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# REFLEXIVITY INDEX AND IRRATIONAL ROTATIONS

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**Abstract**

We determine the reflexivity index of some closed set lattices by constructing maps relative to irrational rotations. For example, various nests of closed balls and some topological spaces, such as even-dimensional spheres and a wedge of two circles, have reflexivity index 2. We also show that a connected double of spheres has reflexivity index at most 2.

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## 1. Introduction

For any topological space $X$, let $\mathcal{S}(X)$ denote the set of all closed subsets of $X$ and let $C(X)$ denote the set of all continuous endomorphisms on $X$, that is, the set of all continuous functions that map $X$ into itself. A subset $A$ of $X$ is invariant under an endomorphism $f$ on $X$ if $f(A) \subseteq A$, that is, $f(x) \in A$ for all $x \in A$. For any $\mathcal{L} \subseteq \mathcal{S}(X)$ and any $\mathcal{F} \subseteq C(X)$, define

$$\begin{aligned} \text{Alg}\,\mathcal{L} &= \{f \in C(X) : f(A) \subseteq A \text{ for all } A \in \mathcal{L}\}, \\ \text{Lat}\,\mathcal{F} &= \{A \in \mathcal{S}(X) : f(A) \subseteq A \text{ for all } f \in \mathcal{F}\}, \end{aligned}$$

that is, Alg $\mathcal{L}$ is the set of all continuous endomorphisms on $X$ that leave each subset in $\mathcal{L}$ invariant and Lat $\mathcal{F}$ is the set of all closed subspaces of $X$ that are invariant under each endomorphism in $\mathcal{F}$.

The set $C(X)$ is a semigroup under the operation of function composition, with an identity $id$, where $id(x) = x$ for all $x \in X$. The topology on $X$ induces a topology on $C(X)$, whose sub-basic open neighbourhoods of $\varphi \in C(X)$ are subsets of $C(X)$ of the form

$$\mathcal{N}(x, \varphi, U) = \{\psi \in C(X) : \psi(x) \in U\},$$

where $U$ is any open neighbourhood of $\varphi(x)$ in $X$. It is easy to verify that for any $\mathcal{L} \subseteq \mathcal{S}(X)$, Alg $\mathcal{L}$ is a closed subsemigroup of $C(X)$, with identity $id$.

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A subset $\mathcal{L}$ of $\mathcal{S}(X)$ is *reflexive* if Lat Alg $\mathcal{L} = \mathcal{L}$. Since Lat $\mathcal{F}$ = Lat Alg Lat $\mathcal{F}$ for any $\mathcal{F} \subseteq C(X)$, $\mathcal{L}$ is reflexive if and only if $\mathcal{L} = \mathrm{Lat}\,\mathcal{F}$ for some $\mathcal{F} \subseteq C(X)$.

The meet and join of any collection $\{A_\omega : \omega \in \Omega\}$ of closed subsets are defined by

$$\bigwedge_{\omega\in\Omega} A_\omega = \bigcap_{\omega\in\Omega} A_\omega \quad \text{and} \quad \bigvee_{\omega\in\Omega} A_\omega = \overline{\bigcup_{\omega\in\Omega} A_\omega},$$

where $\overline{A}$ denotes the closure of the set $A$. With these operations $\mathcal{S}(X)$ is a complete lattice. We call any complete sublattice of $\mathcal{S}(X)$ that contains the trivial subsets $\emptyset$ and $X$ a *closed set lattice*. In particular, Lat $\mathcal{F}$ is a closed set lattice for any $\mathcal{F} \subseteq C(X)$ and so any reflexive family of closed subsets is necessarily a closed set lattice.

For any reflexive closed set lattice $\mathcal{L}$, Alg $\mathcal{L}$ is the largest of all subsets $\mathcal{F}$ of $C(X)$ with the property that Lat $\mathcal{F} = \mathcal{L}$. It is of interest to determine the minimal size of such subsets $\mathcal{F}$.

Definition 1.1. The reflexivity index, $\kappa(\mathcal{L})$, of a reflexive closed set lattice $\mathcal{L}$ is

$$\kappa(\mathcal{L}) = \min\{|\mathcal{F}| : \mathrm{Lat}\,\mathcal{F} = \mathcal{L}\}.$$

If $\mathcal{L} = \{\emptyset, X\}$, we denote the reflexivity index of $\mathcal{L}$ by $\kappa(X)$ for convenience and call it the reflexivity index of the topological space $X$. If $\kappa(X) = 1$, we say that the space $X$ is *transitive* and a map $f$ in $\mathcal{F}$ is a transitive map.

The notion of the reflexivity index was introduced by Zhao in [10] in the context of arbitrary subset lattices. These can be regarded as closed set lattices for the discrete topology on $X$. The results in [10] were generalised in [3]. The notion of reflexivity for lattices of closed subspaces of a Hilbert space was introduced by Halmos in [2] and has received a lot of attention (see [1] for a general reference). The reflexivity index of various types of closed subspace lattices has been calculated (see, for example, [4, 5, 8]). Very little seems to be known about the reflexivity index of an arbitrary reflexive lattice of closed subsets. In this paper we determine the reflexivity index of some closed set lattices by constructing maps relative to irrational rotations.

By an irrational rotation we mean a rotation operator acting on $\mathbb{R}^2$ whose rotation angle is an irrational multiple of $2\pi$. Since the set $\{1, z, z^2, z^3, \ldots\}$ is dense in the unit circle in $\mathbb{C}$ if and only if $z = e^{2\pi i\theta}$, where $\theta$ is an irrational number, the set of powers $\{I, R, R^2, \ldots\}$ is dense in the set of all rotation operators acting on $\mathbb{R}^2$ if and only if $R$ is an irrational rotation.

In Section 2, we determine the reflexivity index of various nests of closed balls in a separable Hilbert space. In Section 3, we determine the reflexivity index of some topological spaces, such as even-dimensional spheres and a wedge of two circles. We also give an upper bound for the reflexivity index of some spaces, such as odd-dimensional spheres and a connected double of spheres.

The following lemma will be useful.

Lemma 1.2. *For each $\mathcal{F} \subseteq C(X)$, Lat $\mathcal{F}$ = Lat $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is the closure in the induced topology of the semigroup of all finite products of elements of $\mathcal{F} \cup \{id\}$.*

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PROOF. Since $\mathcal{F} \subseteq \widehat{\mathcal{F}}$, it follows that $\mathrm{Lat}\,\widehat{\mathcal{F}} \subseteq \mathrm{Lat}\,\mathcal{F}$ and so it is sufficient to show that $\mathrm{Lat}\,\mathcal{F} \subseteq \mathrm{Lat}\,\widehat{\mathcal{F}}$. Suppose that $M \in \mathrm{Lat}\,\mathcal{F}, x \in M$ and $\varphi \in \widehat{\mathcal{F}}$. Suppose also that $U$ is an open neighbourhood of $\varphi(x)$ in $X$. Since $\varphi \in \widehat{\mathcal{F}}$, there exist $\psi_1, \psi_2, \ldots, \psi_n \in \mathcal{F}$ such that $\psi_n\psi_{n-1}\cdots\psi_2\psi_1(x) \in U$. Since $M \in \mathrm{Lat}\,\mathcal{F}$, $\psi_n\cdots\psi_2\psi_1(x) \in M$. So, $U \cap M \neq \varnothing$ and, since $M$ is closed, it follows that $\varphi(x) \in M$. So, $M \in \mathrm{Lat}\,\widehat{\mathcal{F}}$. □

REMARK 1.3. It is easy to see that the closed set $\widehat{\mathcal{F}}$ is itself a subsemigroup of $C(X)$ with identity $id$.

