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Influence of wall roughness on the slip behaviour of viscous fluids

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We consider the stationary equations of a general viscous fluid in an infinite (periodic) slab supplemented with Navier's boundary condition with a friction term on the upper part of the boundary. In addition, we assume that the upper part of the boundary is described by a graph of a function ϕ_ε, where ϕ_ε oscillates in a specific direction with amplitude proportional to ε. We identify the limit problem when $\varepsilon \to 0$, in particular, the effective boundary conditions.

1. Introduction

Recent developments in microfluidic and nanofluidic technologies have renewed interest in the influence of surface roughness on the slip behaviour of viscous fluids (see [19] and the references cited therein). As a matter of fact, this issue has been subjected to discussion for over two centuries by many distinguished scientists who developed the foundations of fluid mechanics, including Bernoulli, Coulomb, Navier, Couette, Poisson and Stokes, to name but a few.

Consider a viscous fluid confined to a domain Ω in the Euclidean physical space R^3, the boundary of which represents a solid wall. Assuming *impermeability of the wall* we have

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\boldsymbol{u}$ is the fluid velocity and $\boldsymbol{n}$ denotes the (outer) normal vector to $\partial\Omega$.

The mostly accepted hypothesis states that there is no relative motion between a viscous fluid and the solid wall $\partial\Omega$, which means that

$$[\boldsymbol{u}]_\tau = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $[\boldsymbol{u}]_\tau$ denotes the tangential component of $\boldsymbol{u}$, provided that the wall is at rest. The *no-slip* boundary condition expressed by (1.2), together with (1.1), has been the most successful hypothesis for reproducing velocity profiles for macroscopic flows.

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In postulating his *slip hypothesis*, Navier suggested replacing (1.2) by

$$\beta[\boldsymbol{u}]_\tau + [\mathbb{S}\boldsymbol{n}]_\tau = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbb{S}$ is the deviatoric viscous stress tensor and β is the friction coefficient. Note that, formally, condition (1.3) reduces to (1.2), provided that $\beta \to \infty$. In the presence of slip, the liquid motion is opposed by a force proportional to the relative velocity between the fluid and the solid wall.

Although it is intuitively clear that (1.3) is much closer to the observed reality than (1.2) whenever the rate of flow is sufficiently strong (turbulent regimes), there has been a common belief that, even if the Navier slip conditions are correct, the corresponding slip length is likely to be too small to influence the motion of macroscopic fluids (for relevant discussion see [19, § 1]).

Recently, numerous experiments and simulations as well as theoretical studies have shown that the classical no-slip assumption can fail when the walls are sufficiently smooth (see, for example, [20, 21]). Strictly speaking, the slip length characterizing the contact between a fluid and a solid wall in relative motion is influenced by many different factors, among which the intrinsic affinity and commensurability between the liquid and solid molecular size as well as the macroscopic surface roughness caused by imperfections and tiny asperities play a significant role.

From the purely mathematical point of view, the Navier (partial) slip boundary conditions yield a correct solution for problems on domains with sharp corners, where the no-slip condition (1.2) yields spurious solutions (see [7, 14]). Moreover, they are relevant on rough walls, where the presence of microscopic asperities reduces considerably the shear stress, leading to a perfect slip on the boundary (see [12]). Given this perspective, there have been several attempts to justify the no-slip boundary behaviour (1.2) as an inevitable consequence of fluid trapping by surface roughness. Richardson [22] showed that the no-slip condition emerges as the *effective* boundary condition for a Stokes flow on domains with periodically undulating boundary (for more general results see also [12]). On the other hand, in order to avoid the complicated description of the fluid behaviour in a boundary layer adjacent to a rough wall on which the no-slip condition (1.2) is imposed, the Navier law (1.3) with a variable coefficient β is prescribed on the mean (flat) surface to facilitate numerical computations (see [11, 15]).

After a series of recent studies by Amirat *et al*. [1, 2] and Casado-Díaz *et al*. [6], it has become clear that the mathematical problems involved are intimately related to the pointwise behaviour of Sobolev functions on 'tiny' sets and may be studied independently of any particular system of equations. Moreover, the weak convergence methods involving the Young measures and their generalizations by Gerard [10] and Tartar [23] have turned out to be a useful tool for describing the influence of roughness on the effective boundary conditions [4, 5].

