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Extensions to the theory of selective withdrawal

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Most reservoirs contain stratified fluid and selective withdrawal is used to obtain water with the desired properties. We initially deal with a layered density distribution. The theory for the critical discharge for a single layer and a point sink is reviewed and extended to cover the case where there is gate discharge (a line sink). The theory for the case when the upper layer depth is large and the flow is coming from both layers is reviewed and it is shown that the valve controls the discharge and a virtual control determines the ratio of the discharge in each layer. This virtual control moves further from the valve as the total discharge increases. We determine the position of the virtual control and the criteria for the maximum for two layers when the upper layer is finite and below a stationary layer. Before this maximum, we show that when the discharge is increased above the critical discharge for the single layer, the finite upper layer does not affect the ratio of the flows from each layer until the virtual control reaches that for the maximum discharge. At this stage, the upper layer becomes tangential to the dam face and this condition and the smoothness of the lower interface determine both the total discharge and the ratio of the flow from each layer. Indeed, at this stage, virtual control and the control of the discharge are at the same section.

In a similar way, with a stationary layer above and below two flowing layers, we derive the maximum discharge from the two flowing layers. For this case, the solution is self-similar. This is then extended to a stable stratified continuous density distribution. The experiments of Gariel (1949) and Lawrence & Imberger (1979) suggest that the predictions of the theory are within the experimental errors.

1. Introduction

In most reservoirs, at some time of the year, the water is stratified and it is well known that the stratification inhibits the vertical motion without restricting the horizontal motion. Given an appropriate intake tower with multiple ports this allows the use of selective withdrawal to manage the water quality from the outlet. This method has been used for some time (Craya 1949; Gariel 1949; Harleman, Morgan & Purple 1959; Jirka & Katavola 1979). There is a very good review of the state-of-the-art in Imberger (1980).

In all cases, the assumptions are that the effect of viscous forces on the flow properties are negligible and the flow is assumed to be steady. These assumptions with the gradually varied flow assumption (or the long-wave assumption) have been used by Wood (1968), Wood & Lai (1972*a*, *b*), Bryant & Wood (1976), Benjamin (1981) and Killworth (1992) for selective withdrawal problems, and Wood (1970), Armi (1986), Armi & Farmer (1986), Farmer & Armi (1986), Dalziel (1991) and Lane-Serff, Smeed & Postlethwaite (2000) for interchange problems. For selective withdrawal problems, Wood (1978) and Lawrence & Imberger (1979) and Lawrence

(1980) followed Craya (1949) and assumed that the velocity profiles have spherical symmetry at points of control. Initially, this paper extends the work when the spherical symmetry assumption is appropriate, the interfaces are relatively sharp and when the outlet is on a vertical reservoir wall.

Before looking at the specific flows, it is appropriate to look at the equations at an interface which has both a sharp density and a velocity step. The elevation of the case considered is illustrated in figure 1(a) and the plan is shown in figure 1(b). The sink is in a vertical face which extends to infinity in all directions and we adopt Craya's (1949) assumption that flow into the point source is independent of θ (figure 1b) and thus the flow is spherically symmetric. The upper interface is labelled 1 between layers 0 and 1 and this interface is defined by r and ϕ_1 and has an elevation in the reservoir Z_1 above the sink. Above this upper interface the density is ρ_0. Similarly, the other interfaces, labelled as $2, 3, \ldots, n$, are defined as functions of $f(r, \phi_n)$ and elevations above the sink of Z_n. The depth at infinity and the densities of the layers are, respectively, $Z_1 - Z_2$ and $\rho_1(\rho_0 + \Delta\rho_1)$ and $Z_2 - Z_3$ and $\rho_2(\rho_1 + \Delta\rho_2)$ and $Z_3 - Z_4$ and $\rho_3(\rho_2 + \Delta\rho_3)$ etc.

For each interface (n) the interface is defined by the radius and ϕ_n (figure 1c). Above the interface, the velocity is v_{n-1}, the density is ρ_{n-1}, the pressure on the interface is p_n and the potential energy of the flow is PE_{n-1}. Applying the Bernoulli equation on the upper side of the interface between the flow and the infinite reservoir (subscript r) we obtain

$$\tfrac{1}{2}\rho_{n-1}v_{n-1}^2 + PE_{n-1} + p_n = PE_{(rn-1)} + p_{rn}. \tag{1}$$

Below the interface, the velocity is v_n and the density is ρ_n, pressure on the interface is still p_n and the potential energy is PE_n

$$\tfrac{1}{2}\rho_n v_n^2 + PE_n + p_n = PE_{r,n} + p_{r,n}. \tag{2}$$

We remove the pressure by subtracting equation (1) from equation (2) and using the assumption of spherical symmetry we obtain

$$\tfrac{1}{2}\rho_n \frac{q_n^2}{\pi^2 r^4(\sin\phi_n - \sin\phi_{n+1})^2} - \tfrac{1}{2}\rho_{n-1}\frac{q_{n-1}^2}{\pi^2 r^4(\sin\phi_{n-1} - \sin\phi_n)^2} + \Delta\rho_n g r \sin\phi_n = \Delta\rho_n g Z_n. \tag{3}$$

If there is no flow in layer $n-1$ then q_{n-1} is zero and we obtain the expression for the geometry of the free surface as a function of q_n. Similarly when there is no flow in the lower layer, we can obtain an expression for the geometry of the lower surface. The form of equation (3) is the same as that used by Armi (1986), Dalziel (1991) and Lane-Serff *et al.* (2000) and is sometimes called the internal energy equation. (It is, however, the energy difference between the two layers.)

2. The critical discharge for a single layer

There are two common engineering problems with selective withdrawal. The first is where an outlet valve can be approximated to a sink in a dam. This is commonly used in selective withdrawal from a deep reservoir. The second case is where we have a gated outlet of significant horizontal dimension and this can be approximated to a line sink. This case is important when we have selective cold water from a power station cooling pond and the critical discharge for this case will be discussed in § 5.

For the first case, assume that we have a point sink below interface 2 (figure 2). We assume the flow is from layer 2, and, at $r = \infty$, the layer extends from Z_2 to $Z = -\infty$. The layers above interface 2 are stationary. If the discharge is small (less

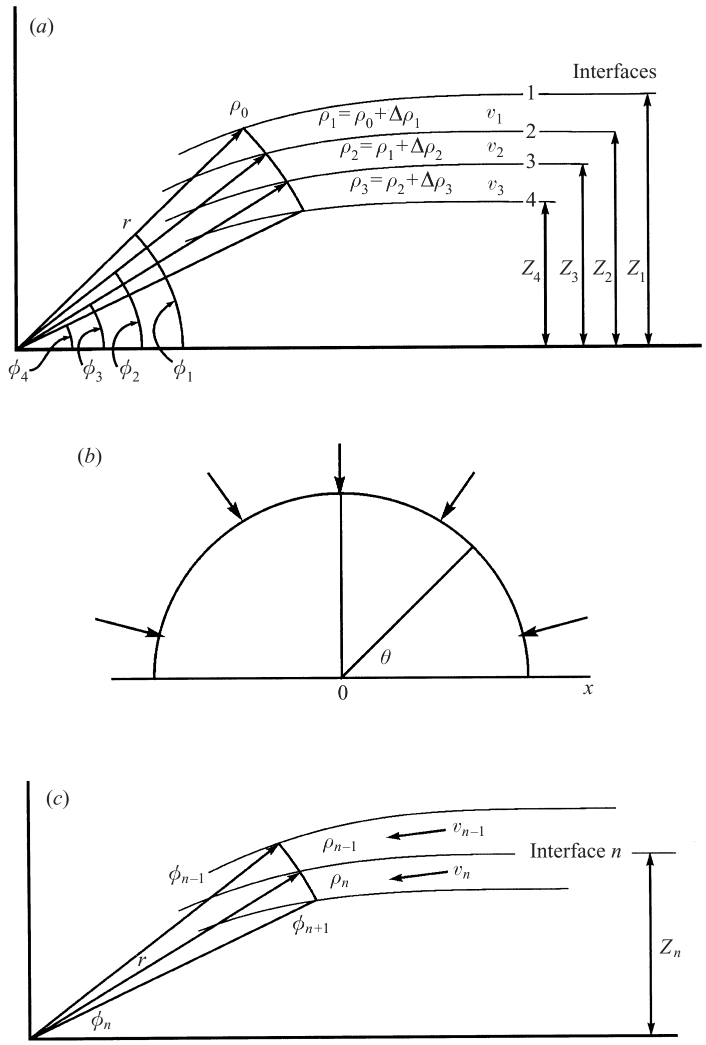

FIGURE 1. (*a*) The nomenclature for the elevation of the layered flow. (*b*) The plan of the sink in the vertical wall. (*c*) The nomenclature for an interface *n*.

