logic $\rhd_i$ as $\rhd$. As stability (8) equals double negation elimination (16), the strong extension $\rhd_i^{g_\bot}$ of intuitionistic logic $\rhd_i$ is classical logic $\rhd_c$.

PROPOSITION 5.4 (Glivenko's theorem [44, 45]). *In propositional logic,*

$$\Gamma \rhd_c \varphi \iff \Gamma \rhd_i \neg\neg\varphi.$$

Proof. Since $\varphi \to \neg\neg\psi \rhd_i \neg\neg(\varphi \to \psi)$ holds, e.g., by [113, lemma 6.2.2], in view of Lemma 5.1(i) R$\to$ is compatible with $\neg\neg$, and we get the claim as an instance of Corollary 3.7. □

That $\neg\neg$ is a $\neg\neg$-translation (Corollary 3.7(b)) is the alternative formulation of Glivenko's theorem

$$\Gamma \rhd_c \varphi \Longrightarrow \neg\neg\Gamma \rhd_i \neg\neg\varphi.$$

5.2.2. Glivenko's theorem in general algebra. We compare our approach with some occurrences of Glivenko's theorem in universal algebra, starting from Birkhoff's presentation [10, chap. IX, theorem 16]:

> *In any pseudo-complemented distributive lattice L, the correspondence $a \mapsto a^{**}$ is a closure operation in L, and a lattice-homomorphism of L onto the complete Boolean algebra of "closed" elements.*

As far as we see, this falls somewhat short of capturing Glivenko's theorem: Any nucleus $j: S \to S$ whatsoever on a frame S is a closure operator and induces a frame homomorphism $S \to S_j$ onto the frame $S_j = \{a \in S: ja = a\}$ in which the join of $C \subseteq S_j$ is $j(\bigvee C)$, where $\bigvee$ denotes the join in S_j, see, e.g., [60, Subsection II.2.2]. As Birkhoff has noticed, moreover, $(a \vee a^*)^{**} = 1$ for every (not necessarily closed) element a, yet we do not see any trace of Glivenko's theorem. The situation changes with later adaptions such as Esakia's [28, Section A.2]:

> *Let H be a Heyting algebra and $Rg(H)$ the set of all its regular elements, i.e.,*
>
> $$Rg(H) = \{\neg a: a \in H\}.$$
>
> *It is known that $Rg(H)$ is a Boolean algebra and that the map $h: H \to Rg(H)$, given by $ha = \neg\neg a$ $(a \in H)$, is a surjective homomorphism of the Heyting algebra H onto the Boolean algebra $Rg(H)$.*

Rather than that the regular elements form a Boolean algebra [8], the crucial issue for Glivenko's theorem is that double negation is a morphism of Heyting algebras, more precisely that double negation commutes with implication:

$$\neg\neg(\varphi \to \varphi) \approx_i \neg\neg\varphi \to \neg\neg\psi. \tag{19}$$

We notice that by Remark 2.3(iv) the left-to-right direction of (19) is a general property of nuclei, whereas the right-to-left direction is equivalent to

$$\varphi \to \neg\neg\psi \rhd_i \neg\neg(\varphi \to \psi), \tag{20}$$

which is nothing but the condition that we have exhibited in Lemma 5.1 for the rule R$\to$ to be compatible with double negation over intuitionistic provability $\rhd_i$.

5.2.3. The minimal case. Take propositional minimal logic $\rhd_m$ as $\rhd$. As observed above, stability (8) equals double negation elimination (16); whence the strong extension $\rhd_m^{g\perp}$ of minimal logic $\rhd_m$ is classical logic $\rhd_c$.

LEMMA 5.5. *Over any given extension $\rhd_*$ of minimal logic, axiom* (13), *viz.*

$$\rhd_* \neg\neg(\perp \to \varphi),$$

is equivalent to (20) *with $\rhd_*$ in place of $\rhd_i$: that is,*

$$\varphi \to \neg\neg\psi \rhd_* \neg\neg(\varphi \to \psi)\,. \tag{21}$$

Proof. We can obtain (13) from the instance

$$\perp \to \neg\neg\varphi \rhd_* \neg\neg(\perp \to \varphi),$$

of (21) by an application of Trans with the derivable $\rhd_* \perp \to \neg\neg\varphi$.