## 2. Reflexivity index of nests of balls

Let $\mathfrak{H}$ denote a separable real Hilbert space. We consider a simple example first. For each $r \geq 0$, let $\mathfrak{N} = \bigcup_{r\geq 0} B_r \cup \mathfrak{H}$, where $B_r = \{x \in \mathfrak{H} : \|x\| \leq r\}$. Then $\mathfrak{N}$ is a totally ordered, closed subset lattice. Note that we choose not to include $\varnothing$ in $\mathfrak{N}$. The subset $B_0 = \{0\}$ is the minimal element of $\mathfrak{N}$. We say that $\mathfrak{N}$ is a *nest*. We shall show that $\mathfrak{N}$ is reflexive and determine its reflexivity index.

PROPOSITION 2.1. *$\mathfrak{N}$ is reflexive.*

PROOF. Let $C_1 = \{f : \|f(x)\| \leq \|x\| \text{ for all } x \in \mathfrak{H}\}$ denote the set of all contractive endomorphisms. It is easy to see that $\mathrm{Alg}\,\mathfrak{N} = C_1$, so $\mathfrak{N} \subseteq \mathrm{Lat}\,\mathrm{Alg}\,\mathfrak{N} = \mathrm{Lat}\,C_1$.

Suppose that $M \in \mathrm{Lat}\,C_1$ and $x \in M\backslash\{0\}$. Suppose also that $\|y\| \leq \|x\|$, and $f(z) = (\|z\|/\|x\|)y$ for each $z \in \mathfrak{H}$. Then $f \in C_1$ and $f(x) = y$, so $y \in M$. It follows that $B_{\|x\|} \subseteq M$. So, $M = \bigcup_{x\in M} B_{\|x\|} \in \mathfrak{N}$ and hence $\mathrm{Lat}\,C_1 \subseteq \mathfrak{N}$. Thus, $\mathfrak{N} = \mathrm{Lat}\,C_1$ and $\mathfrak{N}$ is reflexive. □

PROPOSITION 2.2. *$\kappa(\mathfrak{N}) > 1$.*

PROOF. Suppose that $\mathfrak{N} = \mathrm{Lat}\{f\}$, that is, $\kappa(\mathfrak{N}) = 1$. Then, for each $x \in \mathfrak{H}$, the orbit $O(f,x) = \{x, f(x), f^2(x), \ldots\}$ of $x$ is dense in $B_{\|x\|}$. Here $f^n$ denotes the $n$th iterate of $f$, that is, $f^1 = f$ and $f^{n+1}(x) = f(f^n(x))$. Suppose that $x \neq 0$. Then $\|f^n(x)\| < \|x\|$ for some $n \in \mathbb{N}$. Since $f \in C_1, \|f^m(x)\| \leq \|f^n(x)\|$ for $m \geq n$. But then $O(f,x)$ is not dense in $B_{\|x\|}$. This is a contradiction and so no such function $f$ exists. So, $\kappa(\mathfrak{N}) > 1$. □

PROPOSITION 2.3. *If* $\dim\mathfrak{H} = 1$, *then* $\kappa(\mathfrak{N}) = \aleph_0$.

PROOF. Let $\dim\mathfrak{H} = 1$, so $\mathfrak{H} = \mathbb{R}$. Let $f_r(x) = rx$ for $x \in \mathbb{R}$. Then $f_r \in \mathrm{Alg}\,\mathfrak{N}$ if and only if $|r| \leq 1$. It is easy to see that $\mathfrak{N} = \mathrm{Lat}\,\mathcal{F}$ if $\mathcal{F} = \{f_r : |r| \leq 1 \text{ and } r \text{ is rational}\}$. So, $\kappa(\mathfrak{N}) \leq \aleph_0$. Suppose that $\mathrm{Lat}\,\mathcal{F} = \mathfrak{N}$, where $\mathcal{F}$ is finite. Consider the finite set of functions $S = \{f(x) : x \in \{-1, 1\} \text{ and } f \in \mathcal{F}\}$. Since $\mathcal{F} \subseteq \mathrm{Alg}\,\mathfrak{N} = C_1$, we have $S \subseteq B_1$. If $S \subseteq \{-1, 1\}$, then $\{-1, 1\} \in \mathrm{Lat}\,\mathcal{F}$. But $\{-1, 1\} \notin \mathfrak{N}$, which is a contradiction. Let $r = \max\{|y| : y \in S \text{ and } |y| < 1\}$. Then $B_r \cup \{-1, 1\} \in \mathrm{Lat}\,\mathcal{F}$. But $B_r \cup \{-1, 1\} \notin \mathfrak{N}$. It follows that no such finite set $\mathcal{F}$ exists. So, $\kappa(\mathfrak{N}) = \aleph_0$. □

We shall show that $\dim\mathfrak{H} = 1$ is the exceptional case.

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Assume that $\mathfrak{H}$ is a separable infinite-dimensional Hilbert space with orthonormal basis $\{\xi_n : i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ and each $\theta \in \mathbb{R}$, let $R_n(\theta)$ denote the isometric rotation operator on $\mathfrak{H}$ which satisfies

$$R_n(\theta)\xi_i = \begin{cases} \cos 2\pi\theta\xi_n + \sin 2\pi\theta\xi_{n+1} & \text{if } i = n, \\ -\sin 2\pi\theta\xi_n + \cos 2\pi\theta\xi_{n+1} & \text{if } i = n+1, \\ \xi_i & \text{if } i \notin \{n, n+1\}. \end{cases}$$

Let $\mathfrak{R} = \{R_n(\theta) : n \in \mathbb{N}, \theta \in \mathbb{R}\}$. The following properties of $\widehat{\mathfrak{R}}$ will be useful for our determination of $\kappa(\mathfrak{R})$.

LEMMA 2.4. *Suppose that $x \in \mathfrak{H}$. Then $T(\|x\|\xi_1) = x$ for some $T \in \widehat{\mathfrak{R}}$.*

PROOF. The result is trivially true if $x = 0$. So, suppose that $x = \sum_{j=1}^{\infty} x_j\xi_j \neq 0$. We define sequences of real numbers $(\lambda_n)_{n=0}^{\infty}$ and $(\theta_n)_{n=1}^{\infty}$ recursively as follows. Let $\lambda_0 = \|x\|$, $\theta_1 = \arccos(x_1/\lambda_0)$ and $\lambda_1 = \lambda_0 \sin\theta_1$. For $n \geq 2$, if $\lambda_{n-1} = 0$, let $\theta_n = 0$ and $\lambda_n = 0$, and, if $\lambda_{n-1} \neq 0$, let $\theta_n = \arccos(x_n/\lambda_{n-1})$ and $\lambda_{n+1} = \lambda_n \sin\theta_n$. We shall also require that $x_{n+1}\sin\theta_n \geq 0$. (This requirement is not necessary here but will be needed later.)

