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Rotation sets with non-empty interior and transitivity in the universal covering

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Abstract. Let f be a transitive homeomorphism of the two-dimensional torus in the homotopy class of the identity. We show that a lift of f to the universal covering is transitive if and only if the rotation set of the lift contains the origin in its interior.

1. *Introduction*

Given a homeomorphism f of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is in the homotopy class of the identity, and a lift $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of f to the universal covering, one can associate with $\widehat{f}$ its rotation set $\rho(\widehat{f})$, a convex and compact subset of the plane consisting of all the limit points of sequences of the form $\lim_{k\to\infty}(\widehat{f}^{n_k}(x_k) - x_k)/n_k$, with $n_k \to \infty$ and $x_k \in \mathbb{R}^2$. This definition was introduced by Misiurewicz and Ziemian in [**MZ89**] as a generalization of the rotation number for circle homeomorphisms, and has proved to be a useful tool in the study of the dynamics of these homeomorphisms. In particular, when the rotation set has a non-empty interior it has been shown that f exhibits very rich dynamics, with an abundance of periodic points and positive topological entropy [**Fr89, LM91**]. However, this complex behavior is not restricted to rotation sets with non-empty interiors: there are several examples of homeomorphisms with rich dynamical properties such that their rotation set is a singleton.

In this work, we are concerned with the interplay between transitivity of f, transitivity on the universal covering space and rotation sets. Our motivation in studying transitivity on the universal covering space is to understand when it is possible for the dynamical system to exhibit this form of extreme transitivity, where there exist trajectories that are not only dense on the surface, but also explore all possible loops and directions. This is a strictly more stringent concept, as ergodic rotations of the 2-torus do not lift to transitive homeomorphisms of the plane. There has been some prior work concerning transitivity of the lifted dynamics of surface homeomorphisms. In [**Bo09**], Boyland defines H_1-transitivity for a surface homeomorphism as the property of having an iterate which lifts to the universal abelian covering as a transitive homeomorphism, and he characterized H_1-transitivity for rel pseudo-Anosov maps.

In [**EGLL**], it was proved that a $C^{1+\alpha}$ diffeomorphism of $\mathbb{T}^2$ homotopic to the identity, and with positive entropy, has a rotation set with a non-empty interior if f has a transitive lift to a suitable covering of $\mathbb{T}^2$. In [**AZT1**] and [**AZT2**], homeomorphisms of the annulus with a transitive lift were studied, and it was shown that if there are no fixed points in the boundary, then the rotation set of the lift contains the origin in its interior.

More relevant to this work is the result from [**Ta12**], where it is shown that if f is a homeomorphism of $\mathbb{T}^2$ homotopic to the identity, which has a transitive lift $\widehat{f}$ to $\mathbb{R}^2$, then the origin lies in the interior of $\rho(\widehat{f})$. Our main result is the converse of this fact.

THEOREM 1. *Let f be a transitive homeomorphism of $\mathbb{T}^2$ homotopic to the identity, and let $\widehat{f}$ be a lift of f to $\mathbb{R}^2$. If $(0, 0)$ belongs to the interior of $\rho(\widehat{f})$, then $\widehat{f}$ is transitive.*

Note that since the fundamental group of $\mathbb{T}^2$ is abelian, the notion of H_1-transitivity (as defined in [**Bo09**]) in the case of a homeomorphism f of $\mathbb{T}^2$ is equivalent to saying that some iterate of f has a lift to $\mathbb{R}^2$ which is transitive. As an immediate consequence of Theorem 1, together with the main result of [**Ta12**], we have a characterization of H_1-transitive homeomorphisms of $\mathbb{T}^2$ in the homotopy class of the identity.

COROLLARY 2. *A homeomorphism of $\mathbb{T}^2$ in the homotopy class of the identity is H_1-transitive if and only if it has (a lift with) a rotation set with non-empty interior.*

The key starting point for the proof of Theorem 1 is a recent result from [**KT12**] that rules out the existence of 'unbounded' periodic topological disks. After this, the theorem is reduced to a technical result, Theorem 8, the proof of which is obtained from geometric and combinatorial arguments relying on a well-known theorem of Franks about realization of rational rotation vectors by periodic points [**Fr89**], and some classic facts from Brouwer theory. The next section states these preliminary facts and introduces the necessary notation. In §3, Theorem 8 is stated, and its proof is presented after using it to prove Theorem 1.

2. *Notation and preliminary results*

We denote by $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ the universal covering of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Given a point $x \in \mathbb{R}^2$, we denote by $(x)_1$ the projection of x onto the first coordinate, by $(x)_2$ its projection onto the second coordinate, and

$$\|x\|_\infty = \max\{|(x)_1|, |(x)_2|\}.$$

The following definitions will simplify the notation in the later proofs. For $v \in \mathbb{Z}^2$ and $L \in \mathbb{N}$, define

$$S(v, L) = \{w \in \mathbb{Z}^2 : \|w - v\|_\infty \le L\}$$

and

$$\Sigma(M) = \{(i, v) \in \mathbb{Z} \times \mathbb{Z}^2 : |i| \le M, \|v\|_\infty \le M\}.$$

Given $v \in \mathbb{R}^2$, we denote by $T_v : \mathbb{R}^2 \to \mathbb{R}^2$ the translation $x \mapsto x + v$. If $\gamma : [0, 1] \to \mathbb{R}^2$ is a curve, we denote its image by $[\gamma]$. We say that γ is a *T_v-translation arc* if it joins a point $z = \gamma(0)$ to its image $T_v(z) = \gamma(1)$, and $[\gamma] \cap T_v([\gamma]) = \gamma(1)$.

The following lemma is a direct consequence of [**Br85**, Lemma 3.1].