In line with the preferential setting used in many computational studies, we consider a fluid between two horizontal surfaces periodic with respect to the plane coordinates (x_1, x_2). More specifically, we consider a spatial domain Ω determined by

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ 0 < x_3 < 1 + \Phi(x_1, x_2)\}, \tag{1.4}$$

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where $\mathcal{T}^2 = (0,1)|_{\{0,1\}}$ denotes the two-dimensional torus. We define the two components of the boundary $\partial\Omega$ by

$$\Gamma_{\text{bot}} = \{(x_1, x_2, 0) \mid (x_1, x_2) \in \mathcal{T}^2\}, \tag{1.5}$$

$$\Gamma_{\text{top}} = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2 x_3 = 1 + \Phi(x_1, x_2)\}. \tag{1.6}$$

We assume that the bottom wall moves with a constant velocity $\boldsymbol{V} = (V_1, V_2, 0)$ and that the fluid sticks to it, i.e.

$$\boldsymbol{u}|_{\Gamma_{\text{bot}}} = \boldsymbol{V}. \tag{1.7}$$

On the other hand, we assume impermeability of the upper wall,

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\Gamma_{\text{top}}} = 0, \tag{1.8}$$

together with the Navier slip condition

$$\beta[\boldsymbol{u}]_\tau + [\mathbb{S}\boldsymbol{n}]_\tau|_{\Gamma_{\text{top}}} = 0, \tag{1.9}$$

where the coefficient $\beta \geqslant 0$ may vary with the horizontal coordinates (x_1, x_2).

The viscous stress tensor $\mathbb{S} = \mathbb{S}(\mathbb{D})$,

$$\mathbb{S} : R^{3\times3}_{\text{sym}} \to R^{3\times3}_{\text{sym}}$$

is a function of the symmetric velocity gradient

$$\mathbb{D}[\boldsymbol{u}] = \tfrac{1}{2}(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u}), \tag{1.10}$$

satisfying the standard coercivity hypothesis

$$\mathbb{S}(\mathbb{D}) : \mathbb{D} \geqslant d_1 |\mathbb{D}|^p, \quad d_1 > 0 \text{ for all } \mathbb{D} \in R^{3\times3}_{\text{sym}}, \tag{1.11}$$

together with a technical growth restriction

$$|\mathbb{S}(\mathbb{D})| \leqslant d_2(1 + |\mathbb{D}|^{p-1}) \quad \text{for all } \mathbb{D} \in R^{3\times3}_{\text{sym}} \tag{1.12}$$

for a certain $p \geqslant 2$. In addition, we require $\mathbb{S}$ to be strictly monotone:

$$(\mathbb{S}(\mathbb{D}_1) - \mathbb{S}(\mathbb{D}_2)) : (\mathbb{D}_1 - \mathbb{D}_2) > 0 \quad \text{for any } \mathbb{D}_1 \neq \mathbb{D}_2. \tag{1.13}$$

The best known example is the so-called linearly viscous fluid, where $\mathbb{S}$ is determined through Newton's rheological law $\mathbb{S} = 2\mu\mathbb{D}$, $p = 2$. More examples as well as the relevant mathematical background may be found in [13].

If the fluid is incompressible and in a stationary (time-independent) state, the velocity $\boldsymbol{u}$ and the pressure p satisfy the Navier–Stokes system of equations

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x P = \operatorname{div} \mathbb{S}(\mathbb{D}[\boldsymbol{u}]) + \boldsymbol{f} \quad \text{in } \Omega, \tag{1.14}$$

supplemented with the standard incompressibility constraint

$$\operatorname{div} \boldsymbol{u} = 0. \tag{1.15}$$

Motivated by Priezjev *et al*. [20] and Qiang and Wang [21], we consider a family of solutions $\{\boldsymbol{u}_\varepsilon, p_\varepsilon\}_{\varepsilon>0}$ of problem (1.14), (1.15), supplemented with the boundary conditions (1.7)–1.9), posed on spatial domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ given by (1.4) for $\Phi = \Phi_\varepsilon$. The functions Φ_ε depend only on a single spatial variable, say, $\Phi_\varepsilon = \Phi_\varepsilon(x_1)$, $x_1 \in \mathcal{T}^1 = (0,1)|_{\{0,1\}}$, mimicking a ribbed surface, with the amplitude and typical wavelength of oscillations approaching zero for $\varepsilon \to 0$.

