Proceedings of the Edinburgh Mathematical Society (2024) **67**, 990–1012
doi:10.1017/S0013091524000245

EQUISINGULARITY IN PENCILS OF CURVES ON GERMS OF REDUCED COMPLEX SURFACES

GONZALO BARRANCO MENDOZA AND JAWAD SNOUSSI

Universidad Nacional Autónoma de México, Instituto de Matemáticas, Unidad Cuernavaca, Morelos, Mexico

Corresponding author: Gonzalo Barranco Mendoza, email: gonzalo@im.unam.mx

(Received 27 February 2023)

Abstract We study pencils of curves on a germ of complex reduced surface $(S, 0)$. These are families of curves parametrized by $\mathbb{P}^1$ having 0 as the unique common point. We prove that for $w \in \mathbb{P}^1$, the corresponding curve of the pencil does not have the generic topology if and only if either the corresponding curve of the pulled-back pencil to the normalized surface has a non generic topology or w is a limit value for the function f/g along the singular locus of $(S, 0)$, where f and g are generators of the pencil.

Keywords: surface singularities; pencils of curves; equisingularity; generically reduced curves

2020 Mathematics subject classification: 32Sxx; 14B05

1. Introduction

Let $(S, 0)$ be a germ of reduced complex surface and call $\mathcal{O}_{S,0}$ its local ring of holomorphic functions. Consider two holomorphic functions f and g in $\mathcal{O}_{S,0}$ such that the ideal $I = \langle f, g \rangle$ is primary for the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{S,0}$. The pencil $\Lambda_{f,g}$ generated by f and g on a representative S of $(S, 0)$ is the family of curves λ_w defined on S by the functions $\beta f - \alpha g$, where $[\alpha : \beta] = w \in \mathbb{P}^1$.

We are interested in characterizing the values $w \in \mathbb{P}^1$ for which the curve λ_w does not have the generic topological behaviour.

This study has been done in the case of pencils of curves on $\mathbb{C}^2$ by Lê and Weber in [13]. That work is highly related to the Jacobian conjecture in the complex plane. They express the genericity of the topological type in terms of the minimality of the Milnor number, and they characterize the values for which the Milnor number of λ_w is not minimal. They start by constructing a resolution of the pencil, which consists of removing the indeterminacy of the meromorphic map f/g by a sequence of point blow-ups. Then, they show that the special curves of the pencil are precisely those curves whose strict transform in the resolution contains some special points; they call them 'special values' of the pencil.

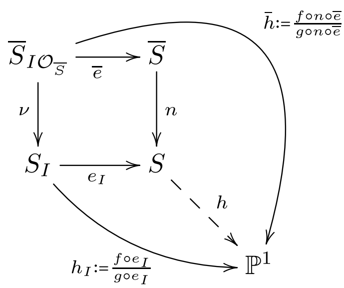
Still in the case of pencils in the complex plane, Maugendre and Delgado gave in [5] a characterization of the special values in terms of the discriminant locus of the finite map $\pi := (f, g) : (S, 0) \to (\mathbb{C}^2, 0)$ associated with the pencil. More precisely, $w \in \mathbb{P}^1$ is special if and only if the line of $\mathbb{C}^2$ corresponding to w is tangent to the discriminant locus of π. Later on, in [6], the same authors generalize this characterization to pencils on normal surfaces. They also give a description of the special values from the minimal resolution of the pencil, in the style of [13].

Bondil and Lê studied in [1] families of curves on normal surfaces. Their results apply for pencils and give a description of the special values.

In the present paper, we consider the case of pencils on germs of reduced complex surfaces. In other words, we allow the surface to have non-isolated singularities and to be non Cohen Macaulay. We proceed to explore the impact of these two allowed phenomena on the topology of elements of pencils of curves.

For a pencil $\Lambda_{f,g}$ on a reduced complex surface S, we will proceed as in [1], [18] and [19] and construct the blow-up $e_I : S_I \to S$ of the ideal I generated by f and g. This modification removes the indeterminacy of the quotient map $h := f/g$. Therefore, we obtain a basepoint-free family of curves with a well-defined map to $\mathbb{P}^1$. For $w \in \mathbb{P}^1$, the curve λ_w does not have the generic topological behaviour if and only if the family of curves $\frac{f\circ e_I}{g\circ e_I} : S_I \to \mathbb{P}^1$ is not topologically trivial at $(0, w)$.

Naturally, the normalization $n : (\overline{S}, \overline{0}) \to (S, 0)$ gives rise to pencils $\Lambda_{f\circ n, g\circ n}$ on each germ of the multi-germ of normal surface $(\overline{S}, \overline{0})$. By blowing-up the ideal $I\mathcal{O}_{\overline{S},\overline{0}} = \langle f \circ n, g \circ n\rangle$, we obtain the following commutative diagram:



This diagram will allow us to compare the topology of the curves in S with the one in $\overline{S}$ and the family of curves in S_I with the one in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$.

Our main result, stated in Theorem 4.1, says that the topology of an element λ_w of the pencil on S is different from the generic one if and only if either the corresponding curve in the normalized surface $\overline{S}$ does not have the generic topology or the point $(0, w)$ of S_I belongs to the strict transform of the singular locus of S by e_I.

This result gives a complete characterization of curves with non generic topology, for pencils on reduced complex surfaces. Indeed, we know how to characterize non-generic curves of a pencil on a normal surface, and we know that points $(0, w)$ of the strict transform of the singular locus of S by e_I correspond to limit values of f/g at 0 along a component of the singular locus of the surface S.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

As a direct consequence of our result, one can notice that for a non-normal surface with an isolated singularity, the topology of the elements of a pencil changes exactly at the same values as for the pulled-back pencil on the normalized surface.

In the process of the proof, it was important to distinguish two types of components of the singular locus $(\Sigma, 0)$ of a surface $(S, 0)$. Namely, let Σ_0 be a component of the singular locus of S and consider the restriction of the normalization $n_0 := n_{|n^{-1}(\Sigma_0)} : n^{-1}(\Sigma_0) \to \Sigma_0$. When n_0 is generically r to 1, we say that Σ_0 is an r to 1 component of the singular locus of S. When Σ_0 is a one-to-one component of the singular locus, we prove that the surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is not topologically trivial at least at one point lying over the intersection point of the exceptional divisor with the strict transform of Σ_0 by e_I. Meanwhile, when Σ_0 is an r to 1 component with $r > 1$, we prove the statement directly on S_I without checking topological triviality on the surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$.

In the first section, we mention generalities on pencils of curves on complex surfaces, such as resolution of pencils and the relation between the pencil and its pull-back to the normalized surface. We also explain how the blow-up of the ideal $\langle f, g\rangle$ generated by the generators f and g of the pencil removes the indeterminacy of the quotient map f/g.

In the second section, in order to simplify the main statement, we define special values of the pencil at the normalization. These are the values w at which the pull-back of the curve λ_w to the normalized surface has a topology different from the generic one. Then, we define what we call special values of a pencil on a reduced surface. These are the special values at the normalization and the values w corresponding to an intersection point $(0, w)$ of the exceptional divisor $e_I^{-1}(0)$ with the strict transform of the singular locus of $(S, 0)$ by e_I.

Section three is dedicated to the proof of the main result. It has been cut into several lemmas dealing with different situations. Finally, we give a series of examples in the last section, illustrating situations that arise along the proof.

2. Pencils

Consider a germ of reduced complex surface $(S, 0)$ with local ring of holomorphic functions $\mathcal{O}_{S,0}$ and call $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{S,0}$. For the rest of this work, S will denote a sufficiently small representative of $(S, 0)$.

Definition 2.1. *Consider two holomorphic functions $f, g \in \mathfrak{m} \subset \mathcal{O}_{S,0}$ such that the ideal $\langle f, g\rangle$ is $\mathfrak{m}$-primary, i.e., $\sqrt{\langle f, g\rangle} = \mathfrak{m}$. Define the curve $\lambda_{[\alpha:\beta]} \subset S$, $[\alpha : \beta] \in \mathbb{P}^1$, to be the zero set on S of the function $\alpha f - \beta g \in \mathcal{O}_{S,0}$, with the ring of holomorphic functions $\mathcal{O}_{[\alpha:\beta]} = \mathcal{O}_{S,0}/\langle \alpha f - \beta g\rangle$.*

The pencil generated by f and g on S is the set:

$$\Lambda_{f,g} = \{\lambda_w \subset S \,|\, w \in \mathbb{P}^1\}.$$

Notice that the condition $\sqrt{\langle f, g\rangle} = \mathfrak{m}$ implies that, in a sufficiently small representative S of $(S, 0)$, the only common point of all the elements λ_w of $\Lambda_{f,g}$ is the origin. In particular, the curves defined by f and g do not have any common branch.

We associate with such a pencil a function with an indeterminacy at the origin.

Definition 2.2. *Given a pencil $\Lambda_{f,g}$ on S, we define the associated map:*

$$\begin{array}{rcl} h_{f,g} : S \setminus \{0\} & \rightarrow & \mathbb{P}^1 \\ x & \longmapsto & [g(x) : f(x)]. \end{array}$$

For simplicity, and when there is no ambiguity, we will sometimes refer to the map $h_{f,g}$ as h.

The elements of the pencil can be compared with fibres of $h_{f,g}$ in the sense that the associated reduced curves are the same:

$$|\lambda_w| = |\overline{h_{f,g}^{-1}(w)}|;$$

notice that, depending on the surface $(S, 0)$, the curve λ_w may have an embedded component at the origin.

