Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–9, *2024*
DOI:10.1017/prm.2024.128

Strongly outer actions of certain torsion-free amenable groups on the Razak–Jacelon algebra

Norio Nawata
Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Yamadaoka 1-5, Suita, Osaka 565-0871, Japan (nawata@ist.osaka-u.ac.jp)

(Received 3 July 2024; revised 11 November 2024; accepted 11 November 2024)

Let $\mathfrak{C}$ be the smallest class of countable discrete groups with the following properties: (i) $\mathfrak{C}$ contains the trivial group, (ii) $\mathfrak{C}$ is closed under isomorphisms, countable increasing unions and extensions by $\mathbb{Z}$. Note that $\mathfrak{C}$ contains all countable discrete torsion-free abelian groups and poly-$\mathbb{Z}$ groups. Also, $\mathfrak{C}$ is a subclass of the class of countable discrete torsion-free elementary amenable groups. In this article, we show that if $\Gamma \in \mathfrak{C}$, then all strongly outer actions of Γ on the Razak–Jacelon algebra $\mathcal{W}$ are cocycle conjugate to each other. This can be regarded as an analogous result of Szabó's result for strongly self-absorbing C*-algebras.

Keywords: first cohomology vanishing type theorem; Kirchberg's central sequence C*-algebra; Razak–Jacelon algebra; Rohlin type theorem; torsion-free amenable groups

1991 *Mathematics Subject Classification:* Primary: 46L55
Secondary: 46L35; 46L40

1. Introduction

Let $\mathcal{W}$ be the Razak–Jacelon algebra studied in [15] (see also [33]). By classification results in [2] and [6] (see also [30]), $\mathcal{W}$ is the unique simple separable nuclear monotracial $\mathcal{Z}$-stable C*-algebra that is KK-equivalent to $\{0\}$. Also, $\mathcal{W}$ is regarded as a stably finite analog of the Cuntz algebra $\mathcal{O}_2$. More generally, we can consider that $\mathcal{W}$ is a non-unital analog of strongly self-absorbing C*-algebras. (Note that every strongly self-absorbing C*-algebra is unital by definition.) In this article, we study group actions on $\mathcal{W}$ and show an analogous result of Szabó's result in [40] for group actions on strongly self-absorbing C*-algebras (see also [12–14, 22–24, 26, 37, 39] for pioneering works). We refer the reader to [11] for the importance and some difficulties of studying group actions on C*-algebras. Gabe and Szabó classified outer actions of countable discrete amenable groups on Kirchberg algebras up to cocycle conjugacy in [7]. In their classification, $\mathcal{O}_2$ and $\mathcal{O}_\infty$ play central roles. Hence it is natural to expect that $\mathcal{W}$ plays a central role in the classification theory of group actions on 'classifiable' stably finite (at least stably projectionless) C*-algebras.

Let $\mathfrak{C}$ be the smallest class of countable discrete groups with the following properties: (i) $\mathfrak{C}$ contains the trivial group, (ii) $\mathfrak{C}$ is closed under isomorphisms, countable increasing unions and extensions by $\mathbb{Z}$. (We say that Γ is an extension by $\mathbb{Z}$ if there exists an exact sequence $1 \to H \to \Gamma \to \mathbb{Z} \to 1$.) Note that $\mathfrak{C}$ is the same class as in [40, definition B]. It is easy to see that $\mathfrak{C}$ contains all countable discrete torsion-free abelian groups and poly-$\mathbb{Z}$ groups, and $\mathfrak{C}$ is a subclass of the class of countable discrete torsion-free elementary amenable groups. Szabó showed that if $\Gamma \in \mathfrak{C}$ and $\mathcal{D}$ is a strongly self-absorbing C*-algebra, then there exists a unique strongly outer action of Γ on $\mathcal{D}$ up to cocycle conjugacy [40, corollary 3.4]. In this article, we show an analogous result of this result. Indeed, the main theorem in this article is the following theorem.

Theorem A (theorem 4.3) *Let Γ be a countable discrete group in $\mathfrak{C}$, and let α be a strongly outer action of Γ on $\mathcal{W}$. Then α is cocycle conjugate to $\mu^{\Gamma} \otimes \mathrm{id}_{\mathcal{W}}$ on $M_{2^\infty} \otimes \mathcal{W}$ where μ^{Γ} is the Bernoulli shift action of Γ on $\bigotimes_{g\in\Gamma} M_{2^\infty} \cong M_{2^\infty}$.*

We say that an action α on $\mathcal{W}$ is *$\mathcal{W}$-absorbing* if there exists a simple separable nuclear monotracial C*-algebra A and an action β on A such that α is cocycle conjugate to $\beta \otimes \mathrm{id}_{\mathcal{W}}$ on $A \otimes \mathcal{W}$. The proof of the main theorem above is based on a characterization in [31] of strongly outer $\mathcal{W}$-absorbing actions of countable discrete amenable groups. Actually, we use the following theorem that is a slight variant of [31, theorem 8.1]. Note that $F(\mathcal{W})$ is Kirchberg's central sequence C*-algebra of $\mathcal{W}$. Furthermore, $F(\mathcal{W})^{\alpha}$ is the fixed point algebra for the action on $F(\mathcal{W})$ induced by an action α on $\mathcal{W}$. Let $\mathrm{Sp}(x)$ denote the spectrum of x.

