PROOF OF A CONJECTURE OF SCHOENBERG ON THE GENERATING FUNCTION OF A TOTALLY POSITIVE SEQUENCE

ALBERT EDREI

Let

$$(1) \qquad a_0, a_1, a_2, \ldots$$

be a sequence of real terms with which we associate the generating power series

$$(2) \qquad a_0 + a_1 z + a_2 z^2 + \ldots = f(z).$$

We consider the following definition due to Schoenberg [7, p. 362]:

DEFINITION 1. *The sequence* (1) *is said to be totally positive if the infinite matrix*

$$(3) \qquad \begin{matrix} a_0 & 0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & 0 & \ldots \\ a_2 & a_1 & a_0 & 0 & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

has only non-negative minors (*of all orders, with any choice of rows and columns*).

Schoenberg [7] proved that the coefficients of the series (2) form a totally positive sequence if $f(z)$ is a function of the form

$$(4) \qquad f(z) = Cz^{\lambda}e^{\gamma z}\prod_{\nu=1}^{\infty}(1 + \alpha_\nu z)\Big/\prod_{\nu=1}^{\infty}(1 - \beta_\nu z),$$

where λ is a non-negative integer and $C \geqslant 0$, $\gamma \geqslant 0$, $\alpha_\nu \geqslant 0$, $\beta_\nu \geqslant 0$, $\sum(\alpha_\nu + \beta_\nu) < + \infty$.

From now on, we shall assume $a_0 \neq 0$ (i.e., $\lambda = 0$); obviously this is only an apparent restriction.

Schoenberg [7, p. 367] also conjectured that functions of the form (4) are the only ones which generate totally positive sequences.

At the International Congress of Mathematicians [Cambridge, Massachusetts, 1950], Aissen, Schoenberg, and Whitney announced that they had proved the following proposition:

A. *If the coefficients of the series* (2) *form a totally positive sequence, then the series* (2) *is necessarily the expansion of a function of the form*

$$f(z) = e^{\phi(z)}\prod_{\nu=1}^{\infty}(1 + \alpha_\nu z)\Big/\prod_{\nu=1}^{\infty}(1 - \beta_\nu z),$$

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where $\phi(z)$ *is an integral function and* $\alpha_\nu \geqslant 0$, $\beta_\nu \geqslant 0$, $\sum(\alpha_\nu + \beta_\nu) < +\infty$.

In trying to prove this proposition, I was able to complete the solution of Schoenberg's problem by showing:

B. *Under the assumptions of* A, $\phi(z)$ *necessarily reduces to a polynomial of the form* $\gamma z + \delta$ $(\gamma \geqslant 0)$.

A detailed account of the results of Aissen, Schoenberg, Whitney, and the author will be given in the Journal d'Analyse Mathématique [**1**; **3**; **8**]. These results yield a proof of Schoenberg's conjecture which differs substantially from my first approach to the subject. My excuse for presenting here this early arrangement is that my method essentially reduces to the construction of the Padé table of the generating function of a totally positive sequence. It discloses a number of surprisingly simple properties of such a table. Their investigation has been continued by R. J. Arms and the author and has led to theorems on the convergence of the Padé table and continued fractions associated with a totally positive sequence.

1. Definition and immediate consequences of the property (P).

DEFINITION 2. *Consider the determinants*

$$A_m^{(n)} = \begin{vmatrix} a_m & a_{m-1} & a_{m-2} & \dots & a_{m-n+1} \\ a_{m+1} & a_m & a_{m-1} & \dots & a_{m-n+2} \\ a_{m+2} & a_{m+1} & a_m & \dots & a_{m-n+3} \\ & \dots & & \dots & \\ a_{m+n-1} & a_{m+n-2} & a_{m+n-3} & \dots & a_m \end{vmatrix}, A_m^{(0)} = 1 \qquad (m \geqslant 0, n \geqslant 0)$$

where $a_\mu = 0$ *for* $\mu < 0$. *We say that the power series* (2) *has the property* (P) *if* $A_m^{(n)} > 0$, *for all* $m \geqslant 0$ *and all* $n \geqslant 0$.