A resolution of the pencil $\Lambda_{f,g}$ consists of removing both the indeterminacy of the map $h_{f,g}$ and the singularities of the surface $(S, 0)$.

Let us recall that a modification $\mu : X \rightarrow S$ of a representative S of $(S, 0)$ is a proper map that induces an isomorphism outside a proper closed nowhere dense subspace of S. It is a resolution of singularities of S when X is smooth.

Definition 2.3. *A modification $r : S' \rightarrow S$ is a resolution of the pencil $\Lambda_{f,g}$ if r is a resolution of singularities of S and $r \circ h$ extends to a well-defined map over S'.*

In this situation, if $w \neq w' \in \mathbb{P}^1$, then the $\overline{r^{-1}(\lambda_w \setminus \{0\})}$ and $\overline{r^{-1}(\lambda_{w'} \setminus \{0\})}$ do not intersect. A resolution of the pencil is then a modification that removes the singularities of S and separates the curves λ_w for all $w \in \mathbb{P}^1$.

Definition 2.4. *A resolution $\rho : \hat{S} \rightarrow S$ of $\Lambda_{f,g}$ is minimal if for any other resolution $r : S' \rightarrow S$ of $\Lambda_{f,g}$ there exists a holomorphic map $\tau : S' \rightarrow \hat{S}$, such that $r = \rho \circ \tau$.*

It is well known that any surface singularity admits a minimal resolution which is unique up to isomorphism; see, for example [12, Theorem 5.9] . Also, in [13, Proposition 2.2], it is proved that any pencil of curves in a small neighbourhood of a point in $\mathbb{C}^2$ has a minimal resolution which is also unique up to isomorphism. A combination of these two assertions leads to:

Proposition 2.5. *Any pencil $\Lambda_{f,g}$ on a surface singularity admits a minimal resolution which is unique up to isomorphism.*

Since the map $h_{f,g} \circ r$ extends to a well-defined map h_r whenever $r : S' \rightarrow S$ is a resolution of the pencil $\Lambda_{f,g}$, we can distinguish two types of components of the exceptional locus $r^{-1}(0)$, the ones where h_r is constant and those where it is not.

Definition 2.6. *Call $E = \bigcup_i E_i$ the decomposition into irreducible components of the exceptional curve $r^{-1}(0)$. We say that E_i is a dicritical component if $h_r|_{E_i}$ is not constant.*

If E_i is a dicritical component of a resolution r of the pencil $\Lambda_{f,g}$ then for any point $p \in E_i$ there exists a unique $w \in \mathbb{P}^1$ such that $p \in \overline{r^{-1}(\lambda_w \setminus \{0\})}$. Conversely, if E_i is dicritical, the image of $h_r|_{E_i}$ is $\mathbb{P}^1$ (since the components E_i are connected). This implies that for any $w \in \mathbb{P}^1$ the strict transform of λ_w by r intersects E_i in at least one point.

Let us now consider a particular modification: the blow-up of the ideal I generated by the functions f and g. Consider the map:

$$\begin{array}{rcl} l_{f,g} : S \setminus \{0\} & \to & \mathbb{P}^1 \\ x & \mapsto & [g(x) : f(x)]. \end{array}$$

Define the surface S_I to be the closure of the graph of $l_{f,g}$ in $S \times \mathbb{P}^1$. Call e_I the restriction to S_I of the projection onto S. The map $e_I : S_I \to S$ is the blow-up of the ideal I in a representative S of the germ $(S, 0)$.

This blow-up satisfies the following universal property with respect to the pencil $\Lambda_{f,g}$: a modification μ of the surface S removes the indeterminacy of the map $h_{f,g}$ induced by a pencil of curves, if and only if it factors through the blow-up of the ideal $\langle f, g\rangle$. It is a direct consequence of the universal property of the blow-up (see [11, Proposition 7.14]).

We have in a more general setting:

Proposition 2.7. *Let $\mu : Y \to X$ be a modification map between reduced complex analytic spaces over a neighbourhood of $0 \in X$. Consider holomorphic functions $f_0, \ldots, f_r \in \mathcal{O}_{X,0}$, for which the common zero locus V is a nowhere dense subset of X. Define the map*

$$\begin{array}{rcl} h : X \setminus V & \to & \mathbb{P}^r \\ x & \mapsto & [f_0(x) : \ldots : f_r(x)], \end{array}$$

whose indeterminacy locus is V.

Then, the composition map $h \circ \mu$ has no indeterminacy on Y if and only if the modification μ factors through the blow-up of the ideal $\langle f_0, \ldots, f_r\rangle$ in X.

In the case of a pencil of curves $\Lambda_{f,g}$ on a representative of a surface singularity $(S, 0)$, we have then:

Corollary 2.8. *Let $e_I : S_I \to S$ be the blow-up of the ideal $I = \langle f, g\rangle$. The projection from S_I to $\mathbb{P}^1$ is an extension of $h_{f,g} \circ e_I$ on the surface S_I.*

Remark 2.9. When the germ of surface $(S, 0)$ is not Cohen–Macaulay, whenever we consider a pencil $\Lambda_{f,g}$, the pair (f, g) is not a regular sequence in $\mathcal{O}_{S,0}$. Therefore, the surface S_I obtained by the blowing-up e_I of the ideal generated by f and g need not be given by the equation $gt - fs = 0$ in $S \times \mathbb{P}^1$, where $[s : t]$ is a system of homogeneous coordinates in $\mathbb{P}^1$.

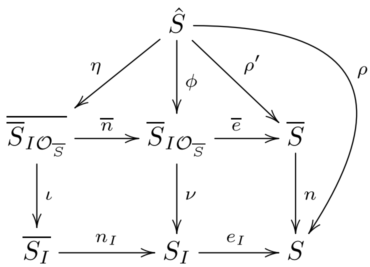
Indeed, let us call $S_1 \subset S \times \mathbb{P}^1$ the surface defined by the equation $gt - fs = 0$. Call π_1 the restriction to S_1 of the projection to $\mathbb{P}^1$. One can easily see that for any $w \in \mathbb{P}^1$, the fibre $\pi_1^{-1}(w)$ is isomorphic to the curve $\lambda_w \subset S$. Since $(S, 0)$ is not Cohen-Macaulay, the curve λ_w has an embedded component at the origin for any $w \in \mathbb{P}^1$. This implies that

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

all the fibres $\pi_1^{-1}(w)$ have an embedded component in the point $(0, w) \in S_1$. The surface S_1 has then an embedded component along the curve $\{0\} \times \mathbb{P}^1$. By construction of the blow-up e_I, the surfaces S_1 and S_I coincide outside $\{0\} \times \mathbb{P}^1$, and S_I is reduced whenever $(S, 0)$ is, so S_I is the reduced surface obtained from S_1.

Corollary 2.8 asserts that the blow-up of the ideal $\langle f, g\rangle$ is the easiest way to separate the elements of the pencil $\Lambda_{f,g}$ and remove the indeterminacy of the map $h_{f,g}$. Theorem 2.10 explains this minimality property and relates it to the minimal resolution of the pencil.

Theorem 2.10. *Given a pencil $\Lambda_{f,g}$ on a representative S of a reduced complex surface singularity $(S, 0)$, we have the following commutative diagram:*



where n is the normalization of S, $\rho = n \circ \rho' : \hat{S} \to S$ is the minimal resolution of the pencil $\Lambda_{f,g}$, e_I is the blow-up of the the ideal $I = \langle f, g\rangle$, and $\overline{e}$ is the blow-up of the pull-back ideal $I\mathcal{O}_{\overline{S}}$. The maps n_I and $\overline{n}$ are respectively the normalizations of S_I and $\overline{S}_{I\mathcal{O}_{\overline{S}}}$.

The map ν is finite, the map ι is an isomorphism, and η is the minimal resolution of the singularities of the normal surface $\overline{\overline{S}_{I\mathcal{O}_S}}$, which is in this case the contraction of all the non-dicritical components of the minimal resolution ρ of the pencil $\Lambda_{f,g}$.

Proof. The minimal resolution ρ of the pencil $\Lambda_{f,g}$ factors naturally through the normalization $n : \overline{S} \to S$. We then have $\rho = n \circ \rho'$ where $\rho' : \hat{S} \to \overline{S}$ is the minimal resolution of the pencil $\Lambda_{f\circ n, g\circ n}$ defined on $\overline{S}$ at each point of $n^{-1}(0)$.

By definition of a resolution of $\Lambda_{f,g}$, the map $h_{f,g} \circ \rho = h_{f\circ n, g\circ n} \circ \rho'$ extends to a well-defined map $\hat{h} : \hat{S} \to \mathbb{P}^1$. Applying Proposition 2.7, there exists a map $\phi : \hat{S} \to \overline{S}_{I\mathcal{O}_{\overline{S}}}$ such that $\rho' = \overline{e} \circ \phi$.

Since $\hat{S}$ is a smooth surface, the map ϕ factors through the normalization $\overline{n}$ of $\overline{S}_{I\mathcal{O}_{\overline{S}}}$, so there exists $\eta : \hat{S} \to \overline{\overline{S}_{I\mathcal{O}_{\overline{S}}}}$ such that $\phi = \overline{n} \circ \eta$, and η is the minimal resolution of the singularities of $\overline{\overline{S}_{I\mathcal{O}_{\overline{S}}}}$.

