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AN ALTERNATIVE CONDITION FOR STOCHASTIC DOMINATION

KENSHI HOSAKA,* *Ritsumeikan University*

Abstract

We will propose an alternative condition for stochastic domination. This condition differs in an essential way from the strong likelihood ratio property. We also show an example which satisfies the new condition, but does not satisfy the strong likelihood ratio property.

Keywords: Strong likelihood ratio property; stochastic domination

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1. Introduction

Let E be a finite set and, for each $e \in E$, let $(\Omega_e, \mathcal{F}_e, \mu_e)$ be a measure space with μ_e a nonnegative σ-finite measure. Suppose that Ω_e is equipped with a total order '$\geq$' that is $\mathcal{F}_t$-measurable. We take as the configuration space the set $\Omega = \prod_{e\in E} \Omega_e$ and the corresponding product σ-algebra $\mathcal{F} = \prod_{e\in E} \mathcal{F}_e$, and we let $\mu = \prod_{e\in E} \mu_e$. The set Ω is a partially ordered set with partial order given by $\omega_1 \leq \omega_2$ if and only if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$. A random variable $X\colon \Omega \to \mathbb{R}$ is called increasing if $X(\omega_1) \leq X(\omega_2)$ whenever $\omega_1 \leq \omega_2$ (see [1, p. 23] or [2, p. 19]). Preston [6] showed that if a pair of probability densities satisfies the condition

$$f_2(\omega_1 \vee \omega_2) f_1(\omega_1 \wedge \omega_2) \geq f_1(\omega_1) f_2(\omega_2), \qquad \omega_1, \omega_2 \in \Omega, \tag{1.1}$$

where $\omega_1 \vee \omega_2 = (\omega_1(e) \vee \omega_2(e)\colon e \in E)$, $\omega_1(e) \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\}$, $\omega_1 \wedge \omega_2 = (\omega_1(e) \wedge \omega_2(e)\colon e \in E)$, and $\omega_1(e) \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\}$, then, for any increasing random variable X on Ω,

$$\int X(\omega) f_2(\omega)\, \mathrm{d}\mu(\omega) \geq \int X(\omega) f_1(\omega)\, \mathrm{d}\mu(\omega).$$

We say that f_1 is stochastically dominated by f_2, written $f_2 \succ_{\mathcal{D}} f_1$ (see [1, p. 23]). We call condition (1.1) the strong likelihood ratio property (see [3] or [4, p. 129]). We also use the following notation for (1.1):

$$f_2 \succ_{TP_2} f_1.$$

In this paper we show an alternative sufficient condition for stochastic domination. However, this sufficient condition is deduced from the strong likelihood ratio property. In Section 2 we give its exact statement. In Section 3 we give a proof of this statement. In Section 4 we give two examples. The first example satisfies our new sufficient condition (and, therefore, satisfies stochastic domination), but it does not satisfy the strong likelihood ratio property (1.1). On the other hand, the second example satisfies the strong likelihood ratio property, but it does not satisfy our new sufficient condition. This example shows that our sufficient condition does not include the strong likelihood ratio property.

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* Postal address: 1-1-1 Noji Higashi, Kusatsu, Shiga, Japan. Email address: khosaka@fc.ritsumei.ac.jp

2. Statement of the result

In this section we give an alternative sufficient condition.

Definition 2.1. Let f_1 and f_2 be probability densities on Ω. We say that f_2 is larger than f_1 in the likelihood difference order, written $f_2 \succ_{\mathrm{ld}} f_1$, if $f_2 - f_1$ is increasing on Ω.

Remark 2.1. A binary relation '$\succ_{\mathrm{ld}}$' is a partial order on the set of all probability densities on Ω.

Theorem 2.1. *Let f_1 and f_2 be probability densities on Ω which satisfy $f_2 \succ_{\mathrm{ld}} f_1$. Then $f_2 \succ_{\mathcal{D}} f_1$.*

Remark 2.2. Definition 2.1 gives a new sufficient condition for stochastic domination (see Example 4.1, below). However, Definition 2.1 is not implied by the strong likelihood ratio property (see Example 4.2, below). Therefore, Definition 2.1 is not a necessary condition for stochastic domination.

3. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1 using the Preston theorem. First, we shall prepare the following notation to prove Theorem 2.1. Let f be a function on Ω given by

$$f(\omega) = f_2(\omega) - f_1(\omega),$$

satisfying $\int f(\omega)\,\mathrm{d}\mu(\omega) = 0$. Note that f is increasing. Let P and N be measurable sets of Ω such that

$$P = \{\omega \in \Omega\colon\ f(\omega) \geq 0\}, \qquad N = \{\omega \in \Omega\colon\ f(\omega) < 0\}.$$

We remark that

$$\int_P f(\omega)\,\mathrm{d}\mu(\omega) = -\int_N f(\omega)\,\mathrm{d}\mu(\omega) \equiv M.$$

We define probability densities f_+ and f_- as follows:

$$f_+(\omega) = \frac{f(\omega) \vee 0}{M}, \quad \omega \in \Omega, \qquad f_-(\omega) = \frac{-f(\omega) \vee 0}{M}, \quad \omega \in \Omega.$$

We note that f_+ is increasing and f_- is decreasing. Therefore, the pair f_+ and f_- satisfies the strong likelihood ratio property

$$f_+(\omega_1 \vee \omega_2) f_-(\omega_1 \wedge \omega_2) \geq f_-(\omega_1) f_+(\omega_2), \qquad \omega_1, \omega_2 \in \Omega.$$

Then, by the Preston theorem [6], we obtain

$$\int X(\omega) f_+(\omega)\,\mathrm{d}\mu(\omega) \geq \int X(\omega) f_-(X)\,\mathrm{d}\mu(\omega)$$

for an arbitrary increasing function X. So, we have

$$\begin{aligned}
&\int X(\omega) f_2(\omega)\,\mathrm{d}\mu(\omega) - \int X(\omega) f_1(\omega)\,\mathrm{d}\mu(\omega)\\
&\quad = \int X(\omega) f(\omega)\,\mathrm{d}\mu(\omega)\\
&\quad = M\left\{\int X(\omega) f_+(\omega)\,\mathrm{d}\mu(\omega) - \int X(\omega) f_-(\omega)\,\mathrm{d}\mu(\omega)\right\}\\
&\quad \geq 0.
\end{aligned}$$

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4. Example

In this section we provide two examples. The first example does not satisfy the strong likelihood ratio property (1.1), but does satisfy the condition of Definition 2.1. The second example is opposite to the first example; it satisfies the strong likelihood ratio property, but does not satisfy Definition 2.1.

Example 4.1. Let μ be the uniform measure on $\Omega = \{0, 1\}^3$. Let f_1 and f_2 be probability densities defined by

$$\begin{aligned} f_2(111) &= \tfrac{1}{8}, \\ f_2(000) &= \tfrac{1}{8} - \delta, \\ f_2(110) = f_2(101) = f_2(011) &= \frac{1}{8} + \frac{\delta}{3}, \\ f_2(100) = f_2(010) = f_2(001) &= \tfrac{1}{8}, \end{aligned}$$

for $\frac{1}{8} \geq \delta > 0$, and define f_1 symmetrically by $f_1(xyz) = f_2([1-x][1-y][1-z])$.

Claim 4.1. *The pair f_1 and f_2 satisfies $f_2 \succ_{\mathrm{ld}} f_1$; however, $f_1 \not\succ_{TP_2} f_2$.*

Proof. It is easy to check the required properties of f_1 and f_2. For example, if we take $\omega_1 = (001)$ and $\omega_2 = (110)$, then we have the following inequality:

$$f_2(\omega_1 \vee \omega_2) f_1(\omega_1 \wedge \omega_2) < f_1(\omega_1) f_2(\omega_2).$$

Example 4.2. Nagahata [5] pointed out the following example. Let $\Omega = \{0, 1\}^2$ and μ be the uniform measure. Let p and q satisfy $\frac{1}{4} \geq p > q \geq 0$. We consider a couple of product Bernoulli densities P_p and P_q. This pair satisfies $P_p \succ_{TP_2} P_q$; however, $P_p \not\succ_{\mathrm{ld}} P_q$. It is easy to check this.

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