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LONG-RANGE ORIENTATIONAL ORDER OF RANDOM NEAR-LATTICE HARD SPHERE AND HARD DISK PROCESSES

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Abstract

We show that a point process of hard spheres exhibits long-range orientational order. This process is designed to be a random perturbation of a three-dimensional lattice that satisfies a specific rigidity property; examples include the FCC and HCP lattices. We also define two-dimensional near-lattice processes by local geometry-dependent hard disk conditions. Earlier results about the existence of long-range orientational order carry over, and we obtain the existence of infinite-volume measures on two-dimensional point configurations that turn out to follow the orientation of a fixed triangular lattice arbitrarily closely.

Keywords: spontaneous symmetry breaking; hard sphere; hard disk; rigidity estimate

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1. Introduction

Random hard disk and hard sphere processes are among the most easily defined physically interesting point processes. Rigorous mathematical results about their behavior at high intensity are limited to two-dimensional systems. It remains an open question whether a phase transition with possibly orientational symmetry breaking occurs in either two or three dimensions. If breaking of rotational symmetry in either of these models could be shown, it would give rise to speculation whether such simple pair interaction could result in crystallization phenomena. In order to simplify the models, we exclude cavities and other crystal defects from the models, and study random hard disk and hard sphere processes that are locally crystals. In our models, being locally crystal implies being a crystal on a long range. There is a lower bound on orientational correlation that is uniform in the distance, and this bound can be made arbitrarily large by taking very 'tight' boundary conditions. Theorem 2.1 is the first rigorous result about hard sphere long-range orientational order to our knowledge.

In the previous work [7], we considered hard disk processes with disks of radius $1/2$ that have the structure of a triangular lattice and neighboring disks having an upper bound on their distance. We showed the existence of natural 'uniform' measures on these permitted configurations that exhibit uniform long-range orientational order. In the first half of this work (Section 2) we show that the same arguments apply to some three-dimensional lattices. In the second half (Section 3) we show that the result in the two-dimensional case can be formulated independently of an underlying triangular lattice structure that was explicitly present in the definition of the probability measures in [7]. Thus we show that being a crystal locally

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implies being a crystal on a greater scale in this particular model. We only require the local geometry-dependent condition that every point has exactly six points in an annulus with radii 1 and $1+\alpha$ around them. In both sections we will have the parameter α, which gives the maximal distance of neighboring points. This α needs to be sufficiently small that some local conditions are fulfilled, but it is on the macroscopic order of about $1/2$, so not particularly small. Fluctuations from the orientation of a fixed lattice, however, can be made arbitrarily small; in particular, they can be made many orders smaller than α.

Similar but not hard-core models were considered in [15] without defects and in [11], and [2] with lattice defects. Models for dislocations were treated in [3] on the mesoscopic scale and in [10] for the Ariza–Ortiz model. It is possible to introduce bounded, separated missing regions as defects into our two-dimensional model using techniques similar to those in [11]. We think it is possible for three dimensions but we have not carried it out. Further, the techniques of Section 3 can possibly be carried out in three dimensions, but an analog of Lemma 3.1 is required together with suitable boundary conditions, since in three dimensions several close-packed lattices are possible analogs of the triangular lattice.

These simplified models with well-defined lattice structure and possible defects are motivated by more natural hard sphere models defined with respect to a Poisson point process at a given intensity $z>0$. The set of Gibbs measures for these natural models is defined similarly to our definition of $\mathcal{G}^z$ in Section 3. They are basically sequential limits of Poisson point processes in bounded domains – as the domains tend to $\mathbb{R}^d$ – conditioned so that no pair of points have distance smaller than one. In these natural models, instead of imposing complex geometry-dependent interactions, only hard-core repulsion is required. As a consequence, even at high intensity, all kinds of possible lattice defects emerge as soon as the domain gets large enough. It is believed that in two or more dimensions there are multiple Gibbs measures in $\mathcal{G}^z$ for sufficiently high intensity z. Their structure is believed to differ in the typical relative orientation of nearby points. It is shown in [16] that in two dimensions any of these measures in $\mathcal{G}^z$ are translation-invariant at any intensity $z>0$, and in [17] a logarithmic lower bound is given on the mean square translational displacement of particles. These results prevent Gibbs measures from having long-range positional order. One strategy for showing that $\mathcal{G}^z$ is not a singleton for sufficiently high $d \geq 2$ and $z>0$ is to search for a measure in $\mathcal{G}^z$ that is not rotation-invariant. The existence of this is called the breaking of rotational symmetry (of the energy function). Showing that such a measure is supported on a perturbed lattice structure with long-range orientational order would be an even stronger result which is connected to the widely studied crystallization problem, even though the crystallization problem is usually studied for different interactions.

We would also like to mention the recent result [12] that at low intensity disagreement percolation results imply the uniqueness of the Gibbs state, whereas at high intensity it is shown in [1] that hard disks percolate with the percolation radius chosen sufficiently large, which was generalized in [14] to arbitrary percolation radii. Percolation is necessary for crystallization, but to our knowledge breaking of rotational symmetry cannot be concluded from it.

2. The three-dimensional enumerated model

In this section we show that the arguments of [7] can be applied to some three-dimensional lattices to obtain results similar to those in [7] about long-range orientational order for random perturbations of such lattices.

2.1. Configuration space

We consider three-dimensional lattices with well-defined distance between nearest neighbors (to be normalized to 1) that fulfill two conditions. Firstly, the nearest-neighbor edges of the lattice have to define a tessellation of $\mathbb{R}^3$ by regular tetrahedra and octahedra. Secondly, the lattice has to be translation-invariant in three linearly independent directions. We remark that regular tetrahedra and octahedra can be replaced by any rigid polyhedron (a polyhedron with all faces being triangles) that satisfies an analog of the rigidity estimates in Lemmas 2.1 and 2.2, and their volume has positive partial derivatives with respect to their edge lengths. We note that by Cauchy's theorem, the volume is uniquely defined for rigid polyhedra when the edge lengths are given.

Examples of such lattices are the face-centered cubic lattice and the hexagonal close-packed lattice. For definitions see [13]. Note that being translation-invariant does not mean that the lattice has to be a Bravais lattice, i.e. of the form $\mathbb{Z}n_1 + \mathbb{Z}n_2 + \mathbb{Z}n_3$ for some vectors $n_i \in \mathbb{R}^3$. Bravais lattices are translation-invariant, but a union of Bravais lattices might still be translation-invariant but no longer a Bravais lattice, for which the hexagonal close-packed lattice serves as an example.

Let the set $I \subset \mathbb{R}^3$ denote one of the lattices that fulfill both criteria. We assume $0 \in I$ and think of I as an index set which is going to be used to parametrize countable point configurations in $\mathbb{R}^3$. Let I have translational symmetry by the linearly independent vectors $t_1, t_2, t_3 \in \mathbb{R}^3$ and define the set $T = \mathbb{Z}t_1 + \mathbb{Z}t_2 + \mathbb{Z}t_3$. Define the quotient space $I_n := I/nT$ for $n \in \mathbb{N}$. We will think of I_n as a specific set of representatives in the half-open parallelepiped U_n spanned by nt_1, nt_2, nt_3, i.e. $U_n = n\{xt_1 + yt_2 + zt_3 \mid x, y, z \in [0, 1)\}$.

A *parametrized point configuration* in $\mathbb{R}^3$ is a map $\omega\colon I \to \mathbb{R}^2$, $x \mapsto \omega(x)$ that determines the point configuration $\{\omega(x) \mid x \in I\} \subset \mathbb{R}^3$. For the set of all parametrized point configurations we introduce the character $\Omega = \{\omega\colon I \to \mathbb{R}^2\}$. Note that a single point configuration $\{\omega(x) \mid x \in I\}$ can be parametrized by many different $\omega \in \Omega$.

Let $\alpha \in (0, 1]$ be an arbitrary but fixed real to be fixed later. An *n-periodic parametrized point configuration* with edge length $l \in (1, 1+\alpha)$ is a parametrized configuration ω which satisfies the boundary conditions

$$\omega(x + nt_i) = \omega(x) + lnt_i \quad \text{for all } x \in I \text{ and } i \in \{1, 2, 3\}. \tag{1}$$

The set of n-periodic parametrized configurations with edge length l is denoted by $\Omega^{\mathrm{per}}_{n,l} \subset \Omega$. From now on we will omit the word parametrized because in this section we are going to work solely with *point configurations* which are parametrized by I. An n-periodic configuration is uniquely determined by its values on I_n. Therefore we identify n-periodic configurations $\omega \in \Omega^{\mathrm{per}}_{n,l}$ with functions $\omega\colon I_n \to \mathbb{R}^2$.

The bond set $E \subset I \times I$ contains index-pairs with Euclidean distance one; this is $E = \{(x, y) \in I \times I \mid |x - y| = 1\}$. We set $E_n = E/nT$; we can think of E_n as a bond set $E_n \subset I_n \times I_n$. Let $\mathcal{T}$ denote the set of convex polyhedra, as in the definition of I, whose edges are in E and provide a tessellation of $\mathbb{R}^3$, which is the Delaunay pre-triangulation; see [13]. Define $\mathcal{T}_n = \mathcal{T}/nT$. Each $\triangle \in \mathcal{T}$ can be triangulated into tetrahedra (not necessarily uniquely); let us fix such a T-periodic triangulation of $\mathcal{T}$. The set of all (necessarily not all regular) tetrahedra created in this way forms a tessellation of $\mathbb{R}^3$ and is denoted by $\mathrm{triang}(\mathcal{T})$. We define $\mathrm{triang}(\mathcal{T}_n) := \mathrm{triang}(\mathcal{T})/nT$.