Let us explain the existence and finiteness of the map ν. By Proposition 2.7, the composition map $h_{f,g} \circ n \circ \overline{e}$ extends to a well-defined map $\overline{h}$ on $\overline{S}_{I\mathcal{O}_{\overline{S}}}$. Again by Proposition 2.7, there exists a map ν such that $e_I \circ \nu = n \circ \overline{e}$. Moreover, the blow-up $\overline{e}$ is the blow-up of the pull-back of the ideal I by the normalization n; therefore, the map ν is the pull-back of the normalization n by e. The normalization being finite, ν is also a finite map.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

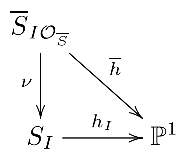
In order to prove that there exists an isomorphism $\iota : \overline{\overline{S}_{I\mathcal{O}_{\overline{S}}}} \to \overline{S_I}$, it is enough to notice that $\nu \circ \overline{n}$ is a finite modification of S_I and that $\overline{\overline{S}_{I\mathcal{O}_{\overline{S}}}}$ is normal; hence, the composition $\nu \circ \overline{n}$ coincides with the normalization n_I up to the isomorphism ι.

Now let us prove our assertion on η. Consider an irreducible component E_i of the exceptional divisor $\rho^{-1}(0)$. If E_i is dicritical for the pencil $\Lambda_{f\circ n, g\circ n}$, then the image $\eta(E_i)$ cannot be a point; otherwise, two (actually an infinity) different elements of the pencil will have their strict transforms by $\overline{e} \circ \overline{n}$ intersecting in a point, which contradicts the fact that $h_{f,g} \circ n \circ \overline{e} \circ \overline{n}$ extends to a well-defined map.

If E_i is not dicritical, we are going to prove that its image by η is a point. Indeed, the map $\overline{h} : \overline{S}_{I\mathcal{O}_{\overline{S}}} \to \mathbb{P}^1$ is not constant on any component of the exceptional divisor. Since the normalization $\overline{n}$ is finite, the map $\overline{h} \circ \overline{n}$ is not constant on any component of the exceptional fibre of $\overline{e} \circ \overline{n}$. However, the extended map $\hat{h} : \hat{S} \to \mathbb{P}^1$ is constant over E_i. Therefore, by the commutativity of the above diagram, the map $\overline{h} \circ \overline{n}$ is constant on the image $\eta(E_i)$ which cannot then be one dimensional. So the non dicritical component E_i contracts to a point by η. $\square$

Theorem 2.10 will allow us to compare the pencils $\Lambda_{f,g}$ on S and $\Lambda_{f\circ n, g\circ n}$ on $\overline{S}$ and their respective special values.

Remark 2.11. The pencil $\Lambda_{f\circ n, g\circ n}$ has one basepoint for each irreducible component of $(S, 0)$ corresponding to a point $O_i \in n^{-1}(0)$. Then, the exceptional divisor of the blow-up $\overline{e} : \overline{S}_{I\mathcal{O}_{\overline{S}}} \to \overline{S}$ has one connected component $\overline{E}_i = \overline{e}^{-1}(O_i)$ for each basepoint of $\Lambda_{f\circ n, g\circ n}$. The restriction $\nu_i : \overline{E}_i = \{O_i\} \times \mathbb{P}^1 \to S_I$ of ν induces the identity map on $\mathbb{P}^1$. Indeed, the extension h_I of $h_{f,g} \circ e_I$ restricted to the exceptional divisor $e_I^{-1}(0)$ induces the identity map on $\mathbb{P}^1$, and the same is true for the extension $\overline{h}$ of $h_{f,g} \circ n \circ \overline{e}$ restricted to $\overline{E}_i$. This statement is a consequence of the commutativity of the diagram:



3. Special values

Our goal is to understand the generic and particular topological behaviour of the elements of a pencil of curves on a germ of complex surface. We will relate them to the behaviour of the strict transforms of the elements of the pencil, either on the minimal resolution of the pencil or on the blow-up of the ideal generated by the generators of the pencil.

In the case of a pencil $\Lambda_{f,g}$ on $\mathbb{C}^2$ or on a normal surface, the change of the topology of the curves was measured by the Milnor number at the origin $\mu(\lambda_w, 0)$ and related to topological triviality on the blown-up surface $S_{\langle f,g\rangle}$. Special values on $\mathbb{P}^1$ are defined in terms of the behaviour of the fibres of $h_{f,g}$ over these values in the minimal resolution. It is then proved that these are precisely the values where the topology changes. We refer

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

to [13] for $\mathbb{C}^2$-case, [18] for a particular case on normal surfaces, [1] for linear systems on normal surfaces and [6] for pencils on normal surfaces.

Following that strategy, we will start by defining special values for a pencil of curves on a germ of a not necessarily normal complex surface $(S, 0)$.

Definition 3.1. *Let $\rho : \hat{S} \to S$ be the minimal resolution of a pencil $\Lambda_{f,g}$. Take E to be the exceptional divisor of the minimal resolution of the normalized surface $\overline{S}$, $\rho' : \hat{S} \to \overline{S}$. Let $E = \cup E_i$ be the decomposition of E into irreducible components. Consider the extension $\hat{h}$ of $h_{f,g} \circ \rho$ to the whole surface $\hat{S}$.*

We say that $w \in \mathbb{P}^1$ is a special value for the pencil $\Lambda_{f,g}$ in the normalization if one or more of the following are fulfilled:

(1) $w = \hat{h}(x)$ *with* $x \in E_i \cap E_j$, $i \neq j$.
(2) $w = \hat{h}(E_i)$ *with* E_i *a non-dicritical component.*
(3) $w = \hat{h}(x)$ *with* x *a critical point of* $\hat{h}|_{E_i}$ *for a dicritical component* E_i.

If w is not special at the normalization, then we say it is a generic value at the normalization.

In other words, a value $w \in \mathbb{P}^1$ is special at the normalization for a pencil $\Lambda_{f,g}$ on a surface S if and only if it is a special value for the pulled-back pencil $\Lambda_{f\circ n, g\circ n}$ on the normal surface $\overline{S}$; see [13], [1] and [6].

Remark 3.2. Since ρ is a resolution of the pencil, there is at least one dicritical component in every connected component of the exceptional divisor by ρ. Therefore, if E_i is a non-dicritical component, it will intersect either a dicritical component or another non-dicritical one. The value $\hat{h}(E_i)$ will be equal to $\hat{h}(x)$ where x is an intersection point of two irreducible components of the exceptional divisor.

The values in the second point of the definition are included in the ones of the first point. However, we prefer to refer to them separately.

When the surface $(S, 0)$ has a one-dimensional singular locus Σ, we need to consider the branches of Σ in the definition of special values of a pencil.

Definition 3.3. *We say that w is a special value for the pencil $\Lambda_{f,g}$ on a representative S of $(S, 0)$ if:*

(1) *w is a special value in the normalization,*
(2) *or* $w = \hat{h}(x)$ *where* $x \in \overline{\rho^{-1}(\Sigma \setminus \{0\})} \cap E$.

When $w \in \mathbb{P}^1$ is not special, we say it is generic for the pencil $\Lambda_{f,g}$ on S.

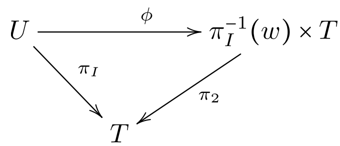
Note that in the case of a germ of non-normal surface with an isolated singularity, the special values of a pencil on the surface coincide with the special values at the normalization.

When the surface is not irreducible, its normalization is a disjoint union of normal germs. The special values at the normalization need then to be considered as special values of pencils on each connected component of the normalized surface.

Our main tool for detecting a change in the topology of the elements of a pencil is the topological triviality in a one-parameter flat family of curves. This is to be considered on the surface obtained after the blow-up of the ideal $I = \langle f, g\rangle$. The reason is that the surface S_I obtained by that blow-up is the minimal one where the elements of the pencil can be viewed as fibres of a well-defined map onto $\mathbb{P}^1$.

Recall that $e_I : S_I \to S$ is the blow-up of $I = \langle f, g\rangle$ and call $\pi_I : S_I \to \mathbb{P}^1$ the induced projection onto $\mathbb{P}^1$ which coincides with the extension h_I of $h_{f,g} \circ e_I$ to S_I.

Definition 3.4. *Let $(0, w) \in e_I^{-1}(0)$. We say that the pencil $\lambda_{f,g}$ is topologically trivial at w if there exist neighbourhoods $(0, w) \in U \subset S_I$ and $w \in T \subset \mathbb{P}^1$ such that the induced family of curves $\pi_I : U \to T$ is a topologically trivial family of curves, i.e., there exists a homeomorphism $\phi : U \to \pi_I^{-1}(w) \times T$ which makes the following diagram commutative:*



where π_2 is the projection onto the second factor of $\pi_I^{-1}(w) \times T$.

In that case, we also say that the surface S_I is topologically trivial at $(0.w)$.

When the surface $(S, 0)$ is Cohen-Macaulay, the blow-up of an ideal generated by a regular sequence gives rise to a Cohen-Macaulay surface. However, when $(S, 0)$ is not Cohen-Macaulay, the blown-up surface S_I need not be Cohen-Macaulay at all points of the exceptional fibre, as shown in Example 5.3. Therefore, some of the fibres $\pi_I^{-1}(w)$ may have an embedded component at the point $(0, w)$. We need then to use equisingularity criteria that consider curves with allowed embedded components. When a curve is possibly with an embedded component and has isolated singularities, we call it a generically reduced curve. We are going to use, along this work, equisingularity criteria on flat families of generically reduced curves as it was done in [7], [17] and [9].

4. Main results

In this section, we will state and prove our main result, which establishes the equivalence between generic values and values where the pencil is topologically trivial, for germs of complex surfaces.