than critical), then assuming the flow is inviscid, the interface stagnation point on the reservoir wall is elevated. Numerical solutions of this case have been obtained by Forbes & Hocking (1990) and Forbes *et al.* (1996). However, as the discharge reaches the critical discharge the interface will suddenly change. At critical discharge, the interface becomes tangential to the dam face and for this case we can determine a steady flow. The steady flow is made up of two parts, one close to the sink where the velocity is dominated by the sink and there is no free surface, and one where the velocity is controlled by the free surface. This critical flow must be transient, as when the critical discharge is exceeded, the flows come from layers 1 and 2. It must be noted that in experiments, since there is always a boundary layer at the interface, the determination of the critical discharge is a matter of judgement.

Craya (1949) used the assumption that the flow was independent of θ (figure 2) and differentiated the Bernoulli equation with respect to the radius to determine the

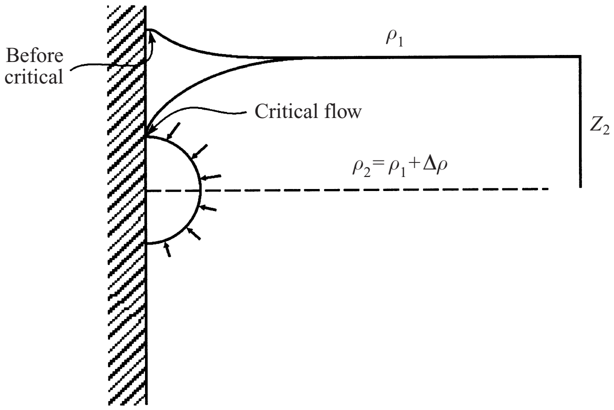

FIGURE 2. The elevation and plan for critical discharge from a single layer and the spherical assumption. Note the change of the interface between near-critical flow to critical flow. This critical flow must be transient as the flow will change to flow in both layers (not to scale).

maximum discharge without the stagnation point on the dam. We determine the same result by assuming that the interface is tangential to the dam face (Wood & Lai 1972*a*). On the interface between the upper layer 1 and the first flowing layer 2, the Bernoulli equation (equation (3) with $q_1 = 0$) and this equation at the dam face is

$$\frac{1}{2}\frac{\rho_2 q_2^2}{\Delta\rho_2 g\pi^2 r^4(1+\sin\phi_2)^2} + r\sin\phi_2 = Z_2. \tag{4}$$

When ϕ_2 is $\frac{1}{2}\pi$, this is the equation for a sink dominated area, and differentiating this with respect to r we obtain Craya's (1949) results. However, we differentiate the equation on the free surface with respect to r and defining the Froude number as

$$Fr_2^2 = \rho_2\frac{q_2^2}{\Delta\rho_2 g\pi^2 r^5(1+\sin\phi_2)^3}, \tag{5}$$

we obtain

$$r\frac{\mathrm{d}\sin\phi_2}{\mathrm{d}r} = \frac{+2Fr_2^2(1+\sin\phi_2)-\sin\phi_2}{(-Fr_2^2+1)}. \tag{6}$$

Now equation (4) can be written as

$$2Fr_2^2(1+\sin\phi_2)r = 4Z_2 - 4r\sin\phi_2, \tag{7}$$

hence,

$$r\frac{\mathrm{d}\sin\phi_2}{\mathrm{d}r} = r\cos\phi_2\frac{\mathrm{d}\phi_2}{\mathrm{d}r} = \frac{4Z_2 - 5r\sin\phi_2}{(-Fr_2^2+1)}. \tag{8}$$

For the interface to become tangential to the dam face, ϕ_2 is $\frac{1}{2}\pi$. Hence, from equation (8)

$$r_c = \tfrac{4}{5}Z_2, \tag{9}$$

where r_c is the control radius. The critical discharge at which the upper layer is drawn into the flow (q_c) occurs when

$$q_2 = 2^{1/2}0.8^{5/2}\pi\left(\frac{\Delta\rho_2 g}{\rho_2}\right)^{1/2} Z_2^{5/2} = 2.54\left(\frac{\Delta\rho_2 g}{\rho_2}\right)^{1/2} Z_2^{5/2}. \tag{10}$$

These results are the same as Craya's (1949) and were verified by the experiments of Gariel (1949). These were very difficult experiments as there were always viscous effects with a boundary layer on the interface. Indeed, even if we had a truly inviscid fluid, the transition between a flow with an elevated stagnation point to a critical discharge with an interface tangential to the dam would be unsteady and move rapidly to a two-layer discharge. (In order to escape the boundary-layer effects, Wood & Lai (1972*a*) used the same theory with a gradually varied contraction and an air–water interface. The theoretical results for the critical discharge had the correct form but were 10% low and it was believed that this was due to surface tension.)

Using the spherical assumption for the free streamline, the backwater for the upper interface could be calculated, but the flow is transient and the results can never be verified.

3. The case when the discharge comes from above and below interface 2

At the critical discharge, the valve discharge equals that determined at the control radius. When the discharge at the valve exceeds this value, some of the fluid must come from above interface 2. Thus, the flow comes from layers 1 and 2. In this case, the discharge is set at the valve and there is a virtual control which sets the discharge ratio. Wood (1978) modified the spherically symmetric assumption for the two layers in a sector of a sphere and determined the virtual control. Lawrence & Imberger (1979) showed experimentally that the spherically symmetric assumption worked reasonably for half a sphere and figure 3 illustrates this case. This theory will now be described. Substituting into equation (3) for interface 2, defining α_{12} as equal to $\Delta\rho_1/\Delta\rho_2$, we obtain

$$\tfrac{1}{2}\rho_2\frac{q_2^2}{\Delta\rho_2 g\pi^2 r^4(1+\sin\phi_2)^2}-\tfrac{1}{2}\alpha_{12}\frac{\rho_1 q_1^2}{\Delta\rho_1 g\pi^2 r^4(1-\sin\phi_2)^2}+r\sin\phi_2=+Z_2. \tag{11}$$

Defining the Froude numbers as

$$Fr_1^2=\rho_1\frac{q_1^2}{\Delta\rho_1 g\pi^2 r^5(1-\sin\phi_2)^3},\quad Fr_2^2=\rho_2\frac{q_2^2}{\Delta\rho_2 g\pi^2 r^5(1+\sin\phi_2)^3}, \tag{12}$$

and differentiating equation (11) with respect to r,

$$\frac{r\mathrm{d}(\sin\phi_2)}{\mathrm{d}r}=\frac{D_1}{D_0}=\frac{(2Fr_2^2(1+\sin\phi_2)-2\alpha_{12}Fr_1^2(1-\sin\phi_2)-\sin\phi_2)}{(1-\alpha_{12}Fr_1^2-Fr_2^2)}. \tag{13}$$