We will show (see theorem 3.1, below) that $\{\boldsymbol{u}_\varepsilon, p_\varepsilon\}_{\varepsilon>0}$ possesses a limit $\{\boldsymbol{u}, p\}$ solving the Navier–Stokes system (1.14), (1.15) on the 'flat' domain $\Omega = \mathcal{T}^2 \times (0,1)$ and satisfying the no-slip boundary conditions (1.7) on the bottom wall Γ_{bot}. In addition, the tangent velocity field $[\boldsymbol{u}]_\tau$ on the upper wall $\Gamma_{\text{top}} = \{x_3 = 1\}$ is parallel to the riblets direction, that means, $[\boldsymbol{u}]_\tau = (0, u_2, 0)$, and satisfies a directional Navier slip condition

$$\tilde{\beta} u_2 + S_{2,3} = 0 \quad \text{on } \{x_3 = 1\}. \tag{1.16}$$

The friction coefficient $\tilde{\beta}$ depends only on a weak limit of $\{\beta_\varepsilon\}_{\varepsilon>0}$, and on σ, a positive quantity which can be computed explicitly in terms of a Young measure associated to horizontal deviations of the normals on $\partial\Omega_\varepsilon$. In particular, we show that the concrete value and even shape of $\tilde{\beta}$ can be 'tuned' choosing a specific distribution of riblets on $\partial\Omega_\varepsilon$, as predicted for similar models by the molecular dynamics approach (cf. [20, 21]). In the particular case when $\beta_\varepsilon = \beta$ are constant, we get

$$\tilde{\beta} > \beta,$$

provided that the amplitude and the frequency of oscillations of $\Gamma_{\text{top}}^\varepsilon$ are of the same order.

Our method is based on the concept of parametrized rugosity measure introduced in [5], which is nothing other than a Young measure associated to the family of gradients $\{\nabla\Phi_\varepsilon\}_{\varepsilon>0}$. Furthermore, we exploit the well-developed theory of Sobolev functions, in particular, the properties of their traces on the boundary. Accordingly, we consider the weak (distributional) solutions to problem (1.14), (1.15). This so-called variational approach seems inevitable in the present situation, as all the refined elliptic estimates yielding regularity of solutions are quite sensitive to the topology of the boundary. Finally, our approach relies on the pressure estimates that can be obtained via a generalized inverse of the div operator. One of the fundamental issues addressed below is the uniformity of these estimates with respect to the parameter $\varepsilon \to 0$.

The paper is organized as follows. In § 2 we recall some preliminary material, including the function spaces framework and a variational formulation of the problem. The main result illustrated by several concrete examples is formulated in § 3. In § 4, we introduce the concept of rugosity measure associated to the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ in order to identify the boundary conditions to be satisfied in the asymptotic limit for $\varepsilon \to 0$. Section 5 is devoted to the basic properties of the so-called Bogovskii operator, div^{-1}. The proof of the main result is completed in § 6 by means of the theory of monotone operators.

2. Preliminaries

To begin with, let us introduce the concept of variational solutions to the problem on Ω_ε.

Definition 2.1. We shall say that functions $\boldsymbol{u}_\varepsilon \in W^{1,p}(\Omega_\varepsilon; R^3)$, $P_\varepsilon \in L^{p'}(\Omega_\varepsilon)$, $1/p + 1/p' = 1$, represent a weak solution to the Navier–Stokes system (1.14), (1.15), supplemented with the boundary conditions (1.7)–(1.9), if $\boldsymbol{u}$ satisfies (1.7), (1.8) in the sense of traces, together with the incompressibility constraints (1.15) almost any (a.a.) in Ω_ε, and the integral identity

$$\int_{\Omega_\varepsilon} ((\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} + P_\varepsilon \operatorname{div} \boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_\varepsilon} \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_\varepsilon} \beta_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma - \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \tag{2.1}$$

holds for any test function $\boldsymbol{\varphi} \in W^{1,p}(\Omega_\varepsilon; R^3)$ such that

$$\boldsymbol{\varphi}|_{\Gamma_{\mathrm{bot}}} = 0, \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n}|_{\Gamma^\varepsilon_{\mathrm{top}}} = 0.$$

