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Computing the critical dimensions of Bratteli–Vershik systems with multiple edges

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Abstract. The critical dimension is an invariant that measures the growth rate of the sums of Radon–Nikodym derivatives for non-singular dynamical systems. We show that for Bratteli–Vershik systems with multiple edges, the critical dimension can be computed by a formula analogous to the Shannon–McMillan–Breiman theorem. This extends earlier results of Dooley and Mortiss on computing the critical dimensions for product and Markov odometers on infinite product spaces.

1. *Introduction*

In this paper, we prove that for Bratteli–Vershik systems with multiple edges, the critical dimension can be computed by a formula analogous to the Shannon–McMillan–Breiman theorem and coincides with the average coordinate entropy.

Throughout the paper, let $(X, \mathcal{B}, \mu, T)$ be a non-singular measurable dynamical system. That is, the transformation T is invertible and bi-measurable, and $\mu \circ T$ and μ are mutually absolutely continuous ($\mu \circ T \sim \mu$). We assume that the system is ergodic in the sense that every T-invariant measurable set or its complement has measure zero. We also assume that the system is conservative, that is, the union of the collection of the wandering sets has measure zero.

Dooley and Hamachi [**3**] proved that every ergodic non-singular system is orbit equivalent to a Bratteli–Vershik system with a Markov odometer; moreover, considered as a G-measure in the sense of [**3**], the system may be taken to be uniquely ergodic. Bratteli–Vershik systems with Markov odometers as defined in [**3**] are none other than Bratteli–Vershik systems satisfying conditions (M1) and (M2) given in §3 of this paper.

For measure-preserving systems, where $\mu \circ T = \mu$, entropy has proved to be an effective invariant under metric isomorphisms. Ornstein [**13**] proved that entropy is a complete invariant for Bernoulli systems. For non-singular systems, however, it is generally believed that no form of entropy can exist, and a new invariant is needed to classify them.

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To address this problem, Mortiss [**12**] introduced the notion of critical dimension for analyzing ergodic, conservative, non-singular dynamical systems. The critical dimension measures the asymptotic growth rate of the sums of Radon–Nikodym derivatives. The critical dimension, when it exists, is an invariant under metric isomorphism [**5**]. Hence, the critical dimension can play a role in distinguishing the behavior of non-singular dynamical systems.

Although it is not entropy, the critical dimension for fundamental Bratteli–Vershik systems is shown to possess some characteristics of entropy. The critical dimension, when it exists, can be calculated for product odometers and Markov odometers on infinite product spaces. It satisfies a version of the Shannon–McMillan–Breiman theorem and coincides with the average coordinate (AC) entropy introduced by Mortiss [**10, 11**].

Now the natural question is to extend these results to Bratteli–Vershik systems. This paper is a step in that direction. We prove that for Bratteli–Vershik systems with multiple edges and a uniform bound on the number of edges with the same source, the critical dimension can be computed by a formula analogous to the Shannon–McMillan–Breiman theorem and coincides with the AC entropy. Our proof in this paper is self contained. Note that Bratteli–Vershik systems with multiple edges include infinite product spaces equipped with Markov odometers as a special case. The condition that there is a bound on the number of edges with a given source was needed even in the product case, where it is a uniform bound on the number of coordinates.

It is of interest to know the extent to which the critical dimension characterizes non-singular dynamical systems. Dooley [**2**] defines the notion of Hurewicz maps, which induce Hurewicz equivalence of non-singular systems. In this paper, we prove that the critical dimension is preserved under Hurewicz equivalence.

This paper is organized as follows. In §2 we define the critical dimension for ergodic, conservative, non-singular dynamical systems and list some of its properties. We define the notion of Hurewicz maps between non-singular systems and show that the critical dimension is preserved under Hurewicz equivalence. In §3 we define Bratteli–Vershik systems. We then explain assumptions on Bratteli–Vershik systems considered in this paper. In §4 we define the average coordinate (AC) entropy. We state a generalized law of large numbers and use it to obtain a convergence formula involving the AC entropy. In §5 we prove that for Bratteli–Vershik systems with multiple edges, the critical dimension can be calculated by a formula analogous to the Shannon–McMillan–Breiman theorem and coincides with the AC entropy.

2. *Critical dimension and Hurewicz equivalence*

The following definitions are taken from [**4, 5**]. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic, conservative, non-singular measurable dynamical system with $\mu(X) = 1$. For x in X, let $\omega_i(x)$ denote the ith Radon–Nikodym derivative of the system, that is, the L^1-function mod μ satisfying $\mu(T^i A) = \int_A \omega_i(x)\, d\mu(x)$ for all $A \in \mathcal{B}$. We write $\omega_i(x) = (d\mu \circ T^i / d\mu)(x)$.

Definition 2.1. Let $\alpha' > 0$ and set

$$X_{\alpha'} = \left\{ x \in X : \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^{\alpha'}} > 0 \right\}.$$

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Notice that $X_{\alpha'}$ is an invariant set. The supremum over the set of α' for which $\mu(X_{\alpha'}) = 1$ is called the *lower critical dimension* α of $(X, \mathcal{B}, \mu, T)$.

Definition 2.2. Let $\beta' > 0$ and set

$$X_{\beta'} = \left\{ x \in X : \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^{\beta'}} = 0 \right\}.$$

Notice that $X_{\beta'}$ is an invariant set. The infimum over the set of β' for which $\mu(X_{\beta'}) = 1$ is called the *upper critical dimension* β of $(X, \mathcal{B}, \mu, T)$.

Note† that for μ-almost every $x \in X$, we have

$$\alpha = \liminf_{n} \log \sum_{i=0}^{n-1} \omega_i(x) / \log n$$

and

$$\beta = \limsup_{n} \log \sum_{i=0}^{n-1} \omega_i(x) / \log n.$$

When $\alpha = \beta$, we say that the system has *critical dimension* α. By a result of Maharam [**9**], it can be shown that the inequalities $0 \le \alpha \le \beta \le 1$ hold. The critical dimension, when it exists, is an invariant under metric isomorphism [**5**].