2.2. Probability space

By definition of Ω and $\Omega^{\mathrm{per}}_{n,l}$, we have $\Omega = (\mathbb{R}^3)^I$ and can identify $\Omega^{\mathrm{per}}_{n,l} = (\mathbb{R}^3)^{I_n}$. Both sets are endowed with the corresponding product σ-algebras $\mathcal{F} = \bigotimes_{x \in I} \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{F}_n =$

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$\bigotimes_{x\in I_n}\mathcal{B}(\mathbb{R}^3)$, where $\mathcal{B}(\mathbb{R}^3)$ denotes the Borel σ-algebra on each factor. The event of admissible n-periodic configurations $\Omega_{n,l}\subset\Omega^{\text{per}}_{n,l}$ is defined by the properties (Ω1)–(Ω4).

(Ω1) $|\omega(x)-\omega(y)|\in(1,\,1+\alpha)$ for all $(x,y)\in E$.

For $\omega\in\Omega$ we define the extension $\hat{\omega}\colon\mathbb{R}^3\to\mathbb{R}^3$ such that $\hat{\omega}(x)=\omega(x)$ if $x\in I$. On the closure of a tetrahedron $\triangle\in\text{triang}(\mathcal{T})$, the map $\hat{\omega}$ is defined to be the unique affine linear extension of the mapping defined on the corners of that tetrahedron.

(Ω2) The map $\hat{\omega}\colon\mathbb{R}^3\to\mathbb{R}^3$ is injective (and thus bijective).

(Ω3) The map $\hat{\omega}$ is almost everywhere orientation-preserving, that is, $\det(\nabla\hat{\omega}(x))>0$ for almost every $x\in\mathbb{R}^3$ with the Jacobian $\nabla\hat{\omega}\colon\mathbb{R}^3\to\mathbb{R}^{3\times3}$.

(Ω4) The image $\hat{\omega}(\triangle)$ of a polyhedron $\triangle\in\mathcal{T}$ is a convex polyhedron.

The conditions (Ω3) and (Ω4) follow from conditions (Ω1) and (Ω2) up to the sign of the determinant in (Ω3), as was also noted in [11, p. 4]. Since the proof is more analytic than stochastic, we also omit the proof and require them as technical conditions. Define the set of *admissible n-periodic configurations with edge length l* as

$$\Omega_{n,l}=\{\omega\in\Omega^{\text{per}}_{n,l}\mid\omega\text{ satisfies }(\Omega1)\text{–}(\Omega4)\}.$$

The set $\Omega_{n,l}$ is open and consists of non-empty subsets of $(\mathbb{R}^3)^{I_n}$. The scaled lattice $\omega_l(x)=lx$ for $x\in I$ and $1<l<1+\alpha$ is an element of $\Omega_{n,l}$. Any configuration $\omega\in\Omega_{n,l}$ is determined by a finite number of locations in $\mathbb{R}^3$. Each property (Ω1)–(Ω4) is satisfied after sufficiently small perturbation of these locations; therefore any $\omega\in\Omega_{n,l}$ has a neighborhood that is fully contained in $\Omega_{n,l}$, hence the openness of $\Omega_{n,l}$.

Clearly $0<\delta_0\otimes\lambda^{I_n\setminus\{0\}}(\Omega_{n,l})<\infty$ with the Lebesgue measure λ on $\mathbb{R}^3$ and the Dirac measure δ_0 in $0\in\mathbb{R}^3$. The lower bound holds because $\Omega^0_{n,l}$ is non-empty and open in $(\mathbb{R}^3)^{I_n\setminus\{0\}}$ (similarly to the case of $\Omega_{n,l}$ above); the upper bound is a consequence of the parameter α in (Ω1). Let the probability measure $\mathbb{P}_{n,l}$ be

$$\mathbb{P}_{n,l}(A)=\frac{\delta_0\otimes\lambda^{I_n\setminus\{0\}}(\Omega_{n,l}\cap A)}{\delta_0\otimes\lambda^{I_n\setminus\{0\}}(\Omega_{n,l})}$$

for any Borel-measurable set $A\in\mathcal{F}_n$, so $\mathbb{P}_{n,l}$ is the uniform distribution on the set $\Omega_{n,l}$ with respect to the *reference measure* $\delta_0\otimes\lambda^{I_n\setminus\{0\}}$. The first factor in this product refers to the component $\omega(0)$ of $\omega\in\Omega$.

2.3. Result

We have the following finite-volume result.

Theorem 2.1. *For α sufficiently small we have*

$$\lim_{l\downarrow1}\sup_{n\in\mathbb{N}}\sup_{\triangle\in\text{triang}(\mathcal{T}_n)}\mathbb{E}_{\mathbb{P}_{n,l}}[|\nabla\hat{\omega}(\triangle)-\text{Id}|^2]=0 \tag{2}$$

with the constant value of the Jacobian $\nabla\hat{\omega}(\triangle)$ on the tetrahedron $\triangle$ from the triangulation of $\mathcal{T}_n$ and some norm $|\cdot|$ on $\mathbb{R}^{3\times3}$.

The choice of α has to be such that the volume of any tetrahedron and octahedron with side lengths in $[1,\,1+\alpha]$ is uniquely minimized by the regular tetrahedron and octahedron with

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side length 1 respectively (see the proof of Theorem 2.1). The central argument is going to be the following rigidity theorem from [5, Theorem 3.1].

Theorem 2.2. (Friesecke, James and Müller) *Let U be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. There exists a constant $C(U)$ with the following property. For each $v \in W^{1,2}(U, \mathbb{R}^d)$ there is an associated rotation $R \in \mathrm{SO}(d)$ such that*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U)\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\|_{L^2(U)}.$$

This is a generalization of Liouville's theorem, which states that a map is necessarily a rotation whose Jacobian is a rotation in every point of its domain. We are going to set $v = \hat{\omega}|_{U_n}$ and $U = U_n$, which is a bounded Lipschitz domain. The function $\hat{\omega}|_{U_n}$ is linear on each triangle $\triangle \in \mathcal{T}_n$, and thus piecewise affine linear on U_n. As a consequence $\hat{\omega}|_{U_n}$ belongs to the class $W^{1,2}(U_n, \mathbb{R}^3)$. The following remark, which also appears in [5] at the end of Section 3, is essential for achieving uniformity in (2) in the parameter n.

Remark 2.1. The constant $C(U)$ in Theorem 2.2 is invariant under scaling: $C(\gamma U) = C(U)$ for all $\gamma > 0$. Indeed, setting $v_\gamma(\gamma x) = \gamma v(x)$ for $x \in U$, we have $\nabla v_\gamma(\gamma x) = \nabla v(x)$ and hence

$$\|\nabla v_\gamma - R\|_{L^2(\gamma U)} = \gamma^{d/2}\|\nabla v - R\|_{L^2(U)}$$

and

$$\|\mathrm{dist}(\nabla v_\gamma, \mathrm{SO}(d))\|_{L^2(\gamma U)} = \gamma^{d/2}\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\|_{L^2(U)}.$$

This implies that for the domains U_n ($n \geq 1$), the corresponding constant $C(U_n)$ can be chosen independently of n.

2.4. Proofs

We are going to show that the L^2-distance of the Jacobian $\nabla\hat{\omega}$ from the scaled identity matrix on U_n can be controlled by the difference of the areas of $\hat{\omega}(U_n)$ and U_n. Because of the periodic boundary conditions, $\lambda(\hat{\omega}(U_n))$ does not depend on configurations ω with (Ω2), so it provides a suitable uniform control on the set $\Omega_{n,l}$. Then we show that the expected square distance of $\nabla\hat{\omega}$ from the scaled identity matrix can be controlled by the expected square deviation of the polyhedra edge lengths from one, with 'one' being associated with the lattice constant of the unscaled lattice.

The following two lemmas from [13] provide the desired rigidity estimate on tetrahedra and octahedra. They state that the distance from SO(3) of a piecewise affine linear map defined on the polyhedron can be controlled by terms that measure how the map deforms the edge lengths of the polyhedron. We conjecture that any convex, rigid polyhedron satisfies such rigidity estimates via Dehn's theorem and the inverse function theorem. However, in this paper, as our main concern is not rigidity theory, we will only consider tetrahedra and octahedra for which these estimates are already proven. Let $|M| = \sqrt{\mathrm{tr}(M^t M)}$ denote the Frobenius norm of a matrix $M \in \mathbb{R}^{3\times 3}$ and $|w|$ the Euclidean norm of $w \in \mathbb{R}^3$.

Lemma 2.1. (Lemma 3.2 of [13]) *There is a positive constant C_1 such that, for all linear maps $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\det(A) > 0$ and*

$$w_1 = (1, 0, 0), \quad w_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right),$$
$$w_3 = w_2 - w_1, \quad w_4 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right),$$
$$w_5 = w_4 - w_2, \quad w_6 = w_4 - w_1, \quad l \geq 1,$$

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the following inequality holds:

$$\operatorname{dist}^2(A,\ \mathrm{SO}(3)) := \inf_{R\in \mathrm{SO}(3)} |A-R|^2 \le C_1 \sum_{i=1}^{6} (|Aw_i|-1)^2.$$

A similar theorem holds for octahedra. Let $\mathcal{O}$ denote an octahedron with vertices P_i, $i \in \{1, \ldots, 6\}$, and edges P_iP_j for $i \ne j \pmod 3$.

Lemma 2.2. (Lemma 3.4 of [13]) *There is a constant $C_2 > 0$ such that*

$$\operatorname{dist}^2(\nabla u,\ \mathrm{SO}(3)) \le C_2 \sum_{i\ne j \,(\mathrm{mod}\ 3)} (|u(P_iP_j)|-1)^2 \quad \text{almost everywhere in } \mathcal{O},$$

for every $u \in \mathcal{C}^0(\mathcal{O};\mathbb{R}^3)$ such that u is piecewise affine with respect to the triangulation determined by cutting $\mathcal{O}$ along the diagonal P_1P_4, $\det(\nabla u) > 0$ *a.e. in $\mathcal{O}$, and $u(\mathcal{O})$ is convex.*

Now we prove the estimate mentioned, which provides control over the L^2-distance of $\nabla\hat{\omega}$ from the scaled identity matrix in terms of the edge length deviations.