Note that we have tacitly assumed that the driving force $\boldsymbol{f}$ is defined on all domains Ω_ε say; $\boldsymbol{f}$ is a restriction of a fixed function belonging to the class $L^\infty(R^3; R^3)$. Note also that, by means of the classical Krasnoselskii theorem, if $\mathbb{S} : R^{3\times3}_{\mathrm{sym}} \to R^{3\times3}_{\mathrm{sym}}$ is continuous, then the associated Nemytskii operator is continuous on the Lebesgue space $L^p(\Omega_\varepsilon; R^{3\times3}_{\mathrm{sym}})$ with values in $L^{p'}(\Omega_\varepsilon; R^{3\times3}_{\mathrm{sym}})$ provided that $\mathbb{S}$ satisfies hypothesis (1.12). Finally, for the traces of Sobolev functions on $\Gamma^\varepsilon_{\mathrm{top}}$ to be well defined, we must assume that Φ_ε are Lipschitz functions on $\mathcal{T}^2$.

Similarly, solutions of the limit problem are defined as follows.

Definition 2.2. A couple $\{\boldsymbol{u}, P\}$ is termed a weak solution to problem (1.14), (1.15) on $\Omega = \mathcal{T}^2 \times (0,1)$, supplemented with the boundary conditions (1.7), (1.8), and

$$u_1 = 0, \qquad \tilde{\beta} u_2 + \mathbb{S}(\mathbb{D}[\boldsymbol{u}])_{2,3} = 0 \quad \text{on } \Gamma_{\mathrm{top}} = \{x_3 = 1\} \tag{2.2}$$

if $\boldsymbol{u} \in W^{1,p}(\Omega; R^3)$, $P \in L^{p'}(\Omega)$, $\boldsymbol{u}$ satisfies (1.7), $u_1 = u_3 = 0$ in the sense of traces on Γ_{top} and the integral identity

$$\int_{\Omega} ((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla_x \boldsymbol{\varphi} + P \operatorname{div} \boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} = \int_{\Omega} \mathbb{S}(\mathbb{D}[\boldsymbol{u}]) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \tilde{\beta} \boldsymbol{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma - \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \tag{2.3}$$

holds for any test function $\boldsymbol{\varphi} \in W^{1,p}(\Omega; R^3)$ such that

$$\boldsymbol{\varphi}|_{\Gamma_{\mathrm{bot}}} = 0, \qquad \varphi_1|_{\Gamma_{\mathrm{top}}} = \varphi_3|_{\Gamma_{\mathrm{top}}} = 0.$$

A remarkable property of both (2.1) and (2.3) is that $\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}$ and $\boldsymbol{u} - \tilde{\boldsymbol{V}}$ respectively represent admissible test functions. This is, of course, related to the fact that the 'convective' terms $\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon$, $\boldsymbol{u} \otimes \boldsymbol{u}$ belong to the Lebesgue space $L^{p'}$, provided that $p \geqslant \frac{9}{5}$. Here, $\tilde{\boldsymbol{V}}$ denotes a *suitable* extension on Ω of the vector field $\boldsymbol{V}$ appearing in the boundary condition (1.7). We can take

$$\tilde{\boldsymbol{V}} = \psi(x_3)\boldsymbol{V} \quad \text{for } \psi \in C^\infty[0,1], \qquad \psi(0) = 1, \qquad \psi(1) = 0. \tag{2.4}$$

In particular, the norm of $\tilde{\boldsymbol{V}}$ can be made arbitrarily small in the Lebesgue space $L^{p'}(\Omega; R^3)$ through a suitable choice of ψ.

3. Main result

The main result of the present paper reads as follows.

THEOREM 3.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains given by (1.4) with $\Phi = \Phi_\varepsilon(x_1)$, where Φ_ε are Lipschitz functions on $\mathcal{T}^1$ such that*

$$0 \leqslant \Phi_\varepsilon \leqslant \varepsilon, \quad |\Phi_\varepsilon'| \leqslant L \text{ on } \mathcal{T}^1, \tag{3.1}$$

with the Lipschitz constant L independent of ε. Furthermore, assume that $\mathbb{S} : R^{3\times 3}_{\text{sym}} \to R^{3\times 3}_{\text{sym}}$ is a continuous mapping satisfying hypotheses (1.11)–(1.13), with $p \geqslant 2$. Let $\{\boldsymbol{u}_\varepsilon, P_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega_\varepsilon; R^3) \times L^{p'}(\Omega_\varepsilon)$, $p \geqslant 2$, be a family of weak solutions to problem (1.14), (1.15), (1.7)–(1.9) specified in definition 2.1. Finally, assume that

$$\beta_\varepsilon \to \beta \text{ weakly-}* \text{ in } L^\infty(\mathcal{T}^2), \tag{3.2}$$

and

$$|\Phi_\varepsilon'| \to \overline{|\Phi'|} \text{ weakly-}* \text{ in } L^\infty(\mathcal{T}^1), \quad \text{where } \overline{|\Phi'|} > 0 \text{ a.a. on } \mathcal{T}. \tag{3.3}$$