Theorem B (theorem 2.4) *Let α be a strongly outer action of a countable discrete amenable group Γ on $\mathcal{W}$. Then α is cocycle conjugate to $\mu^{\Gamma} \otimes \mathrm{id}_{\mathcal{W}}$ on $M_{2^\infty} \otimes \mathcal{W}$ if and only if α satisfies the following properties:*

(i) there exists a unital $*$-homomorphism from $M_2(\mathbb{C})$ to $F(\mathcal{W})^{\alpha}$,
(ii) if x and y are normal elements in $F(\mathcal{W})^{\alpha}$ such that $\mathrm{Sp}(x) = \mathrm{Sp}(y)$ and $0 < \tau_{\mathcal{W},\omega}(f(x)) = \tau_{\mathcal{W},\omega}(f(y))$ for any $f \in C(\mathrm{Sp}(x))_+ \setminus \{0\}$, then x and y are unitary equivalent in $F(\mathcal{W})^{\alpha}$,
(iii) there exists an injective $*$-homomorphism from $\mathcal{W} \rtimes_{\alpha} \Gamma$ to $\mathcal{W}$.

We use a first cohomology vanishing type theorem (corollary 3.4) for showing that if $\Gamma \in \mathfrak{C}$ and α is a strongly outer action of Γ on $\mathcal{W}$, then $F(\mathcal{W})^{\alpha}$ satisfies the properties (i) and (ii) in the theorem above. Kishimoto's techniques for Rohlin type theorems in [19] and [20], Herman-Ocneanu's argument in [9], and homotopy type arguments in [28] enable us to show this first cohomology vanishing type theorem. Also, note that our arguments for $F(\mathcal{W})^{\alpha}$ are based on results that are shown by techniques around (equivariant) property (SI) in [25–27, 34–36, 42].

2. Preliminaries

2.1. Notations and basic definitions

Let α and β be actions of a countable discrete group Γ on C*-algebras A and B, respectively. We say that α is *conjugate to* β if there exists a isomorphism φ from A onto B such that $\varphi \circ \alpha_g = \beta_g \circ \varphi$ for any $g \in \Gamma$. Note that α induces an action on the

https://doi.org/10.1017/prm.2024.128 Published online by Cambridge University Press

multiplier algebra $M(A)$ of A. We denote it by the same symbol α. An α-cocycle on A is a map from Γ to the unitary group of $M(A)$ such that $u_{gh} = u_g\alpha_g(u_h)$ for any $g, h \in \Gamma$. We say that α is *cocycle conjugate to* β if there exist an isomorphism φ from A onto B and a β-cocycle u such that $\varphi \circ \alpha_g = \mathrm{Ad}(u_g) \circ \beta_g \circ \varphi$ for any $g \in \Gamma$. An action α of Γ on A is said to be *outer* if α_g is not an inner automorphism of A for any $g \in \Gamma \setminus \{\iota\}$ where ι is the identity of Γ. We denote by A^α and $A \rtimes_\alpha \Gamma$ the fixed point algebra and the reduced crossed product C*-algebra, respectively.

Assume that A has a unique tracial state τ_A. Let $(\pi_{\tau_A}, H_{\tau_A})$ be the Gelfand–Naimark–Segal representation of τ_A. Then $\pi_{\tau_A}(A)''$ is a finite factor and α induces an action $\tilde{\alpha}$ on $\pi_{\tau_A}(A)''$. We say that α is *strongly outer* if $\tilde{\alpha}$ is an outer action on $\pi_{\tau_A}(A)''$. (We refer the reader to [8] and [26] for the definition of strongly outerness for more general settings.)

We denote by $\mathcal{R}_0$ and M_{2^∞} the injective II_1 factor and the canonical anticommutation relations (CAR) algebra, respectively.

2.2. Fixed point algebras of Kirchberg's central sequence C*-algebras

Let ω be a free ultrafilter on $\mathbb{N}$, and put

$$A^\omega := \ell^\infty(\mathbb{N}, A)/\{\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|x_n\| = 0\}.$$

We denote by $(x_n)_n$ a representative of an element in A^ω. We identify A with the C*-subalgebra of A^ω consisting of equivalence classes of constant sequences. Set

$$\mathrm{Ann}(A, A^\omega) := \{(x_n)_n \in A^\omega \cap A' \mid \lim_{n\to\omega} \|x_n a\| = 0 \text{ for any } a \in A\}.$$

Then $\mathrm{Ann}(A, A^\omega)$ is an closed ideal of $A^\omega \cap A'$, and define

$$F(A) := A^\omega \cap A'/\mathrm{Ann}(A, A^\omega).$$

See [17] for basic properties of $F(A)$. For a finite von Neumann algebra M, put

$$M^\omega := \ell^\infty(\mathbb{N}, M)/\{\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\omega} \|x_n\|_2 = 0\}$$

and

$$M_\omega := M^\omega \cap M'.$$

Note that we identify M with the subalgebra of M^ω consisting of equivalence classes of constant sequences and M_ω is the von Neumann algebraic central sequence algebra (or the asymptotic centralizer) of M.