As (20) is well-known [113], (21) follows from instances of *ex falso sequitur quodlibet* in minimal logic. By using Lj, (21) equally follows from the same instances of (13). In the Appendix 7 we detail a direct proof. □

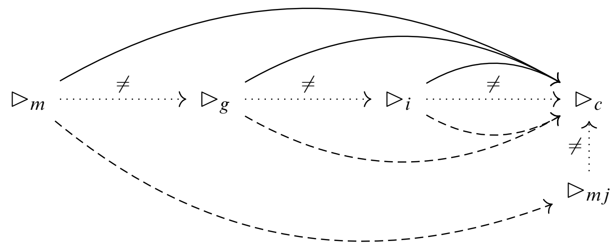

Figure 3. Diagram of the entailment relations involved in the situation of Propositions 5.4 and 5.6. A solid arrow denotes a strong $\neg\neg$-extension, a dashed arrow denotes a weak $\neg\neg$-extension, a dotted arrow denotes a generic extension, and a double line denotes a conservative extension.

The following statement [15, 82][9] is a generalisation of Proposition 5.4 (Figure 3):

Proposition 5.6 (General Glivenko theorem [15, 82]). *Let $\rhd_*$ be an inductive extension of $\rhd_m$ the additional rules of which hold for $\rhd_c$. The following are equivalent*:

(a) $\Gamma \rhd_c \varphi \iff \Gamma \rhd_* \neg\neg\varphi$ *for all* Γ, φ;
(b) $\rhd_g \subseteq \rhd_*$.

Proof. Since $\varphi \to \neg\neg\psi \rhd_i \neg\neg(\varphi \to \psi)$ holds by 5.5, in view of Lemma 5.1(i) the rule R$\to$ is compatible with $\neg\neg$, and we get the claim as an instance of Corollary 3.7. □

Since in minimal logic $\bot$ has no special role, we would get a completely analogous statement if we considered the more general continuation nucleus $g_\alpha : \varphi \mapsto (\varphi \to \alpha) \to \alpha$ in place of $\neg\neg$ and used the axiom

$$\rhd ((\alpha \to \varphi) \to \alpha) \to \alpha$$

in place of (13) to define $\rhd_g$.

5.2.4. The predicate case. Let us go back to considering intuitionistic logic $\rhd_i$ as $\rhd$.

Proposition 5.7 (Gödel's theorem [46]). *Let $\rhd_*$ be an inductive extension of $\rhd_i$ the additional rules of which hold for $\rhd_c$. The following are equivalent in predicate logic*:

(a) $\Gamma \rhd_c \varphi \iff \Gamma \rhd_* \neg\neg\varphi$ *for all* Γ, φ;
(b) $\forall x \neg\neg\varphi \rhd_* \neg\neg\forall x\varphi$ *for all* φ.

Proof. Similar to Proposition 5.4; but apply Theorem 3.5 instead of Corollary 3.7. Lemma 5.1(ii) can be applied since the Glivenko nucleus is compatible with substitution. □

[9] This was observed by Odintsov [82] and Cignoli and Torres [15] apparently simultaneously, in 2004. The latter further showed that our conservation criterion $\varphi \to \neg\neg\psi \rhd_* \neg\neg(\varphi \to \psi)$ is equivalent to the double negation of double negation elimination $\rhd_* \neg\neg(\neg\neg\varphi \to \varphi)$.

Condition (b) in Proposition 5.7 is called *Double Negation Shift* (DNS) and is known to define a proper intermediate logic $\rhd_{\mathrm{DNS}}$, that is, $\rhd_i \subsetneq \rhd_{\mathrm{DNS}} \subsetneq \rhd_c$ [31].[10]

PROPOSITION 5.8. *Let* k, g, J *be respectively the Kolmogorov, Gentzen and Kuroda* $\neg\neg$*-functions. In predicate logic*:

(i) $\Gamma \rhd_c \varphi \iff k\Gamma \rhd_i k\varphi$.
In other words, k is a $\neg\neg$*-translation, known as the* Kolmogorov negative translation [37, 64].

(ii) $\Gamma \rhd_c \varphi \iff g\Gamma \rhd_i g\varphi$.
In other words, g *is a* $\neg\neg$*-translation, known as the* Gentzen negative translation[11] [37, 41, 43].

(iii) $\Gamma \rhd_c \varphi \iff \neg\neg J\Gamma \rhd_i \neg\neg J\varphi \iff J\Gamma \rhd_i \neg\neg J\varphi$.
In other words, $\neg\neg J$ *is a* $\neg\neg$*-translation, known as the* minimal Kuroda negative translation[12] [37, 66, 73, 114].