At the valve, D_0 tends to a large negative value, whereas in the reservoir, D_0 tends to 1. At some point (figure 4) where D_0 equals zero, and to avoid the singularity and have a finite value of the slope, the numerator must also equal zero. At this point, we have the virtual control (r_v). The subtracted Bernoulli equations yields

$$2Fr_2^2 r(1+\sin\phi_2)-2\alpha_{12}Fr_1^2 r(1-\sin\phi_2)=+4Z_2-4r\sin\phi_2. \tag{14}$$

Substituting into equation (13), when D_1 is equal to zero if the interface surfaces are to be smooth

$$r_v\sin\phi_{2v}=+\tfrac{4}{5}Z_2=z_2. \tag{15}$$

When ρ_{12} is ρ_1/ρ_2 then, for each of the discharges we obtain

$$\frac{\rho_{12}q_1^2}{q_c^2}=\frac{(1-\sin\phi_{2v})^3(2+\sin\phi_{2v})}{2^3(\sin\phi_{2v})^5},\quad \frac{q_2^2}{q_c^2}=\frac{(1+\sin\phi_{2v})^3(2-\sin\phi_{2v})}{2^3\sin^5\phi_{2v}}. \tag{16}$$

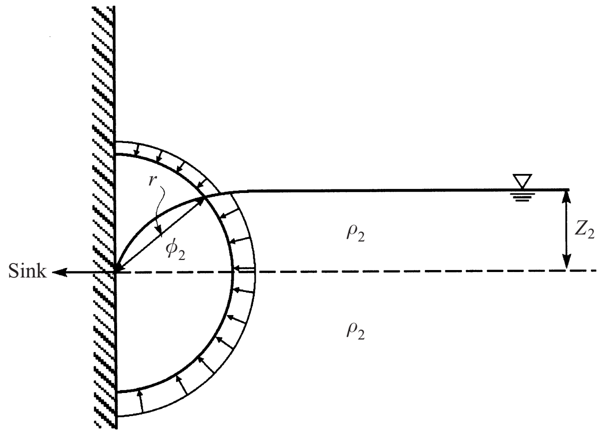

FIGURE 3. The withdrawal from two layers with the upper layer of infinite depth and spherical assumption (not to scale).

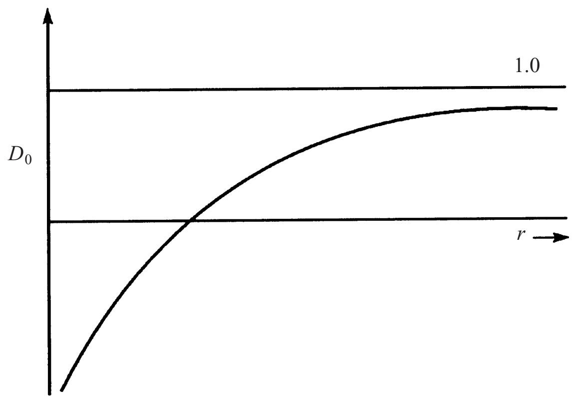

FIGURE 4. The variation of D_1 with radius.

Using equations (16) with the Boussinesq assumption and using $\sin \phi_{2v}$ as the dummy variable, the ratios of the discharges are calculated. Thus, in this flow, we have the control point (the valve) and the virtual control at r_v which sets the ratio of the two discharges.

As already discussed, verification of the result for a single layer (equation (10)) is difficult as the change from one-layer to two-layer flow is unsteady. However, with both layers flowing, the experiments are simpler as the flow is changing very slowly. Lawrence & Imberger (1979) carried out an experiment for a point source in the centre of a reservoir with a density interface. They measured the density profile using conductivity probes, the discharge from the change in the volume of the reservoir, and the ratio of the discharges using the density withdrawal fluid determined by a continuous-flow Anton Parr density-meter. In this very elegant experiment, they measured the ratio of the discharges as a function of the total discharge divided by the critical discharge, and showed that the results are satisfactory. Their results transformed for the case when the sink is on a dam wall are plotted in figure 5. It is noteworthy that as the discharge ratio increases, the value of $\sin \phi_{2v}$ decreases and thus for a particular Z_2 the value of r_v increases and at some time when there is a

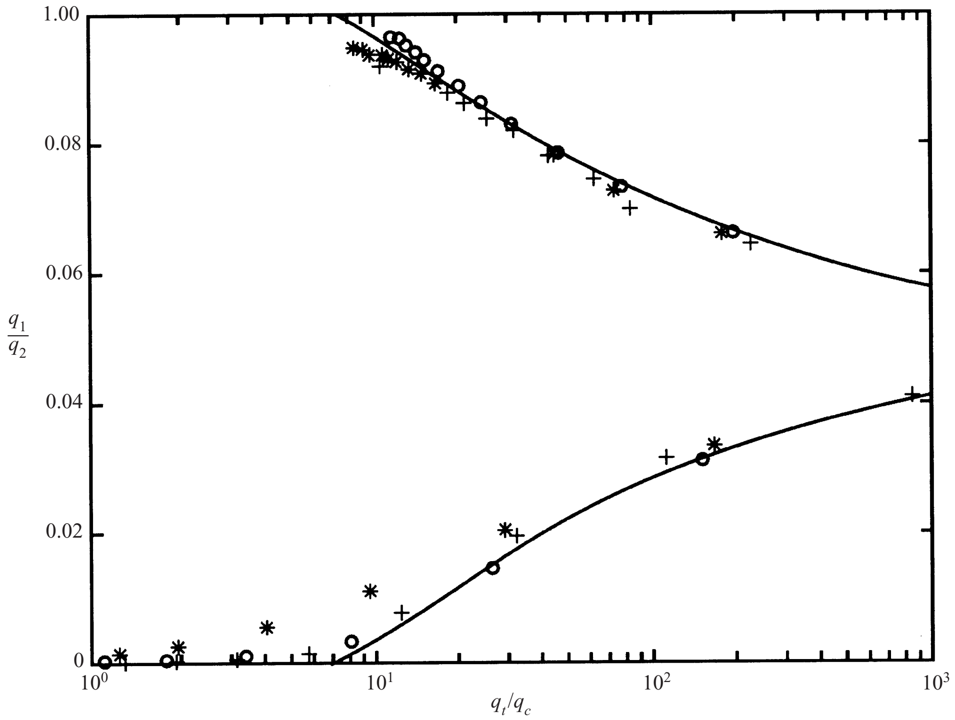

FIGURE 5. A comparison of the experiments of Lawrence & Imberger (1979) with the calculation of q_1/q_2 as function q_t/q_c.

finite layer there will be draw off from the stationary layer above the two flowing layers.