LEMMA 3. *Let $v \in \mathbb{R}^2$ be a non-zero vector and let $K \subset \mathbb{R}^2$ be an arcwise connected set such that $K \cap T_v(K) = \emptyset$. Then $K \cap T_v^i(K) = \emptyset$, for all $i \in \mathbb{Z}$ with $i \neq 0$.*

The next result follows from [**Br85**, Theorem 4.6].

LEMMA 4. *Let $v \in \mathbb{R}^2 \backslash \{(0, 0)\}$ and let K be an arcwise connected subset of $\mathbb{R}^2$ such that $K \cap T_v(K) = \emptyset$, and α a T_v-translation arc disjoint from K. Then, either*

$$K \cap \bigcup_{i \in \mathbb{N}} T_v^i[\alpha] = \emptyset \quad \text{or} \quad K \cap \bigcup_{i \in \mathbb{N}} T_v^{-i}[\alpha] = \emptyset.$$

We will also use the next theorem from [**Fr89**].

THEOREM 5. (Franks) *Let $\widehat{f}$ be a lift to $\mathbb{R}^2$ of a homeomorphism of $\mathbb{T}^2$ homotopic to the identity. If $w \in \mathbb{Z}^2$ and $q \in \mathbb{N}$ are such that w/q lies in the interior of $\rho(\widehat{f})$, then there is an $\widehat{x} \in \mathbb{R}^2$ such that $\widehat{f}^q(\widehat{x}) = \widehat{x} + w$.*

Let us recall some terminology used in [**KT12**]. An open set $U \subset \mathbb{T}^2$ is *inessential* if every loop contained in U is homotopically trivial in $\mathbb{T}^2$. An arbitrary subset of $\mathbb{T}^2$ is called inessential if it has an inessential neighborhood, and is *essential* otherwise. If A is an inessential subset of $\mathbb{T}^2$, and U is an open neighborhood of $\mathbb{T}^2 \backslash A$, then $\mathbb{T}^2 \backslash U$ is a closed inessential set. This implies that every homotopy class of loops of $\mathbb{T}^2$ is represented by some loop contained in U, and, for this reason, any subset of $\mathbb{T}^2$ with an inessential complement is called a *fully essential* set.

If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism, we say that a point $x \in \mathbb{T}^2$ is an *inessential point* for f if there exists $\epsilon > 0$ such that the set $\bigcup_{i \in \mathbb{Z}} f^i(B_\epsilon(x))$ is inessential. The set of all inessential points of f is denoted by $\mathrm{Ine}(f)$. Any point in the set $\mathrm{Ess}(f) = \mathbb{T}^2 \backslash \mathrm{Ine}(f)$ is called an *essential point* for f. It follows from the definitions that $\mathrm{Ine}(f)$ is open and invariant, while $\mathrm{Ess}(f)$ is closed and invariant.

Given an open connected set $U \subset \mathbb{T}^2$, we denote by $\mathcal{D}(U)$ the diameter in $\mathbb{R}^2$ of any connected component of $\pi^{-1}(U)$. If $\mathcal{D}(U) < \infty$, then we say that U is *bounded*.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism homotopic to the identity. We need the following theorem.

THEOREM 6. [**KT12**, Theorem C] *If f is non-wandering and has (a lift with) a rotation set with a non-empty interior, then* $\mathrm{Ine}(f)$ *is a disjoint union of periodic simply connected sets, which are bounded.*

In particular, when f is transitive, one may conclude that $\mathrm{Ine}(f)$ is empty.

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LEMMA 7. *Suppose that f is transitive and has (a lift with) a rotation set with a non-empty interior. Then* $\mathrm{Ess}(f) = \mathbb{T}^2$.

Proof. Suppose, for a contradiction, that $\mathrm{Ine}(f)$ is non-empty. Then there is some point $x \in \mathrm{Ine}(f)$ with a dense orbit. Let D be the connected component of $\mathrm{Ine}(f)$ that contains x. By Theorem 6, we have that D is a periodic simply connected open and bounded set. Thus, any connected component $\widehat{D}$ of $\pi^{-1}(D)$ is bounded.

Let k be such that $f^k(D) = D$ and let $\widehat{f}$ be a lift of f. Then, $\widehat{f}^k(\widehat{D}) = \widehat{D} + v$ for some $v \in \mathbb{Z}^2$, so if $\widehat{g} = T_v^{-1}\widehat{f}^k$, we have $\widehat{g}(\widehat{D}) = \widehat{D}$. The set $\widehat{U} = \bigcup_{i=0}^{k-1} \widehat{f}^i(\widehat{D})$ is then bounded and $\widehat{g}$-invariant (because $\widehat{g}$ commutes with $\widehat{f}$), and therefore $\mathrm{cl}(\widehat{U})$ is also bounded and $\widehat{g}$-invariant. In particular, all points in $\mathrm{cl}(\widehat{U})$ have a bounded $\widehat{g}$-orbit. Since $U = \pi(\widehat{U})$ is the f-orbit of D, which contains the (dense) orbit of x, it follows that $\mathrm{cl}(U) = \mathbb{T}^2$. The boundedness of $\widehat{U}$ then implies that $\pi(\mathrm{cl}(\widehat{U})) = \mathrm{cl}(U) = \mathbb{T}^2$.

It follows from these facts that every point of $\mathbb{R}^2$ has a bounded g-orbit, from which we conclude that $\rho(\widehat{g}) = \{(0, 0)\}$, but it follows from the definition of a rotation set that $\rho(\widehat{g}) = \rho(T_v^{-1}\widehat{f}^k) = k\rho(\widehat{f}) - v$ (see [**MZ89**]), contradicting the fact that $\rho(\widehat{f})$ has a non-empty interior. □

3. *Proof of Theorem 1*

Theorem 1 is a consequence of the following technical result.