The inequalities

$$a_m > 0, \qquad A_m^{(2)} > 0 \qquad (m = 0, 1, 2, \dots),$$

imply

$$\frac{1}{R} = \lim_{m\to\infty} \frac{a_{m+1}}{a_m} < \frac{a_1}{a_0},$$

so that, if (2) has the property (P), its radius of convergence R, is positive.[1] It has been proved by Schoenberg [**6**, p. 558] that, *if* $f(z)$ *has the property* (P), (1) *is totally positive.* On the other hand, it is obvious from the definitions that the converse cannot be true.[2]

[1]By a similar argument [**7**, p. 362[, it is easily verified that the generating series of a totally positive sequence has a positive radius of convergence.

[2]The connection between the property (P) and total positivity is completely clarified by an unpublished result of R. J. Arms: Mr. Arms has shown that, if $f(z)$ denotes the generating function of a totally positive sequence, and if $f(z)$ does not have the property (P), then $f(z)$ reduces to a rational function.

Two observations, of purely formal character play an important part in this investigation.

I. *If $f(z)$ has the property* (P), *then*

$$\frac{1}{f(-z)} = b_0 + b_1 z + b_2 z^2 + \ldots ,$$

also has the property (P).

This is an immediate consequence of the elementary identities

$$a_0^{-n} A_m^{(n)} = b_0^{-m} B_n^{(m)} \qquad (m \geqslant 0, n \geqslant 0),$$

where $B_n^{(m)}$ is the determinant obtained by replacing, in the expression for $A_n^{(m)}$, the a's by the b's.

II. *Let $f(z)$ and $g(z)$ denote generating functions of totally positive sequences. Then the sequence $\{c_n\}$ defined by*

$$f(z)\, g(z) = c_0 + c_1 z + c_2 z^2 + \ldots \qquad (f(0)\, g(0) \neq 0)$$

is totally positive. Moreover $f(z)g(z)$ has the property (P) *if at least one of the two factors f and g has this property.*

It is easily verified, by direct computation, that e^z has the property (P). Hence the function $e^{\epsilon z} f(z)$ has the property (P) as soon as (1) is totally positive and $\epsilon > 0$ $(a_0 \neq 0)$.

2. Poles and zeros of $f(z)$. Let $f(z)$ have the property (P). Consider the polynomials $Q^{(m,n)}(z)$ defined by

$$Q^{(m,n)}(z) = \frac{1}{A_m^{(n)}} \begin{vmatrix} 1 & z & z^2 & \ldots & z^n \\ a_{m+1} & a_m & a_{m-1} & \ldots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \ldots & a_{m-n+2} \\ & \ldots & & \ldots & \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \ldots & a_m \end{vmatrix} \qquad (m \geqslant 0, n \geqslant 0),$$

$$Q^{(m,0)}(z) \equiv 1 \qquad (a_\mu = 0 \text{ for } \mu < 0),$$

$$S^{(m,n)}(z) = Q^{(m,n)}(-z) \qquad (m \geqslant 0, n \geqslant 0).$$

We prove the following seven assertions:

(i) *the coefficients of $S^{(m,n)}$ are all positive;*

(ii) $$\lim_{m\to\infty} Q^{(m,n)}(z) = Q^{(\infty,n)}(z) \text{ exists for all } n \geqslant 0;$$

(iii) $$\lim_{m\to\infty} \frac{A_{m+1}^{(n+1)} \big/ A_m^{(n+1)}}{A_{m+1}^{(n)} \big/ A_m^{(n)}} = \beta_{n+1} > 0 \text{ exists for all } n (\geqq 0);$$

(iv) *the polynomials $Q^{(\infty,n)}(z)$ are connected by the recurrence relations*

$$Q^{(\infty,n+1)}(z) = (1 - \beta_{n+1}z)\, Q^{(\infty,n)}(z) \qquad (n \geqslant 0);$$