4. The determination of the maximum discharge when the two layers are flowing below a stationary layer

In some cases, the depth of the upper layer will be finite and it is important to determine the maximum discharge for the two layers such that the stationary layer is not disturbed (figure 6). Before the maximum discharge, the finite-depth upper layer does not affect the difference between the Bernoulli equation at the interface (equation (3)), and the boundary of layer 1 at the dam face is the stagnation point. Thus, the effect of the confined flow in layer 1 (in the previous case the flow in layer 1 is unconfined) does not affect the virtual control and hence the discharge ratio. However, as the total discharge increases, the position of the virtual control moves further from the valve. With the opening of the valve, we can trace the progress of r_v as it moves until both the discharge and the ratio of the discharges are determined by the fluid in the reservoir by the flow conditions at virtual control. (This implies that r_v and r_c coincide and there is thus only one control and this control is at the junction of the closed conduit and the free surface). At this discharge, the flow at the valve matches the maximum that can be sustained from the two layers and at this stage the flow conditions change in exactly the same manner as the single layer changed from a one-layer to a two-layer flow. Interface 1 will initially have an elevated stagnation point and as the discharge increases to the critical flow will change and become

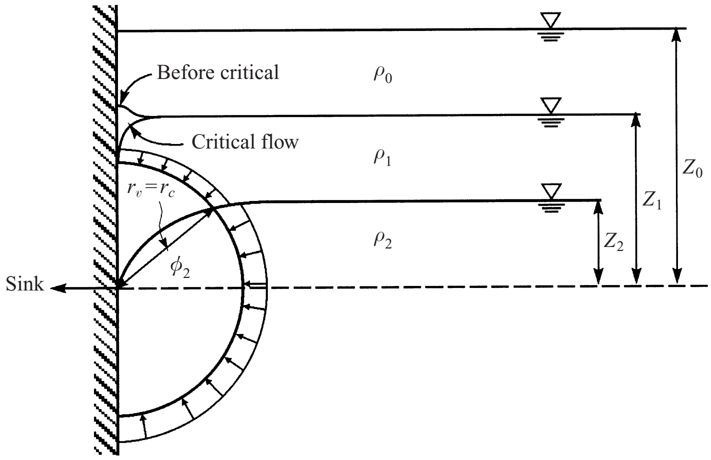

FIGURE 6. The critical discharge from two layers and the spherical assumption. Note the change of the upper interface between near critical flow to critical flow. This critical flow must be transient as the flow will change to flow in three layers (not to scale).

tangent to the dam face. At this point, the value of ϕ_1 is $\frac{1}{2}\pi$. To determine the steady state condition, we use the condition of tangency of the upper interface (as in (2)), and the observed fact that the slope of the lower interface is finite (as in (3)). This change is illustrated in figure 6. Again, it must be noted that the change will be unsteady and we can only determine the final transient steady state.

If we assume a sink level, we obtain for interface 1 for the free surface

$$\frac{1}{2}\rho_1 \frac{q_1^2}{\pi^2 r^4(\sin\phi_1 - \sin\phi_2)^2} + \Delta\rho_1 g r \sin\phi_1 = +\Delta\rho_1 g Z_1. \tag{17}$$

Using equation (3), we obtain

$$\frac{1}{2}\rho_2 \frac{q_2^2}{\Delta\rho_2 g \pi^2 r^4(1+\sin\phi_2)^2} - \frac{1}{2}\rho_1\alpha_{12} \frac{q_1^2}{\Delta\rho_1 g\pi^2 r^4(\sin\phi_1 - \sin\phi_2)^2} + r\sin\phi_2 = +Z_2. \tag{18}$$

Differentiating equations (17) and (18), we obtain

$$a_1 \frac{r\mathrm{d}(\sin\phi_1)}{\mathrm{d}r} + b_1 \frac{r\mathrm{d}\sin\phi_2}{\mathrm{d}r} = c_1, \quad a_2 \frac{r\mathrm{d}(\sin\phi_1)}{\mathrm{d}r} + b_2 \frac{r\mathrm{d}\sin\phi_2}{\mathrm{d}r} = c_2, \tag{19}$$

where

$$\left.\begin{aligned} &a_1 = (1 - Fr_1^2), \quad b_1 = Fr_1^2, \quad c_1 = -\sin\phi_1 + 2Fr_1^2(\sin\phi_1 - \sin\phi_2), \\ &a_2 = \alpha_{12}Fr_1^2, \quad b_2 = [-Fr_2^2 - \alpha_{12}Fr_1^2 + 1], \\ &c_2 = 2Fr_2^2(1+\sin\phi_2) - 2\alpha_{12}Fr_1^2(\sin\phi_1 - \sin\phi_2) - \sin\phi_2, \end{aligned}\right\} \tag{20}$$

now solving for

$$\frac{r\mathrm{d}\sin\phi_1}{\mathrm{d}r} = \frac{D_1}{D_0}, \quad \frac{r\mathrm{d}\sin\phi_2}{\mathrm{d}r} = \frac{D_2}{D_0}, \tag{21}$$

where,

$$D_0 = a_1 b_2 - a_2 b_1 = (1 - Fr_1^2)[+1 - \alpha_{12}Fr_1^2 - Fr_2^2] - \alpha_{12}Fr_1^2 Fr_1^2, \tag{22}$$

in the reservoir, the Froude numbers are zero and D_0 is 1. Close to the valve, the Froude numbers are large and D_0 is positive, and, as discussed, there is only one control at which D_0 is zero. At this point,

$$\left.\begin{aligned} rD_1 &= r(b_2c_1 - b_1c_2) = [-Fr_2^2 - \alpha_{12}Fr_1^2 + 1](+2Fr_1^2 r(\sin\phi_1 - \sin\phi_2) - r\sin\phi_1), \\ &\quad -Fr_1^2[2Fr_2^2 r(1+\sin\phi_2) - 2\alpha_{12}Fr_1^2 r(\sin\phi_1 - \sin\phi_2) - r\sin\phi_2] = 0. \end{aligned}\right\} \tag{23}$$

Using the subtracted Bernoulli equations we obtain

$$\left.\begin{aligned} &2Fr_1^2 r(\sin\phi_1 - \sin\phi_2) = 4Z_1 - 4r\sin\phi_1, \\ &2Fr_2^2 r(1+\sin\phi_2) - 2\alpha_{12}Fr_1^2 r(\sin\phi_1 - \sin\phi_2) = +4Z_2 - 4r\sin\phi_2, \end{aligned}\right\} \tag{24}$$

hence,

$$rD_1 = [-Fr_2^2 - \alpha_{12}Fr_1^2 + 1](+4Z_1 - 5r\sin\phi_1) - Fr_1^2(4Z_2 - 5r\sin\phi_2). \tag{25}$$

Now, for interface 1 to be tangent to the dam face, ϕ_1 is $\frac{1}{2}\pi$, we require rD_1 to be zero, and hence

$$r_v = 0.8Z_1, \quad \sin\phi_2 = 0.8\frac{Z_2}{r_v} = \frac{Z_2}{Z_1}. \tag{26}$$

Now, $rD_1 = 0$ implies that rc_1 and rc_2 are equal to zero, and hence rD_2 must also equal zero. This gives, for the discharges,

$$\left.\begin{aligned} &\frac{q_1}{q_c} = \frac{1}{2}\left(\frac{\alpha_{12}}{\rho_{12}}\right)^{1/2}(Z_{12})(Z_{12}-1)(Z_{12})^{1/2}, \quad \frac{q_2}{q_c} = \tfrac{1}{2}Z_{12}(Z_{12}+1)(1+\alpha_{12}Z_{12})^{1/2}, \\ &\frac{q_t}{q_c} = \tfrac{1}{2}Z_{12}\left[\left(\frac{\alpha_{12}}{\rho_{12}}\right)^{1/2}(Z_{12}-1)(Z_{12})^{1/2} + (Z_{12}+1)(1+\alpha_{12}Z_{12})^{1/2}\right]. \end{aligned}\right\} \tag{27}$$

Where α_{12}, ρ_{12} and Z_{12} is the ratio of the density differences, the densities and the Z terms. The above implies that, at the control radius on the dam face where r_v and r_c coincide, the discharge and the discharge ratio are set. Upstream of the control point, the interfaces of the layers are determined by calculating for any r the values of ϕ_1 and ϕ_2. Using computed discharges from each layer and the Bernoulli equations with the free-surface condition, we obtain two equations for ϕ_1 and ϕ_2. Downstream of the control point, we still have two equations, but the value of ϕ_1 is determined and we have to determine at any r the head on the dam as well as ϕ_2. It is worth noting that when the flow changes from two-layer flow with a stagnation point to two-layer flow with the upper layer being tangential to the dam, the flow is transient. However, with the spherically symmetric assumption, the unsteadiness of the flow is not a consequence of the equations. (Bryant & Wood 1976 used the gradually varied flow equations and concluded that the unsteadiness was a consequence of the equations.)