THEOREM 8. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism homotopic to the identity, and $\widehat{f}$ a lift of f such that* $(0, 0)$ *belongs to the interior of $\rho(\widehat{f})$. Let $O \subset \mathbb{R}^2$ be an open connected set such that $\overline{\pi(O)}$ is inessential and $\bigcup_{n\in\mathbb{Z}} f^n(\pi(O))$ is fully essential. Then, for every $w \in \mathbb{Z}^2$, there exists $n \in \mathbb{N}$ such that $\widehat{f}^n(O) \cap T_w(O) \neq \emptyset$.*

Proof of Theorem 1 assuming Theorem 8. Let O_1, O_2 be two open sets of the plane. Since f is transitive, there exists n_0 such that $f^{n_0}(\pi(O_1))$ intersects $\pi(O_2)$, which implies that there exists $w \in \mathbb{Z}^2$ such that $\widehat{f}^{n_0}(O_1) \cap T_w(O_2) \neq \emptyset$. Let $\widehat{x} \in \mathbb{R}^2$ and $0 < \varepsilon < \frac{1}{2}$ be such that $B_\varepsilon(\widehat{x}) \subset \widehat{f}^{n_0}(O_1) \cap T_w(O_2)$. Note that, as $\varepsilon < \frac{1}{2}$, the set $\pi(B_\varepsilon(\widehat{x}))$ is inessential.

Since f is transitive and $\rho(\widehat{f})$ has an interior, Lemma 7 implies that $\mathrm{Ess}(f) = \mathbb{T}^2$ and, in particular, $U_\varepsilon(x) = \bigcup_{i\in\mathbb{Z}} f^i(B_\varepsilon(x))$ is an essential invariant open set and, since f is non-wandering, the connected component of $\widehat{x}$ in $U_\varepsilon(x)$ is periodic. In $\mathbb{T}^2$, an essential connected periodic open set is either fully essential or contained in a periodic set homeomorphic to an essential annulus, in which case the rotation set is contained in a line segment. Since $\rho(\widehat{f})$ has a non-empty interior, we conclude that $U_\varepsilon(x)$ is fully essential.

Therefore, $B_\varepsilon(\widehat{x})$ satisfies the hypotheses of Theorem 8. Thus, there exists n_1 such that $\widehat{f}^{n_1}(B_\varepsilon(\widehat{x}))$ intersects $T_{-w}(B_\varepsilon(\widehat{x}))$, but this implies that $\widehat{f}^{n_0+n_1}(O_1)$ intersects $T_{-w}(T_w(O_2)) = O_2$, completing the proof. □

3.1. *Proof of Theorem 8.*

We begin by fixing both $\overline{w} \in \mathbb{Z}^2$ and an open connected set $O \subset \mathbb{R}^2$, such that $\bigcup_{n\in\mathbb{Z}} f^n(\pi(O))$ is fully essential and $\overline{\pi(O)}$ is inessential. Our aim is to show that $\widehat{f}^n(O) \cap T_{\overline{w}}(O) \neq \emptyset$ for some $n \in \mathbb{N}$.

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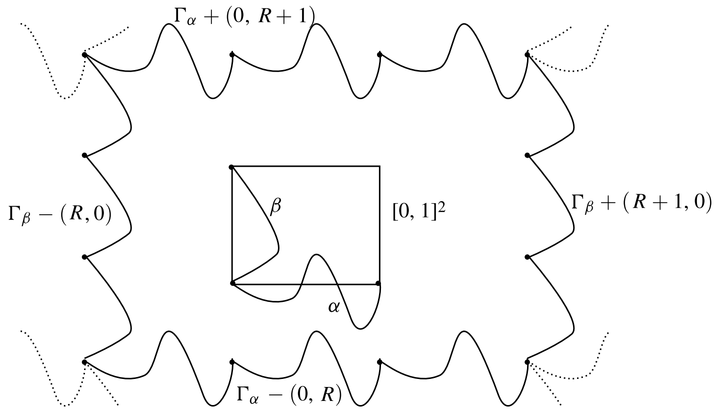

FIGURE 1. Proof of Proposition 9.

Since we are assuming that $\rho(\widehat{f})$ contains (0, 0) in its interior, we may choose $\delta > 0$ (which is fixed from now on) such that $B_\delta((0, 0)) \subset \rho(\widehat{f})$.

PROPOSITION 9. *There exists a positive integer M and a compact set K such that* $[0, 1]^2$ *is contained in a bounded connected component of* $\mathbb{R}^2\backslash K$ *and*

$$K \subset \bigcup_{(i,v)\in\Sigma(M)} T_v(\widehat{f}^i(O)).$$

Proof. Since $\bigcup_{n\in\mathbb{Z}} f^n(\pi(O))$ is fully essential, its preimage by π, which we denote by $\widehat{U}$, is a connected open set invariant by $\mathbb{Z}^2$ translates. Therefore, given a point $\widehat{y}$ in $\widehat{U}$, there exist two connected arcs α and β in $\widehat{U}$ such that α connects $\widehat{y}$ to $\widehat{y} + (1, 0)$ and β connects $\widehat{y}$ to $\widehat{y} + (0, 1)$. Let

$$\Gamma_\alpha = \bigcup_{i=-\infty}^{\infty} T^i_{(1,0)}[\alpha], \quad \Gamma_\beta = \bigcup_{i=-\infty}^{\infty} T^i_{(0,1)}[\beta],$$

and note that as $\widehat{U}$ is $\mathbb{Z}^2$ invariant, all integer translates of Γ_α and Γ_β are contained in $\widehat{U}$.

Fix an integer $R > \max\{\|x\| : x \in [\alpha] \cup [\beta]\}$. Then, since

$$\max\{(x)_2 : x \in \Gamma_\alpha\} = \max\{(x)_2 : x \in [\alpha]\} < R,$$

and $\min\{(x)_2 : x \in \Gamma_\alpha\} > -R$, it follows that $\mathbb{R}^2\backslash\Gamma_\alpha$ has at least two connected components, one containing the semi-plane $\{x : (x)_2 \geq R\}$ and another containing $\{x : (x)_2 \leq -R\}$. Likewise, $\mathbb{R}^2\backslash\Gamma_\beta$ has at least two connected components, one containing $\{x : (x)_1 \geq R\}$ and another containing $\{x : (x)_1 \leq -R\}$.