Theorem 4.1. *A value $w \in \mathbb{P}^1$ is a special value for a pencil $\Lambda_{f,g}$ on a representative S of a germ of reduced complex surface $(S, 0)$, if and only if $\Lambda_{f,g}$ is not topologically trivial at w.*

For convenience, we will split the proof into a series of lemmas dealing with different types of special values and types of fibres.

Consider a pencil $\Lambda_{f,g}$ on a representative S of a germ of reduced complex surface $(S, 0) \subset (\mathbb{C}^N, 0)$. Call $\Sigma = \cup_i \Sigma_i$ the decomposition of the singular locus of S into irreducible components. Consider $\rho : \hat{S} \to S$ the minimal resolution of the pencil $\Lambda_{f,g}$ and call $\hat{h}$ the extension of $h = [g : f]$ to $\hat{S}$.

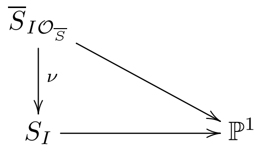
https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

We first consider special values at the normalization:

Lemma 4.2. *Suppose $w \in \mathbb{P}^1$ is a special value in the normalization for $\Lambda_{f,g}$ that is not the image by $\hat{h}$ of an $x \in \Sigma^* \cap E \subseteq \hat{S}$, intersection of $\Sigma^* = \overline{\rho^{-1}(\Sigma \setminus \{0\})}$ with $E = \rho^{-1}(0)$ and suppose that the curve λ_w is generically reduced.*

Then, $\Lambda_{f,g}$ is not topologically trivial at w.

Proof. Suppose first that S is irreducible. Then, the inverse image $n^{-1}(0)$ is a single point, the pull-back by the normalization n of the pencil $\Lambda_{f,g}$ has only one basepoint and the exceptional fibre of $\overline{e}$ is isomorphic to $\mathbb{P}^1$. Since the following diagram commutes:



the morphism ν restricted to the exceptional fibre is the identity. If w satisfies the conditions of the lemma, then there exist neighbourhoods $\overline{V}$ and V of $(0, w)$, respectively, in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ and in S_I where ν induces a homeomorphism. By hypothesis, w is a special value at the normalization, so by [6, Theorem 4] the Milnor number changes at this point. Moreover, the curve $\lambda_w \subset S$ is assumed to be generically reduced, and its strict transform in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is therefore reduced. By [3, Theorem 5.2.2], the surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is not topologically trivial at $(0, w)$ and neither is S_I.

Suppose now that $S = \cup_i S_i$ is a decomposition of S into irreducible components. The normalization $\overline{S}$ of S is a disjoint union

$$\overline{S} = \bigsqcup_i \overline{S}_i,$$

where each surface $\overline{S}_i$ is the normalization of S_i.

The pencil $\Lambda_{f,g}$ induces a pencil $\Lambda^i_{f,g}$ on each component S_i. The surface S_I obtained by the blow-up of $I = \langle f, g \rangle$ is a union:

$$S_I = \bigcup_i S_{i,I},$$

where $S_{i,I}$ is the surface obtained by the blow-up of the ideal $I\mathcal{O}_{S_i}$. Note that the exceptional fibre $\{0\} \times \mathbb{P}^1$ is contained in each component $S_{i,I}$.

The morphism $\nu : \overline{S}_{I\mathcal{O}_{\overline{S}}} \to S_I$ induces morphisms:

$$\nu_i : (\overline{S_i})_{I\mathcal{O}_{\overline{S_i}}} \to S_{i,I}.$$

When w is as in the hypothesis of the lemma, we have seen in the proof of the irreducible case that there exists an index i_0 such that neither $(\overline{S_{i_0}})_{I\mathcal{O}_{\overline{S_{i_0}}}}$ nor $S_{i_0,I}$ is topologically trivial at $(0, w)$.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

We claim then that S_I is not topologically trivial at $(0, w)$. Indeed, if S_I is topologically trivial at $(0, w)$, then there exist neighbourhoods W and U of $(0, w)$ in S_I and of w in $\mathbb{P}^1$, respectively, and a homeomorphism $\phi : W \to \lambda_w \times U$. The homeomorphism ϕ sends every irreducible component of W onto $\lambda_{j,w} \times U$, where $\lambda_{j,w}$ is a branch of λ_w. Each of the intersections $S_{i,I} \cap W$ is a union of irreducible components of W. Therefore, ϕ induces a homeomorphism between $W \cap S_{i,I}$ and $\cup_{j \in J_i} \lambda_{j,w}$, proving that each of the surfaces $S_{i,I}$ is topologically trivial at $(0, w)$. This is a contradiction to the statement in the previous paragraph, and so S_I is not topologically trivial at $(0, w)$. □

In a second step, we propose to deal with points in S_I belonging to the strict transform of the singular locus.

For that purpose, we need to distinguish between two different types of branches of the singular locus of the surface S:

Definition 4.3. *Consider the normalization $n : \overline{S} \to S$ of S. A branch Σ_0 of the singular locus Σ of S is called an r to 1 component of Σ if the normalization is generically r to 1 over Σ_0, with $r \geq 1$.*

We will need to deal separately with the 1 to 1 branches and the r to 1 ones when $r > 1$.

When Σ_0 is an r to 1 component, with $r > 1$ and $\Sigma_0 \subseteq \lambda_w$, the pull-back $\overline{\lambda_w}$ by the normalization may be reduced and the value w may be generic in the normalization (see Example 5.2). But we still can prove that in this case S_I is not topologically trivial at $(0, w)$:

Lemma 4.4. *Let Σ_0 be an r to 1 component of the singular locus of S, with $r > 1$. Suppose there exists a $w \in \mathbb{P}^1$ such that $\Sigma_0 \subseteq \lambda_w$. Then, the surface S_I is not topologically trivial at $(0, w)$.*

Proof. Let Σ_0^* be the strict transform of Σ_0 by the blow-up e_I. From Theorem 2.10, we have the following diagram:

$$\begin{array}{ccccc} \overline{S_I} & \xrightarrow{\overline{n}} & \overline{S}_{I\mathcal{O}_{\overline{S}}} & \xrightarrow{\overline{e}} & \overline{S} \\ & \searrow^{n_I} & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle n} \\ & & S_I & \xrightarrow{e_I} & S \end{array}$$

From the commutativity of the right square we have that a generic point in Σ_0^* has exactly r pre-images by ν. Since the normalization map $\overline{n}$ is an isomorphism over $\overline{S}_{I\mathcal{O}_{\overline{S}}} \setminus \overline{e}^{-1}(n^{-1}(0))$, then by commutativity of the left side of the diagram, a generic point of Σ_0^* has exactly r pre-images by n_I. In other words, for any $x \in \Sigma_0^*$ sufficiently close to $e_I^{-1}(0) \cap \Sigma_0^*$, the surface S_I is not irreducible at x.

On the other hand, if S_I is topologically trivial at $(0, w)$, and since the curve λ_w is irreducible at its generic points, then there is a sufficiently small neighbourhood V of x in S_I which is homeomorphic to a poly-disk of $\mathbb{C}^2$ which is everywhere irreducible. So the surface S_I cannot be topologically trivial at the point $(0, w)$. □

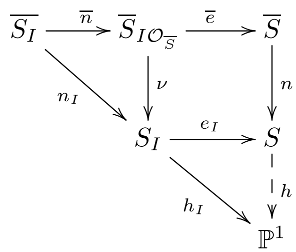
https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

When the r to 1 component of the singular locus is not contained in an element of the pencil, we have:

Lemma 4.5. *Let Σ_0 be an r to 1 branch of the singular locus of S, with $r > 1$, that is not contained in any element λ_w of the pencil. Call $(0, w) \in S_I$ the intersection point of the exceptional divisor $e_I^{-1}(0)$ with the strict transform Σ_0^* of Σ_0 by e_I. Then, the surface S_I is not topologically trivial at $(0, w)$.*

Proof. Recall that the surface $(S, 0) \subset (\mathbb{C}^N, 0)$ is not assumed to be irreducible. Therefore, the normalized surface $\overline{S}$ may be a multi-germ $\sqcup_i(\overline{S}_i, O_i)$, and the blown-up surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is also a disjoint union of surfaces $\overline{S}^{(i)}_{I\mathcal{O}_{\overline{S}}}$ each of them obtained by the blow-up of the ideal $\langle f \circ n, g \circ n\rangle$ in a representative $\overline{S}_i$ of the respective germ $(\overline{S}_i, O_i)$.

We claim that for each index i, there exists an open neighbourhood U_i of (O_i, w) in $\overline{S}_i \times \mathbb{P}^1$ such that the fibre $(h_I \circ \nu)^{-1}(w) \cap U_i$ is contractible and for any $t \in \mathbb{P}^1$, the fibre $(h_I \circ \nu)^{-1}(t) \cap U_i$ is connected; see the commutative diagram below for notation:



In fact, for the contractibility of $(h_I \circ \nu)^{-1}(w) \cap U_i$ it is enough to choose each U_i small enough. Consider now, for each germ $(\overline{S}_i, O_i)$, the projection $\overline{\pi}_i := (f \circ n, g \circ n) : (\overline{S}_i, O_i) \to (\mathbb{C}^2, 0)$. By abuse of notation, we will call $\overline{\pi}_i$ the projection induced on a representative V_i of $(\overline{S}_i, O_i)$. Since the projection $\overline{\pi}_i$ is finite, we can choose the representative V_i small enough so that $\overline{\pi}_i^{-1}(0) = \{O_i\}$.