To date, we have not assumed the Boussinesq approximation and equation (27) should hold for a range of density differences. However, assuming the Boussinesq approximation, we obtain figure 7 for q_t/q_c as a function of α_{12} and Z_{12}. This figure is for a range of ratios of density differences. For each density difference curve, we obtain the maximum value of the total discharge to the critical discharge that can be drawn for a particular value of ratios of Z_{12} without drawing fluid from layer 0. Again, the change from the two-layer flow to the three-layer flow is rapid and we can only calculate transient steady state. This method could be continued for further layers.

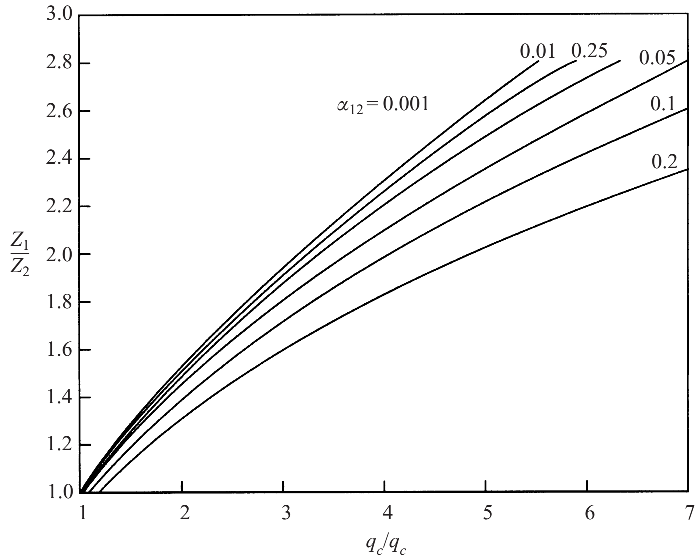

FIGURE 7. The calculation of ratio of Z_1/Z_2 as function q_t/q_c.

5. The determination of the maximum discharge when the layers are surrounded by a stationary upper and lower layer

For two flowing layers surrounded by a stationary upper and lower layer, the analysis is similar to the above. We look for the case where there is only one control. Thus, r_v and r_c are coincident and control the discharge and the ratios of the discharges. We then want the conditions for both upper and lower layers to become tangent to the dam face and the intermediate interface must be smooth. However, for the cases above (2, 3 and 4), the valve level is determined, but with the upper and lower layers stationary there is an extra equation (the equation for interface 3) and this determines the valve level. However, assuming that we know the valve level we obtain equation (17). Substituting equation (17) into the subtracted Bernoulli for interface 2 we obtain

$$\tfrac{1}{2}\rho_2 \frac{q_2^2}{\Delta\rho_2 g 4\pi^2 r^4 (\sin\phi_2 - \sin\phi_3)^2} + \alpha_{12} r \sin\phi_1 + r \sin\phi_2 = +Z_2 + \alpha_{12} Z_1. \qquad (28)$$

Finally, substituting $q_3 = 0$ into the subtracted Bernoulli equations for interface 3 (equation (3)) and substituting for q_2 from (28), we obtain

$$+\Delta\rho_3 g r \sin\phi_3 + \Delta\rho_2 g r \sin\phi_2 + \Delta\rho_1 g r \sin\phi_1 = \Delta\rho_3 g Z_3 + \Delta\rho_2 g Z_2 + \Delta\rho_1 g Z_1. \qquad (29)$$

Differentiating equations (17), (28) and (29) and substituting $r(\sin\phi_3)/\mathrm{d}r$ into equation (29) we obtain expressions for $r(\sin\phi_1)/\mathrm{d}r$ and $r(\sin\phi_2)/\mathrm{d}r$ in the same form as equation (21) except that

$$\left.\begin{aligned} a_1 &= (1 - Fr_1^2), \quad b_1 = Fr_1^2, \quad c_1 = -\sin\phi_1 + 2Fr_1^2(\sin\phi_1 - \sin\phi_2), \\ a_2 &= +\alpha_{12} - \alpha_{13} Fr_2^2, \quad b_2 = 1 - Fr_2^2(1 - \alpha_{23}), \\ c_2 &= Fr_2^2(2(\sin\phi_2 - \sin\phi_3) + \sin\phi_3 + \alpha_{13}\sin\phi_1 + \alpha_{23}\sin\phi_2) \\ &\quad -\alpha_{12}\sin\phi_1 - \sin\phi_2, \end{aligned}\right\} \qquad (30)$$

and

$$D_0 = a_1 b_2 - a_2 b_1 = (1 - Fr_1^2)(1 - Fr_2^2(1-\alpha_{23})) - (\alpha_{12} - \alpha_{13} Fr_2^2) Fr_1^2. \quad (31)$$

This has the same form as equation (22) and again we require D_0 to be zero only once and at this point the control r_c and r_v coincide. Again, we require rD_1 and rD_2 to be equal to zero when D_0 is equal to zero. Thus,

$$\left.\begin{aligned} rD_1 &= b_2 rc_1 - b_1 rc_2 = (1 - Fr_2^2(1-\alpha_{23}))(-r\sin\phi_1 + 2Fr_1^2 r(\sin\phi_1 - \sin\phi_2), \\ &\quad -Fr_1^2 r(Fr_2^2(2(\sin\phi_2 - \sin\phi_3) + \sin\phi_3 + \alpha_{13} r\sin\phi_1 + \alpha_{23} r\sin\phi_2) \\ &\quad -\alpha_{12} r\sin\phi_1 - r\sin\phi_2) = 0. \end{aligned}\right\} \quad (32)$$

Now the subtracted Bernoulli equation for interfaces 1 and 2 can be written as

$$\left.\begin{aligned} 2Fr_1^2 r(\sin\phi_1 - \sin\phi_2) &= 4Z_1 - 4r\sin\phi_1, \\ 2Fr_2^2(\sin\phi_2 - \sin\phi_3) &= 4Z_2 - 4r\sin\phi_2 + \alpha_{12}(4Z_1 - 4r\sin\phi_1). \end{aligned}\right\} \quad (33)$$

Substituting equation (32), we obtain

$$\begin{aligned} rD_1 &= (1 - Fr_2^2(1-\alpha_{23}))(-5r\sin\phi_1 + 4Z_1) \\ &\quad -Fr_1^2 r(4Z_2 - 5r\sin\phi_2 + \alpha_{12}(4Z_1 - 5r\sin\phi_1) \\ &\quad +Fr_2^2(r\sin\phi_3 + \alpha_{13} r\sin\phi_1 + \alpha_{23} r\sin\phi_2)) = 0. \end{aligned} \quad (34)$$

Now, when ϕ_1 is $\frac{1}{2}\pi$, we obtain

$$r_v = 0.8Z_1 = z_v, \quad r_v\sin\phi_2 = 0.8Z_2, \quad (35)$$

and are left to satisfy the last term in equation (34) when ϕ_3 is $-\frac{1}{2}\pi$,

$$-r_v + \alpha_{23} r_v \sin\phi_2 + \alpha_{13} r_v = -0.8Z_1 + 0.8\alpha_{23} Z_2 + 0.8\alpha_{13} Z_1 = 0. \quad (36)$$