Lemma 2.3. *For a polyhedron $\triangle \in \mathcal{T}$, let $\mathcal{E}(\triangle)$ denote the set of edges of $\triangle$. There is a constant $c > 0$ such that, for all $n \ge 1$ and $1 < l < 1+\alpha$, the inequality*

$$\|\nabla\hat{\omega} - l\,\mathrm{Id}\|^2_{L^2(U_n)} \le c \sum_{\triangle\in\mathcal{T}_n} \sum_{\{x,y\}\in\mathcal{E}(\triangle)} (|\omega(x)-\omega(y)|-1)^2 \tag{3}$$

holds for all $\omega \in \Omega_{n,l}$, and hence

$$\mathbb{E}_{\mathbb{P}_{n,l}}[\|\nabla\hat{\omega} - l\,\mathrm{Id}\|^2_{L^2(U_n)}] \le c \sum_{\triangle\in\mathcal{T}_n} \sum_{\{x,y\}\in\mathcal{E}(\triangle)} \mathbb{E}_{\mathbb{P}_{n,l}}[(|\omega(x)-\omega(y)|-1)^2], \tag{4}$$

where the L^2-norm is defined with respect to the scalar product on $\mathbb{R}^{3\times 3}$ that induces the Frobenius norm, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^3$.

Note that the right-hand side of equation (3) is strictly positive because of the boundary conditions (1) and because $l > 1$, whereas the left-hand side is zero for $\omega = \omega_l \in \Omega^{\mathrm{per}}_{n,l}$. Since the measure $\mathbb{P}_{n,l}$ is supported on the set $\Omega_{n,l}$, (4) follows from (3). Also, note that c does not depend on n.

Proof. Let $\omega \in \Omega_{n,l}$ and $\mathcal{E}(\triangle)$ be the set of edges of a polyhedron $\triangle \in \mathcal{T}_n$. By Lemmas 2.1 and 2.2, we conclude that on every polyhedron $\triangle \in \mathcal{T}_n$ we have

$$\operatorname{dist}^2(\nabla\hat{\omega}|_\triangle,\ \mathrm{SO}(3)) \le \max\{C_1, C_2\} \sum_{\{x,y\}\in\mathcal{E}(\triangle)} (|\omega(x)-\omega(y)|-1)^2,$$

where we used (Ω1), (Ω3), and (Ω4) to apply Lemmas 2.1 and 2.2 and with the constants C_1, C_2 from Lemmas 2.1 and 2.2. Orthogonality of functions which are non-zero only on disjoint polyhedra gives

$$\|\operatorname{dist}(\nabla\hat{\omega}, \mathrm{SO}(3))\|^2_{L^2(U_n)} \le C \sum_{\triangle\in\mathcal{T}_n} \sum_{\{x,y\}\in\mathcal{E}(\triangle)} (|\omega(x)-\omega(y)|-1)^2,$$

with constant $C = \max\{C_1,\ C_2\} \max\{\sqrt{2}/12, \sqrt{2}/3\}$, where the second factor is the maximum of the volumes of a regular tetrahedron and octahedron. Applying Theorem 2.2 about geometric

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rigidity, we find an $R(\omega) \in \mathrm{SO}(3)$ such that

$$\|\nabla\hat{\omega} - R(\omega)\|^2_{L^2(U_n)} \leq K\|\mathrm{dist}(\nabla\hat{\omega}, \mathrm{SO}(3))\|^2_{L^2(U_n)},$$

with a constant $K > 0$ that does not depend on n by Remark 2.1. Due to the periodic boundary conditions (1), the function $\hat{\omega} - l\,\mathrm{Id}$ is n-periodic in the directions t_1, t_2, t_3, that is,

$$\hat{\omega}(x + nt_i) - l(x + nt_i) = \hat{\omega}(x) - lx \quad \text{for all } x \in \mathbb{R}^3 \text{ and } i \in \{1, 2, 3\}.$$

By the fundamental theorem of calculus, the gradient of a periodic function is orthogonal to any constant function, and therefore

$$\|\nabla\hat{\omega} - l\,\mathrm{Id}\|^2_{L^2(U_n)} + \|l\,\mathrm{Id} - R(\omega)\|^2_{L^2(U_n)} = \|\nabla\hat{\omega} - R(\omega)\|^2_{L^2(U_n)}$$

by Pythagoras. Since $\mathbb{P}_{n,l}$ is supported on the set $\Omega_{n,l}$, the lemma is established with $c = CK$. □

With Lemma 2.3 we can now prove Theorem 2.1.

Proof of Theorem 2.1. A generalization of Heron's formula for tetrahedra gives the volume $\lambda(\triangle)$ of the tetrahedron $\triangle$ with edge lengths u, v, w, U, V, W (opposite edges denoted by the same letter, lower-case and upper-case):

$$\lambda(\triangle) = \frac{\sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}}{192\,uvw}, \tag{5}$$

with

$$\begin{aligned}
X &= (w - U + v)(U + v + w), & a &= \sqrt{xYZ},\\
x &= (U - v + w)(v - w + U), & b &= \sqrt{yZX},\\
Y &= (u - V + w)(V + w + u), & c &= \sqrt{zXY},\\
y &= (V - w + u)(w - u + V), & d &= \sqrt{xyz},\\
y &= (V - w + u)(w - u + V), & &\\
Z &= (v - W + u)(W + u + v), & &\\
z &= (W - u + v)(u - v + W). & &
\end{aligned}$$

By first-order Taylor approximation of (5) at the regular tetrahedron $\triangle_1$, denoting the edge lengths a_i, $i \in \{1, \dots, 6\}$, we obtain

$$\lambda(\triangle) - \lambda(\triangle_1) = \frac{1}{12\sqrt{2}}\sum_{i=1}^{6}(a_i - 1) + o\left(\sum_{i=1}^{6}|a_i - 1|\right) \quad \text{as } a_i \to 1 \text{ for all } i.$$

For the octahedron, we obtain $1/(6\sqrt{2})$ for the volume derivative in one edge b_1 at $b_1 = 1$ and the remaining 11 edges fixed at $b_i = 1$. This can be achieved by dividing the octahedron into four tetrahedra that all have a common edge d that is a diagonal of the octahedron adjacent to x. Using the formula (5) and some elementary geometry of a regular trapezoid to see that $d = \sqrt{x+1}$, we obtain with the regular octahedron $\bigcirc_1$ of edge length 1:

$$\lambda(\bigcirc) - \lambda(\bigcirc_1) = \frac{1}{6\sqrt{2}}\sum_{i=1}^{12}(b_i - 1) + o\left(\sum_{i=1}^{12}|b_i - 1|\right) \quad \text{as } b_i \to 1 \text{ for all } i.$$

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We only need that the partial derivatives of the volume at $\triangle_1$ and $\bigcirc_1$ are positive. By continuity, in a small neighborhood of the regular polyhedra, increasing one edge length increases the volume. Therefore we can choose $\alpha > 0$ from the definition of permitted configurations so small that the polyhedra of the tessellation obtain minimal volume as the edge lengths go to 1. We choose $c_1 > 12\sqrt{2}$ and a corresponding $\alpha > 0$ so small that the inequalities

$$\begin{aligned} \sum_{i=1}^{6} (a_i - 1) &\leq c_1(\lambda(\triangle) - \lambda(\triangle_1)), \\ \sum_{i=1}^{12} (b_i - 1) &\leq c_1(\lambda(\bigcirc) - \lambda(\bigcirc_1)) \end{aligned} \tag{6}$$

are satisfied whenever $1 < a_i < 1 + \alpha$ and $1 < b_i < 1 + \alpha$. Let us fix such $c_1 > 0$ and $\alpha > 0$ and assume that $\Omega_{n,l}^{\text{per}}$ is defined by means of this α. Using (6) we can also estimate the squared edge length deviations:

$$\begin{aligned} \sum_{i=1}^{6} (a_i - 1)^2 &\leq c_1\ \alpha\ (\lambda(\triangle) - \lambda(\triangle_1)), \\ \sum_{i=1}^{12} (b_i - 1)^2 &\leq c_1\ \alpha\ (\lambda(\bigcirc) - \lambda(\bigcirc_1)). \end{aligned} \tag{7}$$

By equation (3) from Lemma 2.3 and (7), we get an upper bound on $\|\nabla\hat{\omega} - l\ \mathrm{Id}\|^2_{L^2(U_n)}$ in terms of the area differences. By summing up the contributions (7) of the polyhedra $\triangle \in \mathcal{T}_n$, we conclude for all $\omega \in \Omega_{n,l}$ that

$$\|\nabla\hat{\omega} - l\ \mathrm{Id}\|^2_{L^2(U_n)} \leq c_1\ \alpha\ c \sum_{\triangle\in\mathcal{T}_n} (\lambda(\hat{\omega}(\triangle)) - \lambda(\triangle)). \tag{8}$$

As a consequence of (Ω2) and the periodic boundary conditions (1), the right-hand side of (8) does not depend on $\omega \in \Omega_{n,l}$. Hence, with $\omega_l \in \Omega_{n,l}$ we can compute

$$\sum_{\triangle\in\mathcal{T}_n} (\lambda(\hat{\omega}(\triangle)) - \lambda(\triangle)) = \sum_{\triangle\in\mathcal{T}_n} (\lambda(\hat{\omega}_l(\triangle)) - \lambda(\triangle)) = |U_n|(l^3 - 1). \tag{9}$$

Combining the equations (8) and (9) gives

$$\|\nabla\hat{\omega} - l\ \mathrm{Id}\|^2_{L^2(U_n)} \leq c_1\ \alpha\ c\ |U_n|\ (l^3 - 1). \tag{10}$$