A standard notion of equivalence of non-singular dynamical systems is that of orbit equivalence. Two non-singular systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *orbit equivalent* if there exists an invertible bi-measurable map $\Phi : X \to Y$ such that $\nu \circ \Phi \sim \mu$ and a cocycle $\sigma : \mathbb{Z} \times X \to \mathbb{Z}$ satisfying $S^n \Phi x = \Phi T^{\sigma(n,x)} x$ for μ-almost every $x \in X$.

An ergodic non-singular system that is neither atomic nor measure preserving is of type III. Type III systems are divided into type III_λ, $\lambda \in [0, 1]$, systems by the Krieger–Araki–Woods ratio set. Krieger [**8**] proved that there is a unique orbit equivalence class within type III_λ systems, where $\lambda \in (0, 1]$. For each $\lambda \in [0, 1]$ and $\varepsilon \in [0, 1]$, a type III_λ system with critical dimension ε exists [**11**]. Hence, the critical dimension can be regarded as a refinement of orbit equivalence.

It is of interest to define the notion of equivalence that preserves the critical dimension. We would want such equivalence to be finer than orbit equivalence but coarser than metric equivalence. Such a notion is induced by a natural class of maps satisfying the following lemma. Let ω_i^X denote the ith Radon–Nikodym derivative of the system $(X, \mathcal{B}, \mu, T)$.

LEMMA 2.3. [**2**] *Suppose that for μ-almost every $x \in X$,*

$$0 < \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{\sum_{i=0}^{n-1} \omega_{\sigma(i,x)}^X(x)} \le \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{\sum_{i=0}^{n-1} \omega_{\sigma(i,x)}^X(x)} < \infty.$$

Then the upper and lower critical dimensions of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ agree.

Definition 2.4. Call the orbit equivalence Φ a *Hurewicz map* if it satisfies inequalities in Lemma 2.3. The class of Hurewicz maps induces the *Hurewicz equivalence* of non-singular dynamical systems.

† Here, and in what follows, we shall take logarithms to base 2. This is natural in the case of products of two-point spaces. Changing to a different base leaves the critical dimension unchanged.

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Example. A Hurewicz map that is not a metric isomorphism. Let

$$X = Y = \prod_{i=1}^{\infty} \mathbb{Z}_2 = \prod_{i=1}^{\infty} \{0, 1\},$$

$\mathcal{B}$ be the usual Borel σ-algebra on X and Y, and T be the odometer on X (S on Y, respectively). Let ν_1 and ν_2 be distinct measures on $\{0, 1\}$ satisfying $\nu_1(0)\nu_1(1) \neq 0$ and $\nu_2(0)\nu_2(1) \neq 0$. Define product measures μ_1 on X and μ_2 on Y by

$$\mu_1 = \nu_1 \otimes \nu_2 \otimes \nu_1 \otimes \nu_2 \otimes \cdots$$

and

$$\mu_2 = \nu_2 \otimes \nu_1 \otimes \nu_2 \otimes \nu_1 \otimes \cdots .$$

Let the permutation $p : \mathbb{N} \to \mathbb{N}$ be given by $p(2n) = 2n - 1$ and $p(2n - 1) = 2n$ for $n \in \mathbb{N}$, and construct the orbit equivalence $\Phi : X \to Y$ by $\Phi((x_i)) = (x_{p(i)})$.

The map Φ induces an orbit equivalence between $(X, \mathcal{B}, \mu_1, T)$ and $(Y, \mathcal{B}, \mu_2, S)$: indeed, Φ is invertible and bi-measurable; moreover, it satisfies $\mu_2 \circ \Phi = \mu_1$, and the existence of a cocycle σ satisfying $S^n \Phi x = \Phi T^{\sigma(n,x)} x$ for μ_1-almost every $x \in X$ is easy to see.

We show that Φ is a Hurewicz map. Rewriting the denominator of the quotient in Lemma 2.3 in terms of μ_2, we get

$$\frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{\sum_{i=0}^{n-1} \omega_{\sigma(i,x)}^X(x)} = \frac{\sum_{i=0}^{n-1}((d\mu_1 \circ T^i)/d\mu_1)(x)}{\sum_{i=0}^{n-1}((d\mu_1 \circ T^{\sigma(i,x)})/d\mu_1)(x)} = \frac{\sum_{i=0}^{n-1}((d\mu_1 \circ T^i)/d\mu_1)(x)}{\sum_{i=0}^{n-1}((d\mu_2 \circ S^i)/d\mu_2)(\Phi x)}. \tag{1}$$

Proposition 2.1 of [**11**] implies that

$$h_{AC}(\mu_1) = h_{AC}(\mu_2) = \tfrac{1}{2}(h(\nu_1) + h(\nu_2)),$$

where $h(\nu_1)$ is the entropy of the partition $\{\{0\}, \{1\}\}$ of $(\mathbb{Z}_2, \mathcal{B}(\mathbb{Z}_2), \nu_1)$ ($h(\nu_2)$ and $(\mathbb{Z}_2, \mathcal{B}(\mathbb{Z}_2), \nu_2)$, respectively). Write $h_{AC}(\mu_1) = h_{AC}(\mu_2) = \alpha$. Now (1) can be written as

$$\frac{(1/n^\alpha) \sum_{i=0}^{n-1}((d\mu_1 \circ T^i)/d\mu_1)(x)}{(1/n^\alpha) \sum_{i=0}^{n-1}((d\mu_2 \circ S^i)/d\mu_2)(\Phi x)}.$$

By [**5**, Theorems 3.2 and 3.4], both the numerator and the denominator converge to a finite non-zero limit almost everywhere (as α is the critical dimension). Hence, the quotient also converges to a non-zero finite limit as $n \to \infty$. It follows that Φ is a Hurewicz map.

To see that Φ is not a metric isomorphism, take $x = (0, 0, \ldots) \in X$, for example. The set of paths in X whose first two entries are 0 has non-zero measure. Clearly, $T\Phi x = (1, 0, \ldots)$ but $\Phi T x = (0, 1, \ldots)$.