Proof. Since $\bot \approx_i \neg\neg\bot$, it is easy to see that k, g, J all satisfy (9). Hence by Remark 5.2, k and g satisfy the conditions of Theorem 3.13, which shows (i) and (ii). Similarly, by Remark 5.2, J satisfies the conditions of Corollary 3.14, which shows (iii). □

Theorem 3.13 gives results similar to those in Proposition 5.8 also for other negative translations, such as the *Gödel negative translation* [47], the original *Kuroda negative translation* [66], the *Krivine negative translation*—which was introduced by Streicher and Reus [108] based on Krivine's work [65]—and the negative translation by Ferreira and Oliva [37]. We only point out that, unlike the ones presented here, the latter two cannot be generalised—at least not in a direct way—for an arbitrary nucleus j, and hence a counterpart of Remark 5.2 cannot be given.

5.2.5. The infinitary case. The following statement [110][13] is the counterpart of Proposition 5.7 for infinitary logic:

PROPOSITION 5.9. *Let* $\rhd_*$ *be an inductive extension of* $\rhd_i$ *the additional rules of which hold for* $\rhd_c$*. The following are equivalent in infinitary logic*:

(a) $\Gamma \rhd_c \varphi \iff \Gamma \rhd_* \neg\neg\varphi$ *for all* Γ, φ;
(b) $\bigwedge_{i\in\mathbb{N}} \neg\neg\varphi_i \rhd_* \neg\neg \bigwedge_{i\in\mathbb{N}} \varphi_i$ *for all* $\{\varphi_i : i \in \mathbb{N}\}$.

Proof. Similar to Proposition 5.7. □

As above, condition (b) in Proposition 5.9 defines a proper intermediate logic $\rhd_{\mathrm{DNS}_{inf}}$, that is, $\rhd_i \subsetneq \rhd_{\mathrm{DNS}_{inf}} \subsetneq \rhd_c$.

[10] Even though DNS is not valid in intuitionistic logic, it holds constructive value, as noted by Ilik [54].

[11] In literature, this is often referred to as the Gödel–Gentzen negative translation. However, the translation introduced by Gödel [47] was in fact a different one, somewhere in between Kolmogorov's and Gentzen's [33, 37, 112].

[12] This is a variant introduced by Murthy [73] of the original Kuroda negative translation [66]. It has been studied in the literature for having somewhat nicer properties than Kuroda's original version [37, 114].

[13] We are grateful to Matteo Tesi for the advance communication of his result.

5.2.6. The Peirce nucleus and the Clavius nucleus. Let again $\rhd_*$ be an inductive extension of $\rhd_m$, and fix a formula α. The *Peirce nucleus* [60, 62, 93, 114] is

$$p_\alpha \colon \varphi \mapsto (\varphi \to \alpha) \to \varphi \,.$$

As for Lemma 5.3 one readily proves with the generating rules of $\rhd_m$ that p_α is in fact a nucleus. We are especially interested in the particular case of the *Clavius nucleus*:

$$p_\perp \colon \varphi \mapsto \neg\varphi \to \varphi \,.$$

Over intuitionistic logic, it is easy to see that the Glivenko nucleus is equivalent to the Clavius nucleus, i.e., $\neg\neg\varphi \approx_i \neg\varphi \to \varphi$ for every φ, and hence Propositions 5.4 and 5.7–5.9 equally hold for the Clavius nucleus in place of the Glivenko nucleus.

If we consider minimal logic $\rhd_m$ instead, then the Clavius nucleus is no more equivalent to the Glivenko nucleus. As stability (8) for the Clavius nucleus j equals (14), the strong extension $\rhd_m^j$ of minimal logic $\rhd_m$ is nothing but the Clavius logic $\rhd_s$. We thus get the following counterpart of Glivenko's theorem:

PROPOSITION 5.10. *In propositional logic,*

$$\Gamma \rhd_s \varphi \iff \Gamma \rhd_m \neg\varphi \to \varphi.$$

Proof. Since $\varphi \to (\neg\psi \to \psi) \rhd_m \neg(\varphi \to \psi) \to (\varphi \to \psi)$, by Lemma 5.1(i) we get the claim as an instance of Corollary 3.7. □