Let $T_0 = I$ and $T_n = R_n(\theta_n)T_{n-1}$ for $n \geq 1$. Let $P_n$ denote the orthogonal projection with range $\operatorname{span}\{\xi_j : 1 \leq j \leq n\}$. It is easy to show that $T_n(\lambda_0\xi_1) = P_nx + \lambda_n\xi_{n+1}$ and $|\lambda_n| = \|(1 - P_n)x\|$. Since $P_nx \to x$ in norm as $n \to \infty, T_n(\lambda_0\xi_1) = T_n(\|x\|\xi_1) \to x$ as $n \to \infty$. Clearly, $T_n \in \mathfrak{R}$ for each $n \in \mathbb{N}$. Choose $y \in \mathfrak{H}$ and consider the sequence $(T_ny)_{n=1}^{\infty}$. Choose $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|(I - P_N)y\| < \varepsilon$. Since $T_nP_N = T_NP_N$ for all $n > N, \|T_ny - T_my\| \leq \|T_nP_Ny - T_mP_Ny\| + \|(T_n - T_m)(I - P_N)y\| < 2\varepsilon$ if $n > N$ and $m > N$. So, $(T_ny)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{H}$ and hence has a limit. Define $Ty = \lim_{n\to\infty} T_ny$. It is easy to verify that $T_n \to T$ as $n \to \infty$, in both the strong operator topology and the induced topology on $C(\mathfrak{H})$. Since $\widehat{\mathfrak{R}}$ is closed, it follows that $T \in \widehat{\mathfrak{R}}$ and $T(\|x\|\xi_1) = \lim_{n\to\infty} T_n(\|x\|\xi_1) = x$. □

LEMMA 2.5. *Suppose that $x, y \in \mathfrak{H}$ and $\|x\| = \|y\|$. Then $Tx = y$ for some $T \in \widehat{\mathfrak{R}}$.*

PROOF. We may suppose that $x$ and $y$ are nonzero. By Lemma 2.4, $T_1(\|x\|\xi_1) = x$ for some $T_1 \in \widehat{\mathfrak{R}}$ and $T_2(\|y\|\xi_1) = y$ for some $T_2 \in \widehat{\mathfrak{R}}$. Since $R_n(\theta)^{-1} = R_n(-\theta)$ and the rotations $R_n(\theta)$ are isometric, $\widehat{\mathfrak{R}}$ is indeed a group. So, $T_1^{-1} \in \widehat{\mathfrak{R}}$. Let $T = T_2T_1^{-1}$. Then $T \in \widehat{\mathfrak{R}}$ and $Tx = y$. □

Let us introduce a theorem of Kronecker [7]. Let $\mathcal{U}$ denote the set of 'multi-rotations' acting on $\mathfrak{H}$, which are direct sums of rotation operators acting on the two-dimensional subspaces $\operatorname{span}\{\xi_{2n-1}, \xi_{2n}\}$. That is, $R \in \mathcal{U}$ if and only if there are real numbers $\theta_n, n \in \mathbb{N}$, such that, for each $n \in \mathbb{N}$,

$$R\xi_{2n-1} = \cos 2\pi\theta_n\xi_{2n-1} + \sin 2\pi\theta_n\xi_{2n},$$
$$R\xi_{2n} = -\sin 2\pi\theta_n\xi_{2n-1} + \cos 2\pi\theta_n\xi_{2n}.$$

Note that $\mathcal{U}$ contains each of the rotation operators $R_{2n-1}(\theta)$ for $n \in \mathbb{N}$.

From [7], $\mathcal{U}$ is singly generated. That is, there exists $R \in \mathcal{U}$ such that the set of powers $\{I, R, R^2, \ldots\}$ is strongly dense in $\mathcal{U}$. To see this, let $G$ denote the group

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$(\mathbb{R}/\mathbb{Z})^\infty \approx [0,1)^\infty$, where the group action is pointwise addition modulo 1. For each $\theta = (\theta_n)_{n\in\mathbb{N}} \in G$, let $R(\theta)$ denote the multi-rotation $R$ which is defined above. Then $\mathcal{U} = \{R(\theta) : \theta \in G\}$. Note that $\mathcal{U}$ is a commutative group of unitary operators whose identity is the identity operator $I$. Furthermore, we have $R(\theta + \varphi) = R(\theta)R(\varphi)$ and $\|R(\theta) - R(\varphi)\| = \sup_{n\in\mathbb{N}}|2\sin(\pi(\theta_n - \varphi_n)| \le 2\pi\sup_{n\in\mathbb{N}}|\theta_n - \varphi_n|$.

Suppose that $\theta = (\theta_n)_{n\in\mathbb{N}} \in G$. The *orbit* $O(\theta)$ is the set $\{m\theta : m \in \mathbb{N}\}$, that is, the set of all positive integral multiples of $\theta$. The set of numbers $\theta_n, n \in \mathbb{N}$, is rationally independent if a finite sum of the form $q_1\theta_1 + q_2\theta_2 + \cdots + q_N\theta_N$, where $q_n \in \mathbb{Q}$ for each $n$, is 0 if and only if $q_n = 0$ for each $n$. According to Kronecker's theorem, $O(\theta)$ is dense in $G$, with the product topology, if and only if the numbers $\theta_n, n \in \mathbb{N}$, are rationally independent.

Suppose now that $\theta^\# = (\theta^\#_n)_{n\in\mathbb{N}}$, where the numbers $\theta^\#_n, n \in \mathbb{N}$, are rationally independent, and suppose that $R(\varphi) \in \mathcal{U}, \{x_1, x_2, \ldots, x_K\} \subset \mathfrak{H}$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|(I - P_{2N})x_k\| < \varepsilon$ for $1 \le k \le K$. Since $O(\theta^\#)$ is dense in $G$, there exists $m \in \mathbb{N}$ such that $|(m\theta^\#)_n - \varphi_n| < \varepsilon \pmod 1$ for $1 \le n \le 2N$. Let $D = R(m\theta^\#) - R(\varphi)$. Then $\|D\| \le 2$ and $\|DP_{2N}\| \le 2\pi\varepsilon$. So, for $1 \le k \le K$,

$$\|Dx_k\| \le \|DP_{2N}x_k\| + \|D(I - P_{2N})x_k\| \le 2\pi\varepsilon\|x_k\| + 2\varepsilon \le C\varepsilon,$$

where $C = 2\pi\sup_{1\le k\le K}\|x_k\| + 2$. So, the set $\{R(n\theta^\#) : n \in \mathbb{N}\}$ is strongly dense in $\mathcal{U}$. We say that $R(\theta^\#)$ is a *generator* of $\mathcal{U}$.