Then, passing to a suitable subsequence as the case may be, we have

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } W^{1,p}(\Omega; R^3), \qquad P_\varepsilon \to P \quad \text{in } L^{p'}(\Omega), \tag{3.4}$$

where the $\{\boldsymbol{u}, P\}$ solve problem (1.14), (1.15), (1.7), (1.8), (2.2) on Ω in the sense of definition 2.2, where

$$\tilde{\beta} = L^\infty - \text{weak-}* \lim_{\varepsilon\to 0} \beta_\varepsilon \sqrt{1 + |\Phi_\varepsilon'|^2}. \tag{3.5}$$

Note that, in accordance with (3.2), (3.4), the friction coefficient $\tilde{\beta}$ for the limit problem is always greater than a weak limit of $\{\beta_\varepsilon\}_{\varepsilon>0}$. Moreover,

$$\tilde{\beta} > \beta \quad \text{whenever } \beta_\varepsilon \to \beta \in L^1(\Omega). \tag{3.6}$$

As $\Phi_\varepsilon \to 0$ uniformly on $\mathcal{T}^1$, we have

$$\Phi_\varepsilon' \to 0 \quad \text{weakly-}* \text{ in } L^\infty(\mathcal{T}^1), \tag{3.7}$$

while hypothesis (3.3) requires the convergence in (3.7) not to be strong on any subdomain of $\mathcal{T}^1$. In other words, the oscillations of the normal vectors to $\Gamma^\varepsilon_{\text{top}}$ persist in the asymptotic limit $\varepsilon \to 0$.

A sufficient condition for (3.3) to hold reads

$$\liminf_{\varepsilon\to 0} \int_a^b |\Phi_\varepsilon'| \, \mathrm{d}z \geqslant r(b-a) \quad \text{for any } a \leqslant b, \tag{3.8}$$

where $r > 0$ is a constant independent of ε. Observe that

$$\int_a^b |\Phi_\varepsilon'| \, \mathrm{d}z = \operatorname{var}_a^b[\Phi_\varepsilon] = \sup_{a\leqslant z_1 < \cdots < z_n \leqslant b} \sum_{i=1}^{n-1} |\Phi_\varepsilon(z_{i+1}) - \Phi_\varepsilon(z_i)|, \tag{3.9}$$

where the rightmost expression corresponds intuitively to the degree of roughness of $\Gamma^\varepsilon_{\text{top}}$.

Even more interesting example is provided by the co-area formula (see [8, § 3.4.2, theorem 1]):

$$\int_a^b |\Phi_\varepsilon'| \,\mathrm{d}z = \int_0^\varepsilon \#[(a,b) \cap \Phi_\varepsilon^{-1}(y)] \,\mathrm{d}y, \tag{3.10}$$

where $\#[(a,b) \cap \Phi_\varepsilon^{-1}(y)]$ denotes the number of points $x \in (a,b)$ (non-negative integer or ∞) where $\Phi_\varepsilon(x) = y$, in other words, the number of points where the graph $(x, \Phi_\varepsilon(x))$ intersects the straight line (x,y), $x \in \mathcal{T}^1$.

Formulae (3.8)–(3.10) give rise to a number of examples, which are listed below.

(i) Periodically oscillating boundaries. The most frequently studied situation takes

$$\Phi_{\varepsilon_k}(x_1) = \varepsilon_k \Phi\biggl(\frac{x_1}{\varepsilon_k}\biggr) \quad \text{with } \frac{1}{\varepsilon_k} \text{ a positive integer,} \tag{3.11}$$

where $\Phi \in W^{1,\infty}(\mathcal{T}^1)$. It is easy to check that

$$|\Phi_\varepsilon'| \to \int_{\mathcal{T}^1} |\Phi'(z)| \,\mathrm{d}z \quad \text{weakly in } L^1(\mathcal{T}^1);$$

whence (3.3) holds unless Φ is constant.