(v) $\sum_{n=1}^{\infty} \beta_n$ *converges so that*

$$Q(z) = \lim_{n\to\infty} Q^{(\infty,n)}(z) = \prod_{n=1}^{\infty} (1 - \beta_n z)$$

is either a polynomial or an integral function of genus zero;

(vi) $$f(z)\, Q^{(\infty,n)}(z) = p_0^{(n)} + p_1^{(n)}z + p_2^{(n)}z^2 + \dots$$

converges for $|z| < 1/\beta_{n+1}$, *the coefficients* $p_\nu^{(n)}$ *are all positive;*

(vii) $$f(z)\, Q(z) = p_0 + p_1 z + p_2 z^2 + \dots$$

is an integral function, the coefficients p_ν *are non-negative.*

Proof. All the above assertions are easily deduced from the identities[3]

(5) $$S^{(m,n+1)}(z) = S^{(m+1,n)}(z) + zS^{(m,n)}(z)\frac{A_{m+1}^{(n+1)} / A_m^{(n+1)}}{A_{m+1}^{(n)} / A_m^{(n)}},$$

(6) $$S^{(m+1,n+1)}(z) - S^{(m,n+1)}(z) = -\,zS^{(m,n)}(z)\frac{A_{m+1}^{(n+2)} / A_m^{(n+1)}}{A_{m+1}^{(n+1)} / A_m^{(n)}} = -\,z\gamma_{mn}S^{(m,n)}(z).$$

Assertion (i) is a consequence of a theorem of Schoenberg [6, p. 558]; it is also easy to deduce directly from (5). We now prove (ii).

From (6), we obtain

(7) $$S^{(l+1,n+1)}(z) = S^{(m,n+1)}(z) - z\sum_{\mu=m}^{l} \gamma_{\mu n}S^{(\mu,n)}(z) \qquad (\gamma_{\mu n} > 0).$$

As the coefficients of the S's are positive, it follows from (7) that all the coefficients (except the constant term, which is 1) decrease if we increase l and keep n fixed. This proves (ii) and also the convergence of

$$\sum_{\mu=0}^{\infty} \gamma_{\mu n}.$$

Observing that, for a positive z,

$$\lim_{m\to\infty} zS^{(m,n)}(z) > 0,$$

we deduce from (5) the existence of the limit in (iii). This proves (iii) and (iv) simultaneously.

Consider (7) for $l = \infty$, $m = 0$, and compare the coefficients of z; this yields

$$\beta_1 + \beta_2 + \dots + \beta_{n+1} < a_1/a_0,$$

which proves (v).

To prove (vi), we consider the expansion

[3]These formulae are straightforward consequences of Jacobi's theorem on the minors of the adjoint of a given determinant.

$$f(z)Q^{(m,n)}(z) = \sum_{\mu=0}^{\infty} p_\mu^{(m,n)} z^\mu \tag{8}$$

and observe that the well-known identity [5, p. 430]

$$f(z)Q^{(m,n)}(z) = \frac{1}{A_m^{(n)}}\sum_{\mu=0}^{\infty} z^\mu \begin{vmatrix} a_\mu & a_{\mu-1} & a_{\mu-2} & \dots & a_{\mu-n} \\ a_{m+1} & a_m & a_{m-1} & \dots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \dots & a_{m-n+2} \\ & \dots & & \dots & \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \dots & a_m \end{vmatrix}, \tag{9}$$

gives us explicitly the coefficients $p_\mu{}^{(m,n)}$. Now the sequence of functions

$$f(z)\ Q^{(m,n)}(z) \qquad (m = 0, \dots, \infty) \tag{10}$$

is regular in the circle $|z| < R$, where R denotes the radius of convergence of (2).