For a tracial state τ_A on A, define a map $\tau_{A,\omega}$ from $F(A)$ to $\mathbb{C}$ by $\tau_{A,\omega}([(x_n)_n]) = \lim_{n\to\omega} \tau_A(x_n)$ for any $[(x_n)_n] \in F(A)$. Then $\tau_{A,\omega}$ is a well-defined tracial state on $F(A)$ by [28, proposition 2.1]. Put $J_{\tau_A} := \{x \in F(A) \mid \tau_{A,\omega}(x^*x) = 0\}$. If A is separable and τ_A is faithful, then π_{τ_A} induces an isomorphism from $F(A)/J_{\tau_A}$ onto $\pi_{\tau_A}(A)''_\omega$ by essentially the same argument as in the proof of [18, theorem 3.3]. In this article, the reindexing argument and the diagonal argument (or Kirchberg's ε-test [17, lemma A.1]) are frequently used. We refer the reader to [1, Section 1.3] and [32, Chapter 5] for details of these arguments. Every action α of a countable discrete group on A induces an action on $F(A)$. We denote it by the same symbol

https://doi.org/10.1017/prm.2024.128 Published online by Cambridge University Press

α for simplicity. Note that if α on A are cocycle conjugate to β on B, then α on $F(A)$ are conjugate to β on $F(B)$. If A is simple, separable, and monotracial, then $\tilde{\alpha}$ also induces an action on $\pi_{\tau_A}(A)''_{\omega}$. We also denote it by the same symbol $\tilde{\alpha}$. By [31, proposition 3.9], we see that π_{τ_A} induces an isomorphism from $F(A)^{\alpha}/J^{\alpha}_{\tau_A}$ onto $(\pi_{\tau_A}(A)'')^{\tilde{\alpha}}_{\omega}$.

The following proposition is an immediate consequence of [31, theorem 3.6], [31, proposition 3.11], and [31, proposition 3.12]. Note that these propositions are based on results in [25–27, 34–36, 42].

Proposition 2.1. *Let α be an outer action of a countable discrete amenable group on $\mathcal{W}$.*

(1) The Razak–Jacelon algebra $\mathcal{W}$ has property (SI) relative to α, that is, if a and b are positive contractions in $F(\mathcal{W})^{\alpha}$ satisfying $\tau_{\mathcal{W},\omega}(a) = 0$ and $\inf_{m\in\mathbb{N}} \tau_{\mathcal{W},\omega}(b^m) > 0$, then there exists an element s in $F(\mathcal{W})^{\alpha}$ such that $bs = s$ and $s^*s = a$.
(2) The fixed point algebra $F(\mathcal{W})^{\alpha}$ is monotracial.
(3) If a and b are positive elements in $F(\mathcal{W})^{\alpha}$ satisfying $d_{\tau_{\mathcal{W},\omega}}(a) < d_{\tau_{\mathcal{W},\omega}}(b)$, then there exists an element r in $F(\mathcal{W})^{\alpha}$ such that $r^*br = a$.

Definition 2.2. *Let α be an action of a countable discrete group Γ on $\mathcal{W}$. We say that α has property W if α satisfies the following properties:*

(i) there exists a unital $*$-homomorphism from $M_2(\mathbb{C})$ to $F(\mathcal{W})^{\alpha}$,
(ii) if x and y are normal elements in $F(\mathcal{W})^{\alpha}$ such that $\mathrm{Sp}(x) = \mathrm{Sp}(y)$ and $0 < \tau_{\mathcal{W},\omega}(f(x)) = \tau_{\mathcal{W},\omega}(f(y))$ for any $f \in C(\mathrm{Sp}(x))_+ \setminus \{0\}$, then x and y are unitary equivalent in $F(\mathcal{W})^{\alpha}$.

Note that if there exists a unital $*$-homomorphism from $M_2(\mathbb{C})$ to $F(\mathcal{W})^{\alpha}$, then α on $\mathcal{W}$ is cocycle conjugate to $\alpha\otimes\mathrm{id}_{M_{2^\infty}}$ on $\mathcal{W}\otimes M_{2^\infty}$. Indeed, there exists a unital $*$-homomorphism from M_{2^∞} to $F(\mathcal{W})^{\alpha}$ by a similar argument as [17, corollary 1.13]. Hence [38, corollary 3.8] (see also [41]) implies this cocycle conjugacy. Using this observation, proposition 2.1 and definition 2.2 instead of M_{2^∞}-stability of $\mathcal{W}$, [28, proposition 4.1], [28, theorem 5.3], and [28, theorem 5.8], we obtain the following theorem by essentially the same arguments as in the proofs of [28, proposition 4.2], [28, theorem 5.7], and [28, corollary 5.11] (or [29, corollary 5.5]).