This determines the position of the valve. Using equation (29), we have

$$-r_v + \alpha_{23} r_v \sin\phi_2 + \alpha_{13} r_v = -Z_3 + \alpha_{23} Z_2 + \alpha_{13} Z_1. \quad (37)$$

To satisfy equations (36) and (37), we have

$$\left.\begin{aligned} -0.8Z_1 + \alpha_{23} 0.8Z_2 + \alpha_{13} 0.8Z_1 &= -Z_3 + \alpha_{23} Z_2 + \alpha_{13} Z_1, \\ Z_3 - 0.8Z_1 &= +0.2\alpha_{23} Z_2 + 0.2\alpha_{13} Z_1, \end{aligned}\right\} \quad (38)$$

but from equation (36)

$$+Z_3 - 0.8Z_1 = 0.2Z_1, \quad Z_3 = Z_1. \quad (39)$$

Thus, the value of the depth of the intermediate layer in the reservoir (Z_2) is also determined from equation (36). This yields

$$Z_2 = +\frac{Z_1}{\alpha_{23}} - \frac{\alpha_{13}}{\alpha_{23}} Z_1 = Z_1(\alpha_{32} - \alpha_{12}). \quad (40)$$

Now the discharges are

$$\frac{q_1}{q_c} = \tfrac{1}{2}(\alpha_{12}\rho_{21})^{1/2}\left(\frac{Z_1}{Z_2} - 1\right)\left(\frac{Z_1}{Z_2}\right)^{3/2} \quad (41)$$

and

$$\frac{q_2}{q_c} = \frac{1}{2}\left(1 - \frac{Z_3}{Z_2}\right)\left(\frac{Z_1}{Z_2} + \frac{Z_3}{Z_2}\right)\left[\alpha_{12}\left(1 + \frac{Z_1}{Z_2}\right)\right]^{1/2}. \quad (42)$$

The ratio of the velocities squared in each layers at any r is

$$\frac{v_1^2}{v_2^2} = \frac{\rho_2\Delta\rho_1}{\rho_1\Delta\rho_2}\frac{Z_1 - r\sin\phi_1}{(Z_2 - r\sin\phi_2 + \alpha_{12}(Z_1 - r\sin\phi_1))} = \frac{\rho_2\Delta\rho_1}{\rho_1\Delta\rho_2}\frac{Z_1 - z_1}{(Z_2 - z_2 + \alpha_{12}(Z_1 - z_1))}. \tag{43}$$

Thus, we can write the velocity distribution without a function of r and we have a self-similar solution. This makes the calculation for the backwater curve for each layer upstream from the value of r_c relatively simple. This will be discussed in the more general case. It is notable that the ratio of the layer depths at infinity and at the control are constant. Similarly, the ratios of the velocities are also constant and this implies that the two layers upstream of the control are behaving as a single layer of composite density. This is similar to the case for the open channel (Wood 1968) where the control is at the maximum contraction rather than at the dam wall and leads to a similar similarity solution.

It is also simple to show that with the upper and lower layers becoming tangential to the dam and r_v and r_c coinciding, we can obtain the self-similar solution by determining the maximum discharge.

6. The critical flow from a line sink

So far, we have discussed the case for a point sink where the streamlines are radial and the velocity distribution is the same for all θ. For a gated outlet, the flow may be approximated as a line sink. For a line sink below interface 2 extending from $-\frac{1}{2}L$ to $+\frac{1}{2}L$, the streamlines and the velocity distributions are different. The extremes of the velocity distribution are on the centreline ($\theta = \frac{1}{2}\pi$) and on the plane defined by $\frac{1}{2}L$ and to determine the critical discharge we need the maximum discharge for each extreme and select the lower discharge.

Define a point on the plane with r_1 and r_2 (figure 8). For the flow below the free surface, we use the Stokes streamfunction and obtain the streamline (ψ) as

$$\psi = -\frac{q}{4\pi}(r_1 - r_2). \tag{44}$$

These streamlines are ellipses which are rotated about the X-axis (the equipotential lines are hyperbolas, Streeter 1948).

Along the centreline in the ellipses the velocity is radial, giving

$$v_r^2 = \frac{(qL)^2}{(4\pi r)^2}\left(\frac{4}{(r^2 + (\frac{1}{2}L)^2)}\right). \tag{45}$$

Matching this with the gravity dominated flow at the start of the free surface on the centreline, we obtain q_c,

$$v_r^2 = \frac{(q_c L)^2}{(4\pi r)^2}\frac{4}{(r^2 + (\frac{1}{2}L)^2)} = 2\frac{\Delta\rho g}{\rho}(Z_1 - r). \tag{46}$$

Hence,

$$\frac{(\rho Q_c L)^2}{\Delta\rho g Z_2^5} = \frac{(8\pi r')^2}{4}(r'^2 + (\tfrac{1}{2}L')^2)2(1 - r'), \tag{47}$$

where $r' = r/Z_2$ and $L' = L/Z_2$. Following Craya (1949) and looking for the maximum

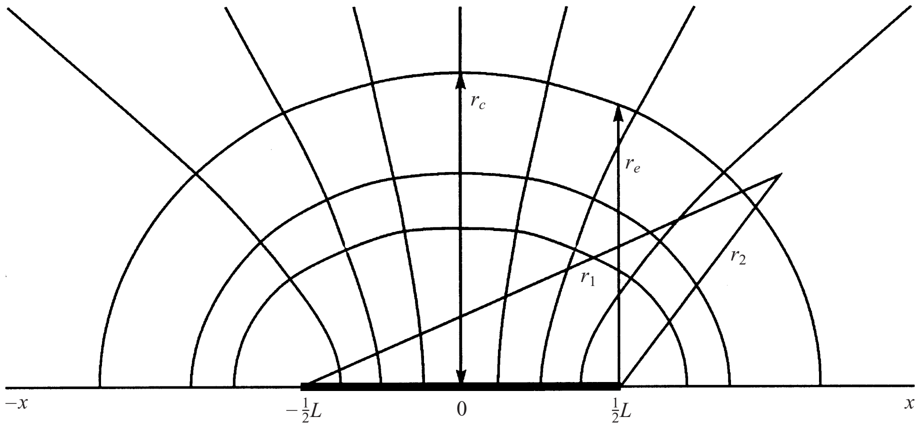

FIGURE 8. The nomenclature for the line sink and the streamlines and the equipotential lines.

q_c, we obtain

$$r_c'^3(4 - 5r_c') + r_c'L'^2(2 - 3r_c') = 0, \tag{48}$$

where r_c' is at the junction of the free surface and the dam face. Thus, when L' tends to zero, $r_c' = 0.8$, and when L' becomes large, $r_c' = 0.6667$. Using Matlab, we obtain the general solution:

$$r_c' = (0.05L^2 + 0.0190 + 0.0010L(135L^4 + 396L^2 + 2048)^{1/2})^{1/3}$$
$$- \frac{0.05L^2 - 0.0711}{(0.05L^2 + 0.0190 + 0.0010L(135L^4 + 396L^2 + 2048)^{1/2})^{1/3}} + 0.2667. \tag{49}$$

This general solution is shown in figure 9, and the discharge is given by

$$\frac{\rho(q_{mc}L)^2}{\Delta\rho g 32\pi^2 Z_1^5} = \frac{\rho Q_{mc}^2}{\Delta\rho g 32\pi^2 Z_1^5} = (r_c')^2(r_c'^2 + (\tfrac{1}{2}L')^2)(1 - r_c'), \tag{50}$$

and when $L' \gg r_c$, the maximum discharge determined on the centreline streamline (Q_{mc}) is

$$\frac{\rho(q_{mc}L)^2}{\Delta\rho g 32\pi^2 Z_1^5} = \frac{\rho Q_{mc}^2}{\Delta\rho g 32\pi^2 Z_1^5} = \left(\frac{2}{3}\right)^2 \frac{1}{3}\frac{L'^2}{4}. \tag{51}$$

So far, we have assumed that the maximum discharge is determined by matching the velocity at the centreline ($\theta = \frac{1}{2}\pi$).