For n fixed and $m \to \infty$, (10) converges uniformly to

$$f(z)\ Q^{(\infty,n)}(z) = p_0{}^{(n)} + p_1{}^{(n)}z^2 + \dots, \tag{11}$$

in any circle $|z| \leqslant R^* < R$.

From (7) we deduce

$$f(z)\ Q^{(\infty,n)}(z) = f(z)\ Q^{(m,n)}(z) + z\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1}\{f(z)\ Q^{(\mu,n-1)}(z)\}. \tag{12}$$

Using (8), (11), (12) and Weierstrass's double-series theorem, we obtain

$$p_k^{(n)} = p_k^{(m,n)} + \sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} p_{k-1}^{(\mu,n-1)} \qquad (k = 1, 2, 3, \dots). \tag{13}$$

The total positivity of (1) implies $p_k^{(m,n)} > 0$, when $k \leqslant m$, so that, (13) yields

$$p_k^{(n)} > p_k^{(m,n)} \qquad (k \leqslant m). \tag{14}$$

Observing that

$$p_{m+1}^{(m,n)} = 0,$$

and using (14), we also obtain

$$p_{m+1}^{(n)} < p_m^{(n-1)}(\gamma_{m,n-1} + \gamma_{m+1,n-1} + \gamma_{m+2,n-1} + \dots) \qquad (m \geqslant 0, n \geqslant 1). \tag{15}$$

Putting

$$\limsup_{m\to\infty} |(p_m^{(n)})^{1/m}| = \sigma_n \qquad (n = 0, 1, 2, \dots), \tag{16}$$

we deduce, from (15),

$$\sigma_n \leqslant \sigma_{n-1} \limsup_{m\to\infty} \left| \left(\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} \right)^{1/m} \right|. \tag{17}$$

Moreover, as all series of the form

$$\sum_{\mu=m}^{\infty} \gamma_{\mu n}$$

are convergent, (17) trivially implies

(18) $$\sigma_n \leqslant \sigma_{n-1} \qquad (n = 1, 2, 3, \ldots).$$

We now verify the inequalities

(19) $$\sigma_k \leqslant \beta_{k+1} \qquad (k = 0, 1, 2, \ldots),$$

by an induction over k.

We note that $p_\mu^{(0)} = a_\mu$, implies $\sigma_0 = \beta_1$, and assume that (19) is true for $k = 0, 1, 2, \ldots, n-1$. We distinguish two cases

(a) $$\beta_1\beta_2 \ldots \beta_n = 0,$$

(b) $$\beta_1\beta_2 \ldots \beta_n \neq 0.$$

The case (a) is immediately settled by observing that our induction assumption, and (18), then yield $\sigma_k = 0$, for $k \geqslant n$.

In the case (b), we observe that

(20) $$\limsup_{m\to\infty} \left| \left(\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} \right)^{1/m} \right| \leqslant \frac{\beta_{n+1}}{\beta_n},$$

is trivial if $\beta_{n+1} \geqslant \beta_n$.

If $\beta_{n+1} < \beta_n$, we return to the formulae (6) which define the quantities γ_{mn} as ratios of determinants.

Observing that (iii) implies[4]

$$\lim_{m\to\infty} \frac{A_{m+1}^{(n+1)}}{A_m^{(n+1)}} = \beta_1\beta_2 \ldots \beta_n\beta_{n+1},$$

(21) $$\lim_{m\to\infty} \left| (A_m^{(n)})^{1/m} \right| = \beta_1\beta_2 \ldots \beta_n,$$

we obtain

$$\lim_{m\to\infty} (\gamma_{m,n-1})^{1/m} = \lim_{m\to\infty} \left(\frac{A_{m+1}^{(n+1)}}{A_{m+1}^{(n)}} \frac{A_m^{(n-1)}}{A_m^{(n)}} \right)^{1/m} = \frac{\beta_{n+1}}{\beta_n}.$$