5.3. Open and closed nuclei. Once more let $\rhd_*$ be an inductive extension of $\rhd_m$, and fix a formula α. The *closed nucleus* c_α and the *open nucleus* o_α [93, 114] are defined by[14]

$$c_\alpha \colon \varphi \mapsto \varphi \vee \alpha, \qquad\qquad o_\alpha \colon \varphi \mapsto \alpha \to \varphi.$$

Still as for Lemma 5.3, with the generating rules of $\rhd_m$ one easily sees that c_α and o_α are nuclei over $\rhd_*$. We notice the following facts about the extensions induced by these nuclei:

- Stability of the open nucleus is equivalent to $\rhd \alpha$, which can be read as "α is derivable"; thus the strong extension $\rhd_*^{o_\alpha}$ is the smallest entailment relation containing $\rhd_*$ in which "α holds". If $\alpha = \perp$, then stability of $o_\perp$ becomes (18), thus the strong extension $\rhd_m^{o_\perp}$ is nothing but negative logic $\rhd_n$.

[14] Why "open" and "closed"? The following is well-known [60]. Let L be a bounded distributive lattice, and let $X = \mathrm{Spec}(L)$ be the *spectrum* of L (i.e., the collection of prime filters of L). Then X is a topological space with basis of opens

$$\mathrm{ext}(a) = \{P \in X \colon a \in P\} \qquad\qquad (a \in L).$$

Consider $a \in L$, write $U = \mathrm{ext}(a)$. Then $X \setminus U = \mathrm{Spec}(L^U)$ and $U = \mathrm{Spec}(L_U)$, where L^U and L_U are L with $\leqslant$ modified:

$$x \leqslant^U y \iff x \leqslant y \vee a, \qquad\qquad \text{(closed case)},$$
$$x \leqslant_U y \iff x \wedge a \leqslant y, \qquad\qquad \text{(open case)}.$$

For instance, $a \leqslant^U 0$ and $1 \leqslant_U a$. If L is a Heyting algebra, then $x \wedge a \leqslant y \iff x \leqslant a \to y$.

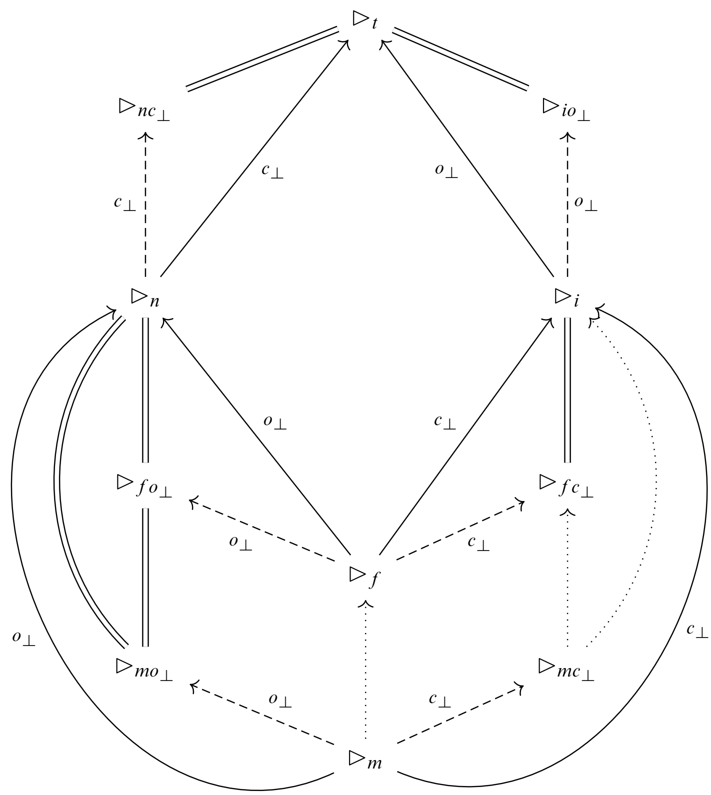

Figure 4. Diagram of the entailment relations induced by open and closed nuclei. A solid arrow denotes a strong extension, a dashed arrow denotes a weak extension, a dotted arrow denotes a generic extension, and a double line denotes a conservative extension.

