SOME OPEN QUESTIONS ON MINIMAL PRIMES OF A KRULL DOMAIN

PAUL M. EAKIN, JR. AND WILLIAM J. HEINZER

Let A be an integral domain and K its quotient field. A is called a Krull domain if there is a set $\{V_\alpha\}$ of rank one discrete valuation rings such that $A = \bigcap_\alpha V_\alpha$ and such that each non-zero element of A is a non-unit in only finitely many of the V_α. The structure of these rings was first investigated by Krull, who called them *endliche discrete Hauptordungen* (**4** or **5**, p. 104). Samuel (**7**), Bourbaki (**1**), and Nagata (**6**) gave an excellent survey of the subject. In terms of the semigroup $D(A)$ of divisors of A, A is a Krull domain if and only if $D(A)$ is an ordered group of the form $Z^{(I)}$ (**1**, p. 8). In fact, if A is a Krull domain, then the minimal positive elements of $D(A)$ generate $D(A)$ and are in one-to-one correspondence with the minimal prime ideals of A. Moreover, as Bourbaki observed in (**1**, p. 83), each divisor of A has the form $\operatorname{div}(Ax + Ay)$ for some elements x and y of K. In particular, if P is a minimal prime of A, then $\operatorname{div}(P) = \operatorname{div}(Ax + Ay)$; hence $P = A : (A : (x, y))$.

The extent to which the minimal primes of a Krull domain are related to finitely generated ideals has not been completely resolved. This question appeared to be partially answered by Bourbaki in (**1**, p. 83) when he indicated a method for constructing a two-dimensional Krull ring with a non-finitely generated minimal prime. Our purpose is to show that a domain constructed in the manner suggested by Bourbaki must be noetherian and thus cannot provide the desired example. We make use of the following result of (**3**):

Let R be a commutative ring with identity and let S be an overring of R which is a finite unitary R-module. Then if S is noetherian, R is noetherian.

Let Z denote the integers and Q the rationals. Define inductively a sequence of algebraic number fields $\{K_i\}_{i=1}^{\infty}$ such that:

(i) $Q = K_0$,

(ii) $K_{i+1} = K_i(y_i)$, where y_i is a root of $Y^2 - 5a_i \in K_i[Y]$,

(iii) the integral closure of Z in $\bigcup K_i$ is Dedekind.

Let $A = Z[[X]]$ and K its quotient field. Set $z_i = (a_iX)^{\frac{1}{2}}$. Bourbaki contends that the integral closure of A in $K(\{z_i\}_{i=1}^{\infty})$ is a Krull domain and that the minimal prime generated by X and the z_i is not finitely generated.[1]

We remark that if the integral closure of Z in $L = Q(\{(5a_i)^{\frac{1}{2}}\}_{i=1}^{\infty})$ is

Received May 3, 1967.

[1]One apparent difficulty with this exercise is controlling the factorization of the prime integer 2 in the desired sequence of algebraic number fields. This difficulty can be avoided by using $Z[\frac{1}{2}]$ instead of Z. We show, however, that a Krull domain obtained by any such modification of Bourbaki's exercise is noetherian.

https://doi.org/10.4153/CJM-1968-122-6 Published online by Cambridge University Press

Dedekind, then the integral closure of Z in $L(5^{\frac{1}{2}})$ is also Dedekind. Thus the following result is applicable to the above example:

Let A_0 be a Dedekind domain with quotient field K_0 and let $J_0 = A_0[[X^2]]$. Suppose that $K = K_0(a_1, a_2, \ldots, a_n, \ldots)$ is a separable algebraic extension of K_0, that A is the integral closure of A_0 in K, and that L_0 is the quotient field of J_0. Then if J is the integral closure of J_0 in $L_0(a_1X, a_2X, \ldots, a_nX, \ldots) = L$, the following are equivalent:

(1) *J is a Krull domain,*
(2) *J is noetherian, and*
(3) *A is a Dedekind domain.*

Proof. (3) $\Rightarrow$ (2). Suppose that A is Dedekind. For $i \geqq 1$, let A_i be the integral closure of A_0 in $K_0(a_1, \ldots, a_i)$. Let $R = \bigcup_i A_i[[X]]$. We will show that R is a noetherian, finite integral overring of J and conclude that J is noetherian by Theorem 2 of (**3**). We first show that R is the integral closure of J_0 in $L(X)$.