We may then compare

$$\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1}$$

to the remainder of a geometric series. This comparison leads again to (20). Combining (17) and (20), we complete the induction which proves (19). As (13) implies $p_\mu^{(n)} > 0$, assertion (vi) is also proved. Assertion (vii) follows from (vi) and Weierstrass's double-series theorem. Although the fact is not used in this paper, we note that (19) can be replaced by

(22) $$\sigma_k = \beta_{k+1} \qquad (k = 0, 1, 2, \ldots).$$

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[4]It should be noted that the validity of (21) is independent of the arguments following (14). Interrupting our proof after (14), and using (21), a direct application of Hadamard's theorem on polar singularities (see, for instance [2, p. 333])would have enabled us to verify that $f(z)$ is a meromorphic function. However, as the wording of Hadamard's theorem does not meet all our requirements, and as we do not need its more delicate parts, I have replaced its use by simple and elegant arguments which I owe to R. J. Arms.

To verify this, we assume that, for a definite value of $k(=n)$, $\sigma_n < \beta_{n+1}$. The point $1/\beta_{n+1}$ would then be interior to the circle of convergence of the expansion of $f(z)Q^{(\infty,n)}(z)$. The function $f(z)Q^{(\infty,n+1)}(z)$ would be regular and would vanish at $z = 1/\beta_{n+1}$. This is impossible in view of the positivity of the coefficients $p_\nu^{(n+1)}$. We finally observe that (18) and (22) imply

$$\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant \dots .$$

3. Proof of Proposition A. The function $e^z f(z)$ certainly possesses the property (P); it is therefore sufficient to prove the proposition for functions which have this property.

By I and the seven assertions, there exist two integral functions $P(z)$ and $Q(z)$ such that

$$P(z) = \prod_{\nu=1}^{\infty} (1 - \alpha_\nu z) \qquad (\alpha_\nu \geqslant 0, \sum \alpha_\nu < +\infty),$$

$$Q(z) = \prod_{\nu=1}^{\infty} (1 - \beta_\nu z) \qquad (\beta_\nu \geqslant 0, \sum \beta_\nu < +\infty)$$

and such that both $P(z)/f(-z)$ and $Q(z)f(z)$ are integral functions. Moreover, when $z \geqslant 0$, the function $P(z)/f(-z)$ does not vanish because, by assertion (vii) the coefficients of its expansion are real, non-negative, and $a_0^{-1} = P(0)/f(0) \neq 0$. Hence $f(z)/P(-z)$ is regular for negative values of z and

$$\frac{f(z)}{P(-z)} Q(z) = \frac{f(z)Q(z)}{P(-z)} = v(z)$$

is obviously an integral function. Similarly, $1/v(z)$ is an integral function so that $v(z)$ has no zeros.

Hence

$$v(z) = e^{\phi(z)},$$

which proves the theorem of Aissen, Schoenberg, and Whitney.

4. Proof of Schoenberg's conjecture. As (1) is totally positive, the coefficients of the function

$$f^*(z) = a_0 + a_2 z + a_4 z^2 + \dots + a_{2n} z^n + \dots .$$

also form a totally positive sequence. Applying Proposition A to both $f(z)$ and $f^*(z)$, we can then assert the existence of

1. two integral functions $\phi(z)$ and $\phi^*(z)$,

2. four convergent series of non-negative terms, say

$$\Sigma\alpha_\nu, \quad \Sigma\beta_\nu, \quad \Sigma\alpha_\nu^*, \quad \Sigma\beta_\nu^*,$$

such that

$$f(z) = e^{\phi(z)} \frac{\prod (1 + \alpha_\nu z)}{\prod (1 - \beta_\nu z)}, \tag{23}$$

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$$f^*(z) = e^{\phi^*(z)} \frac{\prod (1 + \alpha^*_\nu z)}{\prod (1 - \beta^*_\nu z)}. \tag{24}$$