Now, let

$$F = (\Gamma_\alpha - (0, R)) \cup (\Gamma_\alpha + (0, R+1)) \cup (\Gamma_\beta - (R, 0)) \cup (\Gamma_\beta + (R+1, 0)),$$

and note that $F \subset \widehat{U}$ and $\mathbb{R}^2\backslash F$ has a connected component W that contains $[0, 1]^2$ and is contained in $[-2R, 2R+1] \times [-2R, 2R+1]$ (see Figure 1).

Let $K = \partial W$. Then, K is a compact subset of $\widehat{U}$. Since $\widehat{U} = \bigcup_{v\in\mathbb{Z}^2} \bigcup_{i\in\mathbb{Z}} T_v(\widehat{f}^i(O))$ is an open cover of K, choosing a finite subcover, we conclude the existence of M. □

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PROPOSITION 10. *For every* $w \in \mathbb{Z}^2$, *if* $n > \|w\|/\delta$, *then* $\widehat{f}^n([0,1]^2)$ *intersects* $T_w([0,1]^2)$.

Proof. By our choice of δ (at the beginning of this section), w/n is in the interior of $\rho(\widehat{f})$. It follows from Theorem 5 that there is a $\widehat{y} \in \mathbb{R}^2$ such that $\widehat{f}^n(\widehat{y}) = \widehat{y} + w$. Since we may choose $\widehat{y} \in [0,1]^2$ (by using an appropriate integer translation), the proposition follows. □

PROPOSITION 11. *For every* $w \in \mathbb{Z}^2$ *and every* $n > (\|w\|/\delta)$, *there exists* (j_w, v_w) *in* $\Sigma(2M)$ *such that* $\widehat{f}^n(\widehat{f}^{j_w}(T_{v_w}(O))) \cap T_w(O) \neq \emptyset$.

Proof. By Proposition 10, if U is the connected component of $\mathbb{R}^2 \backslash K$ that contains $[0,1]^2$, then $\widehat{f}^n(U)$ intersects $T_w(U)$, and since U is bounded, this implies that $\widehat{f}^n(\partial U)$ intersects $T_w(\partial U)$. Since $\partial U \subset K$, it follows that $\widehat{f}^n(K) \cap T_w(K) \neq \emptyset$. If $\widehat{x}$ is a point in this intersection, then there exist integers $j_1, j_2 \in [-M, M]$ and $v_1, v_2 \in \mathbb{Z}^2$ with $\|v_i\|_\infty < M$, $i \in \{1,2\}$, such that $\widehat{x}$ belongs to both $\widehat{f}^n(\widehat{f}^{j_1}(T_{v_1}(O)))$ and $\widehat{f}^{j_2}(T_{v_2}(O+w))$. Setting $j_w = j_1 - j_2$ and $v_w = v_1 - v_2$ yields the result. □

LEMMA 12. *Let* $v \in \mathbb{R}^2$ *be a non-zero vector, and* K_1, K_2 *be two arcwise connected subsets of* $\mathbb{R}^2$ *such that* $T_v(K_i) \cap K_i = \emptyset$ *for* $i \in \{1,2\}$. *Suppose there are integers* i, j *with* $i \geq 0$, $j > 0$ *such that* $T_v^{-i}(K_1) \cap K_2 \neq \emptyset$ *and* $T_v^j(K_1) \cap K_2 \neq \emptyset$. *Then* K_1 *intersects* K_2. *Moreover, there exists a* T_v*-translation arc* γ *contained in* $K_1 \cup K_2$ *and joining a point* $x \in K_1$ *to* $T_v(x) \in K_2$.

Proof. We may assume that K_1 is compact by replacing it by a compact arc contained in K_1 and joining some point of $T_v^i(K_2)$ to a point of $T_v^{-j}(K_2)$.

Let $\alpha : [0,1] \to K_2$ be a simple arc satisfying $\alpha(0) \in T_v^{-i}(K_1)$ and $\alpha(1) \in T_v^j(K_1)$. Since $[\alpha]$ and K_1 are compact and v is not null, there exists an integer n_0 such that, if $|n| > n_0$, then $T_v^n(K_1)$ is disjoint from $[\alpha]$.

Let

$$s_0 = \max\left\{t \in [0,1] : \alpha(t) \in \bigcup_{n=0}^{n_0} T_v^{-n}(K_1)\right\},$$

$$s_1 = \min\left\{t \in [s_0,1] : \alpha(t) \in \bigcup_{n=1}^{n_0} T_v^{n}(K_1)\right\},$$

and let i_0 and j_0 be integers such that $\alpha(s_0) \in T_v^{-i_0}(K_1)$ and $\alpha(s_1) \in T_v^{j_0}(K_1)$. Finally, let

$$K_3 = T_v^{-i_0}(K_1) \cup \alpha([s_0, s_1]) \cup T_v^{j_0}(K_1),$$

which is a connected set.

We claim that $i_0 = 0$ and $j_0 = 1$. To prove our claim, assume that it does not hold. Then, $i_0 + j_0 > 1$. Note that, by construction, $\alpha((s_0, s_1))$ is disjoint from $\bigcup_{n\in\mathbb{Z}} T_v^n(K_1)$. Since $K_1 \cap T_v(K_1) = \emptyset$, it follows from Lemma 3 that $K_1 \cap T_v^n(K_1) = \emptyset$ for any $n \neq 0$, and so $T_v(T_v^{-i_0}(K_1))$ is disjoint from $T_v^{j_0}(K_1)$ (because $i_0 + j_0 \neq 1$). From these facts and from the hypotheses, it follows that K_3 is disjoint from $T_v(K_3)$, so, again by Lemma 3, we have that K_3 is disjoint from $T_v^n(K_3)$ for all $n \neq 0$. However, $i_0 + j_0 \neq 0$, and clearly $T_v^{j_0+i_0}(K_3)$ intersects K_3. This contradiction shows that $i_0 + j_0 = 1$, i.e. $i_0 = 0$ and $j_0 = 1$.