We can show that certain Bratteli–Vershik systems are not Hurewicz equivalent. Indeed, it will follow from Theorems 5.1 and 5.5 that Bratteli–Vershik systems with different upper and lower AC entropies are not Hurewicz equivalent. (See for example [**11**].)

Proof for Lemma 2.3. The proof uses techniques in [**12**]. Only the proof for the lower critical dimension will be presented; the proof for the upper critical dimension is similar.

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Let ω_i^Y denote the ith Radon–Nikodym derivative of $(Y, \mathcal{C}, \nu, S)$. Note that for μ-almost every $x \in X$,

$$\frac{d\nu \circ \Phi}{d\mu}(T^{\sigma(i,x)}x)\omega^X_{\sigma(i,x)}(x) = \omega_i^Y(\Phi x)\frac{d\nu \circ \Phi}{d\mu}(x)$$

holds. Take a set $A \subset X$ such that $\mu(A) > 0$ and on which $d\nu \circ \Phi / d\mu$ is bounded above and below. For $x \in A$, we have

$$\sum_{i=0}^{n-1} \chi_A(T^{\sigma(i,x)}x)\omega^X_{\sigma(i,x)}(x) \le K \sum_{i=0}^{n-1} \chi_{\Phi A}(S^i \Phi x)\omega_i^Y(\Phi x)$$

for some $K > 0$. By the Hurewicz ergodic theorem, for $0 < \epsilon < \mu(A)$, there exists $N(x) > 0$ such that for $n > N(x)$,

$$\begin{aligned}(\mu(A) - \epsilon)\sum_{i=0}^{n-1} \omega^X_{\sigma(i,x)}(x) &< \sum_{i=0}^{n-1} \chi_A(T^{\sigma(i,x)}x)\omega^X_{\sigma(i,x)}(x) \\ &\le K \sum_{i=0}^{n-1} \omega_i^Y(\Phi x).\end{aligned}$$

Denote by $m(x)$ and $M(x)$ the lim inf and lim sup of

$$\sum_{i=0}^{n-1} \omega_i^X(x) \Big/ \sum_{i=0}^{n-1} \omega^X_{\sigma(i,x)}(x) \quad \text{as } n \to \infty,$$

respectively. By the assumption, the inequalities $0 < m(x) \le M(x) < \infty$ hold for μ-almost every $x \in X$. We have that

$$\begin{aligned}(\mu(A) - \epsilon)m(x) \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega^X_{\sigma(i,x)}(x)}{n^{\alpha'}} &\le (\mu(A) - \epsilon) \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{n^{\alpha'}} \\ &\le (\mu(A) - \epsilon)M(x) \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega^X_{\sigma(i,x)}(x)}{n^{\alpha'}} \\ &\le M(x)K \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i^Y(\Phi x)}{n^{\alpha'}}\end{aligned}$$

for μ-almost every $x \in X$. From here, it is easy to obtain the desired result. □

3. *Bratteli–Vershik systems*

In this section, we give definitions of the components of Bratteli–Vershik systems $(X, \mathcal{B}, \mu, T)$. Much of the following is taken from Herman *et al* [**7**] and [**3**]; we reproduce it here to establish notation. Also, we explain assumptions imposed on the systems.

A vertex set $V = \bigcup_{i\ge0} V^i$ is a disjoint union of finite sets of vertices and an edge set $E = \bigcup_{i\ge1} E^i$ a disjoint union of finite sets of edges. The set V^0 is taken to be a singleton. The sets V and E are related by source and range maps $s, r : E \to V$ satisfying the following: $s(E^i) \subset V^{i-1}$ and $r(E^i) \subset V^i$; $s^{-1}(v) \ne \emptyset$ for any $v \in V$ and $r^{-1}(v) \ne \emptyset$ for any $v \in V \setminus V^0$. The pair (V, E) together with maps s and r is called a *Bratteli–Vershik diagram*.

Throughout this paper, we assume that Bratteli–Vershik diagrams satisfy the following conditions (BV1) and (BV2).

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(BV1) For each $i \geq 1$, every vertex in V^{i-1} is connected to every vertex in V^i by at least one edge.

(BV2) There exists $N > 0$ such that $|s^{-1}(v)| \leq N$ for any $v \in V$.

Diagrams satisfying condition (BV1) will be said to have *multiple edges*; any diagram can be telescoped to a diagram satisfying this condition. Condition (BV2) states that there is a uniform bound on the number of edges with a given vertex. This condition is used critically in the proof of the lower bound below. It was also needed in the case of a product space (the number of coordinates is uniformly bounded). We believe that even in the product case, if the number of coordinates can grow unboundedly, the formulas in Lemma 5.4 below may no longer hold.

If E is equipped with a partial order $\geq$ so that two edges e and e' are comparable if and only if $r(e) = r(e')$, then (V, E) is called an *ordered Bratteli–Vershik diagram*. In this case, for each $i \geq 1$ and $v \in V^i$, the set $E^i(v) = r^{-1}(v)$ is totally ordered. Denote by $e_i{}'(v)$ and $e_i{}''(v)$ the minimal edge and the maximal edge in $E^i(v)$, respectively. We may omit i or v when they are clear from the context.

For $m < n$ in $\mathbb{N}$, define P_m^n to be the set of paths from V^m to V^n,

$$P_m^n = \left\{ (e_{m+1}, \ldots, e_n) : \begin{array}{l} e_i \in E^i \text{ for } i = m+1, \ldots, n, \text{ and} \\ r(e_i) = s(e_{i+1}) \text{ for } i = m+1, \ldots, n-1 \end{array} \right\}.$$

For each $v \in V^n$, define

$$P_m^n(v) = \{(e_{m+1}, \ldots, e_n) \in P_m^n : r(e_n) = v\}.$$

Let $s(n) = |P_0^n|$. Let $X = X(V, E)$ be the associated set of infinite paths,

$$X = \{(x_i)_{i \geq 1} : x_i \in E^i \text{ and } r(x_i) = s(x_{i+1}) \text{ for all } i \geq 1\}.$$

Based on [**3**], we assume that for each $i \geq 1$, there are two distinct vertices v_i' and v_i'' in V^i such that every minimal edge in E^{i+1} starts from v_i' and every maximal edge in E^{i+1} starts from v_i''. Then the ordered Bratteli–Vershik diagram (V, E) is essentially simple in the sense of [**7**]. Denote by $x_{\max}$ and $x_{\min}$ the unique maximal path and the unique minimal path in X, respectively. Note that for each $i \geq 1$, the inequalities $2 \leq |V^i| \leq N$, $2^2 \leq |E^i| \leq N^2$, and $2^i \leq s(i) \leq N^i$ hold.