The other extreme is on the plane defined by $\frac{1}{2}L$. We can calculate the velocities (v_r and v_θ) as a function of $\frac{1}{2}L$ and r_e (figure 8) and matching the total velocity with the free-surface velocity, we obtain

$$\frac{\rho Q^2}{\Delta\rho g 32\pi^2 Z_1'^5} = \frac{L'^2(L'^2 + r_e'^2)(1 - r_e')r_e'^2(L'^2 + 4r_e'^2)}{4[2r_e'(L'^2 + r_e'^2)^{1/2} - L_2' - 2r_e'^2]^2 + L'^2[(L'^2 + r_e'^2)^{1/2} + r_e']^2}, \tag{52}$$

when $r_e' \gg L'$, then

$$\frac{\rho Q_e^2}{\Delta\rho g 32\pi^2 Z_1'^5} = \tfrac{1}{5}(1 - r_e')r_e'^2 L'^2. \tag{53}$$

Seeking for the maximum Q_e determined on the streamline on the dam (Q_{me}) we

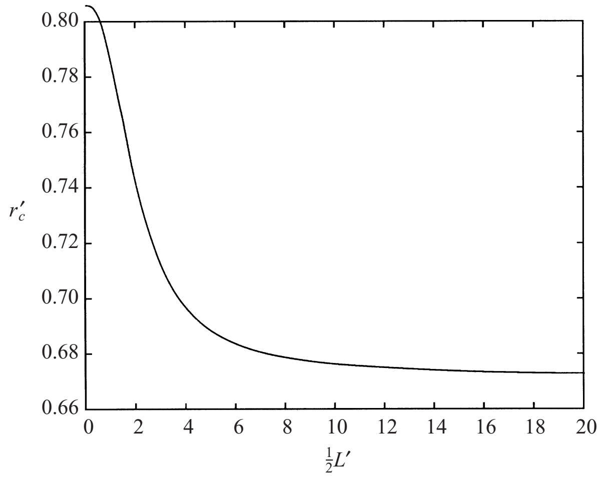

FIGURE 9. The value of r_c' as a function of $\frac{1}{2}L'$.

obtain r_e at the end of the line sink (r_e') as $\frac{2}{3}$ and thus

$$\frac{\rho(q_{me}L)^2}{\Delta\rho g 32\pi^2 Z_1^5} = \frac{\rho Q_{me}^2}{\Delta\rho g 32\pi^2 Z_1^5} = \left(\frac{2}{3}\right)^2 \frac{1}{3}\frac{L'^2}{5}. \tag{54}$$

This implies that the maximum discharge before which some of the flow comes from the upper layer is determined by r_e and thus the first portion of the upper layer comes from close to the dam face.

It should be noted that at $r = r_c$ and when $r = r_e$ the assumption that the flow resembles that of a line sink and the velocity is that of the free surface is satisfied. This is the control point and it determines the discharge.

7. The general solution

The similarity solution in § 4 suggests that there is a general solution for variable density distribution as shown for the density flow through a contraction (Wood 1968). The spherical coordinates used are r and ϕ and it is assumed that density distribution at any r depends only on ϕ. This is illustrated in figure 10 where $UCLL'C'$ represents the distribution of $\Delta\rho$. In this figure, the upper and lower streamlines are defined by $r\sin\phi_1(z_1)$ and $r\sin\phi_u(z_u)$ and the areas between $Z_1 = 0$ and z_1 are stationary and are filled with fluid of density difference $\Delta\rho_1$ and similarly the areas between Z_u and z_u are stationary and are filled with fluid of density difference $\Delta\rho_u$. The flow is steady and the density is constant along the streamline. The pressure on the streamline at the interface s is written as

$$\int_{\phi_s}^{\phi_u} \mathrm{d}p = \int_{\phi_s}^{\phi_u} \Delta\rho g \cos\phi r \,\mathrm{d}\phi = \int_{\phi_s}^{\phi_u} \Delta\rho g \,\mathrm{d}(r\sin\phi)$$

$$= \int_{z_s}^{z_u} \Delta\rho g \,\mathrm{d}z = \int_{z_l}^{z_u} \Delta\rho g \,\mathrm{d}z - \int_{0}^{z_s} \Delta\rho g \,\mathrm{d}z, \tag{55}$$

and on the level of the plane of Z_1 the pressure is constant and this implies

$$\int_{z_l}^{z_u} \Delta\rho g\,\mathrm{d}z + \Delta\rho_l g z_l = \int_{Z_l}^{Z_u} \Delta\rho g\,\mathrm{d}z. \tag{56}$$

Now we assume that for a constant r,

$$\frac{\Delta\rho}{\Delta\rho_l} = f\left(\frac{r\sin\phi_s - r\sin\phi_l}{r\sin\phi_u - r\sin\phi_l}\right) = f\left(\frac{z_s - z_l}{z_m}\right) = f(\eta), \tag{57}$$

where $z_m = (z_u - z_l)$. Thus, making all lengths dimensionless with z_m, we obtain

$$z_l = (Z_m - z_m)\int_0^1 f(\eta)\,\mathrm{d}\eta, \tag{58}$$

and the pressure on the streamline η_s is

$$p_s = \Delta\rho_l g z_m \int_{\eta_s}^1 f(\eta)\,\mathrm{d}\eta. \tag{59}$$

Now the Bernoulli equation is

$$+\tfrac{1}{2}(\rho_u+\Delta\rho_s)v_s^2+\int_{z_s}^{z_u}\Delta\rho g\,\mathrm{d}z+\Delta\rho_s g(z_s-z_l)+\Delta\rho_s g z_l = \int_{z_s}^{z_u}\Delta\rho g\,\mathrm{d}z+\Delta\rho_s g(Z_s-Z_l). \tag{60}$$

Substituting from equations (56) and (57) and rearranging we obtain

$$\frac{(\rho_u+\Delta\rho_s)v_s^2}{\Delta\rho_l g(Z_m - z_m)} = 2\left[\int_{\eta_s}^1 f(\eta)\,\mathrm{d}\eta + \eta_s f(\eta_s) - f(\eta_s)\int_0^1 f(\eta)\,\mathrm{d}\eta\right] = 2g(\eta_s). \tag{61}$$

This gives the velocity distribution on the radius in terms of the density distribution for the similarity solution and is the same expression obtained in Wood (1968). Using the Boussinesq assumption, the discharge is given by

$$\left[\frac{\rho_u}{\Delta\rho_l g(Z_m)Z_m^2}\right]^{1/2}\frac{1}{Z_m^2}q = \left[1-\frac{z_m}{Z_m}\right]^{1/2} 2\pi\frac{z_m}{Z_m}\int_0^1 2g(\eta)^{1/2}\,\mathrm{d}(\eta)$$

$$= \left[1-\frac{z_m}{Z_m}\right]^{1/2} 2\pi\frac{z_m}{Z_m}\frac{z_m}{Z_m}F(1), \tag{62}$$

where $F(1)$ is dependent on density distribution. Wood (1968) next determined the conditions for the flow to go through the contraction and this was equivalent to determining the maximum discharge. For the present case, we determine the maximum value of q^2 and we have to maximize

$$\left[1-\frac{z_m}{Z_m}\right]\left(\frac{z_m}{Z_m}\right)^4, \tag{63}$$

and thus differentiating this, we obtain $z_m = 0.8Z_m$.