Now

$$\frac{f(z) + f(-z)}{2} = f^*(z^2);$$

using (23) and (24), this becomes

$$e^{\phi(z)} \frac{\prod (1 + \alpha_\nu z)}{\prod (1 - \beta_\nu z)} + e^{\phi(-z)} \frac{\prod (1 - \alpha_\nu z)}{\prod (1 + \beta_\nu z)} = 2e^{\phi^*(z^2)} \frac{\prod (1 + \alpha^*_\nu z^2)}{\prod (1 - \beta^*_\nu z^2)}. \tag{25}$$

Put $\phi(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots$ and

$$u(z) = e^{2(\gamma_1 z + \gamma_3 z^3 + \gamma_5 z^5 + \ldots)} \prod_{\nu=1}^{\infty} \frac{(1 + \alpha_\nu z)(1 + \beta_\nu z)}{(1 - \alpha_\nu z)(1 - \beta_\nu z)}; \tag{26}$$

equation (25) is then equivalent to

$$u(z) + 1 = 2e^{\phi^*(z^2) - \phi(-z)} \prod_{\nu=1}^{\infty} \frac{(1 + \alpha^*_\nu z^2)(1 + \beta_\nu z)}{(1 - \beta^*_\nu z^2)(1 - \alpha_\nu z)}. \tag{27}$$

This relation, together with (26), gives us complete information on the distribution of the values 0, ∞, and -1 of the meromorphic function $u(z)$. The value -1 has an exponent of convergence not exceeding 2, whereas both the zeros and the poles have an exponent of convergence not exceeding 1.

By a well-known theorem of Nevanlinna [4, p. 72], the order of $u(z)$ does not exceed 2, so that

$$0 = \gamma_3 = \gamma_5 = \gamma_7 = \ldots.$$

Hence $f(z)$ is of the form

$$f(z) = Ce^{\gamma_1 z + \gamma_2 z^2 + \gamma_4 z^4 + \ldots} \frac{\prod (1 + \alpha_\nu z)}{\prod (1 - \beta_\nu z)} \qquad (C > 0). \tag{28}$$

Now if $\epsilon > 0$, both $e^{\epsilon z} f(z)$ and $e^{\epsilon z}/f(-z)$ have the property (P), so that, by assertion (vii), the power series expansions of

$$u_1(z) = Ce^{(\epsilon + \gamma_1)z + \gamma_2 z^2 + \gamma_4 z^4 + \ldots} \prod_{\nu=1}^{\infty} (1 + \alpha_\nu z),$$

and

$$u_2(z) = C^{-1} e^{(\epsilon + \gamma_1)z - \gamma_2 z^2 - \gamma_4 z^4 - \ldots} \prod_{\nu=1}^{\infty} (1 + \beta_\nu z),$$

have all their coefficients non-negative.

Hence, putting

$$M_1(r) = \max_{|z|=r} |u_1(z)|, \qquad M_2(r) = \max_{|z|=r} |u_2(z)|,$$

we have

$$M_1(r) = u_1(r), \qquad M_2(r) = u_2(r),$$

and

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$$M_1(r)M_2(r) = e^{2(\epsilon+\gamma_1)r} \prod_{\nu=1}^{\infty} (1 + \alpha_\nu r)(1 + \beta_\nu r). \tag{29}$$

As
$$M_2(r) \geqslant M_2(0) = C^{-1},$$
we see that the increase of $M_1(r)$ is dominated by the increase of an integral function of order 1. Hence

$$0 = \gamma_2 = \gamma_4 = \gamma_6 = \dots .$$

We complete the proof of Schoenberg's conjecture by proving $\gamma = \gamma_1 \geqslant 0$. If γ were negative, we could choose $\epsilon > 0$ so that

$$\gamma + \epsilon < 0. \tag{30}$$

Now the increase of

$$\prod_{\nu=1}^{\infty} (1 + \alpha_\nu r)(1 + \beta_\nu r),$$

is at most of the minimal type of order 1. Hence (29) and (30) would imply

$$\lim_{m\to\infty} M_1(r)M_2(r) = 0,$$

which is contrary to the maximum-modulus theorem.

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Syracuse University

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