- Similarly, stability of the closed nucleus is equivalent to $\alpha \rhd \varphi$ for every propositional formula φ, which can be read as "α is inconsistent"; thus the strong extension $\rhd_*^{c_\alpha}$ is the smallest entailment relation containing $\rhd_*$ in which "α is inconsistent". If $\alpha = \perp$, then stability of $c_\perp$ becomes (12), thus the strong extension $\rhd_m^{c_\perp}$ is nothing but intuitionistic logic $\rhd_i$.
- We also note that in $(\rhd^{o_\alpha})^{c_\alpha}$ and $(\rhd^{c_\alpha})^{o_\alpha}$ we have both $\rhd\alpha$ and $\alpha \rhd \varphi$, which by Trans lead to $\rhd\varphi$ for every φ: this means that they both equal trivial logic $\rhd_t$.

We can sum up the situation via the diagram in Figures 4 and 5. We now ask ourselves whether some of the strong extensions are conservative over the corresponding weak extension.

PROPOSITION 5.11.

(i) *Let $\rhd_*$ be $\rhd_m$ plus additional axioms. Then*:

$$\Gamma \rhd_*^{o_\alpha} \varphi \iff \Gamma \rhd_* \alpha \to \varphi \iff \Gamma, \alpha \rhd_* \varphi,$$

for all Γ, φ. In other words, φ is derivable from Γ when assuming that α is derivable, if and only if $\alpha \to \varphi$ is derivable from Γ, if and only if φ is derivable from Γ and α.

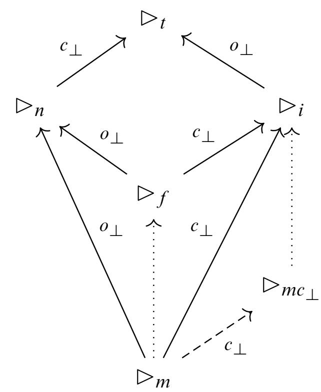

Figure 5. Simplified version of Figure 4, where the weak extensions that coincide with the corresponding strong extension are omitted.

(ii) *Let* $\rhd_*$ *be* $\rhd_m$ *plus additional axioms. Then*

$$\Gamma \rhd_*^{c\alpha} \varphi \iff \Gamma \rhd_* \varphi \vee \alpha$$

for all Γ, φ *if and only if*

$$\rhd_* \alpha \vee (\alpha \to \varphi). \tag{22}$$

Proof.

(i) In view of $\varphi \to (\alpha \to \psi) \rhd_m \alpha \to (\varphi \to \psi)$, the first equivalence holds by means of Theorem 3.5 and Lemma 5.1. The second equivalence is the *deduction theorem* for $\rhd_*$.

(ii) It can easily be shown that (22) is equivalent to $\varphi \to (\psi \vee \alpha) \rhd (\varphi \to \psi) \vee \alpha$; hence the claim follows from Theorem 3.5 and Lemma 5.1. □

Corollary 5.12.

(i) *Let* $\rhd_* \in \{\rhd_m, \rhd_f, \rhd_i\}$. *Then* $\rhd_*^{o_\bot} = \rhd_{*o_\bot}$.
(ii) *Let* $\rhd_* \in \{\rhd_f, \rhd_n\}$. *Then* $\rhd_{*c_\bot} = \rhd_*^{c_\bot}$.
(iii) $\rhd_{mc_\bot} \subsetneq \rhd_c$.

In conclusion, all strong extensions in Figure 4 are conservative over the corresponding weak extension except for $\rhd_c$ over $\rhd_{mc_\bot}$.

5.3.1. A few remarks about the predicate case. In predicate logic, we get an analogous situation by considering the logic $\rhd_F$, defined as the predicate version of the Frobenius logic $\rhd_f$ plus the *dual Frobenius rule* [62]

$$\forall x(\varphi \vee \bot) \rhd (\forall x \varphi) \vee \bot.$$

Functions based on the open and closed nuclei but over predicate intuitionistic logic $\rhd_i$ are known in literature. For instance, having fixed a formula A, Friedman employed the prime c_A-function[15] t_α^C—thus known as *Friedman's A-translation*—to

[15] See §5.1.

prove Markov's rule [38]; and Ishihara and Nemoto used the prime o_A-function t_A^O to prove the independence-of-premiss rule [56]. Note that:

$$t_A^O o_A \varphi = t_A^O A \to t_A^O \varphi, \qquad\qquad t_A^C c_A \varphi = t_A^C \varphi \vee t_A^C A.$$

This means that t_A^O satisfies (8), and hence is an o_A-translation, if and only if "$t_A^O A$ holds" in the sense that $\rhd\, t_A^O A$; this is the case, e.g., when A is atomic. Similarly, t_A^C satisfies (8), and thus is a c_A-translation, precisely when "$t_A^C A$ is inconsistent" in the sense that $t_A^C A \rhd \varphi$ for every φ; this is the case, e.g., when $A = \bot$. In the latter case, however, the strong c_A-extension is trivial.