It now remains to determine the position of the valve at the dam face (z_{dl}),

$$z_{dl} = 0.2Z_u\int_0^1 f\eta\,\mathrm{d}\eta, \tag{64}$$

and the position of the valve is given by $z_{dl} + \frac{1}{2}0.8Z_m$.

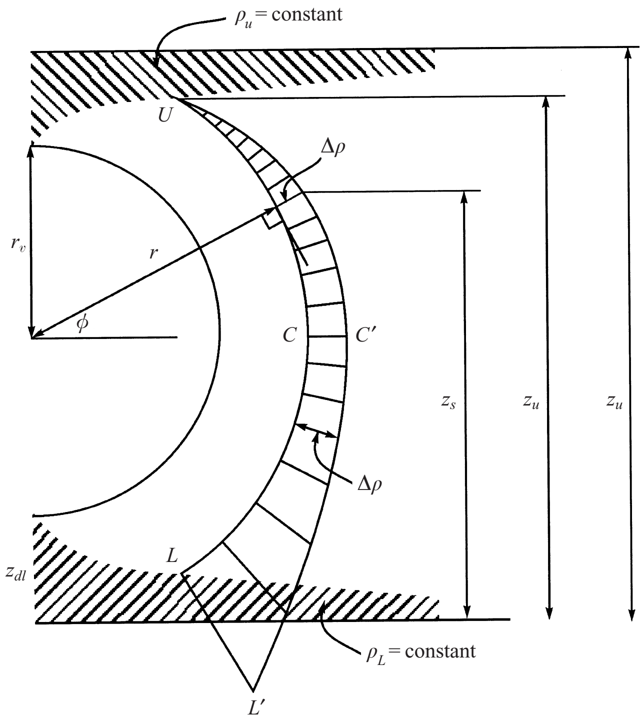

FIGURE 10. The nomenclature for the general case where the density distribution is variable.

Substituting equation (64) into the discharge equation (62), we obtain a solution for any stable density distribution. For a linear density excess, we obtain

$$\frac{q}{\left[\frac{\Delta\rho_l g}{\rho_u}\right]^{1/2}}\frac{1}{Z_m^{5/2}} = (0.8)^2\frac{\pi}{2^{1/2}}\frac{\pi}{8}, \tag{65}$$

using the buoyancy frequency

$$N = \left[\frac{\Delta\rho_l g}{(\rho_u Z_m)}\right]^{1/2}, \tag{66}$$

this can be written as

$$\tfrac{1}{2}Z_m = 0.8\left(\frac{q}{N}\right)^{1/3}. \tag{67}$$

The outline of the flowing layer can be calculated by substituting the value of the discharge (64) into equation (61) and if we define $c = z/Z_m$, we obtain

$$\frac{r}{Z_m} = \frac{[0.2]^{0.5}(0.8)^2}{2[1-c]^{0.5}c}. \tag{68}$$

Using the geometry of the spherical flow for the upper interface, we obtain

$$\sin\phi = \frac{z_m}{2r} = \frac{[1-c]^{1/2}c^2}{[0.2]^{1/2}(0.8)^2}, \tag{69}$$

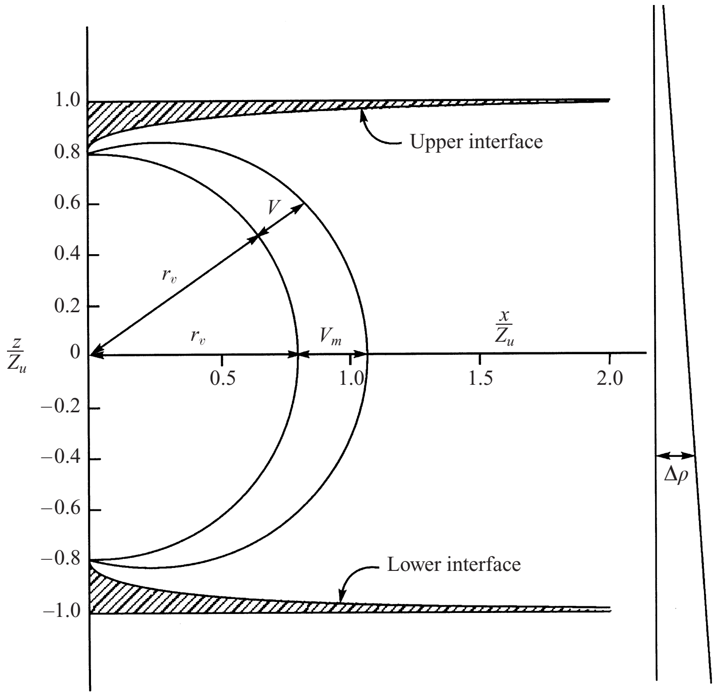

FIGURE 11. The velocity distribution and the upper and lower streamline for the case when the density excess in the reservoir is linear.

hence,

$$\frac{x}{Z_u} = \frac{[0.2]^{1/2}(0.8)^2}{[1-c]^{1/2}c}\left[1-\left(\frac{[1-c]^{1/2}c^2}{[0.2]^{1/2}(0.8)^2}\right)^2\right]^{1/2}. \tag{70}$$

The shape of the stratified flow and the velocity distribution on the critical radius are shown in figure 11.

Experiments with a linearly stratified fluid were carried out by Lawrence & Imberger (1979) with a point source in the centre of a reservoir, and Spigel & Farrant (1984) carried out an extensive set of experiments with a sink in the centre of a wide rectangular tank. The detailed experiments yielded

$$\tfrac{1}{2}Z_m = 1.0\left(\frac{q}{N}\right)^{1/3}. \tag{71}$$

The value of N is based on the initial value of the distribution. This experiment is much more difficult than the two-layer experiments (§ 3). With the two-layer experiments with an overflowing weir, it is relatively easy to obtain a constant upper level and the interface boundary condition in the reservoir changes slowly, but the distribution of density within the layers does not change. With a stratified reservoir which varies with depth without interfaces, as the flow is withdrawn we cannot maintain the constant distribution of density, withdrawal depths in the reservoir change, the distribution of density excess changes and the value of N increases. Thus, as time progresses, the correct value of the constant also increases in equation (55). Indeed, Spigel & Farrant (1984) find that in the case of the finite tank, both theoretically and experimentally, a pynocline develops. In view of this, the fact that the constant in the equation is only

20% below the predicted constant for the experiment suggests that the constant is relatively robust.

8. Conclusions

For a free-surface flow with a contraction, Wood (1968) used the fact that the flow must be smooth and gradually varied to determine the flow of two layers through a contraction. It was assumed that the flow was controlled at contraction and the ratios of the two flows were controlled at some virtual control.

In a similar manner for selective withdrawal at a dam face with an interface dividing two layers of infinite extent, we use the spherical assumption and, assuming the flow interface is smooth, we are able to show that the valve at the dam face controls the total discharge and a virtual control determines the ratio of the two discharges. With a finite depth of the upper layer, we note that until the discharge is the maximum that can be sustained from the two layers, the virtual control is unaffected. At the maximum discharge, both the discharge maximum and the discharge ratio are determined by the conditions at the virtual control.

Again, Wood (1968) showed when the flowing layers were between upper and lower stationary layers that the discharge was controlled at the contraction and there was a special case where the flow was self-similar with a ratio of the depth at the contraction to the depth at infinity of $\frac{2}{3}$. This leads to the general case where there is a general distribution of density excess.

If we wish to draw the maximum discharge from two layers with stationary layers above and below the moving layers, we have to determine the position of the valve, the ratio of the two discharges and the ratio to the total discharge. The equations with the spherical assumption and the smooth flow allow these to be determined and it appears that flow is self-similar with z at the virtual control divided by the Z at infinity having a value of 0.8. As in Wood (1968), this leads to a solution for a general density distribution.

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