We turn now to the problem of determining $\kappa(\mathfrak{N})$. Let $\mathcal{V} = S\mathcal{U}S^*$, where $S$ is the unilateral shift operator on $\mathfrak{H}$ which satisfies $S\xi_n = \xi_{n+1}$ for each $n \in \mathbb{N}$. Note that if $\theta = (\theta_n)_{n\in\mathbb{N}} \in G$, then $SR(\theta)S^*\xi_1 = 0$ and, for each $n \ge 1$,

$$\begin{aligned} SR(\theta)S^*\xi_{2n} &= \cos 2\pi\theta_n\xi_{2n} + \sin 2\pi\theta_n\xi_{2n+1}, \\ SR(\theta)S^*\xi_{2n+1} &= -\sin 2\pi\theta_n\xi_{2n} + \cos 2\pi\theta_n\xi_{2n+1}. \end{aligned}$$

Thus,

$$SR_n(\varphi)S^* = R_{n+1}(\varphi)(I - P_1) \quad \text{for } n \in \mathbb{N} \text{ and } \varphi \in \mathbb{R}.$$

Now let $\mathcal{F} = \{R(\theta^\#), SR(\theta^\#)S^*\}$, where $R(\theta^\#)$ is a generator of $\mathcal{U}$. Note that $SR(\theta^\#)S^*$ is a generator of $\mathcal{V}$. We shall show that $\operatorname{Lat}\widehat{\mathcal{F}} = \mathfrak{N}$, which will imply that $\kappa(\mathfrak{N}) = 2$. The proof relies on several lemmas. For each $k \in \mathbb{N}$, let $\mathfrak{H}_k$ denote $P_k\mathfrak{H}$.

LEMMA 2.6. *Suppose that $x, y \in \mathfrak{H}_1^\perp$ and $\|x\| = \|y\|$. Then $Tx = y$ for some $T \in \widehat{\mathcal{F}}$.*

PROOF. As remarked previously, $\widehat{\mathcal{F}}$ contains all operators of the form $R_{2n-1}(\theta)$. Similarly, $\widehat{\mathcal{F}}$ contains all operators of the form $R_{2n}(\theta)(I - P_1)$. The restrictions of these operators to $\mathfrak{H}_1^\perp$ are all isometric. Furthermore, $\mathfrak{H}_1^\perp$ is invariant for each of these operators, except those of the form $R_1(\theta)$. So, simple modifications of the proofs of Lemmas 2.4 and 2.5 prove this lemma. □

LEMMA 2.7. *Suppose that $x, y \in \mathfrak{H}$ and $\|x\| \ge \|y\|$. Then $Tx = y$ for some $T \in \widehat{\mathcal{F}}$.*

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Proof. We may assume that $x \neq 0$. First suppose that $\|y\| = \|x\|$ and choose $\theta$ and $\theta'$ such that $R_1(\theta)x \in \mathfrak{H}_1^\perp$ and $R_1(\theta')y \in \mathfrak{H}_1^\perp$. By Lemma 2.6, $T(R_1(\theta)x) = R_1(\theta')y$. So, $R_1(-\theta')TR_1(\theta) \in \widehat{\mathcal{F}}$ and $R_1(-\theta')TR_1(\theta)x = y$.

Suppose that $\|x\| \geq \|y\|$. Let $\varphi = \arcsin(\|y\|/\|x\|)$ and $z = \|x\|(\cos\varphi\xi_1 + \sin\varphi\xi_2)$. Then $\|x\| = \|z\|$ and $\|\,\|x\|\sin\varphi\xi_2\| = \|y\|$. So, $T_1x = z$ and $T_2(\|x\|\sin\varphi\xi_2) = y$ for some $T_1, T_2 \in \widehat{\mathcal{F}}$. Now $SS^* = I - P_1 \in \widehat{\mathcal{F}}$ and $SS^*z = \|x\|\sin\varphi\xi_2$. So, $T_2SS^*T_1 \in \widehat{\mathcal{F}}$ and $T_2SS^*T_1x = y$. □

Theorem 2.8. *$\kappa(\mathfrak{N}) = 2$.*

Proof. First we show that $\mathrm{Lat}\,\widehat{\mathcal{F}} = \mathfrak{N}$. Since $\widehat{\mathcal{F}} \subseteq C_1$, $\mathfrak{N} = \mathrm{Lat}\,C_1 \subseteq \mathrm{Lat}\,\widehat{\mathcal{F}}$. Clearly, $\{0\} = B_0 \in \mathrm{Lat}\,\widehat{\mathcal{F}} \cap \mathfrak{N}$. Suppose that $x$ is a nonzero vector in $M \in \mathrm{Lat}\,\widehat{\mathcal{F}}$. By Lemma 2.7, $\|x\| \geq \|y\| \Longrightarrow y \in M$ and so $B_{\|x\|} \subseteq M$. Clearly, $M \subseteq \bigcup_{x\in M} B_{\|x\|}$ and so $M = \bigcup_{x\in M} B_{\|x\|} \in \mathfrak{N}$. Thus, $\mathrm{Lat}\,\widehat{\mathcal{F}} = \mathfrak{N}$.

Since $\mathrm{Lat}\,\widehat{\mathcal{F}} = \mathrm{Lat}\,\mathcal{F}$ by Lemma 1.2, $\kappa(\mathfrak{N}) \leq |\mathcal{F}| = 2$ and we know from Proposition 2.2 that $\kappa(\mathfrak{N}) > 1$. □

Now we show that the reflexivity index of a nest of closed balls in a finite-dimensional Hilbert space is also 2, which can be seen as a corollary to the previous theorem. For $k \in \mathbb{N}$ and $r \geq 0$, let $B_{k,r} = \{x \in \mathfrak{H}_k : \|x\| \leq r\}$ and let

$$\mathfrak{N}_k = \bigcup_{r\geq 0} B_{k,r} \cup \mathfrak{H}_k.$$

Then $\mathfrak{N}_k$ is a nest of closed subsets of the $k$-dimensional Hilbert space $\mathfrak{H}_k$. We shall identify operators of the form $R_n(\theta)$, as defined above, with their restrictions to $\mathfrak{H}_k$. Let $\mathfrak{R}_k = \mathfrak{R}_k^0 \cup \mathfrak{R}_k^1$, where

$$\mathfrak{R}_k^0 = \{R_{2n-1}(\theta) : 1 \leq 2n \leq k, \theta \in \mathbb{R}\},$$
$$\mathfrak{R}_k^1 = \{R_{2n}(\theta)(I - P_1) : 1 \leq 2n \leq k-1, \theta \in \mathbb{R}\}.$$

Corollary 2.9. *$\kappa(\mathfrak{N}_k) = 2$ if $2 \leq k < \infty$.*

Proof. The proof of Lemma 2.7 can easily be amended to show that if $x, y \in \mathfrak{H}_k$ and $\|x\| \geq \|y\|$, then $Tx = y$ for some $T \in \widehat{\mathfrak{R}_k}$. As in the proof of Theorem 2.8, this implies that $\mathrm{Lat}\,\widehat{\mathfrak{R}_k} = \mathfrak{N}_k$.