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Since $\alpha(s_0) \in K_2 \cap T_v^{-i_0}(K_1) = K_2 \cap K_1$, we have shown that K_1 intersects K_2. Note that $\alpha(s_1) \in T_v^{j_0}(K_1) = T_v(K_1)$, and let $\beta : [0, 1] \to K_1$ be a simple arc joining $\alpha(s_1) - v$ to $\alpha(s_0)$. Since $[\beta] \subset K_1$ and $\alpha([s_0, s_1)) \cap (T_v(K_1) \cup T_{-v}(K_1)) = \emptyset$, it follows that $[\beta] \cap T_v(\alpha([s_0, s_1))) = \emptyset = T_v([\beta]) \cap \alpha([s_0, s_1))$. Therefore, letting γ be the concatenation of β with $\alpha|_{[s_0,s_1]}$, we conclude that γ is a T_v-translation arc joining $x = \alpha(s_1) - v \in K_1$ to $T_v(x) \in K_2$. □

Define, for each $(j, v) \in \mathbb{Z} \times \mathbb{Z}^2$, the sets

$$I(j, v) = \{w \in \mathbb{Z}^2 : \ \widehat{f}^j(T_v(O)) \cap T_w(O) \neq \emptyset\}.$$

PROPOSITION 13. *The following properties hold.*

(1) *If $R > 0$ and $n > R/\delta$, then*

$$(\mathbb{Z}^2 \cap B_R((0, 0))) \subset \bigcup_{(j,v)\in\Sigma(2M)} I(j + n, v).$$

(2) *For any $w, v \in \mathbb{Z}^2$ and $j \in \mathbb{Z}$, $T_w(I(j, v)) = I(j, T_w(v))$.*

(3) *Let $u, v \in \mathbb{Z}^2$ and $j \in \mathbb{Z}$. If $S(u, C) \subset I(j, v)$, then $S(u, C - \|v - w\|_\infty) \subset I(j, w)$.*

Proof. Parts (1) and (2) are direct consequences of the definition of $I(j, v)$ and Proposition 11. To prove (3), let $r \in S(u, C - \|v - w\|_\infty)$. Since $r + v - w \in S(u, C) \subset I(j, v)$, it follows from (2) that $r = T_{w-v}(r + v - w) \in I(j, T_{w-v}(v)) = I(j, w)$. □

PROPOSITION 14. *Let i, k_0, k_1, j be integers with $k_0 < k_1$, and $u \in \mathbb{Z}^2$. If both (i, k_0) and (i, k_1) belong to $I(j, u)$, then $(i, k) \in I(j, u)$ for each integer $k \in [k_0, k_1]$. Likewise, if both (k_0, i) and (k_1, i) belong to $I(j, u)$, then $(k, i) \in I(j, u)$ for all integers $k \in [k_0, k_1]$.*

Proof. This is a direct consequence of Lemma 12, choosing $K_1 = O + (i, k)$, $K_2 = f^j(T_u(O))$ and $v = (1, 0)$ for the first case, and an analogous choice for the second case. □

LEMMA 15. *Given $C_1 \in \mathbb{Z}$, there is $C_2 > 0$ such that, for any $w_1 \in \mathbb{Z}^2$, $C > C_2$, and $n > (\|w_1\| + \sqrt{2}C)/\delta$, there exist $w_2 \in \mathbb{Z}^2$ and $\overline{j} \in [-2M, 2M] \cap \mathbb{Z}$ such that*

$$S(w_2, C_1) \subset S(w_1, C) \cap I(n + \overline{j}, \overline{w}).$$

Proof. Define the auxiliary constants $L = (4M + 1)^3 + 1$, which are larger than the cardinality of the set $\Sigma(2M)$, and $R = 2M + \|\overline{w}\|_\infty$. Also, let $D = 2C_1 + 2R + 1$, and let $C_2 = L^L D$.

Note that if $C > C_2$ and n is chosen as in the statement, Proposition 13(1) implies that

$$S(w_1, C) \subset \bigcup_{(j,v)\in\Sigma(2M)} I(n + j, v).$$

Let i_0, k_0 be integers such that $(i_0, k_0) = w_1 - (C, C)$. Define, for $1 \le s \le L - 1$, $k_s = k_0 + sD$. For $1 \le s \le L$, we will define i_s in $\mathbb{Z}$ and $(j_s, v_s) \in \Sigma(2M)$ recursively, satisfying the following properties:

(1) $i_{s-1} \le i_s \le i_{s-1} + C_2/L^{s-1} - C_2/L^s$;

(2) for any $i_s \le i \le i_s + C_2/L^s$, the point $(i, k_{s-1}) \in I(n + j_s, v_s)$.

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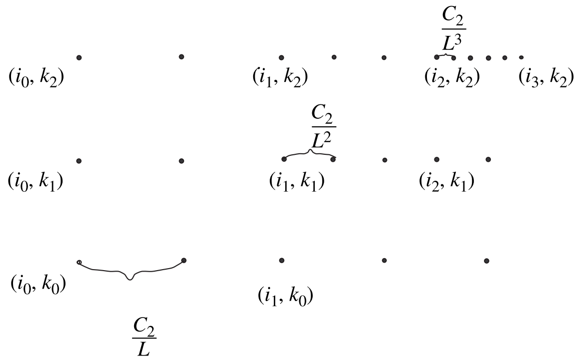

FIGURE 2. Proof of Lemma 15.