Theorem 2.3 *Let α be an outer action of a countable discrete amenable group on $\mathcal{W}$. Assume that α has property W.*

(1) For any $\theta \in [0, 1]$, there exists a projection p in $F(\mathcal{W})^{\alpha}$ such that $\tau_{\mathcal{W},\omega}(p) = \theta$.
(2) For any unitary element u in $F(\mathcal{W})^{\alpha}$, there exists a continuous path of unitaries $U : [0, 1] \to F(\mathcal{W})^{\alpha}$ such that

$$U(0) = 1, \quad U(1) = u \quad \text{and} \quad \mathrm{Lip}(U) \leq 2\pi$$

where $\mathrm{Lip}(U)$ is the Lipschitz constant of U, that is, the smallest positive number satisfying $\|U(t) - U(s)\| \leq \mathrm{Lip}(U)|t - s|$ for any $t, s \in [0, 1]$.

(3) If p and q are projections in $F(\mathcal{W})^\alpha$ such that $0 < \tau_{\mathcal{W},\omega}(p) = \tau_{\mathcal{W},\omega}(q)$, then p and q are Murray–von Neumann equivalent.

For any countable discrete group Γ, let μ^Γ be the Bernoulli shift action of Γ on $\bigotimes_{g\in\Gamma} M_{2^\infty} \cong M_{2^\infty}$. The following theorem is a slight variant of [31, theorem 8.1]. Note that one of the main techniques in the proof of [31, theorem 8.1] is Szabó's approximate cocycle intertwining argument [43] (see also [5]).

THEOREM 2.4 *Let α be a strongly outer action of a countable discrete amenable group Γ on $\mathcal{W}$. Then α is cocycle conjugate to $\mu^\Gamma \otimes \mathrm{id}_{\mathcal{W}}$ on $M_{2^\infty} \otimes \mathcal{W}$ if and only if α has property W and there exists an injective $*$-homomorphism from $\mathcal{W} \rtimes_\alpha \Gamma$ to $\mathcal{W}$.*

Proof. [31, proposition 4.2], [31, theorem 4.5], and [31, theorem 8.1] imply the only if part. The if part is an immediate consequence of [31, theorem 8.1] and theorem 2.3. □

3. First cohomology vanishing type theorem

In this section, we shall show a first cohomology vanishing type theorem (corollary 3.4). This is a corollary of a Rohlin type theorem (theorem 3.3).

The following lemma is well-known among experts. See, for example, [16, theorem 4.8] for a similar (but not the same) result. For the reader's convenience, we shall give a proof based on Ocneanu's classification theorem [32, corollary 1.4].

LEMMA 3.1. *Let Γ be a countable discrete amenable group, and let N be a normal subgroup of Γ. If γ is an outer action of Γ on the injective II$_1$ factor $\mathcal{R}_0$ and $g_0 \notin N$, then γ_{g_0} induces a properly outer automorphism of $(\mathcal{R}_0)_\omega^{\gamma|N}$.*

Proof. Since N is a normal subgroup, it is clear that γ_{g_0} induces an automorphism of $(\mathcal{R}_0)_\omega^{\gamma|N}$. First, we shall show that γ_{g_0} is not trivial as an automorphism of $(\mathcal{R}_0)_\omega^{\gamma|N}$. Let π be the quotient map from Γ to Γ/N, and let β be the Bernoulli shift action of Γ/N on $\mathcal{R}_0 \cong \bigotimes_{\pi(g)\in\Gamma/N} \mathcal{R}_0$. Define an action δ of Γ on $\mathcal{R}_0 \cong \mathcal{R}_0 \bar{\otimes} \mathcal{R}_0$ by $\delta_g := \gamma_g \otimes \beta_{\pi(g)}$ for any $g \in \Gamma$. By Ocneanu's classification theorem [32, corollary 1.4], γ on $\mathcal{R}_0$ and δ on $\mathcal{R}_0 \bar{\otimes} \mathcal{R}_0$ are cocycle conjugate. Hence there exists an isomorphism Φ from $(\mathcal{R}_0)_\omega$ onto $(\mathcal{R}_0 \bar{\otimes} \mathcal{R}_0)_\omega$ such that $\Phi \circ \gamma_g = \delta_g \circ \Phi$ for any $g \in \Gamma$. Since $\beta_{\pi(g_0)}$ is an outer automorphism of $\mathcal{R}_0$, there exists an element $(x_n)_n$ in $(\mathcal{R}_0)_\omega$ such that $(\beta_{\pi(g_0)}(x_n))_n \neq (x_n)_n$ by [4, theorem 3.2]. Put $(y_n)_n := \Phi^{-1}((1 \otimes x_n)_n) \in (\mathcal{R}_0)_\omega$. Then it is easy to see that we have $(y_n)_n \in (\mathcal{R}_0)_\omega^{\gamma|N}$ and $(\gamma_{g_0}(y_n))_n \neq (y_n)_n$. Finally, we shall show that γ_{g_0} is properly outer as an automorphism of $(\mathcal{R}_0)_\omega^{\gamma|N}$. Since $(\mathcal{R}_0)_\omega^{\gamma|N}$ is a factor (see, for example, [26, lemma 4.1]), it is enough to show that γ_{g_0} is outer as an automorphism of $(\mathcal{R}_0)_\omega^{\gamma|N}$. In particular, we shall show that for any element $(u_n)_n$ in $(\mathcal{R}_0)_\omega^{\gamma|N}$, there exists an element $(z_n)_n$ in $(\mathcal{R}_0)_\omega^{\gamma|N}$ such that $(u_n z_n)_n = (z_n u_n)_n$ and $(\gamma_{g_0}(z_n))_n \neq (z_n)_n$. Taking a suitable subsequence of $(y_n)_n$ (or the reindexing argument), we obtain the desired element $(z_n)_n$. Consequently, the proof is complete. □