We have to make some small changes to the definitions of $\mathcal{U}$ and $\mathcal{V}$ so that the new definitions are suitable for the finite-dimensional case. Let $\mathcal{U}_k$ denote the set of direct sums of rotation operators acting on the two-dimensional subspaces $\mathrm{span}\{\xi_{2n-1}, \xi_{2n}\}$ for $2 \leq 2n \leq k$. We also let $R\xi_k = \xi_k$ if $k$ is odd. Similarly, let $\mathcal{V}_k$ denote the set of direct sums of rotation operators acting on the subspaces $\mathrm{span}\{\xi_{2n}, \xi_{2n+1}\}$ for $2 \leq 2n \leq 2k+1$. We also let $R\xi_1 = 0$ and $R\xi_k = \xi_k$ if $k$ is even.

Then $\mathfrak{R}_k^0 \subseteq \mathcal{U}_k$ and $\mathfrak{R}_k^1 \subseteq \mathcal{V}_k$ and it is easy to see that both of $\mathcal{U}_k$ and $\mathcal{V}_k$ are singly generated. Let $\mathcal{F} = \{R, R'\}$, where $R$ and $R'$ are generators of $\mathfrak{R}_k^0$ and $\mathfrak{R}_k^1$. It follows from Lemma 1.2 that $\mathfrak{N} = \mathrm{Lat}\,\mathcal{F}$ and so $\kappa(\mathfrak{N}_k) \leq 2$. But $\kappa(\mathfrak{N}_k) > 1$ by Proposition 2.2 and so $\kappa(\mathfrak{N}_k) = 2$. □

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We extend these results by determining the reflexivity index of a larger class of nests of balls. Suppose that $\Lambda$ is a closed subset of $\mathbb{R}^+$ and set

$$\mathfrak{N}_\Lambda = \{B_r : r \in \Lambda\} \cup \mathfrak{H}.$$

Then $\mathfrak{N}_\Lambda$ is a nest of closed balls in $\mathfrak{H}$. In the special case $\Lambda = \mathbb{R}^+$, we have $\mathfrak{N}_\Lambda = \mathfrak{N}$, where $\mathfrak{N}$ is as defined before. We shall assume that $0 \in \Lambda$ and that $\Lambda$ is unbounded.

Suppose that $\varphi$ is a strictly increasing function in $C(\mathbb{R}^+)$ such that

$$\varphi(t) = t \quad \text{if } t \in \Lambda \quad \text{and} \quad \varphi(t) > t \quad \text{if } t \notin \Lambda.$$

For example, we could define $\varphi(t) = a + \sqrt{(b-a)(t-a)}$ for all $t \in (a, b)$ for each component $(a, b)$ of $\mathbb{R}^+ \backslash \Lambda$. Now define $\Phi \in C(\mathfrak{H})$ by $\Phi(0) = 0$ and

$$\Phi(x) = \varphi(\|x\|)\frac{x}{\|x\|} \quad \text{if } x \neq 0.$$

Let $A = R(\theta^\#)$ and $B = SR(\theta^\#)S^*\Phi$, where $R(\theta^\#)$ is a generator of $\mathcal{U}$. Observe that $Bx = (\varphi(x)/\|x\|)SR(\theta^\#)S^\# x$ for each $x \neq 0$, and that $Bx = SR(\theta^\#)S^\# x$ if $\|x\| \in \Lambda$. Let $\mathcal{F} = \{A, B\}$. It is easy to see that $\mathcal{F} \subset \operatorname{Alg}\mathfrak{N}_\Lambda$. We shall show that $\operatorname{Lat}\mathcal{F} = \mathfrak{N}_\Lambda$. The proof proceeds with several lemmas.

LEMMA 2.10. *$B^n S = SA^n\Phi^n = SR(n\theta^\#)\Phi^n$ for $n \in \mathbb{N}$.*

PROOF. The proof follows from the repeated application of the identities $\Phi S = S\Phi$, $\Phi A = A\Phi$, $S^*S = I$ and $A^n = R(n\theta^\#)$. □

For $t \geq 0$, let

$$\varphi^\infty(t) = \inf\{u \in \Lambda : u \geq t\}.$$

Since $\Lambda$ is bounded, $\varphi^\infty$ is well defined. Now define $\Phi^\infty$ by $\Phi^\infty(0) = 0$ and

$$\Phi^\infty(x) = \varphi^\infty(\|x\|)\frac{x}{\|x\|} \quad \text{if } x \neq 0.$$

Note that $\varphi^n(t) \uparrow \varphi^\infty(t)$ as $n \to \infty$ for each $t \in \mathbb{R}^+$ and $\Phi^n(x) \to \Phi^\infty(x)$ as $n \to \infty$ for each $x \in \mathfrak{H}$.

LEMMA 2.11. *Suppose that $M \in \operatorname{Lat}\mathcal{F}$ and that $x \in M \cap \mathfrak{H}_1^\perp$. Then $\Phi^\infty(x) \in M \cap \mathfrak{H}_1^\perp$.*

PROOF. Note that $x = SS^*x$ since $x \in \mathfrak{H}_1^\perp$ and $B^n x = B^n SS^* x = SR(n\theta^\#)\Phi^n(S^*x) \in M$ for all $n \in \mathbb{N}$ since $M \in \operatorname{Lat}\mathcal{F}$. Now

$$\Phi^n(S^*x) \to \Phi^\infty(S^*x) = \varphi^\infty(\|S^*x\|)\frac{S^*x}{\|S^*x\|} = \varphi^\infty(\|x\|)\frac{S^*x}{\|x\|} \quad \text{as } n \to \infty.$$

Since $R(\theta^\#)$ generates $\mathcal{U}$, it follows that $I$ is the strong limit of a subsequence of the operators $R(n\theta^\#)$, $n \in \mathbb{N}$. Since $M$ is closed,

$$SI\Phi^\infty(S^*x) = \varphi^\infty(\|x\|)\frac{SS^*x}{\|x\|} = \varphi^\infty(\|x\|)\frac{x}{\|x\|} = \Phi^\infty(x) \in M. \qquad \square$$

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LEMMA 2.12. *Suppose that* $M \in \operatorname{Lat} \mathcal{F}, x \in M \cap \mathfrak{H}_1^{\perp}$ *and* $\|x\| \in \Lambda$. *Suppose also that* $y \in \mathfrak{H}$ *and* $\|y\| \leq \|x\|$. *Then* $y \in M$.

PROOF. Let $S_r = \{u \in \mathfrak{H} : \|u\| = r\}$ and observe that $S_r \cap \mathfrak{H}_1^{\perp}$ is an invariant set for both $A$ and $B$. Let $z = z_2\xi_2 + (I - P_2)y$, where $|z_2|^2 = \|x\|^2 - \|(I - P_2)y\|^2$. Then $z \in S_{\|x\|} \cap \mathfrak{H}_1^{\perp}$ and the proofs of Lemmas 2.4 and 2.5 can be easily modified to show that $Tx = z$ for some $T \in \widehat{\mathcal{F}}$. So, $z \in M$.