Suppose we have already defined i_s and (j_s, v_s). Consider the L different points of the form $(i_{s,r}, k_s)$ with $0 \le r \le L-1$, where $i_{s,r} = i_s + rC_2/L^{s+1}$. Note that, from the recursion hypothesis (1), it follows that $i_s \le i_0 + C_2 - C_2/L^s$, so

$$i_{s,r} \le i_0 + C_2 - C_2/L^s + (L-1)C_2/L^{s+1} = i_0 + C_2 - C_2/L^{s+1} \le i_0 + C.$$

In particular, $(i_{s,r}, k_s) \in S(w_1, C)$ for each $r \in \{0, \ldots, L-1\}$, so there exists $(j_{s,r}, v_{s,r}) \in \Sigma(2M)$ satisfying $(i_{s,r}, k_s) \in I(n + j_{s,r}, v_{s,r})$. Since L is greater than the cardinality of $\Sigma(2M)$, by the pigeonhole principle, there exists $r_1 < r_2$ such that $(j_{s,r_1}, v_{s,r_1}) = (j_{s,r_2}, v_{s,r_2})$. Define $i_{s+1} = i_{s,r_1}$ and $(j_{s+1}, v_{s+1}) = (j_{s,r_1}, v_{s,r_1})$. Since both (i_{s,r_1}, k_s) and (i_{s,r_2}, k_s) belong to $I(n + j_{s+1}, v_{s+1})$, it follows from Proposition 14 that $(i, k_s) \in I(n + j_{s+1}, v_{s+1})$, whenever $i_{s,r_1} \le i \le i_{s,r_2}$. Note also that $i_{s+1} \ge i_s$, and that $i_{s+1} + C_2/L^{s+1} \le i_{s,r_2} \le i_s + C_2/L^s$. So, i_{s+1} and (j_{s+1}, v_{s+1}) satisfy properties (1) and (2) (see Figure 2).

Having defined i_s and (j_s, v_s) for $1 \le s \le L$, we can again use the pigeonhole principle to find integers $1 \le s_1 < s_2 \le L$ such that $(j_{s_1}, v_{s_1}) = (j_{s_2}, v_{s_2})$.

By property (2) in the recursion, for each integer $i \in [i_{s_2}, i_{s_2} + C_2/L^{s_2}]$, the point $(i_{s_2} + i, k_{s_2-1})$ belongs to $I(n + j_{s_2}, v_{s_2})$.

As $i_{s_1} \le i_{s_2}$ and $i_{s_2} + C_2/L^{s_2} < i_{s_1} + C_2/L^{s_1}$, then the interval $[i_{s_2}, i_{s_2} + C_2/L^{s_2}]$ is contained in $[i_{s_1}, i_{s_1} + C_2/L^{s_1}]$, so, again by property (2), for each integer $i \in [i_{s_2}, i_{s_2} + C_2/L^{s_2}]$, the point $(i_{s_2} + i, k_{s_1-1})$ belongs to $I(n + j_{s_1}, v_{s_1}) = I(n + j_{s_2}, v_{s_2})$.

Hence, for all $0 \le i \le C_2/L^{s_2}$, both $(i_{s_2} + i, k_{s_1-1})$ and $(i_{s_2} + i, k_{s_2-1})$ belong to $I(n + j_{s_2}, v_{s_2})$. From Proposition 14, applied to the latter pair of points for each i, it follows that

$$\left\{(i, k) : i_{s_2} \le i \le i_{s_2} + \frac{C_2}{L^{s_2}}, k_{s_1-1} \le k \le k_{s_2-1}\right\} \subset I(n + j_{s_2}, v_{s_2}).$$

Note that $k_{s_2-1} - k_{s_1-1} \ge D$, and also $C_2/L^{s_2} \ge D$. Therefore, if we define $w_2 = (i_{s_2}, k_{s_1-1}) + (R + C_1 + 1, R + C_1 + 1)$, then $S(w_2, R + C_1) \subset I(n + j_{s_2}, v_{s_2})$, and $S(w_2, R + C_1) \subset S(w_1, C)$.

Finally, since $\|v_{s_2} - \overline{w}\|_\infty \le R$, if $\overline{j} = j_{s_2}$, we have, by Proposition 13, $S(w_2, C_1) \subset I(n + \overline{j}, \overline{w})$. □

PROPOSITION 16. *For every $R > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n > n_0$, then $\widehat{f}^n(O)$ intersects each of the four sets*

$$U_1 = \{y : (y)_1 > R, (y)_2 > R\}, \quad U_2 = \{y : (y)_1 < -R, (y)_2 > R\},$$
$$U_3 = \{y : (y)_1 > R, (y)_2 < -R\}, \quad U_4 = \{y : (y)_1 < -R, (y)_2 < -R\}.$$

Proof. Let C_1 be such that $O \subset B_{C_1}((0, 0))$, and let $C_2 = \max_{x\in\mathbb{R}^2} \|\widehat{f}(x) - x\|$, which is finite since f is homotopic to the identity.

Let $C > 2M(C_2 + 1) + C_1 + R$ be an integer, and consider $w_1 = (C, C)$, $w_2 = (C, -C)$, $w_3 = (-C, C)$ and $w_4 = (-C, -C)$.