5.4. Propositional lax logic. In propositional lax logic [32] the modality $\bigcirc$ is characterised by rules corresponding [32, p. 2, Section 1] to the ones of a nucleus. In our setting, we define $\rhd_L$ by adding a unary relation symbol $\bigcirc$ to the language of $\rhd_i$, and the axiom

$$\varphi \to \bigcirc\psi \approx \bigcirc\varphi \to \bigcirc\psi$$

to the inductive definition of $\rhd_i$. It is evident that $j = \bigcirc$ is a nucleus over $\rhd_L$. This trivially extends to every inductive extension of $\rhd_L$ since L$\bigcirc$ is a generating rule.

The strong $\bigcirc$-extension $\rhd_L^{\bigcirc}$ of $\rhd_L$ is $\rhd_i$ plus

$$\varphi \approx \bigcirc\varphi,$$

i.e., plus an *identity operator* $\bigcirc$ on formulae. We also define $\rhd_{L*}$ as $\rhd_L$ plus

$$\varphi \to \bigcirc\psi \rhd \bigcirc(\varphi \to \psi),$$

i.e., the smallest extension of $\rhd_L$ in which the nucleus $\bigcirc$ is compatible with R$\to$ (Theorem 3.5).

Given a formula φ in the language of $\rhd_L$, we denote by φ' the formula obtained from φ by removing all occurrences of $\bigcirc$. Formally, the definition of φ' is given inductively:

$$\begin{array}{ccc} P' = P, & \top' = \top, & \bot' = \bot, \\ (\varphi \wedge \psi)' = \varphi' \wedge \psi', & (\varphi \vee \psi)' = \varphi' \vee \psi', & (\varphi \to \psi)' = \varphi' \to \psi', \\ (\forall x\varphi)' = \forall x\varphi', & (\exists x\varphi)' = \exists x\varphi', & (\bigcirc\varphi)' = \varphi'. \end{array}$$

We obtain a somewhat more general version of *strong conservativity* [32, theorem 2.4]:

PROPOSITION 5.13. *The following are equivalent*:

(a) $\Gamma \rhd_{L*} \bigcirc\varphi$,
(b) $\Gamma \rhd_L^{\bigcirc} \varphi$,
(c) $\Gamma' \rhd_L^{\bigcirc} \varphi'$,
(d) $\Gamma' \rhd_i \varphi'$.

In particular, if $\Gamma \rhd_L \varphi$, *then* $\Gamma' \rhd_i \varphi'$.

Proof. Items (a) and (b) are equivalent by Theorem 3.5. The fact that (d) implies (c) holds since $\rhd_L^{\bigcirc}$ extends $\rhd_i$. The directions from (c) to (b) and from (b) to (d) are proved straightforwardly by structural induction. □

§6. Future work. Orevkov [84] has established some well-known conservativity results of classical logic over intuitionistic and minimal first-order logics with equality.

In particular, he isolates seven classes of single-succedent sequents—the so-called *Glivenko sequent classes*—defined in terms of the absence of positive or negative occurrences of particular logical symbols (in a first-order language with equality) where classical derivability implies intuitionistic derivability. The same article also shows that these classes are optimal: any class of sequents for which classical derivability implies intuitionistic derivability is contained in one of these seven classes. In recent years simpler proofs of conservativity results for some Glivenko sequent classes have been given [55, 74, 100]. An extremely simple and purely logical proof of the first Glivenko class for coherent theories has been obtained by Negri [77] by means of **G3**-style sequent calculi; this has been extended to geometric theories in [79] and then to cover all other first-order Glivenko sequent classes in [34, 78]. We want to investigate whether a generalisation of these results is possible in our setting.

§7. Appendix Proof details.

We give here a direct proof of Lemma 5.5.

LEMMA 7.1. *Over any given extension $\rhd_*$ of minimal logic, the following are equivalent*:

(a) $\rhd_* \neg\neg(\bot \to \varphi)$,
(b) $\varphi \to \neg\neg\psi \rhd_* \neg\neg(\varphi \to \psi)$.