Let us denote by $\Lambda^{(i)}$ the pencil induced by $\Lambda_{f,g}$ on $\overline{S}_i$. Every element of this pencil is the inverse image by $\overline{\pi}_i$ of a line through the origin in $\mathbb{C}^2$. For $t \in \mathbb{P}^1$ call $0 \in L_t$ the line in $\mathbb{C}^2$ with slope t. Since $\overline{\pi}_i^{-1}(0) \cap V_i = \{O_i\}$, the curve $\lambda_t^i := \overline{\pi}_i^{-1}(L_t) \cap V_i$ is connected. If we choose U_i to be an open set contained in $V_i \times \mathbb{P}^1$, then the fibre $(h_I \circ \nu)^{-1}(t) \cap U_i$, which is isomorphic to its image in V_i, is connected as claimed above.

We will now prove that since Σ_0 is an r to 1 component of the singular locus, then for any $t \in \mathbb{P}^1$ close enough to w, the fibre $h_I^{-1}(t)$ contains a cycle.

Indeed, the curve Σ_0 is not contained in any element of the pencil $\Lambda_{f,g}$. The function h_I is then non-constant on the strict transform Σ_0^* of Σ_0 by e_I. We can then choose neighbourhoods U of 0 in $\mathbb{C}^N$ and T of w in $\mathbb{P}^1$ such that for any $t \in T$ we have

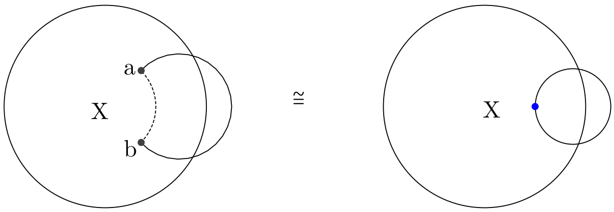
$$h_I^{-1}(t) \cap (U \times T) \cap \Sigma_0^* \neq \emptyset \neq h_I^{-1}(t) \cap (U \times T) \cap e_I^{-1}(0).$$

We will furthermore require that $S_I \cap (U \times T)$ is contained in the image by ν of each component $\overline{S}_{I\mathcal{O}_{\overline{S}}} \cap U_i$ defined above.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

Let us fix a $t \in T \setminus \{w\}$. We will call $\{x\} := h_I^{-1}(t) \cap (U \times T) \cap e_I^{-1}(0)$ and choose $y \in h_I^{-1}(t) \cap (U \times T) \cap \Sigma_0^*$. Since Σ_0 is an r to 1 component, with $r \geq 2$, there exist at least two points, y_1 and y_2 in $\nu^{-1}(y)$. These two points may be in the same open set U_0 defined in the first part of this proof or in different open sets U_1 and U_2.

In the first case, call x_0 a point in $\nu^{-1}(0) \cap U_0$ and recall that the fibre $(h_I \circ \nu)^{-1}(t) \cap U_0$ is connected. There exist then two continuous paths connecting x_0 to y_1 and x_0 to y_2. The image of these two paths is a non-trivial loop in the fibre $h_I^{-1}(t) \cap (U \times T)$. Indeed, for any topological space X, identifying two points a, b results in a space that is homotopically equivalent to the original space with a loop attached:



Then, by an appropriate decomposition and using Van Kampen theorem, we see that the original space and the one with the loop are not homeomorphic.

In the second case, choose two points $x_1 \in \nu^{-1}(x) \cap U_1$ and $x_2 \in \nu^{-1}(x) \cap U_2$. Again the fibres $(h_I \circ \nu)^{-1}(t) \cap U_i$ are connected for $i = 1, 2$. Choose two paths on the fibre $(h_I \circ \nu)^{-1}(t)$, one in U_1 connecting x_1 to y_1 and the other in U_2 connecting x_2 to y_2. The image of these two paths by ν is again a non-trivial loop in $h_I^{-1}(t) \cap (U \times T)$.

In both cases, the fibre $h_I^{-1}(t) \cap (U \times T)$ is not contractible; meanwhile, the fibre $h_I^{-1}(w) \cap (U \times T)$ is contractible. The surface S_I is, therefore, not topologically trivial at $(0, w)$. □

When Σ_0 is a one-to-one branch of the singular locus of S, the preceding argument does not hold anymore. But still we are going to prove that such a component produces a special value at the normalization.

Consider for that a finite map $\pi : S \to \mathbb{C}^2$, defined on a sufficiently small representative of $(S, 0)$. Recall that we denote by $n : \overline{S} \to S$ the normalization of S.

Lemma 4.6. *Let Σ_0 be a one-to-one component of the singular locus of S. Then, the inverse image $n^{-1}(\Sigma_0)$ is contained in the critical locus of $\pi \circ n$.*

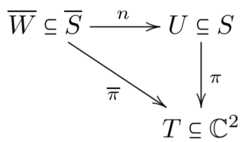
Proof. Take a point $p \in n^{-1}(\Sigma_0)$. If p is a singular point of $\overline{S}$, then by definition it is in the critical locus of $\pi \circ n$. Then, we can assume that p is not singular.

Since Σ_0 is a 1 to 1 component, there exist neighbourhoods $U \subseteq S$ of $n(p)$ and $\overline{U} \subset \overline{S}$ of p, such that n induces a homeomorphism $\overline{U} \to U$.

Let $\overline{\pi} := \pi \circ n$, and suppose p is not critical for $\overline{\pi}$. Then, there exist neighbourhoods $\overline{W} \subset \overline{U} \subset \overline{S}$ of p and $T \subset \mathbb{C}^2$ of $\overline{\pi}(p)$ where $\overline{\pi}$ induces an isomorphism: $\overline{W} \to T$.

The composition map $n \circ \overline{\pi}^{-1}$ defined from $T \cap \overline{\pi}(\overline{U}) \to U \cap n(\overline{W})$ is the inverse map of the restriction of π to $U \cap n(\overline{W})$, with the fact that $n(p)$ is singular for S. Therefore, p is critical for $\overline{\pi}$. □

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press



In [6, Theorem 3], Delgado and Maugendre established a relation between critical loci of projections to $\mathbb{C}^2$ and special values in the case of normal surfaces. More precisely, if S is a representative of a normal surface germ $(S, 0)$ and $\Lambda_{u,v}$ is a pencil on it, then consider the finite map $\pi = (u, v)$ and call $C(\pi)$ its critical locus. Consider the blow-up $e_{u,v}$ of the ideal generated by u and v, then the intersection points of $e_{u,v}^{-1}(0)$ with the strict transform of $C(\pi)$ by $e_{u,v}$ are special values of the pencil.

We are going to use this relation to prove the following lemma:

Lemma 4.7. *Let Σ_0 be a one-to-one component of the singular locus of S and call Σ_0^* its strict transform by e_I. Then, the image by h_I of the intersection point $\{(0, w)\} = \Sigma_0^* \cap e_I^{-1}(0)$ is a special value in the normalization.*

When the corresponding curve λ_w is generically reduced, the surface S_I is not topologically trivial at $(0, w)$.

Proof. The first statement is a direct consequence of Lemma 4.6 and [6, Theorem 3] applied to each component of the normalization of S and to the pencil $\Lambda_{f\circ n, g\circ n}$.

When the curve λ_w is generically reduced, its strict transform on $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is reduced. We have then a flat family of reduced curves with non-constant Milnor number. The surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is then not topologically trivial at $(0, w)$.

We can assume that no r to 1 component of the singular locus with $r > 1$ has its strict transform in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ passing through $(0, w)$; otherwise, by Lemma 4.5, the pencil will be already non topologically trivial at w. There is then a neighbourhood U of $(0, w)$ in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ on which ν induces a homeomorphism to a neighbourhood V of $(0, w)$ in S_I. This homeomorphism is compatible with the projections to $\mathbb{P}^1$. The surface S_I is then not topologically trivial at $(0, w)$. □

We still need to deal with two cases: when λ_w contains a one-to-one component of the singular locus, and the case when the curve λ_w is not generically reduced. In both cases, the strict transform of the curve λ_w by the normalization is not reduced. Indeed we have:

Lemma 4.8. *Let Σ_0 be a one-to-one component of the singular locus of S and let λ_w be an element of the pencil such that $\Sigma_0 \subseteq \lambda_w$. Then, the pre-image, $\overline{\lambda_w}$, of λ_w by the normalization is non reduced.*

Proof. Let S be a sufficiently small representative of $(S, 0)$. Let $x \in \Sigma_0$ be a generic point different from 0 and sufficiently close to 0. Since Σ_0 is a one-to-one component of Σ, $y := n^{-1}(x)$ is a unique point.

Consider the inclusion of integral domains induced by the normalization, $n^* : \mathcal{O}_{S,x} \to \mathcal{O}_{\overline{S},y}$. It makes $\mathcal{O}_{\overline{S},y}$ into a finite $\mathcal{O}_{S,x}$-module. Call $h_w = \alpha f - \beta g$, where $w = [\alpha : \beta]$.

Applying [14, Theorem 14.8] to the ideal $\langle h_w \rangle$, we obtain:

$$e(\langle n^*(h_w)\rangle, \mathcal{O}_{\overline{S},y}) = e(\langle h_w\rangle, \mathcal{O}_{S,x}),$$

where $e(J,R)$ refers to the multiplicity of the ideal J in the local ring R.