If $n > \sqrt{2}C/\delta$, then, by Proposition 11, for each $i \in \{1, 2, 3, 4\}$, there exists $(j_i, v_i) \in \Sigma(2M)$ such that $w_i \in I(n + j_i, v_i)$. This implies that $\widehat{f}^{n+j_i}(T_{v_i}(O))$ intersects $T_{w_i}(O) \subset T_{w_i}(B_{C_1}((0, 0))) = B_{C_1}(w_i)$, for each $i \in \mathbb{N}$. Since the definition of C_2 implies that, for every $x \in \mathbb{R}^2$,

$$\|\widehat{f}^{n+j_i}(x) - \widehat{f}^n(x)\| \le |j_i|C_2 \le 2MC_2,$$

we conclude that $\widehat{f}^n(T_{v_i}(O))$ intersects $B_{C_1+2MC_2}(w_i)$. This means that $\widehat{f}^n(O)$ intersects $T_{v_i}^{-1}(B_{C_1+2MC_2}(w_i))$. Since $\|v_i\|_\infty \le 2M$, it follows that

$$T_{v_i}^{-1}(B_{C_1+2MC_2}(w_i)) \subset B_{C_1+2M(C_2+1)}(w_i) \subset U_i,$$

so $\widehat{f}^n(O) \cap U_i \neq \emptyset$. □

CLAIM 1. *Suppose that $\widehat{f}^n(O) \cap T_{\overline{w}}(O) = \emptyset$ for all $n \in \mathbb{Z}$. Let $k \in \mathbb{N}$ and $v \in \mathbb{Z}^2$ be such that $S(v, 1) \subset I(k, \overline{w})$. Then, for every sufficiently large n, we have $v \in I(n, (0, 0))$.*

Proof. Assume, by contradiction, that there exists a sequence $(n_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} n_i = \infty$, and such that $v \notin I(n_i, (0, 0))$ for all $i \in \mathbb{N}$. Then, by Proposition 14, we cannot have that both $v - (1, 0)$ and $v + (1, 0)$ belong to $I(n_i, (0, 0))$ simultaneously, and neither can $v - (0, 1)$ and $v + (0, 1)$. Thus, we may assume, with no loss of generality, that for infinitely many values of i, neither $v + (1, 0)$ nor $v + (0, 1)$ belong to $I(n_i, (0, 0))$ (since the other cases are analogous). This means that

$$\widehat{f}^{n_i}(O) \cap \big(T_v(O) \cup T_{v+(1,0)}(O) \cup T_{v+(0,1)}(O)\big) = \emptyset.$$

Extracting a subsequence of $(n_i)_{i\in\mathbb{N}}$, we may (and will) assume that the latter holds for *all* $i \in \mathbb{N}$.

Let $K_2 = \widehat{f}^k(T_{\overline{w}}(O))$. Since $S(v, 1) \subset I(k, \overline{w})$, it follows that K_2 intersects the open connected sets $T_v(O)$ and $T_{(1,0)}(T_v(O))$, and therefore Lemma 12 (with $K_1 = T_v(O)$) implies that there exists a $T_{(1,0)}$-translation arc α contained in $T_v(O) \cup K_2$ joining a point $x_1 \in T_v(O)$ to $T_{(1,0)}(x_1)$. Note that the assumption that $\widehat{f}^n(O)$ is disjoint of $T_{\overline{w}}(O)$ for all $n \in \mathbb{Z}$ implies that $K_2 \cap \widehat{f}^{n_i}(O) = \emptyset$ for all $i \in \mathbb{N}$. Furthermore, since $v \notin I(n_i, (0, 0))$, the set $\widehat{f}^{n_i}(O)$ is also disjoint from $T_v(O)$. Thus, $[\alpha] \cap \widehat{f}^{n_i}(O) = \emptyset$ for all $i \in \mathbb{N}$.

A similar reasoning shows that there exists a translation arc $\beta \subset T_v(O) \cup K_2$, joining a point $x_2 \in T_v(O)$ to $T_{(0,1)}(x_2)$, such that $[\beta] \cap \widehat{f}^{n_i}(O) = \emptyset$ for all $i \in \mathbb{N}$.

Let $\gamma : [0, 1] \to T_v(O)$ be an arc joining x_1 to x_2, and define

$$\alpha^+ = \bigcup_{i\in\mathbb{N}} T_{(1,0)}^i([\alpha]), \quad \alpha^- = \bigcup_{i\in\mathbb{N}} T_{(1,0)}^{-i}([\alpha]),$$
$$\beta^+ = \bigcup_{i\in\mathbb{N}} T_{(0,1)}^i([\beta]), \quad \beta^- = \bigcup_{i\in\mathbb{N}} T_{(0,1)}^{-i}([\beta]).$$

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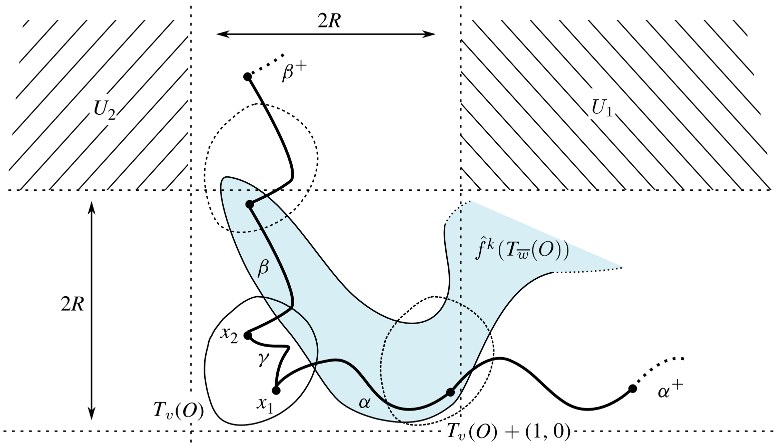

FIGURE 3. Proof of Claim 1.

Since $\widehat{f}^{n_i}(O)$ is disjoint from $[\alpha] \cup [\beta] \cup [\gamma]$, it follows from Lemma 4 that $\widehat{f}^{n_i}(O)$ is disjoint from at least one of the following four connected sets:

$$F_1 = \alpha^+ \cup \beta^+ \cup [\gamma], \quad F_2 = \alpha^+ \cup \beta^- \cup [\gamma],$$
$$F_3 = \alpha^- \cup \beta^+ \cup [\gamma], \quad F_4 = \alpha^- \cup \beta^- \cup [\gamma].$$

In particular, there is $j \in \{1, 2, 3, 4\}$ such that $\widehat{f}^{n_i}(O)$ is disjoint from F_j for infinitely many values of $i \in \mathbb{N}$. We assume that $j = 1$, since the other cases are analogous. Again extracting a subsequence of $(n_i)_{i \in \mathbb{N}}$, we may assume that $f^{n_i}(O)$ is disjoint from F_1 for all $i \in \mathbb{N}$.

