Mathematical Structures in Computer Science (2021), **31**, pp. 1003–1005
doi:10.1017/S0960129520000286

CAMBRIDGE
UNIVERSITY PRESS

PAPER

The genesis of the groupoid model

Thomas Streicher

Fachbereich Mathematik, Technische Universität Darmstadt, Darmstadt, Germany
Email: streicher@mathematik.tu-darmstadt.de

(Received 1 May 2020; revised 19 October 2020; accepted 2 December 2020; first published online 29 December 2020)

Abstract
I recall how Martin Hofmann and I found the groupoid model of type theory in the early 1990s.

Keywords: Type theory; categorical semantics; homotopy

Dedicated to the Memory of my Dear Friend Martin Hofmann

I still remember vividly when I met Martin Hofmann the first time at University of Passau. I cannot remember the precise date but it was early summer 1990 or 1991 but in any case before he left for Edinburgh to start his PhD studies there. I was working in Martin Wirsing's research group and Martin Hofmann was trying to find out whether there was a PhD position for him in our research group. This did not work out since Martin W. managed to convice Martin H. to do his PhD in Edinburgh which those days was one of the most vibrant places for doing research in theoretical computer science. But Martin H. was already in contact with Thorsten Altenkirch who had already started his PhD in Edinburgh and certainly told him from another perspective how good a place it was. That Don Sanella became Martin H.'s supervisor is presumably due also to Martin W.'s recommendation since Sanella and Wirsing knew each other quite well from common collaboration.

Starting from the mid 1980s, I was working on the categorical semantics of Martin-Löf Type Theory and its realizability models which eventually lead to my doctoral thesis (Streicher, 1989). Actually, my thesis was on the impredicative Calculus of Constructions devised by Th. Coquand and G. Huet in the first half of the 1980s. In this calculus, identity types do not show up prominently but it was clear that in realizability models so-called *Leibniz equality* on a type A, that is, x and y of type A are equal iff they share the same properties, coincides with the diagonal $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$, the interpretation of Martin-Löf's identity types in a locally cartesian closed category as devised by R. Seely in his JSL paper (Seely, 1984). It was an unpleasant surprise for me that this very natural interpretation of type theory was not supported by recent proof checkers for type theory as, for example, the LEGO system developed at Univ. Edinburgh by R. Pollack which I started to play with those days.

When I met Martin the first time, we immediately started discussing type theory and I was deeply impressed how much he knew already before having even started his PhD. Not only scientifically I was impressed by him but we soon got closer also personally when continuing our discussions in a pub... We, of course, kept contact when he left for Edinburgh starting his PhD. When visiting him in Edinburgh I also got to know Thorsten Altenkirch who had started his PhD a year before Martin (and lived at the same place). I learned a lot from discussions with them and they helped me a lot to eventually understand what were the reasons for the change from extensional to intensional type theory. For a dyed in the wool categorical logician like me, it was difficult to understand what could be wrong about the most natural idea of interpreting equality on A as the diagonal δ_A as common in categorical logic in general and categorical semantics of *extensional* type theory in particular. With their help I eventually understood that extensional type theory was problematic because its identification of *judgmental* and *propositional* equality has the consequence that type checking becomes undecidable. I still do not think that conceptually this necessarily has to be considered as a problem but they convinced me that at least pragmatically this is inconvenient and thus was not supported by proof checkers like LEGO (as opposed to Constable's NuPrL system implementing extensional type theory where one has to pay the price of carrying around derivations of judgments). I got so obsessed by these questions that I started to construct models of intensional type theory based on variations of realizability models which faithfully reflected most of the "defects" of intensional type theory on a semantical level which work formed the main part of my habilitation thesis (Streicher, 1994).

Though in this work I managed to build models which (in my eyes) in most respects were "as bad as syntax" there was one problem left which I could not solve, namely whether intensional type theory proves the principle UIP (Uniqueness of Identity Proofs), i.e.

$$(\Pi x, y : A)(\Pi p, q : \mathrm{Id}_A(x, y)\ \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p, q))$$

for all types A. This principle remained valid in all the models I constructed in Streicher (1994). Moreover, one could prove UIP when adding to the eliminator J for Martin-Löf's identity types a further eliminator K for $x : A \vdash \mathrm{Id}_A(x, x)$. Adding this K did not violate the computational character of intensional type theory. So K is very natural and conceptually sound but there remained the nagging question whether this additional eliminator K could be implemented in terms of J.

Of course, I often discussed this question with Martin and we both had observed that Id_A endows every type A with the structure of a weak kind of groupoid in the following way. The elements of A are the objects of the groupoid and for $x, y : A$ the type of morphisms from x to y is given by $\mathrm{Id}_A(x, y)$. Composition is given by the proof of transitivity, identity on x is given by the canonical element $r_A(x) : \mathrm{Id}_A(x, x)$ (proof of reflexivity), and the proof of symmetry for Id_A gives rise to inverses. However, these data validate the groupoid laws only in the sense of propositional and not in the sense of judgmental equality. Now at some point – and that was very typical for his original and audacious mind – Martin came up with the brilliant idea to turn this observation upside down and construct a model of intensional type theory where (families of) types are (fibrations of) groupoids and, most importantly, identity types $\mathrm{Id}_A(x, y)$ are interpreted as the discrete groupoid (i.e. set) $A(x, y)$ of morphism from x to y in A. Then, obviously, if A is a nontrivial group the corresponding type violates UIP since A has just one element $*$ which is equal to itself in as many ways as there are elements of the group A.