The curve λ_w is the zero set of h_w on S. Since Σ_0 is a non-reduced component of λ_w, the multiplicity $e(\langle h_w\rangle, \mathcal{O}_{S,x})$ is at least two. Therefore, the multiplicity $e(\langle n^*(h_w)\rangle, \mathcal{O}_{\overline{S},y})$ is also at least 2, which implies that the curve $n^{-1}(\Sigma_0)$ is a non-reduced component of the pre-image , $\overline{\lambda_w}$, of λ_w by the normalization n. □

In general, when λ_w is a non generically reduced element, we loose topological triviality. More precisely:

Lemma 4.9. *Let $\Lambda_{f,g}$ be a pencil on a representative S of a reduced surface germ $(S,0)$. If λ_{w_0} is not generically reduced for some $w_0 \in \mathbb{P}^1$, then $\Lambda_{f,g}$ is not topologically trivial at w_0.*

Proof. Let λ_{w_0} be a non generically reduced element of the pencil $\Lambda_{f,g}$ on S. Call δ_0 a non reduced component of λ_{w_0}. If δ_0 is an r to 1 component of the singular locus of S, then by Lemma 4.4, the blown-up surface S_I is not topologically trivial at $(0,w_0)$.

We can assume the surface S_I to be irreducible at $(0,w_0)$. Indeed, suppose that S_I is not irreducible at $(0,w_0)$. Two cases may occur. In the first case, Σ_i^* will be the strict transform by e_I of an r to 1 component Σ_i of the singular locus of S, with $r>1$. We will be then in the situation of the Lemma 4.5. The second case is when the components of S_I at $(0,w_0)$ intersect only along the exceptional divisor $e_I^{-1}(0)$. In this case, the surface S_I satisfies the hypothesis of Theorem 4.11 of [8], i.e., S_I is topologically trivial at $(0,w_0)$ if and only if each of its irreducible components at $(0,w_0)$ is. Then, we have reduced the situation to the case when S_I is irreducible at $(0,w_0)$.

We are now in the situation where the curve $\delta_0 \subset \lambda_{w_0}$ is either a one-to-one component of the singular locus of S or is not contained in the singular locus of S. In both cases, the normalization n_I of S_I is a homeomorphism in a neighbourhood of $(0,w_0)$.

Suppose now that S_I is topologically trivial in a neighbourhood U of $(0,w_0)$, that is we have a homeomorphism $\phi: S_I\cap U \to (\lambda_{w_0}\cap U)\times V$, with $V\subseteq \mathbb{C}$ an open disc. Since S_I is irreducible at $(0,w_0)$ so is the curve λ_{w_0}; in other words, we have $\lambda_{w_0}=\delta_0$. The normalization $n_0: W\times V \to (\lambda_{w_0}\cap U)\times V$ is then also a homeomorphism, W being a disc in $\mathbb{C}$.

So we have:

$$\begin{array}{ccc} \overline{S_I}\cap U & \xrightarrow{n_0^{-1}\circ\phi\circ n_I} & W\times V \\ {\scriptstyle n_I}\downarrow & & \downarrow{\scriptstyle n_0} \\ S_I\cap U & \xrightarrow{\phi} & (\lambda_{w_0}\cap U)\times V \end{array}$$

The normalized surface $\overline{S_I}\cap U$ is then homeomorphic to $W\times V$. By a Theorem of Mumford [15, p. 5], a normal surface is homeomorphic to a non-singular space if and only if it is non-singular.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

Consequently, we may assume the surface $\overline{S_I}$ to be non singular in a sufficiently small neighbourhood U of $n_I^{-1}(0, w_0)$.

Let us call $\overline{E_0}$ the reduced curve associated to the exceptional divisor in $\overline{S_I}$. By hypothesis, λ_{w_0} is not generically reduced. If λ_{w_0} is a one-to-one component of the singular locus, then by Lemma 4.8, the strict transform of λ_{w_0} by n_I, $\overline{\lambda_{w_0}}$, is also non-reduced. If λ_{w_0} is not part of the singular locus, n_I being a normalization is an isomorphism outside the singular locus. This implies again that $\overline{\lambda_{w_0}}$ is non-reduced. Then, the intersection number $i(\overline{\lambda_{w_0}}, \overline{E_0}) > 1$. Since $\overline{S_I}$ is non-singular at $n_I^{-1}(0)$, we can use Theorem 3.14 of [10]. Therefore, if t is in a neighbourhood of w_0, the intersection number of $\overline{\lambda_t}$ with $\overline{E_0}$ is greater than 1. Notice that $\overline{\lambda_t}$ is non-singular and the intersection of $\overline{\lambda_t}$ with $\overline{E_0}$ is transversal.

Remember that the projection $h_I : S_I \to \mathbb{P}^1$ restricted to E_0 is one to one, where E_0 is the reduced curve associated to the exceptional divisor $e_I^{-1}(0)$. The previous argument implies that t has two pre-images by $h_I \circ n_I$. Then, $(h_I|_{E_0})^{-1}(t)$ has two preimages by n_I. Since n_I is a normalization, we have that λ_t is not irreducible. So it cannot be homeomorphic to λ_{w_0}, which contradicts the topological triviality of S_I at $(0, w_0)$. □

Now the missing implication:

Lemma 4.10. *If w is a generic value, then $\Lambda_{f,g}$ is topologically trivial at w.*

Proof. Suppose that w is a generic value. Call $x := (0, w)$ the corresponding point in the blown-up surface S_I. By Definition 3.3, w is generic at the normalization and x is not a point of the strict transform by e_I of the singular locus Σ of S.

Let us first treat the case when $(S, 0)$ is irreducible. The surface $\overline{S}$ is then connected, and the pencil $\Lambda_{f\circ n, g\circ n}$ has a unique indeterminacy point at $n^{-1}(0)$. The blow-up $\overline{e} : \overline{S}_{I\mathcal{O}_{\overline{S}}} \to \overline{S}$ produces an irreducible (connected) exceptional divisor $\overline{E}$. The map $\nu : \overline{S}_{I\mathcal{O}_{\overline{S}}} \to S_I$ induces a homeomorphism outside the strict transform of the singular locus Σ. The inverse image $\nu^{-1}(x)$ is a single point $y \in \overline{S}_{I\mathcal{O}_{\overline{S}}}$ satisfying $(h_I \circ \nu)(y) = w$.

Each element of the pencil $\Lambda_{f\circ n, g\circ n}$ is isomorphic to the corresponding fibre of $h_I \circ \nu$. Since w is generic in the normalization, [6, Theorem 4] implies that the Milnor number of the corresponding elements of the pencil $\Lambda_{f\circ n, g\circ n}$ is equal to the minimal value. Then by [3, Theorem 5.2.2], $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is topologically trivial in a neighbourhood of y.

By the homeomorphism induced by ν, S_I is also topologically trivial in a neighbourhood of x.

Now suppose $(S, 0)$ is reducible. The normalized surface $\overline{S}$ is a disjoint union of l components with $l \geq 2$ and so is the blown-up surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$. Call $\{y_1, \ldots, y_l\} := \nu^{-1}(x)$. Since w is generic at the normalization, each connected component of $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ is topologically trivial at y_i.

Furthermore, none of the y_i's is a point of the strict transform of the singular locus Σ of S by $n \circ \overline{e}$. One can then choose sufficiently small neighbourhoods W_i's of y_i's, in such a way that the restriction of ν to $\bigcup_i W_i$ maps each connected component of $(n\circ\overline{e})^{-1}(0) \cap W_i$ homeomorphically onto $e_I^{-1}(0) \cap U$, for some neighbourhood U of x in S_I. The surface S_I is then topologically trivial at x [8, Proposition 4.11]. □

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

Now, putting together all the previous lemmas, we have a proof of Theorem 4.1.

Proof. (of Theorem 4.1) Let $w \in \mathbb{P}^1$ be a special value for the pencil $\Lambda_{f,g}$. If the curve λ_w is not generically reduced, then by Lemma 4.9 the surface S_I is not topologically trivial at $(0, w)$; this situation includes the case when λ_w contains a component of the singular locus of S.

Assume now that the curve λ_w is generically reduced. If the point $(0, w) \in S_I$ belongs to the strict transform by e_I of an r to 1 component of the singular locus of S with $r > 1$, then by Lemma 4.5, the surface S_I is not topologically trivial at $(0, w)$. If the point $(0, w)$ belongs to the strict transform by e_I of a one-to-one component of the singular locus, then by 4.7, the surface S_I is not topologically trivial at $(0, w)$.

Suppose now w is a special value at the normalization with λ_w generically reduced and $(0, w) \in S_I$ does not belong to the strict transform by e_I of any component of the singular locus of S. Then, Lemma 4.2 shows that S_I is not topologically trivial at $(0, w)$.

So every special value corresponds to a point where the pencil is not topologically trivial.

Conversely, if the pencil is not topologically trivial at w, then Lemma 4.10 shows that w is a special value for the pencil. □

One can observe in the proof of Theorem 4.1 that the singular locus of the surface contributes to special values of the pencil in two different ways. The branches of the singular locus that are one-to-one components produce special values at the normalization; meanwhile, the r to 1 components with $r > 1$ produce special values of the pencil that may not be special at the normalization.

Also, note that when the original surface $(S, 0)$ is not Cohen-Macaulay, then the surface S_I obtained by the blow-up of the ideal $I = \langle f, g \rangle$ need not be Cohen-Macaulay in all points of the exceptional fibre. The proof of Theorem 4.1 shows that a point $(w, 0) \in e_I^{-1}(0)$ where the surface S_I is not Cohen-Macaulay does not correspond necessarily to a special value of the pencil, unless it is a special value at the normalization or a point of the strict transform of the singular locus of S.

In particular, when the surface $(S, 0)$ has an isolated singularity, then for any pencil of curves $\Lambda_{f,g}$ on S, the special values are exactly the special values at the normalization.