Now choose $\theta_1 \in \mathbb{R}$ such that $|z_2| \cos\theta_1 = \|P_2 y\|$. Since $\{R_1(\theta_1), SS^*\} \subset \widehat{\mathcal{F}}$,

$$SS^* R_1(\theta_1)z = (I - P_1)R_1(\theta_1)z = \|P_2 y\|\xi_2 + (I - P_2)y \in M.$$

Finally, choose $\theta_2$ such that $R_1(\theta_2)\|P_2 y\|\xi_2 = P_2 y$. Then

$$R_1(\theta_2(\|P_2 y\|\xi_2 + (I - P_2)y) = P_2 y + (I - P_2)y = y \in M. \qquad \square$$

THEOREM 2.13. *Suppose that* $\Lambda$ *is a closed, unbounded subset of* $\mathbb{R}^+$ *and* $0 \in \Lambda$. *Then* $\kappa(\mathfrak{N}_\Lambda) = 2$.

PROOF. Suppose that $M \in \operatorname{Lat}\mathcal{F}$ and that $x \in M\backslash\{0\}$. By Lemma 2.11, $\Phi^\infty(x) \in M$ and $\|\Phi^\infty(x)\| = \varphi^\infty(\|x\|) \in \Lambda$. So, by Lemma 2.12, $y \in M$ for all $y$ satisfying $\|y\| \leq \|\Phi^\infty(x)\|$. Then $M = \bigcup\{B_{\|\Phi^\infty(x)\|} : x \in M\}$. Since $\Lambda$ is closed, it follows that $M = B_r$ for some $r \in \Lambda$, or $M = \mathfrak{H}$.

Therefore, $\operatorname{Lat}\mathcal{F} = \mathfrak{N}_\Lambda$ and hence $\kappa(\mathfrak{N}_\Lambda) \leq 2$. A modification of Proposition 2.2 shows that $\kappa(\mathfrak{N}_\Lambda) > 1$. So, $\kappa(\mathfrak{N}_\Lambda) = 2$. $\square$

REMARK 2.14. It would be interesting to determine $\kappa(\mathfrak{N}_\Lambda)$ when $0 \notin \Lambda$.

## 3. Reflexivity index of some topological spaces

In this section we determine the reflexivity index of some topological spaces.

PROPOSITION 3.1. *The reflexivity index is a topological invariant.*

PROOF. Suppose that $\mathcal{F} \subset C(X)$ and that $\phi$ is a homeomorphism acting on $X$. Suppose also that $\mathcal{L} = \operatorname{Lat}\mathcal{F}$. Then $\phi\mathcal{L} = \phi\operatorname{Lat}\mathcal{F} = \operatorname{Lat}(\phi\mathcal{F}\phi^{-1})$ and so $\phi\mathcal{L}$ is reflexive. Suppose also that $|\mathcal{F}| = \kappa(\mathcal{L})$. Since $|\mathcal{F}| = |\phi\mathcal{F}\phi^{-1}|$, we have $\kappa(\phi\mathcal{L}) \leq \kappa(\mathcal{L})$. To complete the proof, observe that $\phi^{-1}$ is also a homeomorphism and that $\phi^{-1}(\phi\mathcal{L}) = \mathcal{L}$. $\square$

Let $S^n$ denote the $n$-sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. It is not difficult to see that $S^n$ is reflexive. From the property of irrational rotations, we clearly have $\kappa(S^1) = 1$. We show that $\kappa(S^n) = 2$ when $n \geq 2$ is even.

LEMMA 3.2. $\kappa(S^n) \leq 2$ *for all* $n \geq 2$.

PROOF. Let us make some small changes to the definitions of $\mathcal{U}$ and $\mathcal{V}$ once again. Let $\mathcal{U}'$ be the set of direct sums of rotation operators acting on $\operatorname{span}\{\xi_{2n-1}, \xi_{2n}\}$, with the additional requirement that $R\xi_{n+1} = \xi_{n+1}$ if $n$ is even. Let $\mathcal{V}'$ denote the set of direct sums of rotation operators acting on $\operatorname{span}\{\xi_{2n}, \xi_{2n+1}\}$, with the additional requirement that $R\xi_1 = \xi_1$ and $R\xi_{n+1} = \xi_{n+1}$ if $n$ is odd. It is clear that $\mathcal{U}'$ and $\mathcal{V}'$ are both singly

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generated. Let $\mathcal{F} = \{\mathfrak{U}, \mathfrak{V}\}$, where $\mathfrak{U}$ is a generator of $\mathcal{U}'$ and $\mathfrak{V}$ is a generator of $\mathcal{V}'$. It suffices to show that $\operatorname{Lat}\widehat{\mathcal{F}} = S^n$. From Lemma 2.5, we see that for any $x, y \in S^n$, there exists $T \in \widehat{\mathcal{F}}$ such that $Tx = y$. So, a proof similar to that in Proposition 2.1 yields $S^n = \operatorname{Lat}\mathcal{F} = \operatorname{Lat}\widehat{\mathcal{F}}$ and $\kappa(S^n) \leq 2$. □

We introduce some new notation before coming to the next theorem. We recommend [6] as a general reference. Given any continuous map between topological spaces $f : X \to Y$, we have an induced map $f_{n*} : H_n(X) \to H_n(Y)$ in the $n$th homology group. For simplicity we just write $f_*$ instead of $f_{n*}$. Since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$ and any homomorphism from $\mathbb{Z}$ to itself is of the form $r \mapsto mr$, where $m$ is an integer, we may call $m$ the *degree* of the map $f : S^n \to S^n$, denoted by $d(f) = m$. If $x = (x_1, x_2, \ldots, x_{n+1}) \in S^n$, its antipode is $-x = (-x_1, -x_2, \ldots, -x_{n+1})$. The antipodal map is defined as $a : x \mapsto -x$.

Proofs of the following propositions can be found in [9].

PROPOSITION 3.3. *If $f, g : S^n \to S^n$ are continuous maps, then:*

(i) $d(f \circ g) = d(f)d(g)$;
(ii) $d(1_{S^n}) = 1$, *where $1_{S^n}$ is the identity map;*
(iii) *$f$ is homotopic to $g$ if and only if $d(f) = d(g)$.*

PROPOSITION 3.4. *If $n \geq 1$, then the antipodal map $a^n : S^n \to S^n$ has degree $(-1)^{n+1}$.*

PROPOSITION 3.5. *If $f : S^n \to S^n$ has no fixed points, then $f$ is homotopic to the antipodal map $a^n$.*

THEOREM 3.6. *$\kappa(S^n) = 2$ when $n > 0$ is even.*

PROOF. We show that $\kappa(S^n) \neq 1$. If instead $\kappa(S^n) = 1$, then we can take a transitive continuous map $f \in C(S^n)$. Denote by $f^2$ the composition $f \circ f$. Since $f$ is transitive, $f$ and $f^2$ both have no fixed points. Then we see from Proposition 3.5 that $f$ and $f^2$ are both homotopic to the antipodal map $a^n$. Thus, $f$ is homotopic to $f^2$ and, from Proposition 3.3, $d(f) = d(f^2)$. However, from Proposition 3.4, $d(f) = (-1)^{n+1} = -1$ while $d(f^2) = d(f)d(f) = (-1)^2 = 1 \neq -1$, which is a contradiction.

From Lemma 3.2, $\kappa(S^n) \leq 2$ and it follows that $\kappa(S^n) = 2$. □

REMARK 3.7. It would be interesting to determine $\kappa(S^n)$ when $n \geq 3$ is odd.