For each $(e_1, \ldots, e_n) \in P_0^n$, define a cylinder set of length n by

$$[e_1, \ldots, e_n]_1^n = \{(x_i)_{i \geq 1} \in X : x_i = e_i \text{ for } i = 1, \ldots, n\}.$$

Cylinder sets of length n generate cylinder sets of the form

$$[e_m, \ldots, e_n]_m^n = \{(x_i)_{i \geq 1} \in X : x_i = e_i \text{ for } i = m, \ldots, n\}.$$

The σ-algebra $\mathcal{B}$ is generated by the cylinder sets. Let $\mathcal{P}_n$ denote the partition of X by cylinder sets of length n.

The *Vershik transformation* $T : X \to X$ is defined as follows. For $x_{\max} \in X$, let $T x_{\max} = x_{\min}$. For $x \in X$ but $x \neq x_{\max}$, at least one edge in x is not maximal. Let x_i be the non-maximal edge with the smallest coordinate i. Denote by e_i the edge in $E^i(r(x_i))$ next largest to x_i and by $(e_1, \ldots, e_{i-1})$ the unique minimal path in $P_0^{i-1}(s(e_i))$.

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Let $Tx = (e_1, \ldots, e_{i-1}, e_i, x_{i+1}, \ldots)$. It is easy to show that the Vershik transformation is invertible on X and bi-measurable with respect to $\mathcal{B}$.

A matrix $P^i = (P^i_{v,e})$, where $(v, e) \in V^{i-1} \times E^i$, is stochastic if it satisfies the following conditions (i) and (ii):

(i) $P^i_{v,e} > 0$ if and only if $s(e) = v$;

(ii) $\sum_{s(e)=v}^{e \in E^i} P^i_{v,e} = 1$ for each $v \in V^{i-1}$.

Non-zero entries $P^i_{v,e}$ are called weights on edges $e \in E^i$. Given a sequence of stochastic transition matrices P^i, define a *Markov measure* μ on cylinder sets by

$$\mu([e_1, \ldots, e_n]_1^n) = P^1_{s(e_1),e_1} \cdots P^n_{s(e_n),e_n}. \tag{2}$$

Define the push-forward measure ν^n on V^n by

$$\nu^n(v) = \mu(\{(x_i)_{i \geq 1} \in X : r(x_n) = v\}) \quad \text{for each } v \in V^n.$$

We assume that the measure μ satisfies the following conditions (M1) and (M2).

(M1) For μ-almost every $x \in X$, there exist an integer $n \geq 1$ and a block $e_1 e_2 \cdots e_n$ such that $x_n < e_n$ and $\mu([e_1, e_2, \ldots, e_n, x_{n+1}]_1^{n+1}) > 0$.

(M2) For μ-almost every $x \in X$, there exist an integer $m \geq 1$ and a block $f_1 f_2 \cdots f_m$ such that $f_m < x_m$ and $\mu([f_1, f_2, \ldots, f_m, x_{m+1}]_1^{m+1}) > 0$.

Then $n = \sup\{i \geq 1 : (Tx)_i \neq x_i\} < \infty$ for μ-almost every $x \in X$. The system is non-singular; in fact,

$$\frac{d\mu \circ T}{d\mu}(x) = \frac{P^1_{s(e_1),e_1} \cdots P^n_{s(e_n),e_n}}{P^1_{s(x_1),x_1} \cdots P^n_{s(x_n),x_n}},$$

where $e_i = (Tx)_i$ for $i = 1, \ldots, n$. The measure μ is ergodic, conservative, and non-atomic (see Aaronson [**1**]).

4. *Average coordinate entropy and the law of large numbers*

We define the average coordinate entropy. We state Rosenblatt-Roth's law of large numbers for Markov measures on product spaces. We use it to obtain a convergence formula involving the average coordinate entropy. In the following argument, order is irrelevant and hence is not considered.

Let $(X, \mathcal{B}, \mu, T)$ be a Bratteli–Vershik system defined in §3. Let $H(\mathcal{P}_n)$ denote the entropy of μ by the partition $\mathcal{P}_n$.

Definition 4.1. The *lower average coordinate entropy* of $(X, \mathcal{B}, \mu, T)$ is defined by

$$\underline{h}_{AC}(\mu) = \liminf_{n \to \infty} \frac{H(\mathcal{P}_n)}{\log s(n)}.$$

Definition 4.2. The *upper average coordinate entropy* of $(X, \mathcal{B}, \mu, T)$ is defined by

$$\overline{h}_{AC}(\mu) = \limsup_{n \to \infty} \frac{H(\mathcal{P}_n)}{\log s(n)}.$$

When the upper and lower average coordinate entropies coincide with each other, the common value is called the *average coordinate entropy* of the system.

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Before giving the Rosenblatt-Roth result, we discuss when a Bratteli–Vershik diagram is an infinite product space. A Bratteli–Vershik diagram $Y = Y(V_Y, E_Y)$ is a product space if for all $i \geq 1$, every vertex in V_Y^{i-1} is connected to every vertex in V_Y^i by exactly one edge. In this case, each edge $e \in E_Y^i$ is indexed by the pair $(s(e), r(e))$, and its weight is written by $P^i_{s(e),r(e)}$. Write $V_Y^i = \mathbb{Z}_{l(i)} = \{0, 1, \ldots, l(i) - 1\}$. Then each path $y \in Y$ is realized as a sequence $(y_i)_{i \geq 0}$ of coordinates $y_i \in \mathbb{Z}_{l(i)}$. Transition matrices P^i are realized as $l(i-1) \times l(i)$ stochastic matrices with (y_{i-1}, y_i)th entry $P^i_{y_{i-1},y_i}$. See [**3**] for details.