Let $R_1 = \max_{y \in \alpha} \|y\|$, $R_2 = \max_{y \in \beta} \|y\|$, $R_3 = \max_{y \in \gamma} \|y\|$, and note that $\max_{y \in \alpha^+}(y)_2 \leq R_1$, and $\max_{y \in \beta^+}(y)_1 \leq R_2$.

Finally, let $R = \max\{R_1, R_2, R_3\}$, so that

$$F_1 \subset \{y : |(y)_1| \leq R\} \cup \{y : |(y)_2| \leq R\},$$

and observe that the sets

$$U_1 = \{y : (y)_1 > R, (y)_2 > R\}, \quad U_2 = \{y : (y)_1 < -R, (y)_2 > R\}$$

lie in distinct connected components of $\mathbb{R}^2 \backslash F_1$ (see Figure 3).

As $\widehat{f}^{n_i}(O)$ is connected and contained in $\mathbb{R}^2 \backslash F_1$, for each $i \in \mathbb{N}$, the set $\widehat{f}^{n_i}(O)$ is disjoint from either U_1 or U_2. This contradicts Proposition 16, completing the proof of the claim. □

Let us denote by $R_{\overline{w}}(v) = 2\overline{w} - v = T^2_{\overline{w}-v}(v)$ the symmetric point of v with respect to $\overline{w}$.

CLAIM 2. *There exist* $k_1, k_2 \in \mathbb{Z}$ *and* $\overline{v} \in \mathbb{Z}^2$ *such that* $S(\overline{v}, 1) \subset I(k_1, \overline{w})$ *and* $S(R_{\overline{w}}(\overline{v}), 1) \subset I(k_2, \overline{w})$.

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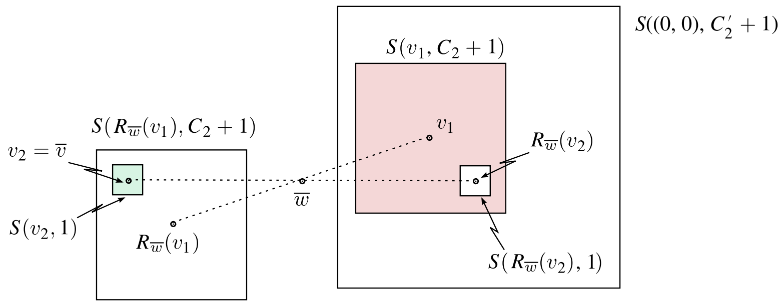

FIGURE 4. Proof of Claim 2.

Proof. Let $C_2 > 0$ be such that the conclusion of Lemma 15 holds with $C_1 = 1$. Again using Lemma 15, but setting $C_1 = C_2 + 1$ and $w_1 = (0, 0)$, there exist $C'_2 > 0$, $n_1 \in \mathbb{N}$, an integer $j_1 \in [-2M, 2M]$, and $v_1 \in \mathbb{Z}^2$ such that

$$S(v_1, C_2 + 1) \subset S((0, 0), C'_2 + 1) \cap I(n_1 + j_1, \overline{w}).$$

As a result of our choice of C_2, Lemma 15, applied with $C_1 = 1$ and $w_1 = R_{\overline{w}}(v_1)$, implies that there exist $n_2 \in \mathbb{N}$, an integer $j_2 \in [-2M, 2M]$, and $v_2 \in \mathbb{Z}^2$ such that

$$S(v_2, 1) \subset S(R_{\overline{w}}(v_1), C_2 + 1) \cap I(n_2 + j_2, \overline{w}).$$

Let $\overline{v} = v_2$, $k_1 = n_2 + j_2$ and $k_2 = n_1 + j_1$. Then, $S(\overline{v}, 1) \subset I(k_1, \overline{w})$, while

$$S(R_{\overline{w}}(\overline{v}), 1) = R_{\overline{w}}(S(\overline{v}, 1)) \subset R_{\overline{w}}(S(R_{\overline{w}}(v_1), C_2 + 1)) = S(v_1, C_2 + 1) \subset I(k_2, \overline{w}),$$

so the claim follows (see Figure 4). □

To complete the proof of Theorem 8, suppose that $\widehat{f}^n(O) \cap T_{\overline{w}}(O) = \emptyset$ for all $n \in \mathbb{Z}$. Claims 1 and 2 together imply that, for sufficiently large n, both $\overline{v} \in I(n, (0, 0))$ and $2\overline{w} - \overline{v} = R_{\overline{w}}(\overline{v}) \in I(n, (0, 0))$. This means that $\widehat{f}^n(O) \cap T_{\overline{v}}(O) \neq \emptyset$ and $\widehat{f}^n(O) \cap T_{2\overline{w}-\overline{v}}(O) \neq \emptyset$. Letting $K_1 = T_{\overline{w}}(O)$ and $K_2 = \widehat{f}^n(O)$, it follows that $T^{-1}_{\overline{w}-\overline{v}}(K_1) \cap K_2 \neq \emptyset$ and $T_{\overline{w}-\overline{v}}(K_1) \cap K_2 \neq \emptyset$. Thus, Lemma 12 (with $v = \overline{w} - \overline{v}$) implies that K_1 intersects K_2, i.e. $\widehat{f}^n(O) \cap T_{\overline{w}}(O) \neq \emptyset$. This contradicts our assumption at the beginning of this paragraph, completing the proof. □

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