The reference measure $\delta_0 \otimes \lambda^{I_n\setminus\{0\}}$ and the set of permitted configurations $\Omega_{n,l}$ are invariant under the translations

$$\psi_b \colon \Omega_{n,l}^{\text{per}} \to \Omega_{n,l}^{\text{per}} \quad (\omega(x))_{x\in I} \mapsto (\omega(x+b) - \omega(b))_{x\in I}$$

for $b \in T$. As a consequence the matrix-valued random variables $\nabla(\hat{\omega}(\triangle))$ are identically distributed for $\triangle, \widetilde{\triangle} \in \mathrm{triang}(\mathcal{T}_n)$ such that $\triangle = \widetilde{\triangle}$ (mod T). Thus, for any $\triangle \in \mathrm{triang}(\mathcal{T}_1)$, the random variables $\nabla(\hat{\omega}(\triangle + t))_{t\in T}$ are identically distributed. Therefore

$$\mathbb{E}_{\mathbb{P}_{n,l}}[\|\nabla\hat{\omega} - l\ \mathrm{Id}\|^2_{L^2(U_n)}] = \sum_{\triangle\in\mathrm{triang}(\mathcal{T}_1)} |U_n(\triangle)|\ \mathbb{E}_{\mathbb{P}_{n,l}}[|\nabla\hat{\omega}(\triangle) - l\ \mathrm{Id}|^2],$$

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with the regions $U_n(\triangle)$ of U_n taken up by T-translates of $\triangle$. As the proportions $|U_n(\triangle)|/|U_n|$ are independent of n for any $\triangle \in \mathrm{triang}(\mathcal{T}_1)$, this equation together with (10) implies

$$\limsup_{l\downarrow 1}\sup_{n\in\mathbb{N}}\sup_{\triangle\in\mathrm{triang}(\mathcal{T}_n)}\mathbb{E}_{\mathbb{P}_{n,l}}[|\nabla\hat{\omega}(\triangle)-l\,\mathrm{Id}|^2]=0.$$

By means of the triangle inequality, we see that, for all $\triangle \in \mathrm{triang}(\mathcal{T}_n)$ and $\omega \in \Omega_{n,l}$,

$$|\nabla\hat{\omega}(\triangle)-\mathrm{Id}|^2 \le |\nabla\hat{\omega}(\triangle)-l\,\mathrm{Id}|^2+c_2^2(l-1)^2+2c_2\,|l-1|\,|\nabla\hat{\omega}(\triangle)-l\,\mathrm{Id}|,$$

with $c_2 = |\mathrm{Id}| > 0$. For $\omega \in \Omega_{n,l}$, the term $|\nabla\hat{\omega}(\triangle) - l\,\mathrm{Id}|$ is uniformly bounded for $l \in (1, \alpha)$ and $n \in \mathbb{N}$, which proves the theorem. □

3. Two-dimensional model with local geometry-dependent interactions

In this section we extend the result of [7] about long-range orientational order in that we get rid of the *a priori* enumeration of two-dimensional hard disk configurations by an underlying triangular lattice and merely impose local geometry-dependent conditions by means of a Hamiltonian H. The conditions impose that hard disks have exactly six neighbors that are not too far away. We show that long-range orientational order carries over to infinite-volume Gibbsian point processes defined by H.

3.1 Definitions

Let us cite some definitions from [4]. We equip the plane $\mathbb{R}^2$ with its Borel σ-algebra $\mathcal{B}(\mathbb{R}^2)$ and we let λ denote the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. The characters Λ and Δ will always denote measurable regions in $\mathbb{R}^2$ and the notation $\Delta \Subset \mathbb{R}^2$ means that in addition Δ is bounded. Consider the set $\mathcal{X} \subset 2^{(\mathbb{R}^2)}$ of locally finite point configurations in $\mathbb{R}^2$. That means $X \in \mathcal{X}$ is a subset $X \subset \mathbb{R}^2$, and for any $\Delta \Subset \mathbb{R}^2$, the intersection $X_\Delta := \mathrm{pr}_\Delta(X) := X \cap \Delta$ has finite cardinality $|X_\Delta| < \infty$. The counting variables $N_\Delta(X) := |X_\Delta|$ generate a σ-algebra $\mathcal{A} := \sigma(N_\Delta\colon \Delta \Subset \mathbb{R}^2)$ on $\mathcal{X}$. The union of $X, Y \in \mathcal{X}$ will be denoted by XY; this will be used when defining the configuration $X_\Lambda Y_{\Lambda^c}$ that agrees with X on Λ and with Y on the complement of Λ. In a sequence of set operations, unions XY are evaluated first in order to reduce brackets. On the measurable space $(\mathcal{X}, \mathcal{A})$, we consider the Poisson point process Π^z with intensity $z > 0$. The measure Π^z is uniquely characterized by the properties that, for all $\Delta \Subset \mathbb{R}^2$ under Π^z, (i) N_Δ is Poisson-distributed with parameter $z\lambda(\Delta)$, and (ii) conditional on $N_\Delta = n$, the n points in Δ are independently and uniformly distributed on Δ for each integer $n \ge 1$. Similarly, configurations $\mathcal{X}_\Lambda = \{X_\Lambda : X \in \mathcal{X}\}$ in the set Λ carry the trace σ-algebra $\mathcal{A}_\Lambda' := \mathcal{A}|_{\mathcal{X}_\Lambda}$ and the reference measure Π^z_Λ, which is the law of X_Λ if X is distributed according to Π^z. We will also need the pullback of $\mathcal{A}_\Lambda'$ to $\mathcal{X}$ defined by $\mathcal{A}_\Lambda := \mathrm{pr}_\Lambda^{-1}\mathcal{A}_\Lambda' \subset \mathcal{A}$. Finally we define the shift group $\Theta = \{\theta_r\colon r \in \mathbb{R}^2\}$, where $\theta_r\colon \mathcal{X} \to \mathcal{X}$ is the translation by $-r \in \mathbb{R}^2$, consequently $N_\Delta(\theta_r X) = N_{\Delta+r}(X)$ for all $\Delta \Subset \mathbb{R}^2$.

We fix $\alpha > 0$ small enough; the size of α will be specified later. We change the notation of [7] from ϵ to α at this point to emphasize that α is fixed and not particularly small. Let

$$\Lambda^{1+\alpha} := \{x \in \mathbb{R}^2\colon |x-y| < 1+\alpha \text{ for some } y \in \Lambda\}$$

be the $(1+\alpha)$-enlargement of Λ. For $X \in \mathcal{X}$ we define the Hamiltonian $H_{\Lambda,Y}$ in Λ with boundary condition $Y \in \mathcal{X}$ by

$$H_{\Lambda,Y}(X) := \begin{cases} 0 & \text{if } |x-y| > 1 \text{ whenever } x \in X_\Lambda Y_{\Lambda^{1+\alpha}\setminus\Lambda} \text{ and } y \in X_\Lambda Y_{\Lambda^c} \text{ for } x \ne y, \\ & \text{and for all } x \in X_\Lambda Y_{\Lambda^{1+\alpha}\setminus\Lambda}\colon |X_\Lambda Y_{\Lambda^c} \cap A_{1,1+\alpha}(x)| = 6, \\ \infty & \text{otherwise.} \end{cases}$$

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That is, $H_{\Lambda,Y}(X) \in \{0, \infty\}$ takes the value 0 if and only if every point of $X_{\Lambda^{1+\alpha}}$ has distance greater than one from points in $X_\Lambda Y_{\Lambda^c}$ and has exactly six $X_\Lambda Y_{\Lambda^c}$-neighbors in the annulus $A_{1,1+\alpha}(x) = \{y \in \mathbb{R}^2 : |y - x| \in (1, 1+\alpha)\}$; otherwise H is defined to be infinity. Note that the only part of the boundary condition Y relevant for $H_{\Lambda,Y}(X)$ is in the region $\Lambda^{2(1+\alpha)} \setminus \Lambda$.

Definition 3.1. We define the partition function $Z^z_{\Lambda,Y}$ by

$$Z^z_{\Lambda,Y} := \Pi^z_\Lambda\{X_\Lambda : H_{\Lambda,Y}(X_\Lambda) = 0\} = \int \mathrm{e}^{-H_{\Lambda,Y}(X)} \Pi^z_\Lambda(\mathrm{d}X).$$

We call a boundary condition $Y \in \mathcal{X}$ admissible for the region $\Lambda \Subset \mathbb{R}^2$ if $0 < Z^z_{\Lambda,Y}$. We write $\mathcal{X}^{\Lambda,z}_*$ for the set of all these Y.

The set of admissible boundary conditions $\mathcal{X}^{\Lambda,z}_*$ is never empty as the $l \in (1, 1+\alpha)$ scaling of a triangular lattice with lattice constant one is always in $\mathcal{X}^{\Lambda,z}_*$. We note that $H_{\Lambda,Y}(\emptyset) = 0$ for $Y_{\Lambda^{1+\alpha}} = \emptyset$ and also for specifically chosen Λ and possibly non-empty Y. The partition function $Z^z_{\Lambda,Y}$ is zero if neither $Y_{\Lambda^{1+\alpha}\setminus\Lambda} = \emptyset$ nor the boundary condition $Y_{\Lambda^{1+\alpha}\setminus\Lambda}$ can be extended to a near-triangular lattice configuration in $\Lambda^{1+\alpha}$.

Definition 3.2. For $Y \in \mathcal{X}^{\Lambda,z}_*$, we define the Gibbs distribution in the region $\Lambda \Subset \mathbb{R}^2$ with boundary condition Y by the formula

$$\gamma^z_\Lambda(F|Y) = \int_{\mathcal{X}_\Lambda} \mathbb{1}_F(XY_{\Lambda^c})\, \mathrm{e}^{-H_{\Lambda,Y}(X)} \Pi^z_\Lambda(\mathrm{d}X)/Z^z_{\Lambda,Y},$$

where $F \in \mathcal{A}$. Note that $\gamma^z_\Lambda(\cdot\,|Y)$ is a measure on the whole space $(\mathcal{X}, \mathcal{A})$.