Suppose that a matrix $P = (P_{i,j})$ is stochastic. Then

$$\tau(P) = \frac{1}{2} \max_{i,j} \sum_k |P_{i,k} - P_{j,k}| = 1 - \min_{i,j} \sum_k \{P_{i,k}, P_{j,k}\}$$

is known as the ergodic coefficient of P. It is easy to see that given a sequence of transition matrices P^i, the sequence of ergodic coefficients $\tau(P^i)$ is bounded above by some $\rho < 1$ if and only if all entries $P^i_{j,k}$ are bounded below by some $\eta > 0$ [**6**].

THEOREM 4.3. [**14**] *Let ν be a Markov measure on the product space Y. Suppose that the transition matrices P^i satisfy $\tau(P^i) < \rho < 1$ for all $i \geq 1$. For each i, let $f_i(y) = f_i(y_i)$ be a function that depends only on the ith coordinate of $y \in Y$. Suppose that the variances $\mathbf{E}({f_i}^2)$ of the f_i satisfy $\sum_{i=1}^{\infty} \mathbf{E}({f_i}^2)^2 / i^2 < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} f_i(y) - \sum_{i=1}^{n} \mathbf{E}(f_i) \right) = 0 \quad \textit{for } \nu\textit{-almost every } y \in Y.$$

We also use a result of Wen [**15**] on the convergence of the functions $-\log P^{i+1}_{y_i,y_{i+1}}$. Let ν be a Markov measure on the product space Y. For $y \in Y$, define

$$H^i_\nu(y) = H(P^{i+1}_{y_i,0}, \ldots, P^{i+1}_{y_i,l(i+1)-1}) = -\sum_{k=0}^{l(i+1)-1} P^{i+1}_{y_i,k} \log P^{i+1}_{y_i,k}.$$

THEOREM 4.4. [**15**] *For a non-homogeneous Markov information measure with alphabet $\{0, 1, \ldots, s-1\}$ and transition matrices $P^i = (P^i_{j,k})$ with $P^i_{j,k} > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\log P^{i+1}_{y_i,y_{i+1}} + H^i_\nu(y)) = 0 \quad \textit{for } \nu\textit{-almost every } y \in Y.$$

It should be noted that Theorem 4.4 does not require that the entries of the transition matrices should be bounded below by some $\eta > 0$.

We extend the definition of H^i_ν to the setting of Markov measures μ on Bratteli–Vershik diagrams $X = X(V_X, E_X)$. For $x \in X$, define

$$\begin{aligned} H^i_\mu(x) &= H(\{P^{i+1}_{s(e),e} : e \in E^{i+1}_X \text{ and } s(e) = r(x_i)\}) \\ &= -\sum_{\substack{e \in E^{i+1}_X \\ s(e) = r(x_i)}} P^{i+1}_{s(e),e} \log P^{i+1}_{s(e),e}. \end{aligned}$$

Clearly, $H(\mathcal{P}_n) = \sum_{i=0}^{n-1} \mathbf{E}(H^i_\mu)$ is the usual entropy of μ by the partition $\mathcal{P}_n$. It is easy to extend the result of Theorem 4.4 to apply to Markov measures on Bratteli–Vershik

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diagrams providing that for all $i \geq 1$, the number of edges in E_X^i having a common source in V_X^{i-1} is bounded above by some $N > 0$. It is in this form that we use Theorem 4.4 later.

We use Theorem 4.3 to obtain the convergence of the H_μ^i. To this end, given a Bratteli–Vershik diagram $X = X(V_X, E_X)$ with a Markov measure μ, we construct a product space $Y = Y(V_Y, E_Y)$ (without putting an order) with a Markov measure ν as follows. Let Y have the same set of vertices at any level as X does: $V_Y^i = V_X^i$ for all $i \geq 0$. We may use the same letter to denote the corresponding vertices in V_X^i and V_Y^i. Every vertex in V_Y^i is connected to every vertex in V_Y^{i+1} by exactly one edge. The measure ν on Y is induced by μ by letting the weight of each edge $e \in E_Y$ be the sum of the weights of the edges in E_X with source $s(e) \in V_X$ and range $r(e) \in V_X$.

Assume that the transition matrices P^i of Y satisfy $\tau(P^i) < \rho < 1$ for all $i \geq 1$. Recall that each path y in a product space Y is realized as a sequence $(y_i)_{i\geq 0}$ of coordinates $y_i \in V_Y^i$. Let

$$f_i(y) = -\sum_{s(e)=y_i \in V_X^i}^{e\in E_X^{i+1}} P_{s(e),e}^{i+1} \log P_{s(e),e}^{i+1}.$$

Notice that $f_i(y)$ depends only on the ith coordinate of y. It is easy to show that the variances $\mathbf{E}({f_i}^2)$ of the f_i satisfy $\sum_{i=1}^\infty \mathbf{E}({f_i}^2)^2/i^2 < \infty$. By Theorem 4.3,

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^n f_i(y) - \sum_{i=1}^n \mathbf{E}(f_i)\right) = 0 \quad \text{for } \nu\text{-almost every } y \in Y.$$

Note that $H_\mu^i(x) = f_i(y)$, where

$$x = (x_i)_{i\geq 1} \in X, \qquad y = (s(x_i))_{i\geq 1} \in Y.$$

Also,

$$\mathbf{E}(H_\mu^i) = \mathbf{E}(f_i) \quad \text{and} \quad \mathbf{E}({H_\mu^i}^2) = \mathbf{E}({f_i}^2)$$

hold. By the construction of ν, we have that

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^n H_\mu^i(x) - \sum_{i=1}^n \mathbf{E}(H_\mu^i)\right) = 0 \quad \text{for } \mu\text{-almost every } x \in X.$$

5. *Computation of the critical dimensions*

Our goal is to prove the following theorem.