Remark 4.11. In [16], the authors give conditions for a Rees algebra to be Cohen-Macaulay. From their work, one can expect to extract conditions for a pencil $\Lambda_{f,g}$ on a non Cohen-Macaulay surface so that the blown-up surface S_I is Cohen-Macaulay. But there are cases where the Rees algebra is not Cohen-Macaulay, but the analytic space S_I is Cohen-Macaulay.

We are now able to describe special curves of a pencil on a reduced complex surface singularity in terms of their topology on the surface.

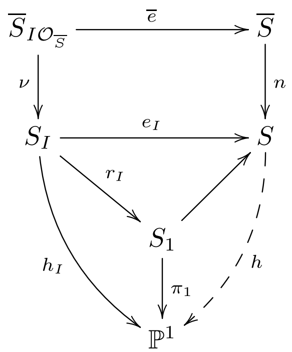
Corollary 4.12. *Consider a pencil of curves $\Lambda_{f,g}$ on a representative of a germ of reduced complex surface singularity $(S, 0) \subset (\mathbb{C}^N, 0)$. A point $w \in \mathbb{P}^1$ is a special value if and only if either the corresponding curve λ_w is not generically reduced or there exists a neighbourhood U of the origin in $\mathbb{C}^N$ such that the curve $\lambda_w \cap U$ is contractible and the curves $\lambda_t \cap U$ are not, for any $t \in \mathbb{P}^1$ sufficiently close to w.*

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

Proof. Consider the surface $S_1 \subset S \times \mathbb{P}^1$ defined in Remark 2.9 by the equation $gt - fs = 0$, where $(s\colon t)$ is a system of homogeneous coordinates in $\mathbb{P}^1$ and S is a sufficiently small representative of $(S, 0)$. Call π_1 the restriction to S_1 of the projection $S \times \mathbb{P}^1 \to \mathbb{P}^1$. Then, for any $t \in \mathbb{P}^1$, the fibre $\pi_1^{-1}(t) \subset S \times \{t\}$ is isomorphic to $\lambda_t \subset S$. Therefore, in order to compare the topology of the curves λ_t in a representative S of $(S, 0)$, it is enough to compare the topology of the fibres $\pi_1^{-1}(t)$ in S_1.

On the other hand, the surface S_I obtained as the blow-up of the ideal $I = \langle f, g\rangle$ is the reduced surface associated to S_1. The reduction map $r_I : S_I \to S_1$ is a homeomorphism that commutes with the respective projections π_1 and h_I to $\mathbb{P}^1$. The fibres $h_I^{-1}(t) \subset S_I$ are then homeomorphic to the corresponding elements $\lambda_t \subset S$.

We have a commutative diagram:



Now, let us proceed to prove the equivalence asserted in the Corollary.

If a curve λ_w is not generically reduced. We have two possibilities: either the curve $n^{-1}(\lambda_w)$ is reduced or it is not. In the first case, λ_w contains a component of the singular locus, so w is a special value. In the second case, the Milnor number of λ_w at the origin is not minimal, and hence, w is a special value at the normalization and therefore a special value for the pencil.

If now λ_w is generically reduced, its strict transform in S_I is also generically reduced. Suppose there exists a neighbourhood U of 0 in $\mathbb{C}^N$ such that $\lambda_w \cap U$ is contractible and $\lambda_t \cap U$ is not for $t \in \mathbb{P}^1$ sufficiently close to w. The fibres $h_I^{-1}(t) \cap (U \times \{t\})$ and $h_I^{-1}(w) \cap (U \times \{w\})$ cannot be homeomorphic, so that the surface S_I is not topologically trivial at $(0, w)$, and therefore, by Theorem 4.1, w is a special value.

Conversely, fix a special value $w \in \mathbb{P}^1$, and assume λ_w is generically reduced. Then, in particular, the curve λ_w does not contain any component of the singular locus Σ of S. We will distinguish two cases:

If $(0, w)$ is a point of the strict transform by e_I of an r to 1 component of Σ, with $r > 1$, then we have seen in the proof of Lemma 4.5, that in a fixed neighbourhood $U \times T \subset (\mathbb{C}^N \times \mathbb{P}^1)$ of $(0, w)$, the fibre $h_I^{-1}(w)$ is contractible and $h_I^{-1}(t)$ is not for $t \neq w$ sufficiently close to w. The corresponding curve $\lambda_w \cap U$ will be then contractible; meanwhile, $\lambda_t \cap U$ will not, for $t \neq w$ sufficiently close to w.

If $(0, w)$ is either a point of the strict transform by e_I of a one-to-one component of Σ, or does not belong to the strict transform by e_I of Σ, then, by Lemma 4.7 and Definition 3.3, w is a special value at the normalization. Call $\overline{\lambda}_w$ the corresponding curve of the pencil

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

on the normalized surface that might be a disjoint union of normal surfaces, and let $\overline{S} = \sqcup \overline{S_i}$ be the decomposition of $\overline{S}$ into irreducible components of $\overline{S}$. By [6, Theorem 4], there exists a connected component $\overline{S}_i$ of $\overline{S}$ such that the curve $\overline{\lambda}_{w,i}$ of the pencil on $\overline{S}_i$ has Milnor number strictly bigger then the minimal one among the curves of the pencil on $\overline{S}_i$.

Consider now the surface $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ obtained by the blow-up $\overline{e}$ of the ideal $I\mathcal{O}_{\overline{S}} = \langle f \circ n, g \circ n \rangle$. It is a disjoint union of surfaces obtained by the blow-up of the corresponding ideals in $\overline{S}_j$. The strict transform of $\overline{\lambda}_{w,i}$ by $\overline{e}$ intersects the exceptional divisor at a point $x_i \in \nu^{-1}(0, w)$. By hypothesis on $(0, w)$, we can choose a connected neighbourhood V_i of x_i in $\overline{S}_{I\mathcal{O}_{\overline{S}}}$ such that the restriction $\nu|_{V_i}$ is a homeomorphism onto its image, the restriction $p_i := (h_I \circ \nu)|_{V_i} : V_i \to T \subset \mathbb{P}^1$ has all its fibres reduced, where T is a suitable neighbourhood of w in $\mathbb{P}^1$ and the fibre $p_i^{-1}(w) \subset V_i$ is contractible.

We can then choose a section $\sigma : T \to V_i$, whose image is the exceptional divisor in V_i such that the family of curves $p_i : V_i \to T$ is a family of reduced curves where each fibre $p_i^{-1}(t)$ is non-singular outside $\sigma(t)$.

Moreover, each fibre of p_i is isomorphic to its image in the corresponding $\overline{S}_i$. The Milnor number $\mu(p_i^{-1}(w), x_i)$ is then strictly bigger than $\mu(p_i^{-1}(t), \sigma(t))$ for $t \in T$ and $t \neq w$.

Using the characterization of topologically trivial families of reduced curves given in [3, Theorem 4.2.4], we conclude that the fibre $p_i^{-1}(t)$ is not contractible in V_i. Recall that the morphism ν induces a homeomorphism between V_i and its image W_i in S_I compatible with the projections to $\mathbb{P}^1$. Therefore, the curve $h_I^{-1}(t) \cap W_i$ is not contractible for $t \in T$ and $t \neq w$; meanwhile, $h_I^{-1}(w) \cap W_i$ is contractible. Since these curves are homeomorphic to their respective images in S, there exists a neighbourhood U of 0 in $\mathbb{C}^N$ such that $\lambda_t \cap U$ is not contractible for $t \neq w$ and t sufficiently close to w and $\lambda_w \cap U$ is contractible. □

5. Examples

We give now some examples of pencils on different types of surfaces. Most of the calculations were made using the SINGULAR software [4].

The first example we give is to illustrate how it is possible to have a special value which is not special at the normalization:

Example 5.1. Consider the polynomial in $\mathbb{C}[x, y, z]$:

$$G(x, y, z) = 256x^3 - 27y^4 - 128x^2z^2 + 144xy^2z + 16xz^4 - 4y^2z^3.$$

Its zero set $V(G)$ is the surface known as the Swallow tail. The normalization of $V(G)$ is given by the parametrization:

$$\begin{aligned} n &: \mathbb{C}^2 \to \mathbb{C}^3 \\ (s, t) &\mapsto \left(\frac{t^2(3t^2 + 4s)}{16}, \frac{t^3 + 2ts}{2}, s \right). \end{aligned}$$

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

The singular locus of this surface has two components. Call Σ_1 the one given by the image of $V\left(s+\frac{3}{2}t^2\right)$. It is parametrized by the restriction of n:

$$\begin{aligned}\sigma_1 : \mathbb{C} &\to \mathbb{C}^3\\ (t) &\mapsto \left(-\frac{3t^4}{16}, -t^3, -\frac{3}{2}t^2\right).\end{aligned}$$

Here, we can see that Σ_1 is one-to-one component of the singular locus.

Call Σ_2 the second component of the singular locus; it is given by the image of $V\left(s+\frac{1}{2}t^2\right)$ and the parametrization:

$$\begin{aligned}\sigma_2 : \mathbb{C} &\to \mathbb{C}^3\\ (t) &\mapsto \left(\frac{t^4}{16}, 0, -\frac{t^2}{2}\right).\end{aligned}$$

This component is a two-to-one component of the singular locus of the surface.

Consider now the pencil on $V(G)$ generated by the functions $f = y-x^2$ and $g = x-z^2$. The associated rational function with indeterminacy at 0 is defined by $h(x,y,z) = [y-x^2 : x-z^2]$.