Proof. As for (b) $\Longrightarrow$ (a):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\bot, \neg\varphi \rhd_* \bot}\,\text{Refl}}{\bot \rhd_* \neg\neg\varphi}\,\text{R}\!\to}{\rhd_* \bot \to \neg\neg\varphi}\,\text{R}\!\to \qquad \dfrac{}{\bot \to \neg\neg\varphi \rhd_* \neg\neg(\bot \to \varphi)}\,\text{assumption}}{\rhd_* \neg\neg(\bot \to \varphi)}\,\text{Trans}$$

As for (a) $\Longrightarrow$ (b):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to \neg\neg\psi, \varphi \rhd_* \neg\neg\psi}\,\text{axiom}}{\varphi \to \neg\neg\psi, \neg\neg\varphi \rhd_* \neg\neg\psi}\,\text{L}\neg\neg}{\varphi \to \neg\neg\psi \rhd_* \neg\neg\varphi \to \neg\neg\psi}\,\text{R}\!\to \qquad \dfrac{\dfrac{\overset{\text{(see below)}}{\neg(\varphi \to \psi) \rhd_* \neg\neg\varphi} \qquad \dfrac{\dfrac{\dfrac{}{\psi, \bot \rhd_* \bot}\,\text{Refl} \qquad \dfrac{\dfrac{}{\psi, \varphi \rhd_* \psi}\,\text{Refl}}{\psi \rhd_* \varphi \to \psi}\,\text{R}\!\to}{\psi, \neg(\varphi \to \psi) \rhd_* \bot}\,\text{L}\!\to}{\neg\neg\psi, \neg(\varphi \to \psi) \rhd_* \bot}\,\text{L}\neg\neg}{\neg\neg\varphi \to \neg\neg\psi, \neg(\varphi \to \psi) \rhd_* \bot}\,\text{L}\!\to}{\neg\neg\varphi \to \neg\neg\psi \rhd_* \neg\neg(\varphi \to \psi)}\,\text{R}\!\to}{\varphi \to \neg\neg\psi \rhd_* \neg\neg(\varphi \to \psi)}\,\text{Trans}$$

where $\neg(\varphi \to \psi) \rhd_* \neg\neg\varphi$ is derived as follows:

$$\dfrac{\dfrac{}{\rhd_* \neg\neg(\bot \to \psi)}\,\text{(a)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bot \to \psi, \bot, \neg\varphi \rhd_* \bot}\,\text{Refl} \qquad \dfrac{\dfrac{\dfrac{}{\psi, \neg\varphi, \varphi \rhd_* \psi}\,\text{Refl} \qquad \dfrac{}{\neg\varphi, \varphi \rhd_* \bot}\,\text{axiom}}{\bot \to \psi, \neg\varphi, \varphi \rhd_* \psi}\,\text{L}\!\to}{\bot \to \psi, \neg\varphi \rhd_* \varphi \to \psi}\,\text{R}\!\to}{\bot \to \psi, \neg(\varphi \to \psi), \neg\varphi \rhd_* \bot}}{\bot \to \psi, \neg(\varphi \to \psi) \rhd_* \neg\neg\varphi}\,\text{R}\!\to}{\neg\neg(\bot \to \psi), \neg(\varphi \to \psi) \rhd_* \neg\neg\varphi}\,\text{L}\neg\neg}{\neg(\varphi \to \psi) \rhd_* \neg\neg\varphi}\,\text{Trans}$$

□

Acknowledgements. The authors wish to thank Benno van den Berg, Ingo Blechschmidt, Hajime Ishihara, Sara Negri, Paulo Oliva, Mario Piazza, Matteo Tesi and Tarmo Uustalu for their interest and suggestions. Last but not least, the authors are grateful to Daniel Misselbeck-Wessel whose original idea set the basis for this work.

Funding. The present study was carried out within the projects "A New Dawn of Intuitionism: Mathematical and Philosophical Advances" (ID 60842) funded by the John Templeton Foundation, and "Reducing complexity in algebra, logic, combinatorics - REDCOM" belonging to the programme "Ricerca Scientifica di Eccellenza 2018" of the Fondazione Cariverona. Both authors are members of the "Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni" (GNSAGA) of the Istituto Nazionale di Alta Matematica (INdAM).

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