https://doi.org/10.1017/prm.2024.128 Published online by Cambridge University Press

Consider a semidirect product group $N \rtimes \mathbb{Z}$. For $g \in N$ and $m \in \mathbb{Z}$, let (g, m) denote an element in $N \rtimes \mathbb{Z}$. Note that we have $N \rtimes \mathbb{Z} = \{(g, m) \mid g \in N, m \in \mathbb{Z}\}$. The following lemma is an analogous lemma of [28, lemma 6.2]. See also [24, theorem 3.4].

Lemma 3.2. *Let Γ be a semidirect product $N \rtimes \mathbb{Z}$ where N is a countable discrete amenable group, and let α be a strongly outer action of Γ on $\mathcal{W}$. Then for any $k \in \mathbb{N}$, there exists a positive contraction f in $F(\mathcal{W})^{\alpha|_N}$ such that*

$$\tau_{\mathcal{W},\omega}(f) = \frac{1}{k} \quad \textit{and} \quad f\alpha_{(\iota,j)}(f) = 0$$

for any $1 \leq j \leq k-1$.

Proof. Since $\pi_{\tau_{\mathcal{W}}}(\mathcal{W})''$ is isomorphic to the injective II_1 factor, lemma 3.1 implies that $\tilde{\alpha}_{(\iota,1)}$ is an aperiodic automorphism of $(\pi_{\tau_{\mathcal{W}}}(\mathcal{W})'')_\omega^{\tilde{\alpha}|_N}$. Hence it follows from [3, theorem 1.2.5] that there exists a partition of unity $\{P_j\}_{j=1}^k$ consisting of projections in $(\pi_{\tau_{\mathcal{W}}}(\mathcal{W})'')_\omega^{\tilde{\alpha}|_N}$ such that $\tilde{\alpha}_{(\iota,1)}(P_j) = P_{j+1}$ for any $1 \leq j \leq k-1$. Since $\pi_{\tau_{\mathcal{W}}}$ induces an isomorphism from $F(\mathcal{W})^{\alpha|_N}/J_{\tau_{\mathcal{W}}}^{\alpha|_N}$ onto $(\pi_{\tau_{\mathcal{W}}}(\mathcal{W})'')_\omega^{\tilde{\alpha}|_N}$ (see §2.2), there exists a positive contraction $[(e_n)_n]$ in $F(\mathcal{W})^{\alpha|_N}$ such that $(\pi_{\tau_{\mathcal{W}}}(e_n))_n = P_1$ in $(\pi_{\tau_{\mathcal{W}}}(\mathcal{W})'')_\omega^{\tilde{\alpha}|_N}$. Then we have

$$\lim_{n\to\omega} \|\pi_{\tau_{\mathcal{W}}}(e_n\alpha_{(\iota,j)}(e_n))\|_2 = 0 \quad \text{and} \quad \tau_{\mathcal{W},\omega}([(e_n)_n]) = \tilde{\tau}_{\mathcal{W},\omega}(P_1) = \frac{1}{k}$$

for any $1 \leq j \leq k-1$, where $\tilde{\tau}_{\mathcal{W},\omega}$ is the induced tracial state on $(\pi_{\tau_{\mathcal{W}}}(\mathcal{W})'')_\omega^{\tilde{\alpha}|_N}$ by $\tau_{\mathcal{W}}$. The rest of the proof is the same as [28, lemma 6.2]. (See also [24, proposition 3.3].) □

Using proposition 2.1, theorem 2.3 (we need to assume that $\alpha|_N$ has property W), and lemma 3.2 instead of [28, proposition 4.1], [28, proposition 4.2], [28, theorem 5.8], and [28, lemma 6.2], we obtain the following Rohlin type theorem by essentially the same arguments in the proofs of [28, lemma 6.3] and [28, theorem 6.4]. Note that these arguments are based on [19] and [20].