We next introduce wedges of two circles.

DEFINITION 3.8. We say that a topological space $X$ is a connected double of a space $Y$ if $X$ can be written as a union of two connected subspaces $X = A \cup B$, where:

(i) $A$ and $B$ are both homeomorphic to $Y$;
(ii) $A \cap B$ is a proper subspace of $A$ and $B$;
(iii) there exists a homeomorphism $f : A \to B$ such that $f|_{A \cap B} = 1$, where 1 denotes the identity map.

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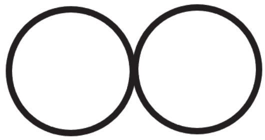

FIGURE 1. Wedge of two circles.

EXAMPLE 3.9. It is not difficult to see that the wedge of two circles (see [6]), also called '*figure 8*', is a connected double of a circle. See Figure 1.

It is clear that if we let $A, B$ denote the two circles in the example above, then $A \cap B$ is a unique point $Z$. If $P \in A$ and $Q \in B$, then any connected topological subspace of $A \cup B$ containing $P$ and $Q$ contains $Z$.

THEOREM 3.10. *Let $X$ be a connected double of an $n$-sphere ($n \geq 1$). Then $\kappa(X) \leq 2$.*

PROOF. Let $X = A \cup B$, where $A$ and $B$ are both homeomorphic to $S^n$. We study the case when $n \geq 2$ first. We retain the notation $\mathcal{U}'$, $\mathcal{V}'$, $\mathfrak{U}$ and $\mathfrak{V}$ defined in Lemma 3.2. Recall that $\mathrm{Lat}\,\{\widehat{\mathfrak{U}, \mathfrak{V}}\} = S^n$.

Let $f : S^n \to A$ be a homeomorphism and $g$ be the composition of $f$ and the homeomorphism from $A$ to $B$. Then $g : S^n \to B$ is also a homeomorphism. Since $f^{-1}(P) = g^{-1}(P)$ for all $P \in A \cap B$, we have a well-defined map $h : X \to S^n$ by setting

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \in B. \end{cases}$$

.

Now define maps $F : X \to X$, $G : X \to X$ as follows:

$$F(P) = \begin{cases} f\mathfrak{U}f^{-1}(P) & \text{if } P \in A, \\ f\mathfrak{U}g^{-1}(P) & \text{if } P \in B, \end{cases} \quad \text{and} \quad G(P) = \begin{cases} g\mathfrak{V}f^{-1}(P) & \text{if } P \in A, \\ g\mathfrak{V}g^{-1}(P) & \text{if } P \in B. \end{cases}$$

We see that $F$ and $G$ are both continuous.

Next we will show that $\mathrm{Lat}\,\{\widehat{F, G}\} = X$. It suffices to show that for any $x, y \in X$, there exists a map $H \in \{\widehat{F, G}\}$ such that $Hx = y$.

Assume without loss of generality that $y \in B$. Denote by $M$ the group of elements of finite order generated by $\mathfrak{U}$ and $\mathfrak{V}$, and by $N$ the group of elements of finite order generated by $F$ and $G$. By mapping $\mathfrak{U}$ to $F$ and $\mathfrak{V}$ to $G$, we have a group isomorphism $j : M \to N$.

Given any $\epsilon > 0$, there exists an open ball $B_r$ of radius $r$ at $h(y)$ such that $\|g(t) - y\| < \epsilon$ whenever $t \in B_r \cap S^n$. Moreover, there exists an element $H_1 \in M$ such that $\|H_1(h(x)) - h(y)\| < r/2$. From the definition of $\mathfrak{V}$, we see that there exists a positive number $Q$ such that $\|I - \mathfrak{V}^Q\| < r/2$. Now $\|\mathfrak{V}^Q H_1(h(x)) - h(y)\| < r/2 + r/2 = r$ and $\mathrm{Im}(j(\mathfrak{V}^Q H_1)) = \mathrm{Im}(j(\mathfrak{V})j(\mathfrak{V}^{Q-1}H_1)) \subset \mathrm{Im}(j\mathfrak{V}) \subset \mathrm{Im}(g) \subset B$. With $H = j(\mathfrak{V}^Q H_1)$, we have $\|Hx - y\| < \epsilon$.

Since $X \subset \mathrm{Lat}\{\widehat{F, G}\}$ and $\mathrm{Lat}\{\widehat{F, G}\} \subset X$, it follows that $\kappa(X) \leq 2$.

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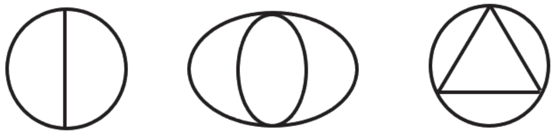

FIGURE 2. Examples of connected doubles of $S^1$.

When $n = 1$, let $\mathfrak{U}$ be an irrational rotation on the circle $S^1$ and let $F, G$ be

$$F(P) = \begin{cases} f\mathfrak{U}f^{-1}(P) & \text{if } P \in A, \\ f\mathfrak{U}g^{-1}(P) & \text{if } P \in B, \end{cases} \quad \text{and} \quad G(P) = \begin{cases} g\mathfrak{U}f^{-1}(P) & \text{if } P \in A, \\ g\mathfrak{U}g^{-1}(P) & \text{if } P \in B. \end{cases}$$

Again $F$ and $G$ are continuous. A similar proof gives $\mathrm{Lat}\,\{\widehat{F, G}\} = X$ and the reflexivity index is again at most 2. □

COROLLARY 3.11. *The spaces shown in Figure 2, as connected doubles of $S^1$, all have reflexivity index $\leq 2$.*

Finally, we show that the reflexivity index of the wedge of two circles is 2. From [9], we see that if $f : (X, x_0) \to (S^1, 1)$ is a continuous pointed map between topological spaces and $t_0 \in \mathbb{Z}$, then we have a unique lifting map $f' : (X, x_0) \to (\mathbb{R}, t_0)$ with $\exp(f') = f$. Here $\exp(t)$ denotes $e^{2\pi i t}$.

THEOREM 3.12. *The reflexivity index of the wedge of two circles is 2.*

PROOF. Let $X$ denote the 'figure 8' space and let $A, B$ denote the two circles of $X$. Let $A \cap B = Z$, the unique intersection point.

We argue by contradiction. If the reflexivity index is 1, let $f$ be the transitive map. Assume without loss of generality that $f(Z) = S \in A$. Let $H : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the map defined by $H(j) = j + \mathbb{Z}$ and let $H'$ be the map from $\mathbb{R}/\mathbb{Z}$ to $[0, 1)$ defined by $j + \mathbb{Z} \mapsto j - [j]$, where $[j]$ denotes the largest integer not greater than $j$. Since there is a homeomorphism $\theta + \mathbb{Z} \mapsto e^{2\pi i\theta}$ from $\mathbb{R}/\mathbb{Z}$ to the complex unit circle, we may construct a homeomorphism $h$ from $\mathbb{R}/\mathbb{Z}$ to $A$ satisfying $h(0 + \mathbb{Z}) = Z$.