THEOREM 5.1. *Suppose that there exists some $\eta > 0$ such that the sum of the weights of the edges having the common source and range is bounded below by η. The lower critical dimension α is given by*

$$\alpha = \liminf_{n\to\infty} -\frac{\sum_{i=1}^n \log P_{s(x_i),x_i}^i}{\log s(n)} = \underline{h}_{AC}(\mu) \quad \textit{for } \mu\textit{-almost every } x \in X.$$

To prove Theorem 5.1, we need some definitions and lemmas. Given a Bratteli–Vershik system $(X, \mathcal{B}, \mu, T)$, define quantities n_p, r_p^i, and I_p as follows; using the Borel–Cantelli lemma, these quantities can be shown to exist for μ-almost every $x \in X$. Take $n_p(x) = n_p$ be the index for the pth edge of x for which $x_{n_p} \notin E_{\max}$ (and ∞ if no such place exists).

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Let $r_p^i(x) = r_p^i$ be the ith non-negative integer k for which

$$(T^k x)_j \in E_{\max}, \quad 1 \le j \le n_p - 1,$$
$$r((T^k x)_{n_p}) = r(x_{n_p}) \quad \text{and} \quad (T^k x)_{n_p} \notin E_{\max},$$
$$(T^k x)_j = x_j, \quad n_p + 1 \le j.$$

Then $r_p^i + 1$ is the odometer power that changes the n_pth edge in the forward orbit of x for the ith time. Note that

$$r_p^{i+1}(x) = r_p^i(x) + |P_0^{n_p - 1}(s((T^{r_p^i + 1} x)_{n_p}))|$$

and that, for a fixed $p \in \mathbb{N}$, there are

$$q(p) = |\{e \in E^{n_p}(r(x_{n_p})) : x_{n_p} \le e < e''\}|$$

entries in the sequence $r_p^i(x)$.

Define $I_p(x) = I_p$ to be the first integer k for which $(T^k x)_j \in E_{\max}$, $1 \le j \le n_p$. Observe that

$$I_p(x) = r_p^{q(p)}(x) + |P_0^{n_p - 1}(s((T^{r_p^{q(p)} + 1} x)_{n_p}))|$$

and

$$I_p(x) = r_{p+1}^1(x)$$

and that, by the connectivity assumption (BV1), $s(n_p - 2) \le I_p(x) \le s(n_p)$ holds†.

LEMMA 5.2. *The following limits hold.*

(i) $$\lim_{i \to \infty} -\frac{\log \nu^i(r(x_i))}{i} = 0 \quad \textit{for } \mu\textit{-almost every } x \in X.$$

(ii) $$\lim_{i \to \infty} -\frac{\log P^i_{s(x_i), x_i}}{i} = 0 \quad \textit{for } \mu\textit{-almost every } x \in X.$$

Proof. Given $\epsilon > 0$, define $A_i = \{x \in X : -\log \nu^i(r(x_i))/i \ge \epsilon\}$. Then $\mu(A_i) \le N \cdot 2^{-\epsilon i}$ and $\sum \mu(A_i) \le N \cdot \sum 2^{-\epsilon i}$, which is a summable series. The Borel–Cantelli lemma implies that for μ-almost every $x \in X$, we have that $-\log \nu^i(r(x_i))/i < \epsilon$ for all but a finite number of i. The proof for (ii) is similar. □

LEMMA 5.3. *The lower critical dimension α satisfies the following.*

(i) $$\alpha \le \liminf_{p \to \infty} -\frac{\sum_{i=1}^{n_p} \log P^i_{s(x_i), x_i}}{\log s(n_p - 2)} \quad \textit{for } \mu\textit{-almost every } x \in X.$$

(ii) $$\alpha \ge \liminf_{p \to \infty} -\frac{\sum_{i=1}^{n_{k(p-1)}} \log P^i_{s(x_i), x_i}}{\log s(n_p)} \quad \textit{for } \mu\textit{-almost every } x \in X.$$

Here $n_{k(p-1)}$ is the largest integer $j < n_{p-1}$ for which $x_j \notin E_{\max}$ and

$$\sum_{\substack{e \in E^j(r(x_j)) \\ x_j < e \le e''}} \mu([e]_j^j)/\nu^j(r(x_j)) \ge 1/(j+1)^2 ‡.$$

† This is the crucial point where we use (BV1). In fact, the inequalities $s(n_p - 2) \le I_p(x) \le s(n_p)$ are weaker than (BV1) and sufficient to ensure that Theorems 5.1 and 5.5 hold. The inequalities can be further weakened to $s(n_p - k) \le I_p(x) \le s(n_p)$ for some fixed $k > 0$, for example, with the theorems still holding.

‡ Here, and in what follows, $1/(j+1)^2$ can be replaced with any summable sequence $\{a_j\}$ for which $\log a_j / j \to 0$ as $j \to \infty$.

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Proof. Fix $j > 0$. For μ-almost every $x \in X$,

$$\begin{aligned}
\sum_{i=r_j^1+1}^{I_j} \omega_i(x) &= \sum_{k=1}^{q(j)} \sum_{i=r_j^k+1}^{r_j^k+|P_0^{n_j-1}(s((T^{r_j^{k+1}}x)_{n_j}))|} \omega_i(x) \\
&= \sum_{x_{n_j}<e\le e''}^{e\in E^{n_j}(r(x_{n_j}))} \frac{\mu([e]_{n_j}^{n_j})}{\mu([x]_1^{n_j})} \\
&\le \frac{\nu^{n_j}(r(x_{n_j}))}{\mu([x]_1^{n_j})} \\
&\le \frac{1}{\mu([x]_1^{n_j})}. \qquad (3)
\end{aligned}$$