So there just remained the task of checking that this actually gives rise to a model of Intensional Martin-Löf Type Theory. The result of this effort we published as Hofmann and Streicher (1994) and which 20 years later won us the *Test of Time Award* for LiCS papers having some longer term effect on research. A year later we presented an extended version of this work at a conference in Venice celebrating the 25th anniversary of Martin-Löf type theory published one year later (Hofmann and Streicher 1996). Though our original intention was to obtain a negative metamathematical result, namely the underivability of UIP in intensional type theory, we added some discussion speculating that the (mis)interpretation of equality of types as isomorphism might implement the category theorist's implicit idea that *isomorphic types are equal*. We also speculated that UIP for iterated identity types might fail when replacing ordinary groupoids by *higher dimensional groupoids*. This idea was worked out in detail later on in M. Warren's PhD Thesis supervised by Steve Awodey. Moreover, we suggested that the failure of groupoid laws formulated in terms of judgmental equality might be obtained semantically when replacing higher dimensional groupoids by a weak version where the required equalities only hold up to isomorphism.

https://doi.org/10.1017/S0960129520000286 Published online by Cambridge University Press

Alas, we did not know those days that such a notion was introduced by D. Kan as particular simplicial sets, nowadays called *Kan complexes*, roughly at the time when the author of these lines was a born (end of 1950s). I learned of these gadgets more than a decade after my work with Martin when trying to understand some early texts of A. Joyal on *quasicategories* which relate to Kan complexes just the way as ordinary categories relate to ordinary groupoids. At a conference in Gothenburg in fall 2006, I gave a talk observing that Kan complexes provide a model of intensional type theory refuting UIP also for iterated identity types and where the groupoid laws of Id-types only hold in the sense of propositional equality and not in the sense of judgmental equality. Similar but independent such observations were presented at the same meeting in a talk by R. Garner. At this meeting, Awodey and Warren also presented their work on interpreting identity types in model structures the most prominent example of which are simplicial sets.

But, as we all learnt soon after this meeting, Voevodsky had seen this before and formulated his program of *Univalent Type Theory* based on the idea that types are spaces and equal when they are *weakly equivalent*, that is, isomorphic in a somewhat generalized sense.[1] Actually, the intention of this meeting was that Voevodsky would present his very recent ideas in this vein to the assembled type theorists and algebraic topologists but, alas, he could not attend. However, his ideas spread out quickly and very soon we saw the advent of *Homotopy Type Theory* aka *Univalent Foundations*.

Both Martin and I were somewhat surprised how many people started to work on this program which we had (sort of) anticipated for the much more modest purpose of proving a simple independence result in type theory which was considered as sort of arcane when we wrote our paper in 1993. That generalizations of the groupoid model might give rise to an alternative somewhat "extremist" foundation of mathematics in the spirit of category theory was sketched in a speculative paragraph in Hofmann and Streicher (1996).[2]

Martin shifted his interest to slightly more applied problems in computer science still using methods from type theory and various other semantically oriented theories. Accordingly, we unfortunately met less often but when we did as always it was a mere pleasure to see his sparkling mind at work when discussing in a bar or at the occasion of mutual visits. It is a deep sorrow that this will not be possible anymore...

Notes

1 Similar ideas were ventilated in an unpublished report by F. Lamarche from the early 1990s which also at least subconsciously influenced Martin and me!

2 I am still sceptical whether equality of objects can always be ignored even in higher category theory. This is in accordance with Joyal and Lurie's generalization of Grothendieck fibrations to quasicategories where equality of objects is still indispensible!

References

Hofmann, M. and Streicher, T. (1994). The groupoid model refutes uniqueness of identity proofs. In: *Proceedings of LiCS94*, 208–212.

Hofmann, M. and Streicher, T. (1996). The groupoid interpretation of type theory. In: Sambin, G. (ed.) *Twenty-Five Years of Constructive Type Theory*, OUP.

Seely, R. A. G. (1984). Locally cartesian closed categories and type theory. *Mathematical Proceedings of the Cambridge Philosophical Society* **95**(1) 33–48.

Streicher, T. (1989). *Semantics of Type Theory*. Thesis University of Passau.

Streicher, T. (1994). *Investigations into Intensional Type Theory*. Habilitation Thesis, Ludwig Maximilians Universität, Munich.

Cite this article: Streicher T (2021). The Genesis of the Groupoid Model. *Mathematical Structures in Computer Science* **31**, 1003–1005. https://doi.org/10.1017/S0960129520000286