Theorem 3.3 *Let Γ be a semidirect product $N \rtimes \mathbb{Z}$ where N is a countable discrete amenable group, and let α be a strongly outer action of Γ on $\mathcal{W}$. Assume that $\alpha|_N$ has property W. Then for any $k \in \mathbb{N}$, there exists a partition on unity $\{p_{1,i}\}_{i=0}^{k-1} \cup \{p_{2,j}\}_{j=0}^{k}$ consisting of projections in $F(\mathcal{W})^{\alpha|_N}$ such that*

$$\alpha_{(\iota,1)}(p_{1,i}) = p_{1,i+1} \quad \textit{and} \quad \alpha_{(\iota,1)}(p_{2,j}) = p_{2,j+1}$$

for any $0 \leq i \leq k-2$ and $0 \leq j \leq k-1$.

Theorem 2.3, theorem 3.3, and Herman–Ocneanu's argument [9, theorem 1] (see also remarks after [9, lemma 1] and [10, 21]) imply the following corollary.

Corollary 3.4. *Let Γ be a semidirect product $N \rtimes \mathbb{Z}$ where N is a countable discrete amenable group, and let α be a strongly outer action of Γ on $\mathcal{W}$. Assume*

https://doi.org/10.1017/prm.2024.128 Published online by Cambridge University Press

that $\alpha|_N$ has property W and S is a countable subset in $F(\mathcal{W})^{\alpha|_N}$. For any unitary element u in $F(\mathcal{W})^{\alpha|_N}\cap S'$, there exists a unitary element v in $F(\mathcal{W})^{\alpha|_N}\cap S'$ such that $u = v\alpha_{(\iota,1)}(v)^*$.

4. Main theorem

In this section, we shall show the main theorem. Recall that $\mathfrak{C}$ is the smallest class of countable discrete groups with the following properties: (i) $\mathfrak{C}$ contains the trivial group, (ii) $\mathfrak{C}$ is closed under isomorphisms, countable increasing unions and extensions by $\mathbb{Z}$. Note that if Γ is an extension of N by $\mathbb{Z}$, then Γ is isomorphic to a semidirect product $N \rtimes \mathbb{Z}$.

The following lemma is an easy consequence of the definition of property W and the diagonal argument.

LEMMA 4.1. *Let Γ be an increasing union $\bigcup_{m\in\mathbb{N}}\Gamma_m$ of discrete countable groups Γ_m, and let α be an action of Γ on $\mathcal{W}$. If $\alpha|_{\Gamma_m}$ has property W for any $m\in\mathbb{N}$, then α has property W.*

The following lemma is an application of corollary 3.4.

LEMMA 4.2. *Let Γ be a semidirect product $N \rtimes \mathbb{Z}$ where N is a countable discrete amenable group, and let α be a strongly outer action of Γ on $\mathcal{W}$. If $\alpha|_N$ has property W, then α has property W.*

Proof. (i) There exists a unital $*$-homomorphism φ from $M_2(\mathbb{C})$ to $F(\mathcal{W})^{\alpha|_N}$ by the assumption. Let $\{e_{ij}\}_{i,j=1}^2$ be the standard matrix units of $M_2(\mathbb{C})$. Since we have $0 < \tau_{\mathcal{W},\omega}(\varphi(e_{11})) = \tau_{\mathcal{W},\omega}(\alpha_{(\iota,1)}(\varphi(e_{11})))$, there exists an element w in $F(\mathcal{W})^{\alpha|_N}$ such that $w^*w = \alpha_{(\iota,1)}(\varphi(e_{11}))$ and $ww^* = \varphi(e_{11})$ by theorem 2.3. Put $u := \sum_{i=1}^2 \varphi(e_{i1})w\alpha_{(\iota,1)}(\varphi(e_{1i}))$. Then u is a unitary element in $F(\mathcal{W})^{\alpha|_N}$ such that $\alpha_{(\iota,1)}(\varphi(x)) = u^*\varphi(x)u$ for any $x\in M_2(\mathbb{C})$. By corollary 3.4, there exists a unitary element v in $F(\mathcal{W})^{\alpha|_N}$ such that $u = v\alpha_{(\iota,1)}(v)^*$. We have $\alpha_{(\iota,1)}(v^*\varphi(x)v) = v^*\varphi(x)v$ for any $x\in M_2(\mathbb{C})$. Hence the map ψ defined by $\psi(x) := v^*\varphi(x)v$ for any $x\in M_2(\mathbb{C})$ is a unital $*$-homomorphism from $M_2(\mathbb{C})$ to $F(\mathcal{W})^{\alpha}$. (ii) Let x and y be normal elements in $F(\mathcal{W})^{\alpha}$ such that $\mathrm{Sp}(x) = \mathrm{Sp}(y)$ and $0 < \tau_{\mathcal{W},\omega}(f(x)) = \tau_{\mathcal{W},\omega}(f(y))$ for any $f\in C(\mathrm{Sp}(x))_+ \setminus \{0\}$. Since x and y are also elements in $F(\mathcal{W})^{\alpha|_N}$, there exists a unitary element u in $F(\mathcal{W})^{\alpha|_N}$ such that $uxu^* = y$ by the assumption. Note that $u\alpha_{(\iota,1)}(u)^*$ is a unitary element in $F(\mathcal{W})^{\alpha|_N}\cap\{y\}'$. Hence corollary 3.4 implies that there exists a unitary element v in $F(\mathcal{W})^{\alpha|_N}\cap\{y\}'$ such that $u\alpha_{(\iota,1)}(u)^* = v\alpha_{(\iota,1)}(v)^*$. We have $\alpha_{(\iota,1)}(v^*u) = v^*u$ and $v^*uxu^*v = v^*yv = y$. Therefore, x and y are unitary equivalent in $F(\mathcal{W})^{\alpha}$. By (i) and (ii), α has property W. □