In the case of $Y_{\Lambda^\alpha\setminus\Lambda} \neq \emptyset$, the $\mathcal{X}_\Lambda$-marginal of the measure $\gamma^z_\Lambda(\cdot\,|Y)$ is uniform on the configurations in $\mathcal{X}_\Lambda$ that extended $Y_{\Lambda^\alpha\setminus\Lambda}$ to a near-triangular lattice configuration in Λ^α. Otherwise if $Y_{\Lambda^\alpha\setminus\Lambda} = \emptyset$, then $\gamma^z_\Lambda(\cdot\,|Y) = \delta_{Y_{\Lambda^c}}$. Note that $(F, Y) \in (\mathcal{A}, \mathcal{X}) \mapsto \gamma^z_\Lambda(F|Y)$ is a probability kernel from $(\mathcal{X}, \mathcal{A}_{\Lambda^c})$ to $(\mathcal{X}, \mathcal{A})$, but the distribution $\gamma^z_\Lambda(\cdot\,|Y)$ has $\delta_{Y_{\Lambda^c}}$ as its marginal on $(\mathcal{X}_{\Lambda^c}, \mathcal{A}'_{\Lambda^c})$.

Definition 3.3. (*Infinite-volume Gibbs measure*) A probability measure $\mathbb{P}$ on $(\mathcal{X}, \mathcal{A})$ is an infinite-volume Gibbs measure for $z > 0$ if $\mathbb{P}(\mathcal{X}^{\Lambda,z}_*) = 1$ and

$$\int f \mathrm{d}\mathbb{P} = \int_{\mathcal{X}^{\Lambda,z}_*} \frac{1}{Z^z_{\Lambda,Y}} \int_{\mathcal{X}_\Lambda} f(XY_{\Lambda^c})\, \mathrm{e}^{-H_{\Lambda,Y}(X)} \Pi^z_\Lambda(\mathrm{d}X)\mathbb{P}(\mathrm{d}Y)$$

for every $\Lambda \Subset \mathbb{R}^2$ and every measurable $f : \mathcal{X} \to [0, \infty)$. We denote the set of infinite-volume Gibbs measures by $\mathcal{G}^z$.

Note that the right-hand side of the defining equality is equal to $\mathbb{E}_\mathbb{P}[\gamma^z_\Lambda(f|\cdot)]$. Therefore a measure $\mathbb{P}$ is an infinite-volume Gibbs measure if and only if $\mathbb{P}\gamma^z_\Lambda = \mathbb{P}$ for every $\Lambda \Subset \mathbb{R}^2$, where the product is understood as taking the average with $\mathbb{P}$ in the second variable of γ^z_Λ. We can easily obtain a degenerate measure $\delta_\emptyset \in \mathcal{G}^z$, but we will consider more interesting Gibbs measures. In fact, as soon as $\mathbb{P}(\emptyset) = 0$ for a measure $\mathbb{P} \in \mathcal{G}^z$, we have that $\mathbb{P}$ is supported on hard disk configurations with infinitely many disks.

The Hamiltonian H implements an example of a k-nearest-neighbor interaction, as explained in [4, Chapter 4.2.1]. Therefore, by [4, Lemma 5.1], the kernels γ^z_Λ, γ^z_Δ for $\Lambda \subset \Delta \Subset \mathbb{R}^2$ and $Y \in \mathcal{X}^{\Lambda,z}_*$ satisfy the consistency conditions $\gamma^z_\Lambda(\mathcal{X}^{\Lambda,z}_*|Y) = 1$ and $\gamma^z_\Delta\gamma^z_\Lambda = \gamma^z_\Delta$, where the product is understood as the product of probability kernels.

3.2 Results

We show the following generalization of [7, Theorem 4.1]. The wording of Theorem 3.1 up to some minor modification in the definition of H was suggested by Franz Merkl in a talk at a conference ('Trends in Mathematical Crystallization') held at Warwick University in May 2016.

Theorem 3.1. *Let* $0 < \alpha$ *be small enough (such that Lemma 3.1 and Theorem 3.2 hold true for the choice of this* α*). Then, for every* $2/(\sqrt{3}(1+\alpha)^2) < \rho < 2/\sqrt{3}$ *(the density of centers in the densest packing of disks with diameter 1), there is a measure* $\mathbb{P}_\rho \in \cap_{z>0}\mathcal{G}^z$ *such that we have the following.*

(i) Density $= \rho$. *For any* $\Lambda \Subset \mathbb{R}^2$*, we have* $\mathbb{E}_{\mathbb{P}_\rho}[N_\Lambda] = \rho\lambda(\Lambda)$.

(ii) Translational invariance. *The measure* $\mathbb{P}_\rho$ *is translation-invariant in any direction in* $\mathbb{R}^2$*, i.e.* $\mathbb{P}_\rho \circ \theta_r^{-1} = \mathbb{P}_\rho$ *for any* $r \in \mathbb{R}^2$.

(iii) Long-range orientational order. *Let* $x \in X$ *be the point with the smallest distance from the origin. It is a.s. unique. We have* $\mathbb{P}_\rho(N_{A_{1,1+\alpha}(x)} = 6) = 1$. *Choose a random neighbor* $y \in X$ *of* x *(i.e.* $1 < |y-x| < 1+\alpha$*) uniformly distributed among all six neighbors. Then, as* $\rho \uparrow 2/\sqrt{3}$*, the law of* $y - x$ *with respect to* $\mathbb{P}_\rho$ *converges weakly to the uniform distribution on the 6th roots of unity in* $\mathbb{C} \mathrel{\hat{=}} \mathbb{R}^2$.

Note that by translational invariance of $\mathbb{P}_\rho$, property (iii) holds when initially picking the closest point x to any reference point $x_0 \in \mathbb{R}^2$ instead of the origin. Hence the long-range orientational order, as neighbors of x position themselves close to translates of the 6th roots of unity. The choice of α will be made somewhat explicit in the proof of Lemma 3.1. The set of Gibbs measures $\mathcal{G}^z$ is most likely independent of $z > 0$; however, we will not pursue the proof of this statement as it leads to geometric considerations that are not the focus of our analysis.

3.3. Proofs

For a configuration $X \in \mathcal{X}$, we say that $H(X) = 0$ if, for all $x, y \in X$, we have $|x-y| > 1$ and $|X \cap A_{1,1+\alpha}(x)| = 6$. This is the same as having $H_{\Lambda,X}(X) = 0$ for any $\Lambda \Subset \mathbb{R}^2$. For a configuration $\emptyset \neq X \in \mathcal{X}$ with $H(X) = 0$, we can define a simplicial complex $K(X)$ consisting of zero, one, and two cells defined as follows. The set of zero-cells $K_0(X)$ is $X \subset \mathbb{R}^2$. The set of one-cells $K_1(X)$ are edges between zero-cells of distance between 1 and $1+\alpha$, and the two-cells are triangles with sides in $K_1(X)$. We will see in the following lemma that, by definition of H and some geometric considerations, for α small enough, the graph defined by the one- and two-skeleton of this complex is locally – and therefore also globally – isomorphic to the triangular lattice $I = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau = \mathrm{e}^{\mathrm{i}\pi/3}$ with edge set $E = \{\{i, j\} \subset I : |i-j| = 1\}$. The set of triangles surrounded by three edges in E is denoted by $\mathcal{T}$; these are two-cells if we regard I as a simplicial complex.

The most important lemma linking the theorem above to [7, Theorem 4.1] is as follows.

Lemma 3.1. *With the choice of a small enough* α*, for any configuration* $X \in \mathcal{X}$ *with* $H(X) = 0$ *we have that the graph defined by the one- and two-skeletons of* $K(X)$ *is isomorphic to the triangular lattice* I*. In other words, there is a bijective map* $\omega : I \to X$ *such that, for all* $i, j \in I$*,* $|i-j| = 1$ *if and only if* $|\omega(i) - \omega(j)| \in (1, 1+\alpha)$.

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Later on, we will choose α sufficiently small that Lemma 3.1 and Theorem 3.1 both work for that α. From the proof of the lemma it will be obvious that the choice of α does not need to be particularly small for it (and any smaller choice) to work.

Proof. For $i \in I$ we define its closest neighborhood $N(i) \subset I$ by $N(i) = \{j \in I\colon |i - j| \leq 1\}$. Let $X \in \mathcal{X}$ such that $H(X) = 0$. A map $\omega\colon N(i) \to X$ is called a local isomorphism at i if, for all $j, k \in N(i)$, we have $|j - k| = 1$ if and only if $|\omega(j) - \omega(k)| \in (1, 1 + \alpha)$. By taking $\alpha > 0$ small enough, we can ensure that for all $i \in I$ and $x \in X$ there is a local isomorphism ω at i such that $\omega(i) = x$. To see this, observe that as $\alpha \to 0$, for every $y \in A_{1,1+\alpha}(x)$ there are exactly two points $y_1, y_2 \in A_{1,1+\alpha}(x) \setminus \{y\}$ such that $|y_i - y| \to 1$; for other $z \in A_{1,1+\alpha}(x) \setminus \{y\}$, we have $\liminf_{\alpha\to 0} |z - y| \geq \sqrt{3}$. Since we know that $|X \cap A_{1,1+\alpha}(y)| = 6$, a simple geometric consideration related to the kissing problem gives that $y_1, y_2 \in A_{1,1+\alpha}(y)$, since if $y_i \notin A_{1,1+\alpha}(y)$ for $i \in \{1, 2\}$, for α small enough there was not enough space to place six points in $A_{1,1+\alpha}(y)$ having distance larger than 1 from each other and from y_i. To be more precise, for all $i \in I$ and $x \in X$ there will be twelve such local isomorphisms taking rotations and reflection into account. We fix α sufficiently small that the local isomorphism property holds.