By computing limits:

$$\lim_{t\to 0} h\circ\sigma_1 = \lim_{t\to 0}[-t^3 - \frac{9}{16^2}t^8 : -\frac{3}{16}t^4 - \frac{9}{4}t^4] = [1:0]$$

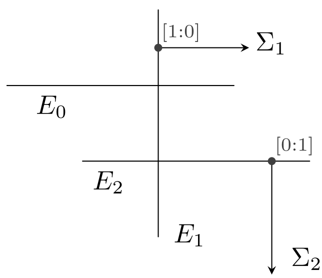
we obtain that the strict transform of Σ_1 passes through the point $(0, 1:0) \in S_I \subseteq S\times\mathbb{P}^1$.

In a similar way:

$$\lim_{t\to 0} h\circ\sigma_2 = \lim_{t\to 0}[-\frac{1}{16^2}t^8 : -\frac{1}{16}t^4 - \frac{1}{4}t^4] = [0:1].$$

Then, the strict transform of Σ_2 passes through the point $(0, 0:1) \in S_I \subseteq S\times\mathbb{P}^1$.

We obtain the minimal resolution of the pulled back pencil $\Lambda_{f\circ n, g\circ n}$ by blowing-up 3 times. And we get the following diagram:



where E_i are the exceptional divisors of the blow-ups.

The component E_0 is dicritical. The restriction to E_0 of $\hat{h}$ is $[s : -t]$, so it has no critical point on E_0.

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

The component E_1 is non dicritical and $h|_{E_1} \equiv [1,0]$. The last one, E_2, is dicritical. The intersection of Σ_2 with E_2 maps to $[0,1]$. And near this intersection, the map $\hat{h}$ is defined by $h(s,t) =$

$$[\frac{1}{2}s - \frac{1}{4}t^4 + \frac{5}{4}st^5 - \frac{39}{16}s^2t^6 + \frac{19}{8}s^3t^7 - \frac{79}{64}s^4t^8 + \frac{21}{64}s^5t^9 - \frac{9}{256}s^6t^{10} : -\frac{3}{4} - \frac{1}{2}st + \frac{3}{16}s^2t^2].$$

Here, the component of the exceptional divisor is defined by $t=0$. So we can see that the restriction of $\hat{h}$ to this component does not have critical points.

In conclusion, the pencil has two special values: $[1:0]$ which is also special at the normalization and $[0,1]$ which is not a special value at the normalization.

Example 5.2. Consider the Whitney umbrella $S = V(x^2 - zy^2)$ and the pencil generated by $f = z$ and $g = y$. Here, the singular locus Σ coincides with the reduced space associated to $\lambda_{[1:0]} = V(y)$. A normalization of S is given by the parametrization:

$$n : \mathbb{C}^2 \to \mathbb{C}^3$$
$$(s,t) \mapsto (st, s, t^2).$$

The pull-back of the pencil by n is generated by s and t^2. The inverse image $n^{-1}(\lambda_{[1:0]})$ is defined by $s=0$; it is a reduced curve. We can then see that the pull-back of a non-reduced curve can be reduced. Since the normalization is 2 to 1 over the singular locus Σ, then by Lemma 4.4, the pencil Λ is not topologically trivial at $[1:0]$ which is not a special value at the normalization. The other special value of Λ is $[0:1]$ which is special at the normalization.

Example 5.3. Consider the surface defined as the following zero set:

$$S = V(\langle z - x^2, w - y^2\rangle \cap \langle y + w^2, z + x^2\rangle) \subseteq \mathbb{C}^4.$$

It is the union of two smooth surfaces intersecting at one point in a sufficiently small neighbourhood of the origin. It is then a non Cohen-Macaulay surface with an isolated singularity.

Consider the ideal $I = \langle x, y + w\rangle$ generated by two functions that define a pencil on a sufficiently small representative of $(S, 0)$.

The blow-up of the ideal I in S is a surface in $\mathbb{C}^4 \times \mathbb{P}^1$ defined in the chart $t \neq 0$, where $[s,t]$ is a system of homogeneous coordinates in $\mathbb{P}^1$, by the intersection of the ideals:

$$I_1 = \langle w^2 - 2w^3 + zs^2 + w^4, y + w^2, y + w - xs, xw + zs - xw^2, z + x^2\rangle \text{ and}$$

$$I_2 = \langle w^2 - 2w^3 + w^4 - 2zws^2 - 2zw^2s^2 + z^2s^4, w + 2yw + w^2 - zs^2, yw + 2w^2 + yw^2 - yzs^2, w - y^2, y + w - xs, xw - xw^2 - yzs + zws, xy + xw - zs, z - x^2\rangle.$$

https://doi.org/10.1017/S0013091524000245 Published online by Cambridge University Press

Intersecting with the hyperplane defined by $s = 0$, we obtain the curve defined by the ideal:

$$J = \langle y + w, z2 - x4, xw, zw + 2x^2y + x^2w - zw^2 + x^2w^2, w^2\rangle.$$

The primary decomposition of J is:

$$J = \langle w, z - x^2, y\rangle \cap \langle w, z + x^2, y\rangle \cap \langle w^2, z, xw, x^4, y + w\rangle.$$

The ideal $J_3 := \langle w^2, z, xw, x^4, y + w\rangle$ defines an embedded component of the section $S_I \cap \{s = 0\}$. The surface S_I is then non Cohen-Macaulay at the origin of this chart.

The pencil $\Lambda_{x,y+w}$ has no special values. One can check that the surface S_I is in fact topologically trivial. In order to do that, in this context, we can compute the Milnor number of the fibres. A generic fibre is a curve with two irreducible smooth non tangent components. Its Milnor number is 1. The fibre defined by $s = 0$ has an embedded component. Applying the formula of Milnor number of generically reduced curves given in [2], we obtain that the Milnor number is also 1. Since all the fibres are connected, as seen in the proof of Lemma 4.5, [9, Theorem 9.3] implies that S_I is topologically trivial along $\{0\} \times \mathbb{P}^1$.

Notice that the fibre given by $s = 0$ in S_I has two smooth and tangent irreducible components. Still the family is topologically trivial. The Milnor number stays constant thanks to the embedded component.

Acknowledgements. The authors wish to thank Mark Spivakovsky for his availability and insightful talks, Anne Pichon, Otoniel Nogueira da Silva, Pablo Portilla Cuadrado and Baldur Sigur∂sson for their time and good ideas. The first Author would like to thank Instituto de Matemáticas UNAN, Unidad Cuernavaca, for financial support and the second author would like to thank the grand PAPIIT117523 by DGAPA-UNAM.

Competing interests. The authors declare not to have any conflict of interests.

References

(1) R. Bondil and D. T. Lê, Caractérisations des éléments superficiels d'un idéal, *C. R. Acad. Sci., Paris, Sér. I, Math.* **332**(8) (2001), 717–722.

(2) C. Brücker and G.-M. Greuel, Deformationen isolierter Kurvensingularitäten mit eingebetteten Komponenten. (Deformations of isolated curve singularities with embedded components), *Manuscr. Math.* **70**(1) (1990), 93–114.

(3) R.-O. Buchweitz and G.-M. Greuel, The Milnor number and deformations of complex curve singularities, *Invent. Math.* **58** (1980), 241–281.

(4) W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, *Singular 4-2-0 – A Computer Algebra System for Polynomial Computations.* 2020, http://www.singular.uni-kl.de.

(5) F. Delgado and H. Maugendre, Special fibres and critical locus for a pencil of plane curve singularities, *Compositio Math.* **136**(1) (2003), 69–87.

(6) F. Delgado and H. Maugendre, Pencils and critical loci on normal surfaces, *Rev. Mat. Complut.* **34**(3) (2021), 691–714.

(7) J. Fernández de Bobadilla, J. Snoussi and M. Spivakovsky, Equisingularity in one-parameter families of generically reduced curves, *Int. Math. Res. Not.* **2017**(5) (2017), 1589–1609.

(8) A. Giles Flores, O. N.Silva and J. Snoussi, On tangency in equisingular families of curves and surfaces, *Q. J. Math.* **71**(2) (2020), 485–505.

(9) G.-M. Greuel, Equisingular and equinormalizable deformations of isolated non-normal singularities, *Methods Appl. Anal.* **24**(2) (2017), 215–276.

(10) G.-M. Greuel, C. Lossen and E. Shustin, *Introduction to Singularities and Deformations*, Springer, Berlin 2007.

(11) R. Hartshorne, *Algebraic Geometry. Corr. 3rd Printing*, 52. Springer, Cham, 1983.

(12) H. B. Laufer, *Normal Two-dimensional Singularities*, 71. Princeton University Press Princeton, NJ 1971.

(13) D. T. Lê, and C. Weber, Équisingularité dans les pinceaux de germes de courbes planes et C^0-suffisance, *Enseign. Math.* **43**(3–4) (1997), 355–380.

(14) H. Matsumura, Commutative Ring Theory. Transl. From the Japanese by M. Reid, 8. Cambridge University Press, Cambridge, 1986.

(15) D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, *Publ. Math., Inst. Hautes Étud. Sci.* **9** (1961), 5–22.

(16) T. Ngô Viêt and S. Ikeda, When is the Rees algebra Cohen-Macaulay? *Commun. Algebra* **17**(12) (1989), 2893–2922.

(17) O. N. Silva and J. Snoussi, Whitney equisingularity in families of generically reduced curves, *Manuscr. Math.* **163**(3–4) (2020), 463–479.

(18) J. Snoussi, *Limites d'espaces tangents à une surface normale.* PhD thesis, Université de Provence France, 1998.

(19) J. Snoussi, Limites d'espaces tangents à une surface normale, *Comment. Math. Helv.* **76**(1) (2001), 61–88.