Thus, for any $\alpha' > 0$,

$$\begin{aligned}
\frac{\sum_{i=0}^{I_p}\omega_i(x)}{(I_p+1)^{\alpha'}} &= \frac{1}{(I_p+1)^{\alpha'}}\bigg(\sum_{i=0}^{r_1^1}\omega_i(x) + \sum_{j=1}^{p}\sum_{i=r_j^1+1}^{I_j}\omega_i(x)\bigg) \\
&\le \frac{1}{(I_p+1)^{\alpha'}}\bigg(\sum_{i=0}^{r_1^1}\omega_i(x) + \sum_{j=1}^{p}\frac{1}{\mu([x]_1^{n_j})}\bigg) \\
&\le \frac{1}{(I_p+1)^{\alpha'}}\bigg(\sum_{i=0}^{r_1^1}\omega_i(x) + \sum_{j=1}^{p}\frac{1}{\mu([x]_1^{n_p})}\bigg) \\
&\le \frac{1}{s(n_p-2)^{\alpha'}}\bigg(\sum_{i=0}^{r_1^1}\omega_i(x) + \frac{p}{\mu([x]_1^{n_p})}\bigg). \qquad (4)
\end{aligned}$$

Rewrite part of the right-hand side of (4) as follows:

$$\frac{1}{s(n_p-2)^{\alpha'}}\frac{p}{\mu([x]_1^{n_p})} = 2^{\log s(n_p-2)\left(-\alpha' - \frac{\log\mu([x]_1^{n_p})}{\log s(n_p-2)} + \frac{\log p}{\log s(n_p-2)}\right)}.$$

If $\alpha' > \liminf_{p\to\infty}(-\log\mu([x]_1^{n_p}) + \log p)/\log s(n_p-2)$, then taking lim inf as $p \to \infty$ in (4) we obtain that

$$\liminf_{p\to\infty}\frac{\sum_{i=0}^{I_p}\omega_i(x)}{(I_p+1)^{\alpha'}} = 0.$$

So, α' is strictly greater than the lower critical dimension α. Thus, α satisfies

$$\alpha \le \liminf_{p\to\infty}\bigg(-\frac{\log\mu([x]_1^{n_p})}{\log s(n_p-2)} + \frac{\log p}{\log s(n_p-2)}\bigg) \quad \text{for } \mu\text{-almost every } x \in X.$$

Since $2^{n_p-2} \le s(n_p-2)$ and $p \le n_p$, we have that

$$\alpha \le \liminf_{p\to\infty} -\frac{\log\mu([x]_1^{n_p})}{\log s(n_p-2)} \quad \text{for } \mu\text{-almost every } x \in X.$$

This proves Lemma 5.3(i).

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To prove Lemma 5.3(ii), suppose that $I_{p-1}(x)+1 \leq n \leq I_p(x) \leq s(n_p)$. Then, for any $\alpha' > 0$,

$$\frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^{\alpha'}} \geq \frac{\sum_{i=0}^{I_{p-1}} \omega_i(x)}{s(n_p)^{\alpha'}}. \tag{5}$$

As $n_{k(p-1)} = n_{p'}$ for some $p' < p-1$, it follows from (3) that

$$\begin{aligned}\frac{\sum_{i=0}^{I_{p-1}} \omega_i(x)}{s(n_p)^{\alpha'}} &\geq \frac{\sum_{i=r_{p'}^1+1}^{I_{p'}} \omega_i(x)}{s(n_p)^{\alpha'}} \\ &= \frac{1}{s(n_p)^{\alpha'}}\Bigg(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \frac{\mu([e]_{n_{p'}}^{n_{p'}})}{\mu([x]_1^{n_{p'}})}\Bigg) \\ &= 2^{\log s(n_p)\left(-\alpha' - \frac{\log \mu([x]_1^{n_{p'}})}{\log s(n_p)} + \frac{\log\left(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}})\right)}{\log s(n_p)}\right)}.\end{aligned} \tag{6}$$

If

$$\alpha' < \liminf_{p\to\infty}\Bigg(-\log \mu([x]_1^{n_{p'}}) + \log\Bigg(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}})\Bigg)\Bigg)\Big/\log s(n_p),$$

then taking lim inf as $p \to \infty$ in (6), we obtain that

$$\liminf_{p\to\infty} \frac{\sum_{i=0}^{I_{p-1}} \omega_i(x)}{s(n_p)^{\alpha'}} = \infty.$$

From (5), we see that α' is strictly less than the lower critical dimension α. Thus, α satisfies

$$\alpha \geq \liminf_{p\to\infty}\Bigg(-\frac{\log \mu([x]_1^{n_{p'}})}{\log s(n_p)} + \frac{\log(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}}))}{\log s(n_p)}\Bigg). \tag{7}$$

We claim that

$$\lim_{p\to\infty} \frac{\log(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}}))}{\log s(n_p)} = 0.$$

From the assumption on $n_{p'} = n_{k(p-1)}$,

$$\frac{1}{(n_{p'}+1)^2} \leq \frac{\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}})}{\nu^{n_{p'}}(r(x_{n_{p'}}))} \leq 1.$$

Taking the logarithm and then dividing every term by $\log s(n_p)$, we obtain that

$$-\frac{2\log(n_{p'}+1)}{\log s(n_p)} \leq \frac{\log(\sum_{x_{n_{p'}}<e\leq e''}^{e\in E^{n_{p'}}(r(x_{n_{p'}}))} \mu([e]_{n_{p'}}^{n_{p'}}))}{\log s(n_p)} - \frac{\log \nu^{n_{p'}}(r(x_{n_{p'}}))}{\log s(n_p)} \leq 0.$$

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Since $n_{p'} = n_{k(p-1)} < n_p$ and $2^{n_p} \le s(n_p)$, we see that $\log(n_{p'}+1)/\log s(n_p)$ tends to 0 as $p \to \infty$. This, combined with Lemma 5.2(i), completes the proof for the claim.

From inequality (7), the lower critical dimension α satisfies

$$\alpha \ge \liminf_{p\to\infty} -\frac{\log \mu([x]_1^{n_{p'}})}{\log s(n_p)} \quad \text{for } \mu\text{-almost every } x \in X.$$

This completes the proof for Lemma 5.3(ii). □

By Lemma 5.3, the lower critical dimension α satisfies

$$\liminf_{p\to\infty} -\frac{\sum_{i=1}^{n_{k(p-1)}} \log P^i_{s(x_i),x_i}}{\log s(n_p)} \le \alpha \le \liminf_{p\to\infty} -\frac{\sum_{i=1}^{n_p} \log P^i_{s(x_i),x_i}}{\log s(n_p - 2)}$$

for μ-almost every $x \in X$.