The following theorem is the main theorem in this article.

THEOREM 4.3 *Let Γ be a countable discrete group in $\mathfrak{C}$, and let α be a strongly outer action of Γ on $\mathcal{W}$. Then α is cocycle conjugate to $\mu^{\Gamma}\otimes \mathrm{id}_{\mathcal{W}}$ on $M_{2^\infty}\otimes\mathcal{W}$.*

Proof. Every action of the trivial group on $\mathcal{W}$ has property W by results in [28] (or [30, theorem 3.8]). By lemmas 4.1 and 4.2, we see that α has property W. Note that this implies that $\mathcal{W} \rtimes_\alpha \Gamma$ is M_{2^∞}-stable because α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{M_{2^\infty}}$ on $\mathcal{W} \otimes M_{2^\infty}$. Since the class of separable nuclear C*-algebras that are KK-equivalent to $\{0\}$ is closed under countable inductive limits and crossed products by $\mathbb{Z}$, [30, theorem 6.1] implies that $\mathcal{W} \rtimes_\alpha \Gamma$ is isomorphic to $\mathcal{W}$. Therefore, we obtain the conclusion by theorem 2.4. □

The following corollary is an immediate consequence of the theorem above.

COROLLARY 4.4. *Let Γ be a countable discrete group in $\mathfrak{C}$. Then there exists a unique strongly outer action of Γ on $\mathcal{W}$ up to cocycle conjugacy.*

Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 20K03630.

References

[1] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White and W. Winter. Covering dimension of C*-algebras and 2-coloured classification. *Mem. Amer. Math. Soc.* **257** (2016), 1233.

[2] J. Castillejos and S. Evington. Nuclear dimension of simple stably projectionless C*-algebras. *Anal. PDE* **13** (2020), 2205–2240.

[3] A. Connes. Outer conjugacy classes of automorphisms of factors. *Ann. Sci. Ecole Norm. Sup. (4)* **8** (1975), 383–419.

[4] A. Connes. Periodic automorphisms of the hyperfinite factor of type II_1. *Acta Sci. Math. (Szeged)* **39** (1977), 39–66.

[5] G. A. Elliott. Towards a theory of classification. *Adv. Math.* **223** (2010), 30–48.

[6] G. A. Elliott, G. Gong, H. Lin and Z. Niu. The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension. *J. Geom. Phys.* **158** (2020), 51.

[7] J. Gabe and G. Szabó. The dynamical Kirchberg-Phillips theorem. *Acta Math.* **232** (2024), 1–77.

[8] E. Gardella, I. Hirshberg and A. Vaccaro. Strongly outer actions of amenable groups on $\mathcal{Z}$-stable nuclear C*-algebras. *J. Math. Pures Appl.* **9** (2022), 76–123.

[9] R. H. Herman and A. Ocneanu. Stability for integer actions on UHF C*-algebras. *J. Funct. Anal.* **59** (1984), 132–144.

[10] M. Izumi. The Rohlin property for automorphisms of C*-algebras, *Mathematical Physics in Mathematics and Physics (Siena, 2000)* Fields Inst. Commun., Vol. 30, pp. 191–206 (Amer. Math. Soc, Providence, RI, 2001).

[11] M. Izumi. Group actions on operator algebras, Proceedings of the International Congress of Mathematicians, Vol. III, pp. 1528–1548 (Hyderabad, India, New Delhi: Hindustan Book Agency, 2010).

[12] M. Izumi and H. Matui. $\mathbb{Z}^2$-actions on Kirchberg algebras. *Adv. Math.* **224** (2010), 355–400.

[13] M. Izumi and H. Matui. Poly-$\mathbb{Z}$ group actions on Kirchberg algebras I. *Int. Math. Res. Not. IMRN* **16** (2021), 12077–12154.

[14] M. Izumi and H. Matui. Poly-$\mathbb{Z}$ group actions on Kirchberg algebras II. *Invent. Math.* **224** (2021), 699–766.