*Case 1.* $f(B) \subset A$. If $f(A) \subset A$, then $A$ is an invariant subspace of $X$, which is a contradiction. So, $f(A) \cap (X \backslash A) \neq \varnothing$ and there is a point $R \in A$ with $f(R) \in B$. Choose $r \in (0, 1)$ such that $h^{-1}(R) = H(r)$. Since $f$ is continuous and $Z$ is the unique point that belongs to both $A$ and $B$, there exist $p_0 \in (0, r)$ and $q_0 \in (r, 1)$ such that $fhH(p_0) = fhH(q_0) = Z$.

Let $p_1$ be the minimum value of all such $p_0 \in (0, r)$ and let $q_1$ be the maximum value of all such $q_0 \in (q, 1)$, respectively. Clearly, $s$ is not equal to $p_1$ or $q_1$, for otherwise $f^2(Z) = Z$, which is a contradiction. It suffices to consider the case when $q_1 > s$, for otherwise we may apply the homeomorphism $j + \mathbb{Z} \mapsto (1 - j) + \mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$.

We claim that $s < p_1$. Suppose on the contrary that $s > p_1$. Since $f$ is continuous, for any $\epsilon > 0$, we may find $\delta > 0$ such that $H'h^{-1}fhH(j)$ belongs to $(1 - \epsilon, 1)$ or $(0, \epsilon)$ whenever $j \in (p_1 - \delta, p_1)$. If $H'h^{-1}fhH(j) \in (0, \epsilon)$, by the intermediate value theorem,

$H'h^{-1}fhH(j)$ has a fixed point in $(0, p_1)$. Thus, $H'h^{-1}fhH(j) \in (1-\epsilon, 1)$ whenever $j \in (p_1 - \delta, p_1)$. Similarly, there exists $\delta'$ such that $H'h^{-1}fhH(j) \in (0, \epsilon)$ whenever $j \in (q_1, q_1 + \delta')$. Now we can find $u_1 \in (0, p_1)$ such that $H'h^{-1}fhH(u_1) = q_1$. Let $u$ be the maximum of all such $u_1$. Then $H'h^{-1}fhH(j) \in (q_1, 1)$ whenever $j \in (u, p_1)$ and $H'h^{-1}f^2hH(j) \in (0, p_1)$. Again by the intermediate value theorem, $H'h^{-1}f^2hH(j)$ has a fixed point $u_0$ in $(0, p_1)$. But then $hH(u_0)$ is a fixed point of $f^2$, contradicting the assumption that $f$ is transitive. Thus, $s > p_1$.

Now let $p$ be the maximum value of all such $p_0 \in (0, r)$ and let $q$ be the minimum value of all such $q_0 \in (q, 1)$. For any $j \in [p, q]$, we have $fhH(j) \in B$. Thus, by assumption, $f^2hH(j) \in A$.

Let $J : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism defined by $J(j + \mathbb{Z}) = j - s + \mathbb{Z}$. From [9], we see that the map $Jh^{-1}f^2hH : [p, q] \to \mathbb{R}/\mathbb{Z}$ can be lifted to a map $F : [p, q] \to \mathbb{R}$ such that $F(p) = 0$ and $HF(j) = Jh^{-1}f^2hH(j)$ for all $j \in [p, q]$.

Clearly, $F(q) \in \{0, 1, -1\}$. If $F(q) = 1$, let $F_1(j) = F(j) + s$, so that $F_1(p) = s < p$ and $F_1(q) = 1 + s > q$. By the intermediate value theorem, there exists $j_0 \in (p, q)$ such that $F_1(j_0) = j_0$. That is, $f^2hH(j_0) = hH(j_0)$ and $hH(j_0)$ is a fixed point of $f^2$. If $F(q) = -1$, let $F_2(j) = F(j) + s + 1$. Since $F_2(p) = 1 + s > p, F_2(q) = s < q$, again by the intermediate value theorem, we can find $j_0$ in $(p, q)$ such that $F_2(j_0) = j_0$. Now $HF(j_0) = j_0 - s - 1 + \mathbb{Z} = j_0 - s + \mathbb{Z} = Jh^{-1}f^2hH(j_0)$. It follows that $f^2hH(j_0) = hH(j_0)$ and that $hH(j_0)$ is a fixed point.

If $F(q) = 0$, we can find a point $j' \in [p, q]$ such that $|F(j)|$ attains its maximum value at $j'$. Since $f^2$ is continuous, for any $j_0 \in (p, j')$, we can find $j_2 \in (j', q)$ such that $H'h^{-1}f^2hH(j_0) = H'h^{-1}f^2hH(j_2)$. Moreover, there exists a positive number $K$ such that $H'h^{-1}f^KhH(p) \in (p, j')$ since $f$ is transitive. Since $f^K$ has no fixed point, we can find $j_1 \in (p, q)$ such that $H'h^{-1}f^KhH(j_1) = q$. Thus, by the intermediate value theorem, we can find a point $w \in (p, j_1)$ such that $H'h^{-1}f^{K+2}hH(w) = H'h^{-1}f^2hH(w)$. That is, $f^2hH(w)$ is a fixed point of $f^K$, which is a contradiction.

*Case 2.* $f(B) \cap (X - A) \neq \emptyset$. The proof is similar to that in Case 1. Here we let $i$ be the homeomorphism from $\mathbb{R}/\mathbb{Z}$ to $B$ satisfying $i(0 + \mathbb{Z}) = Z$. By assumption, there exists $k \in (0, 1)$ such that $fiH(k) \in B$. There exist $k_1' \in (0, k)$ and $k_2' \in (k, 1)$ such that $fiH(k_1') = fiH(k_2') = Z$. Let $k_1$ be the largest of all such $k_1'$ and $k_2$ the minimum of all such $k_2'$. Then $fiH(k_0) \in B$ for all $k_0 \in [k_1, k_2]$. Since $H'i^{-1}fiH(k_1) = 0$, the map $i^{-1}fiH : [k_1, k_2] \to \mathbb{R}/\mathbb{Z}$ may be lifted to a map $F' : [k_1, k_2] \to \mathbb{R}$ satisfying $F'(k_1) = 0$ and $HF(k_0) = i^{-1}fiH(k_0)$ for all $k_0 \in [k_1, k_2]$. Now $H'i^{-1}fiH(k_2) \in \{0, 1, -1\}$. If $H'i^{-1}fiH(k_2) \in \{1, -1\}$, using a similar proof to Case 1, $f$ has a fixed point. If $H'i^{-1}fiH(k_2) = 0$, again we can find $k_3 \in (k_1, k_2)$ such that $|F'(k_3)|$ attains its maximum value. There exists a positive number $K'$ such that $f^{K'}(k_1) \in (k_1, k_3)$ and we can find $k_4 \in (k_1, k_2)$ such that $f^{K'+1}iH(k_4) = fiH(k_4)$. Then $fiH(k_4)$ is a fixed point of $f^{K'}$, which is a contradiction.

It is clear that the above two cases cover all possibilities. Thus, the reflexivity index of the wedge of two circles is not 1. Combining this result with Theorem 3.10, we see that the reflexivity index is 2. □

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