LEMMA 5.4. *In the setting of Lemma 5.3, on a set of positive measure,*

$$\liminf_{p\to\infty} -\frac{\sum_{i=1}^{n_{k(p-1)}} \log P^i_{s(x_i),x_i}}{\log s(n_p)} = \alpha = \liminf_{p\to\infty} -\frac{\sum_{i=1}^{n_p} \log P^i_{s(x_i),x_i}}{\log s(n_p - 2)}.$$

Proof. Define the probability measure ν^i_v on $E^i(v)$ for $i \ge 1$ and $v \in V^i$ as follows:

$$\nu^i_v(e) = \frac{\mu([e]^i_i)}{\nu^i(v)}, \quad e \in E^i(v).$$

We will say that the measure ν^i_v on $E^i(v)$ is *e-bad* ($e' \le e < e''$) if

$$\nu^i_v\bigg(\bigcup_{e<f\le e''} f\bigg) < \frac{1}{(i+1)^2}.$$

Otherwise, we will say that ν^i_v is *e-good*. If $\nu^i_v(e'') \ge 1/(i+1)^2$, we say that ν^i_v is *all good*. Consider the set

$$\overline{X} = \{(x_n)_{n\ge1} \in X : e' \le x_n \le t_v(n) \text{ when } r(x_n) = v, n \ge 1\},$$

where $t_v(i)$ is e'' if ν^i_v is all good, otherwise $t_v(i)$ is the smallest edge $e \in E^i(v)$, $e' \le e < e''$, for which ν^i_v is e-bad (this may be e'). Now $\mu(\overline{X}) > 0$.

Given $\epsilon > 0$, let B_n be the event defined by $x \in B_n$ if and only if $x \in \overline{X}$, $x_n \notin E_{\max}$, and

$$-\frac{\sum_{i=k(n)+1}^{n} \log P^i_{s(x_i),x_i}}{n} \ge \epsilon,$$

where $k(n)$ is the largest integer $j < n$ for which $x_j \notin E_{\max}$ and $\nu^j_{r(x_j)}$ is x_j-good.

This last condition implies that if $x \in B_n$, then it belongs to the cylinder set

$$C_{n,x} = [t_{r(x_{k(n)+1})}(k(n)+1), \ldots, t_{r(x_{n-1})}(n-1), x_n]^n_{k(n)+1}$$

with $\mu(C_{n,x}) \le 2^{-\epsilon n}$. Recall that $2 \le |V^n|$ and the number of edges in E^n sharing a common source does not exceed $N > 0$ for all $n \ge 1$. Hence, there are at most $N(N^2 - 2)(n-1)$ of these cylinder sets that cover B_n, and thus $\mu(B_n) \le N(N^2-2)(n-1)\cdot 2^{-\epsilon n}$, which is a summable sequence. It follows from the Borel–Cantelli lemma that for almost every x in $\overline{X}$, either $x_n \in E_{\max}$ or $-\sum_{i=k(n)+1}^{n} \log P^i_{s(x_i),x_i}/n < \epsilon$ for all but a finite number of n. By the essential simplicity of X and non-singularity of μ, we have our result. □

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Proof for Theorem 5.1. To prove the first equality, all that is left to verify is the claim

$$\lim_{p\to\infty} -\frac{\sum_{i=n_{p-1}+1}^{n_p} \log P^i_{s(x_i),x_i}}{\log s(n_p)} = 0$$

on the set $\overline{X}$ of positive measure since, if this limit holds, then

$$\alpha = \liminf_{p\to\infty} -\frac{\sum_{i=1}^{n_p} \log P^i_{s(x_i),x_i}}{\log s(n_p)} = \liminf_{n\to\infty} -\frac{\sum_{i=1}^{n} \log P^i_{s(x_i),x_i}}{\log s(n)}$$

on a set of positive measure by Lemma 5.4, and the result follows from ergodicity.

Given $\epsilon > 0$, let $D_{u,v}$ be the set defined by $x \in D_{u,v}$ if and only if $x \in \overline{X}$, $x_i \in E_{\max}$ for $i = u, u+1, \ldots, u+v$, and

$$-\frac{\sum_{i=u}^{u+v} \log P^i_{s(x_i),x_i}}{u+v} \geq \epsilon.$$

Then $\mu(D_{u,v}) \leq 2^{-\epsilon(u+v)}$ and

$$\mu(\overline{D_u}) = \mu\bigg(\bigcup_{v=1}^{\infty} D_{u,v}\bigg) \leq 2^{-\epsilon u}\frac{2^{-\epsilon}}{1-2^{-\epsilon}}.$$

Thus, the sequence $\mu(\overline{D_u})$ is summable. This, combined with Lemma 5.2(ii), completes the proof for the claim.

To prove the second equality, notice that by the triangle inequality we have that

$$\frac{1}{\log s(n)}\bigg|-\sum_{i=1}^{n} \log P^i_{s(x_i),x_i} - H(\mathcal{P}_n)\bigg|$$
$$\leq \frac{1}{\log s(n)}\bigg|-\sum_{i=1}^{n} \log P^i_{s(x_i),x_i} - \sum_{i=0}^{n-1} H^i_\mu(x)\bigg| + \frac{1}{\log s(n)}\bigg|\sum_{i=0}^{n-1} H^i_\mu(x) - H(\mathcal{P}_n)\bigg|.$$

The first term converges by Theorem 4.4, and the second by Theorem 4.3. □

Entirely similar methods give a result for the upper critical dimension.

THEOREM 5.5. *Suppose that there exists some $\eta > 0$ such that the sum of the weights of the edges having the common source and range is bounded below by η. The upper critical dimension β is given by*

$$\beta = \limsup_{n\to\infty} -\frac{\sum_{i=1}^{n} \log P^i_{s(x_i),x_i}}{\log s(n)} = \overline{h}_{AC}(\mu) \quad \textit{for } \mu\textit{-almost every } x \in X.$$

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