[15] B. Jacelon. A simple, monotracial, stably projectionless C*-algebra. *J. Lond. Math. Soc. (2)* **87** (2013), 365–383.

[16] Y. Katayama. A duality for an action of a countable amenable group on a hyperfinite II_1-factor. *J. Operator Theory* **21** (1989), 297–314.

[17] E. Kirchberg. Central sequences in C*-algebras and strongly purely infinite algebras, *Operator Algebras: The Abel Symposium 2004*, Vol. 1, pp. 175–231 (Springer, Berlin, 2006).

[18] E. Kirchberg and M. Rørdam. Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra. *J. Reine Angew. Math.* **695** (2014), 175–214.

https://doi.org/10.1017/prm.2024.128 Published online by Cambridge University Press

[19] A. Kishimoto. The Rohlin property for automorphisms of UHF algebras. *J. Reine Angew. Math.* **465** (1995), 183–196.

[20] A. Kishimoto. The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras. *J. Funct. Anal.* **140** (1996), 100–123.

[21] A. Kishimoto. The Rohlin property for shift automorphisms. *Rev. Math. Phys.* **12** (2000), 965–980.

[22] H. Matui. Classification of outer actions of $\mathbb{Z}^N$ on $\mathcal{O}_2$. *Adv. Math.* **217** (2008), 2872–2896.

[23] H. Matui. $\mathbb{Z}^N$-actions on UHF algebras of infinite type. *J. Reine Angew. Math.* **657** (2011), 225–244.

[24] H. Matui and Y. Sato. $\mathcal{Z}$-stability of crossed products by strongly outer actions. *Comm. Math. Phys.* **314** (2012), 193–228.

[25] H. Matui and Y. Sato. Strict comparison and $\mathcal{Z}$-absorption of nuclear C*-algebras. *Acta Math.* **209** (2012), 179–196.

[26] H. Matui and Y. Sato. $\mathcal{Z}$-stability of crossed products by strongly outer actions II. *Amer. J. Math.* **136** (2014), 1441–1496.

[27] H. Matui and Y. Sato. Decomposition rank of UHF-absorbing C*-algebras. *Duke Math. J.* **163** (2014), 2687–2708.

[28] N. Nawata. Trace scaling automorphisms of the stabilized Razak-Jacelon algebra. *Proc. Lond. Math. Soc. (3)* **118** (2019), 545–576.

[29] N. Nawata. Rohlin actions of finite groups on the Razak-Jacelon algebra. *Int. Math. Res. Not. IMRN* **4** (2021), 2991–3020.

[30] N. Nawata. A characterization of the Razak-Jacelon algebra. *Anal. PDE* **16** (2023), 1799–1824.

[31] N. Nawata. Equivariant Kirchberg-Phillips type absorption for the Razak-Jacelon algebra. *J. Funct. Anal.* **285** (2023), Paper No. 110088, 44 pp.

[32] A. Ocneanu. *Actions of Discrete Amenable Groups on von Neumann Algebras* Lecture Notes in Mathematics. Vol. 1138 (Springer-Verlag, Berlin, 1985).

[33] S. Razak. On the classification of simple stably projectionless C*-algebras. *Canad. J. Math.* **54** (2002), 138–224.

[34] Y. Sato. Certain aperiodic automorphisms of unital simple projectionless C*-algebras. *Internat. J. Math.* **20** (2009), 1233–1261.

[35] Y. Sato. The Rohlin property for automorphisms of the Jiang-Su algebra. *J. Funct. Anal.* **259** (2010), 453–476.

[36] Y. Sato. Actions of amenable groups and crossed products of $\mathcal{Z}$-absorbing C*-algebras, *Operator Algebras and Mathematical physics* Adv. Stud. Pure Math., Vol. 80, pp. 189–210 (Math. Soc., Japan, Tokyo, 2019).

[37] G. Szabó. Strongly self-absorbing C*-dynamical systems III. *Adv. Math.* **316** (2017), 356–380.

[38] G. Szabó. Strongly self-absorbing C*-dynamical systems. *Trans. Amer. Math. Soc.* **370** (2018), 99–130.

[39] G. Szabó. Equivariant Kirchberg-Phillips-type absorption for amenable group actions. *Comm. Math. Phys.* **361** (2018), 193–228.

[40] G. Szabó. Actions of certain torsion-free elementary amenable groups on strongly self-absorbing C*-algebras. *Comm. Math. Phys.* **371** (2019), 267–284.

[41] G. Szabó. Corrigendum to "Strongly self-absorbing C*-dynamical systems". *Trans. Amer. Math. Soc.* **373** (2020), 7527–7531.

[42] G. Szabó. Equivariant property (SI) revisited. *Anal. PDE* **14** (2021), 1199–1232.

[43] G. Szabó. On a categorical framework for classifying C*-dynamics up to cocycle conjugacy. *J. Funct. Anal.* **280** (2021), 108927, 66 pp.