Let us construct a map $\omega\colon I \to X$ as follows. We fix an arbitrary $x_0 \in X$ and define $\omega|_{N(0)}$ to be one of the six orientation-preserving local isomorphisms at 0 with $\omega(0) = x_0$. Fix a spanning tree T of I. For each $i \in I$, there is a unique path on nearest neighbors in T connecting 0 to i. Since there are local isomorphisms at each pair of points of I and X, we can uniquely extend ω to vertices of T by choosing the unique one of the six orientation-preserving local isomorphisms that is consistent with T. That is to say, if, for a neighbor i of j in T, we have already assigned a point $\omega(i)$, then we choose a local isomorphism at i with $i \mapsto \omega(i)$. Let us assign j to the point in X that is determined by this local isomorphism. Now there is only one local isomorphism at j which is consistent with the local isomorphism chosen at i in the sense that i has identical images under the two local isomorphisms. We use this local isomorphism to proceed with the construction and map all neighbors of j in T into X.

It remains to show that the map $\omega\colon I \to X$ is an isomorphism. To conclude that ω is an isomorphism onto its image, we fix a loop γ starting and ending in $i \in I$ composed of a path in T and an edge between i and one of its neighbors in I to which it is not connected in T. We need to show that the map induced along γ with the initial orientation-preserving local isomorphism $\omega|_{N(i)}$ at i maps to a loop in $K(X)$ starting and ending in $\omega(i)$. To this end, we can show a seemingly more general but equivalent statement. Take any loop $\gamma = (i_0, i_1, i_2, \dots, i_n)$ at $0 \in I$ (i.e. $i_0 = i_n = 0$) and $x \in X$, fix a local isomorphism at 0 with $0 \mapsto x$, and show that the map induced along γ maps γ to a loop $\omega(\gamma)$ in X at x. Here ω is locally defined along the curve γ.

We can deform the loop γ to the boundary of a two-cell that contains 0 by successively 'removing' two-cells that intersect γ and are inside it. By removing a two-cell, we mean one of the following. Two subsequent edges (i_{k-1}, i_k), (i_k, i_{k+1}) of γ, can be exchanged for the unique edge (i_{k-1}, i_{k+1}) if $|i_{k-1} - i_{k+1}| = 1$, or we can exchange one edge (i_k, i_{k+1}) of γ for two edges (i_k, j) and (j, i_{k+1}) in I. For every such transformation of γ, we obtain a modified γ' and a map ω' that is uniquely determined by the local isomorphism at i_k and is the unique extension of the local isomorphism at 0 along γ'. Note that $\omega = \omega'$ on the domain that they are both defined and $\omega(\gamma)$ is closed if and only if $\omega'(\gamma')$ is. After removing finitely many two-cells, we arrive at $\gamma' = (0, i, j, 0)$ being the boundary of a two-cell that contains the origin. Since $\omega'|_{\gamma'}$ should be the unique extension of the local isomorphism at 0 along γ, we see that $\omega'(\gamma')$ is closed and therefore so is $\omega(\gamma)$.

We showed that for neighbors i, j in I, $\omega(i)$, $\omega(j)$ are also neighbors in X. To obtain the converse statement and the injectivity of ω, we repeat the above procedure for the same map ω but with exchanged roles of I and X. This concludes the proof that ω is an isomorphism onto its image.

It remains to show that ω is surjective. Now take a curve $\hat{\gamma}$ in $K(X)$ from x_0 to some $y \in K(X)$. Note that $K(X)$ is a connected graph, as for small enough α and $x \neq y$ we can always find a neighbor z of x which is closer to y than x. The curve $\hat{\gamma}$ corresponds to a curve γ in I from 0 to some $i \in I$. Applying the above procedure to the concatenation of the path from 0 to i in T and the reverse of γ, we see that $\omega(i) = y$. □

This lemma can also be proved with the formalism of Čech cohomology using the de Rham isomorphism and can be generalized to configurations with point defects (missing points). The usefulness of the Čech cohomology and de Rham's theorem was pointed out to us by Franz Merkl. We decided to give another proof using less formalism.

To construct $\mathbb{P}_\rho$, we use measures on periodic configurations. For $l > 1$ and $n \in \mathbb{N}$, let us define measures $P_{n,l}$ on n-periodic configurations as in [7]. A periodic, enumerated configuration $\omega \in \Omega^{\text{per}}_{n,l}$ is a map $I \to \mathbb{R}^2$ such that Theorem 3.2 holds true for this choice of α. Thus

$$\omega(i + nj) = \omega(i) + lnj \quad \text{for all } i, j \in I. \tag{11}$$

It suffices to define an n-periodic, enumerated configuration on a set of n^2 representatives $I_n \subset I$ as equation (11) uniquely defines the configuration on the complement $(I_n)^c$. The event of admissible, n-periodic, enumerated configurations $\Omega_{n,l} \subset \Omega^{\text{per}}_{n,l}$ is defined by the properties (Ω1)–(Ω3).

(Ω1) $|\omega(i) - \omega(j)| \in (1,\ 1 + \alpha)$ for all $\{i, j\} \in E$.

For $\omega \in \Omega$ we define the extension $\hat{\omega} \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{\omega}(i) = \omega(i)$ if $i \in I$, and on the closure of any triangle $\triangle \in \mathcal{T}$, the map $\hat{\omega}$ is defined to be the unique affine linear extension of the mapping defined on the corners of $\triangle$.

(Ω2) The map $\hat{\omega} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is injective.

(Ω3) The map $\hat{\omega}$ is orientation-preserving, that is, $\det(\nabla\hat{\omega}(x)) > 0$ for all $\triangle \in \mathcal{T}$ and $x \in \triangle$ with the Jacobian $\nabla\hat{\omega} \colon \cup\mathcal{T} \to \mathbb{R}^{2\times 2}$.

Define the set of *admissible, n-periodic, enumerated configurations* as

$$\Omega_{n,l} = \{\omega \in \Omega^{\text{per}}_{n,l} \mid \omega \text{ satisfies } (\Omega1)\text{–}(\Omega3)\}.$$

Let the probability measure $\mathbb{P}_{n,l}$ be

$$\mathbb{P}_{n,l}(A) = \frac{\delta_0 \otimes \lambda^{I_n \setminus \{0\}}(\Omega_{n,l} \cap A)}{\delta_0 \otimes \lambda^{I_n \setminus \{0\}}(\Omega_{n,l})}$$

for any Borel-measurable set $A \in \mathcal{F}_n = \bigotimes_{i \in I_n} \mathcal{B}(\mathbb{R}^2)$; thus $\mathbb{P}_{n,l}$ is the uniform distribution on the set $\Omega_{n,l}$ with respect to the *reference measure* $\delta_0 \otimes \lambda^{I_n \setminus \{0\}}$. The first factor in this product refers to the component $\omega(0)$. The parameter l in the definition of $\Omega_{n,l}$ and $\mathbb{P}_{n,l}$ controls the density of periodic configurations such that $\rho = 2/(l^2\sqrt{3})$. We quote Theorem 4.1 from [7], which will be the major ingredient of the proof of Theorem 3.1.

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Theorem 3.2. *For any $\alpha > 0$ small enough we have*

$$\lim_{l \downarrow 1} \sup_{n \in \mathbb{N}} \sup_{\triangle \in \mathcal{T}} \mathbb{E}_{\mathbb{P}_{n,l}}[|\nabla \hat{\omega}(\triangle) - \mathrm{Id}|^2] = 0,$$

with the constant value of the Jacobian $\nabla \hat{\omega}(\triangle)$ on the set $\triangle \in \mathcal{T}$.

We note that the theorem holds for any $\alpha \in (0, \sqrt{3} - 1)$, but we omit the proof, which is just a more careful consideration of arguments in the proof of [7, Theorem 4.1], and will refer to small enough α. The main observation needed for this explicit range of α where the theorem holds is that the area of triangles with side lengths in the range $[1, \sqrt{3})$ is uniquely minimized by the regular triangle with side length 1. This observation is then utilized as in the similar proof of Theorem 2.1 in the three-dimensional case. We note that Theorem 3.2 might work with $\alpha \geq \sqrt{3} - 1$, but looking for the optimal upper bound is not the concern of this paper.

In the following we construct $\mathbb{P}_\rho$ as a limit of translation-invariant versions of $\mathbb{P}_{n,l}$ and show that this measure is a Gibbs measure in $\mathcal{G}^z$ for any $z > 0$. We follow ideas from [4] to construct a limiting measure. Fix $l > 1$ and define the measures $\mathbb{G}_n$ on $(\mathcal{X}, \mathcal{A})$ by specifying its marginal $(\mathbb{G}_n)_{\Lambda_n}$ on $(\mathcal{X}_{\Lambda_n}, \mathcal{A}_{\Lambda_n}{}')$:

$$(\mathbb{G}_n)_{\Lambda_n} = \left(\frac{1}{\lambda(\Lambda_n)} \int_{\Lambda_n} \mathrm{Im}[\mathbb{P}_{n,l}] \circ \theta_r \, \mathrm{d}r \right)_{\Lambda_n},$$

with the image measure $\mathrm{Im}[\mathbb{P}_{n,l}]$ of $\mathbb{P}_{n,l}$ under the map $\mathrm{Im}\colon \omega \mapsto \{\omega(x) \colon x \in I\}$ and the domain $\Lambda_n = l\{x + y\tau : x, y \in [-n/2, n/2)\}$. The averaging over $r \in \Lambda_n$ is necessary to obtain a translation-invariant measure on the torus, since $\omega(0) = 0$ holds $\mathbb{P}_{n,l}$-a.s. The measure $\mathbb{G}_n$ is then defined by having independent and identically distributed projections on the sets $\{\Lambda_n + inl\}_{i \in I}$, which form a tiling of $\mathbb{R}^2$. In order to have translation-invariant probability measures on $(\mathcal{X}, \mathcal{A})$, we consider the averaged measures

$$\hat{\mathbb{G}}_n = \frac{1}{\lambda(\Lambda_n)} \int_{\Lambda_n} \mathbb{G}_n \circ \theta_r \, \mathrm{d}r.$$

By definition and the periodicity of $\mathbb{G}_n$, $\hat{\mathbb{G}}_n$ are translation-invariant. We will show that the sequence $(\hat{\mathbb{G}}_n)_{n \in \mathbb{N}}$ is tight in the *topology of local convergence* on translation-invariant probability measures on $\mathcal{X}$ generated by $\mathbb{P} \to \int f \, \mathrm{d}\mathbb{P}$ for functions f that are $\mathcal{A}_\Lambda$-measurable for some $\Lambda \Subset \mathbb{R}^2$. We call such functions local, and denote the set of local functions by $\mathcal{L}$.