Let J_i be the integral closure of J_0 in $L_0(a_1X, \ldots, a_iX) = L_i$. Note that $A_0[[X]]$ is the integral closure of J_0 in $L_0(X)$ (in fact, $[1, X]$ is an integral basis). It follows that $A_i[[X]]$ is the integral closure of J_i in $L_i(X)$. Because of separability, there exist $u_1, \ldots, u_k$, a module basis for A_i over A_0. If $b = \sum_{i=0}^{\infty} d_i X^i \in A_i[[X]]$, then $d_i = \sum_{j=1}^{k} \lambda_j{}^i u_j$ with $\lambda_j{}^i \in A_0$. Substituting, we can write $b = \sum_{j=1}^{k} u_j (\sum_{i=0}^{\infty} \lambda_j{}^i X^i)$ and conclude[2] that

$$A_i[[X]] \subseteq A_0[[X]]\{u_1, \ldots, u_k\}.$$

Since containment the other way is obvious, we have equality. Since A_i is Dedekind, $A_i[[X]]$ is integrally closed and is therefore the integral closure of J_i in $L_i(X)$. We conclude that the integral closure of $J = \bigcup_i J_i$ in $L(X) = \bigcup_i L_i(X)$ contains R. But $R = \bigcup_i A_i[[X]]$ is integrally closed and has quotient field $L(X)$. Therefore R is the integral closure in $L(X)$ of J_0 and hence of J.

We now show that R is noetherian. By Cohen's theorem (**2**, p. 29) it is sufficient to show that the primes of R are finitely generated. Let Q be a prime of R and let $Q(0) = \{d \in A \mid d$ is the constant term of some $q \in Q\}$. Since $Q(0)$ is an ideal of the Dedekind domain A, it has a basis of two elements, say f_0 and g_0. If $X \in Q$, it follows that Q is generated by X, f_0, and g_0. Let f and g be elements of Q having f_0 and g_0, respectively, as constant terms. If $X \notin Q$, we show that f and g generate Q. If $f_0, g_0 \in A_j$, then since A_i is Dedekind, $A_i f_0 + A_i g_0 = Q(0) \cap A_i$ for each $i \geqq j$. Suppose that $q = \sum_{j=0}^{\infty} d_j X^j \in Q$, then for i sufficiently large, q, f, and g are in $A_i[[X]]$. Since $d_0 \in Q(0) \cap A_i$ there exist ω_0 and β_0 in A_i such that $d_0 = \omega_0 f_0 + \beta_0 g_0$. Thus

$$q - (\omega_0 f + \beta_0 g) = Xq_1 \in Q \cap A_i[[X]].$$

[2]For a ring R, $R\{S_1, \ldots, S_n\}$ denotes the R-module generated by $S_1, \ldots, S_n$.

Since $X \notin Q$, we have that $q_1 \in Q$ and $q_1 = \omega_1 f + \beta_1 g + Xq_2$. Combining these we have that $q = (\omega_0 f + \beta_0 g) + (\omega_1 f + \beta_1 g)X + X^2 q_2$. Proceeding inductively we conclude that

$$q = \sum_{j=0}^{\infty}(\omega_j f + \beta_j g)X^j = f(\sum_{j=0}^{\infty}\omega_j X^j) + g(\sum_{j=0}^{\infty}\beta_j X^j)$$

with $\omega_j, \beta_j \in A_i$ for all j. Thus f and g generate Q and we have established that R is noetherian.

Finally, we show that R is a finite module over J. Let $A' = A \cap J$. Then A' is a Dedekind domain. (It is a one-dimensional Krull domain (**8**, p. 84) since $A' = A \cap (R \cap L)$ and $A' \subseteq A$ implies A' is integral over A_0.) Furthermore, A is a finite A' module. To see this we observe that $a_i X$, $a_j X$, and X^2 are in L for each pair i, j and therefore $a_i a_j \in L$. Since $a_i a_j$ is algebraic over K_0, there exists $h_{ij} \in A_0$ such that $h_{ij}a_i a_j \in A$. Letting K' be the quotient field of A', we have that $a_i a_j \in K'$ for every pair i, j. Thus $K'(a_1) = K$. Therefore A is the integral closure of A' in a finite, separable algebraic extension of K' and it follows that A is a finite A' module. Let $t_1, t_2, \ldots, t_n$ be a module basis for A over A'. We claim that $R = J\{X, t_1, t_2, \ldots, t_n\}$. Clearly, $J\{X, t_1, t_2, \ldots, t_n\} \subseteq R$. As we have already seen, $A_0[[X]]$ is contained in $J[X]$. For a fixed i, let $b_1, b_2, \ldots, b_m$ be a module basis for A_i over A_0. Then $A_i[[X]] \subset J[X]\{b_1, b_2, \ldots, b_m\}$; but each $b_r \in A'\{t_1, t_2, \ldots, t_n\}$. Thus

$$A_i[[X]] \subseteq J\{X, t_1, t_2, \ldots, t_n\}$$

and it follows that $R = \bigcup A_i[[X]] \subseteq J\{X, t_1, t_2, \ldots, t_n\}$. Therefore, R is a finite J module and from the fact that R is noetherian we conclude that J is noetherian.

(2) $\Rightarrow$ (1). If J is noetherian, it is then a noetherian integrally closed domain and is therefore a Krull domain (**8**, p. 82).

(1) $\Rightarrow$ (3). If J is Krull, then $A' = J \cap K$ is a one-dimensional Krull ring and hence is Dedekind. A is the integral closure of A' in a finite algebraic extension and is therefore Dedekind.

We note that in proving R noetherian we have essentially shown the following lemma.

LEMMA. *Let S be a commutative ring with identity and let Q be a prime ideal of $S[[X]]$. Then Q is finitely generated provided $Q(0) = \{c \in S \mid c$ is the constant term of some $q \in Q\}$ is a finitely generated ideal of S. Moreover, if $Q(0)$ is generated by n elements, then Q is generated by $n + 1$ elements if $X \in Q$ and by n elements if $X \notin Q$.*

As immediate corollaries we have the following well-known results.

COROLLARY 1. *If S is a commutative, noetherian ring with identity, then $S[[X]]$ is noetherian.*

COROLLARY 2. *If S is a principal ideal domain, then $S[[X]]$ is a unique factorization domain (u.f.d.).*

Proof. By Corollary 1, $S[[X]]$ is noetherian and by the lemma, its minimal primes are principal. Thus $S[[X]]$ is a u.f.d. (**6**, p. 42).

Although the Bourbaki exercise is not valid, its contention is at least partly correct; there do exist Krull domains with minimal primes that are not finitely generated.

Let K be a field and let $\{X_i\}_{i=1}^{\infty}$ be a set of elements algebraically independent over K. Let $A = K[X_1, X_2, \ldots]$, $A' = K[\{X_iX_j\}]$ with L and L' the respective quotient fields. A' consists of all polynomials f in $K[X_1, X_2, \ldots]$ such that each monomial in f has as total degree an even integer. We show that A' is a Krull domain with a non-finitely generated minimal prime. To see that A' is Krull we observe that it is the intersection of the Krull ring A and the field L'. It is clear that $A' \subseteq A \cap L'$. And if $h \in A \cap L'$, then $h \in L'$ implies that $h = f/g$, where f and g are polynomials in $K[X_1, X_2, \ldots]$ in which each monomial has even total degree. The equality $h \cdot g = f$ implies that each monomial in h has even total degree and $h \in A'$. Thus A' is Krull. The essential valuations of A' are simply the restrictions of the essential valuations of A to L'. In particular, consider P, the centre of the X_1-adic valuation on A'. We see that $P = (X_1^2, X_1X_2, \ldots, X_1X_n, \ldots)$ and using the algebraic independence of the X_j one can show that no proper subset of $\{X_1X_j\}_{j=1}^{\infty}$ generates P. Thus P cannot be finitely generated.

We observe that even though P is not finitely generated, it remains (in some sense) related to the finitely generated ideals. For the second symbolic power of P, $P^{(2)} = (X_1^2)$. In particular, P is the radical of a finitely generated ideal. We also note that the above example is infinite-dimensional. No finite-dimensional example of such a ring is known to us and in fact we know of no non-noetherian, two-dimensional Krull ring. The nature of the relationship between the minimal primes of a Krull ring and the finitely generated ideals of the ring remains, for the most part, unknown.

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Louisiana State University,
Baton Rouge, Louisiana