The only difference to the definitions after [4, Lemma 5.1] are in the nature of the measures $(\mathbb{G}_n)_{\Lambda_n}$. In our case $(\mathbb{G}_n)_{\Lambda_n}$ are measures that inherit geometric constraints from the structure of $\mathbb{P}_{n,l}$ that are defined on tori of different sizes. In [4], in contrast, the authors use measures $\mathbb{G}^z_{\Lambda_n, \bar{\omega}}$ that have fixed boundary condition $\bar{\omega}$ on the complement of Λ_n.

For a shift-invariant probability measure $\mathbb{P}$ on $(\mathcal{X}, \mathcal{A})$ and $\Lambda \Subset \mathbb{R}^2$, define the measure $\mathbb{P}_\Lambda := \mathbb{P} \circ \mathrm{pr}_\Lambda^{-1}$ and the *relative entropy* with respect to Π^z_Λ as

$$I(\mathbb{P}_\Lambda \mid \Pi^z_\Lambda) := \begin{cases} \displaystyle\int f \ln f \, \mathrm{d}\Pi^z_\Lambda & \text{if } \mathbb{P}_\Lambda \ll \Pi^z_\Lambda \text{ with density } f, \\ \infty & \text{otherwise.} \end{cases}$$

The *specific entropy* of $\mathbb{P}$ with respect to Π^z is then defined by

$$I(\mathbb{P}) := \lim_{n \to \infty} \frac{1}{\lambda(\Delta_n)} I\big(\mathbb{P}_{\Delta_n} \mid \Pi^z_{\Delta_n}\big),$$

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where $\Delta_n \Subset \mathbb{R}^2$ is a cofinite increasing sequence of sets. We refer to [8] and [9] for existence and properties of the specific entropy. We will set $z = 1$ and compute entropies relative to $\Pi^1_{\Delta_n}$. By [9, Proposition 2.6], the sublevel sets of I are sequentially compact in the topology of local convergence. Therefore we only need to show that the specific entropies of the measures $\{\hat{\mathbb{G}}_n\}_{n\in\mathbb{N}}$ are bounded by some constant. We start with a proposition that also appeared in [6, Lemma 5.2] and provides a lower bound on the partition sum.

Proposition 3.1. *For all $\alpha \in (0, 1]$ and $l \in (1, 1+\alpha)$, there is an $r = r(\alpha, l) \in (0, 1/2)$ such that for $n \in \mathbb{N}$ we have*

$$\delta_0 \otimes \lambda^{I_n\setminus\{0\}}(\Omega_{n,l}) \geq (\pi r^2)^{|I_n|-1}.$$

Proof. For $r > 0$, we define, like in (3.2) in [11], the set of configurations which are close to the scaled, enumerated, standard configuration $\omega_l(i) = li$ for $i \in I$:

$$S_{n,l,r} = \{\omega \in \Omega^{\text{per}}_{n,l} \mid |\omega(i) - \omega_l(i)| < r \text{ for all } i \in I\}.$$

For sufficiently small $r > 0$, depending on α and l, we conclude, like in the proof of [11, Lemma 3.1], that $S_{n,l,r} \subset \Omega_{n,l}$. To prove this inclusion, we have to show the properties (Ω1)–(Ω3) for all $\omega \in S_{n,l,r}$. For $(i, j) \in E$ and $\omega \in S_{n,l,r}$, let us compute

$$\begin{aligned} \big|\, |\omega(i) - \omega(j)| - l \big| &= \big|\, |\omega(i) - \omega(j)| - |\omega_l(i) - \omega_l(j)| \,\big| \\ &\leq |\omega(i) - \omega_l(i)| + |\omega(j) - \omega_l(j)| \\ &< 2r. \end{aligned}$$

If we choose $2r < \max\{l-1, 1+\alpha-l\} < 1$, then ω satisfies (Ω1). Condition (Ω2) is a consequence of the inequality $\langle v, \nabla\hat{\omega}(x)v\rangle > 0$ for all $v \in \mathbb{R}\setminus\{0\}$, and for all $x \in \mathbb{R}^2$ where $\hat{\omega}$ is differentiable. This inequality holds for small enough r since $\nabla\hat{\omega}$ is close to the identity uniformly on $\mathbb{R}^2$. Hence $\hat{\omega}$ is a bijection onto its image. Here we applied a theorem from analysis which states that a $\mathcal{C}^1$-map f from an open convex domain $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with $\langle v, \nabla f(x)v\rangle > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$ and $x \in U$ is a diffeomorphism onto its image. However, $\nabla\hat{\omega}(x)$ is only piecewise differentiable, but on the straight line L connecting $x, y \in \mathbb{R}^2$ with $x \neq y$ there are only finitely many points $z \in \mathbb{R}^2 \cap L$ where the curve $(\hat{\omega}(ty + (1-t)x))_{t\in(0,1)}$ is not differentiable. Assume that $\langle v, \nabla\hat{\omega}(x)v\rangle > 0$ holds whenever $\hat{\omega}$ is differentiable in x. The curve is piecewise linear, and on each of these pieces, the derivative of the curve forms an acute angle with $y - x$, so the curve cannot be closed. Thus the condition (Ω2) is satisfied in the case of a sufficiently small r. Furthermore, condition (Ω3) is satisfied by ω_l, and therefore also by ω if r is sufficiently small. Hence $S_{n,l,r} \subset \Omega_{n,l}$ for some $r \in (0, 1/2)$, and we conclude that

$$\delta_0 \otimes \lambda^{I_n\setminus\{0\}}(\Omega_{n,l}) \geq \delta_0 \otimes \lambda^{I_n\setminus\{0\}}(S_{n,l,r}) = (\pi r^2)^{|I_n|-1},$$

where the last equality is obtained by integrating over each $\omega(i)$ with $i \neq 0$ successively along a fixed spanning tree of I_n which gives a factor πr^2, and considering that $\omega_l(0) = 0$ and that the measure $\delta_0 \otimes \lambda^{I_n\setminus\{0\}}$ fixes $\omega(0) = 0$. □

Proposition 3.2. *The set $\{I(\hat{\mathbb{G}}_n) : n \in \mathbb{N}\}$ is bounded, thus the set $\{\hat{\mathbb{G}}_n : n \in \mathbb{N}\}$ is sequentially compact in the topology of local convergence. Therefore there is a sequence $n_k \to \infty$ and a shift-invariant measure $\mathbb{P}_\rho$ on $(\mathcal{X}, \mathcal{A})$ such that*

$$\lim_{k\to\infty} \int f \, d\hat{\mathbb{G}}_{n_k} = \int f \, d\mathbb{P}_\rho \quad \text{for any } f \in \mathcal{L}.$$

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Proof. As also noted in the proof of [4, Proposition 5.3], the definition of $\hat{\mathbb{G}}_n$ implies that

$$I^z(\hat{\mathbb{G}}_n) = \frac{1}{\lambda(\Lambda_n)} I\big((\mathbb{G}_n)_{\Lambda_n} \mid \Pi^1_{\Lambda_n}\big).$$

The relative entropy $I((\mathbb{G}_n)_{\Lambda_n} \mid \Pi^1_{\Lambda_n})$ can be explicitly computed as follows. The measure $(\mathbb{G}_n)_{\Lambda_n}$ is supported on configurations that have n^2 points in Λ_n, and if Λ_n is folded into a torus, then each point x has exactly six neighbors in the annulus $A_{1,1+\alpha}(x)$ around it and no points closer than distance one. These configurations $\mathcal{X}_{n,l}$ are images of enumerated configurations $\mathcal{X}_{n,l} = (\mathrm{Im}\ \Omega_{n,l})_{\Lambda_n}$. By Lemma 3.1, $(\mathbb{G}_n)_{\Lambda_n}$ is the uniform distribution on these configurations with respect to $\Pi^1_{\Lambda_n}$. The density of $(\mathbb{G}_n)_{\Lambda_n}$ with respect to $\Pi^1_{\Lambda_n}$ is given by $f = \mathbb{1}_{\mathcal{X}_{n,l}} / \Pi^1_{\Lambda_n}(\mathcal{X}_{n,l})$. To find the constant $\Pi^1_{\Lambda_n}(\mathcal{X}_{n,l})$ more explicitly, consider the expectation

$$\Pi^1_{\Lambda_n}[g] = \mathrm{e}^{-\lambda(\Lambda_n)} \sum_{k=0}^{\infty} \int_{\Lambda_n^k} \frac{1}{k!} g(\{x_1, \ldots, x_k\})\, \lambda^k|_{\Lambda_n^k}(\mathrm{d}x_1, \ldots, \mathrm{d}x_k).$$

Consequently we have

$$\Pi^1_{\Lambda_n}(\mathcal{X}_{n,l}) = \frac{\mathrm{e}^{-\lambda(\Lambda_n)}}{n^2} \lambda(\Lambda_n)\, \delta_0 \otimes \lambda^{I_n \setminus \{0\}}(\Omega_{n,l}).$$

This follows since a factor $\mathrm{e}^{-\lambda(\Lambda_n)}/(n^2)!$ comes from the density of $\Pi^1_{\Lambda_n}$ conditioned on n^2 points with respect to

$$\lambda^{(n^2)}\big|_{\Lambda_n^{(n^2)}}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n^2).$$

Then, conditioned on the position of x_1, the volume of the permitted configurations by their shift invariance on the torus is $(n^2 - 1)!\ \delta_0 \otimes \lambda^{I_n \setminus \{0\}}(\Omega_{n,l})$; furthermore, the first point can be distributed uniformly in Λ_n. The relative entropy is $I((\mathbb{G}_n)_{\Lambda_n} \mid \Pi^1_{\Lambda_n}) = -\ln(\Pi^1_{\Lambda_n}(\mathcal{X}_{n,l}))$ and the specific entropy can be bounded using Proposition 3.1 and $\lambda(\Lambda_n) = n^2 l^2 \sqrt{3}/2$ for sufficiently large n, we obtain

$$\begin{aligned}
I((\mathbb{G}_n)_{\Lambda_n}) &= -\frac{\ln(\Pi^1_{\Lambda_n}(\mathcal{X}_{n,l}))}{\lambda(\Lambda_n)} \\
&= 1 + \frac{n^2}{\lambda(\Lambda_n)} - \frac{\ln(\lambda(\Lambda_n))}{\lambda(\Lambda_n)} - \frac{\ln(\delta_0 \otimes \lambda^{I_n \setminus \{0\}}(\Omega_{n,l}))}{\lambda(\Lambda_n)} \\
&\leq 1 + \frac{n^2}{\lambda(\Lambda_n)} - \frac{\ln(\lambda(\Lambda_n))}{\lambda(\Lambda_n)} - \frac{|I_n - 1| \ln(\pi r^2)}{\lambda(\Lambda_n)} \\
&\leq 1 + \frac{2 - 2\ln(\pi r^2)}{l^2 \sqrt{3}}.
\end{aligned}$$

□

The next proposition shows that $\mathbb{P}_\rho$ is an infinite-volume Gibbs measure. Note that $\hat{\mathbb{G}}_n$ and Λ_n depend on $l > 1$, which we fixed previously.

Proposition 3.3. *The measure $\mathbb{P}_\rho$ is an infinite-volume Gibbs measure $\mathbb{P}_\rho \in \cap_{z>0} \mathcal{G}^z$.*

Proof. Fix $\Lambda \Subset \mathbb{R}^2$, $z > 0$ and $\rho < 2/\sqrt{3}$ sufficiently large that $2/(\sqrt{3}(1+\alpha)^2) < \rho$, where α is such that Lemma 3.1 holds with that α. Let $l > 1$ such that $\rho = 2/(l^2\sqrt{3})$. For $X \in \mathcal{X}$, let

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$\widetilde{X}_n$ be the periodic extension of X_{Λ_n} to $\mathcal{X}$, i.e. $\widetilde{X}_n = \cup_{i\in I} X_{\Lambda_n} + lni$. Let $\kappa > 0$ be so large that $\Lambda^\kappa \setminus \Lambda$ contains a connected ring of triangles from $K_2(\widetilde{X}_n)$ for $\mathbb{G}_n$-almost all X for all $n \in \mathbb{N}$. Consequently, for all $n \in \mathbb{N}$ sufficiently large that $\Lambda^\kappa \subset \Lambda_n$, the number of points in Λ conditioned on X_{Λ^c}, is $\mathbb{G}_n$-a.s. determined by the configuration in $\Lambda^\kappa \setminus \Lambda$. The measure $(\mathbb{G}_n)_{\Lambda_n}$ is the uniform distribution of enumerable permitted configurations with n^2 points on the torus. By Lemma 3.1, the conditional distribution of X_Λ given X_{Λ^c} under $\mathbb{G}_n$ is therefore the uniform distribution on configurations X_Λ such that $H_{\Lambda, X_{\Lambda^c}}(X_\Lambda) = 0$. Uniform distribution makes sense, as the number of points in Λ is almost surely constant with respect to the conditioned measure. Therefore the factorized version of the conditional distribution of $\mathbb{G}_n$ given $\mathcal{A}_{\Lambda^c}$ is given by $\gamma_\Lambda(\cdot \mid \cdot)$, that is,

$$\mathbb{G}_n(F) = \int_{\mathcal{X}} \gamma_\Lambda(F \mid Y)\mathbb{G}_n(\mathrm{d}Y) \tag{12}$$

for any $F \in \mathcal{A}$ and $n \in \mathbb{N}$ large enough for $\Lambda^\kappa \subset \Lambda_n$. Since z is fixed, we can omit it as a superscript in γ^z.

The rest of the proof is as the proof of [4, Proposition 5.5]. Define

$$\Lambda_n^\circ := \{r \in \mathbb{R}^2 \colon \Lambda^\kappa + r \subset \Lambda_n\}$$

and the (subprobability) measures

$$\bar{\mathbb{G}}_n := \frac{1}{|\Lambda_n|} \int_{\Lambda_n^\circ} \mathbb{G}_n \circ \theta_r^{-1}\, \mathrm{d}r.$$

Then

$$\int f\, \mathrm{d}\hat{\mathbb{G}}_n - \int f\, \mathrm{d}\bar{\mathbb{G}}_n \to 0$$

by the same argument as in [9, Lemma 5.7], so $\mathbb{P}_\rho$ can also be seen as an accumulation point of the sequence $(\bar{\mathbb{G}}_n)$. Let $F \in \cup_{\Delta \Subset \mathbb{R}^2} \mathcal{A}_\Delta$ be a local set; using (12), we obtain for $r \in \Lambda_n^\circ$

$$\mathbb{G}_n \circ \theta_r^{-1}(F) = \int_{\mathcal{X}} \gamma_\Lambda(F \mid Y)\mathbb{G}_n \circ \theta_r^{-1}(\mathrm{d}Y).$$

Therefore averaging over $r \in \Lambda_n^\circ$ gives

$$\bar{\mathbb{G}}_n(F) = \int_{\mathcal{X}} \gamma_\Lambda(F \mid Y)\bar{\mathbb{G}}_n(\mathrm{d}Y). \tag{13}$$

Since the integrand on the right is a local function of Y, we can set $n = n_k$ and let $k \to \infty$, which gives (13) for $\mathbb{P}_\rho$ instead of $\bar{\mathbb{G}}_n$. Since local sets generate the σ-algebra $\mathcal{A}$, (13) holds for $\mathbb{P}_\rho$ and $F \in \mathcal{A}$, which by monotone convergence shows that $\mathbb{P}_\rho$ is an infinite-volume Gibbs measure. □

Proof of Theorem 3.1. In Propositions 3.3 and 3.2 we showed the existence of a translation-invariant measure $\mathbb{P}_\rho \in \cap_{z>0} \mathcal{G}^z$ which is the local limit of the measures $(\mathbb{G}_{n_k})_{k\geq 1}$, so $\mathbb{P}_\rho$ satisfies property (ii). Property (i) holds as it can be expressed by a local function and $\mathbb{E}_{\mathbb{G}_{n_k}}[|X \cap B|] = \rho\lambda(B)$ for any $k \geq 1$ by the periodic boundary conditions. Similarly, property (iii) can be expressed by local functions depending on $\{x_0, x_1, \ldots, x_6\} \cap \Lambda_n$, where x_0 is the closest random point to the origin and x_i is the ith closest point to x_0. For n large enough we have

$$\mathbb{G}_{n_k}(|\{x_0, x_1, \ldots, x_6\} \cap \Lambda_n| = 7) = 1$$

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for any $k \geq 1$ and therefore

$$\mathbb{P}_\rho(|\{x_0, x_1, \ldots, x_6\} \cap \Lambda_n| = 7) = 1.$$

By Theorem 3.2 we have

$$\lim_{\rho \uparrow 2/\sqrt{3}} \sup_{k \geq 1} \mathbb{E}_{\mathbb{G}_{n_k}} \left[\sum_{i=1}^{6} |\nabla\hat{\omega}(\triangle_i) - \mathrm{Id}|^2 \right] = 0, \tag{14}$$

where $\{\triangle_i\}_{1 \leq i \leq 6}$ are the random six triangles in $\mathcal{T}$ such that one of their vertices is mapped to x_0 under ω. Let $f \colon \mathbb{C}^6 \to \mathbb{R}$ be continuous, bounded, and permutation-invariant. We use the natural identification of topological spaces $\mathbb{C} \hat{=} \mathbb{R}^2$. Let $y_i = x_i - x_0$. By continuity of f, there is a constant $c > 0$ such that

$$|f(y_1, \ldots, y_6) - f(\mathrm{e}^{\mathrm{i}\pi/3}, \mathrm{e}^{\mathrm{i}2\pi/3}, \ldots, \mathrm{e}^{\mathrm{i}2\pi})| \leq c \sum_{i=1}^{6} |\nabla\hat{\omega}(\triangle_i) - \mathrm{Id}|^2 \tag{15}$$

$\mathbb{G}_{n_k}$-a.s. for any $k \geq 1$. Combining equations (14) and (15), we obtain that

$$\lim_{\rho \uparrow 2/\sqrt{3}} \mathbb{E}_{\mathbb{P}_\rho} \left[|f(y_1, \ldots, y_6) - f\left(\mathrm{e}^{\mathrm{i}\pi/3}, \mathrm{e}^{\mathrm{i}2\pi/3}, \ldots, \mathrm{e}^{\mathrm{i}2\pi}\right)| \right] = 0,$$

which concludes the proof of property (iii). □

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