(ii) The crystalline case. Assume that $\Phi_\varepsilon' \in K$ a.a. on $\mathcal{T}^1$, where $K \subset R^1$ is a finite set $0 \notin K$. Then we can take

$$r = \min_K |\Phi_\varepsilon'| > 0$$

in (3.8) in order to conclude that (3.3) holds.

(iii) The generalized crystalline case. As a matter of fact, we only need

$$\operatorname*{ess\,inf}_{\mathcal{T}^1} |\Phi_\varepsilon'| \geqslant r > 0 \quad \text{for all } \varepsilon > 0$$

to arrive at the same conclusion as in the previous case.

(iv) Oscillatory boundaries. Assume that, for any y_1, y_2 such that $\Phi_\varepsilon(y_1) = \Phi_\varepsilon(y_2) = 0$, $y_1 < y_2$, there exists $y_3 \in (y_1, y_2)$ such that $\Phi_\varepsilon(y_3) = \varepsilon$. In agreement with (3.10), we get

$$\int_a^b |\Phi_\varepsilon'| \,\mathrm{d}z = \int_0^\varepsilon \#[(a,b) \cap \Phi_\varepsilon^{-1}(y)] \,\mathrm{d}y \geqslant \varepsilon \#[x \in (a,b), \Phi_\varepsilon(x) = 0].$$

Consequently, the family $\{\Phi_\varepsilon\}_{\varepsilon>0}$ satisfies (3.3) as soon as

$$\liminf_{\varepsilon \to 0} \{\varepsilon \#[x \in (a,b), \Phi_\varepsilon(x) = 0]\} \geqslant r(b-a)$$

for any $a < b$ and a certain $r > 0$.

(v) Boundaries with asperities. A function $\mathcal{A} \in W^{1,\infty}(\mathcal{T}^1)$ is termed the *asperity of amplitude* $h > 0$ if

$$0 = \min_{\mathcal{T}^1} \mathcal{A} < \max_{\mathcal{T}^1} \mathcal{A} = h. \tag{3.12}$$

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Assume that

$$\Phi_\varepsilon = \sum_i \mathcal{A}_i^\varepsilon,$$

where $\mathcal{A}_i^\varepsilon$ are asperities such that $\operatorname{supp}[\mathcal{A}_i] \cap \operatorname{supp}[\mathcal{A}_j] = \emptyset$ for $i \neq j$.

It is easy to check that

$$\int_a^b |\Phi_\varepsilon'| \, dz = \int_0^\varepsilon \#[(a,b) \cap \Phi_\varepsilon^{-1}(y)] \, dy \\ \geqslant \sum_h h\#[\text{asperities } \mathcal{A}_i^\varepsilon \text{ of amplitude } h, \ \operatorname{supp} \mathcal{A}_i^\varepsilon \cap (a,b) \neq \emptyset],$$

where the sum contains at most a countable number of terms. Consequently, in order to obtain (3.3), we must assume that

$$\liminf_{\varepsilon \to 0} \left\{ \sum_h h\#[\text{asperities } \mathcal{A}_i^\varepsilon \text{ of amplitude } h, \ \operatorname{supp} \mathcal{A}_i^\varepsilon \cap (a,b) \neq \emptyset] \right\} \\ \geqslant r(b-a)$$

for a certain $r > 0$.

The rest of the paper is devoted to the proof of theorem 3.1.

4. Parametrized measures of rugosity

For the time being assume that we have already shown that

$$\sup_{\varepsilon > 0} \int_{\Omega_\varepsilon} (|\nabla_x \boldsymbol{u}_\varepsilon|^p + |\boldsymbol{u}_\varepsilon|^p) \, d\boldsymbol{x} < \infty. \tag{4.1}$$

Consequently, we can assume that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{weakly in } W^{1,p}(\Omega; R^3), \tag{4.2}$$

where $\boldsymbol{u}$ satisfies the boundary condition (1.7) on Γ_{bot} in the sense of traces.

To begin with, observe that the impermeability condition (1.1) is stable with respect to a rather general family of converging domains $\{\Omega_\varepsilon\}_{\varepsilon > 0}$. Note that (1.1) can be restated in the form of an integral identity:

$$\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, d\boldsymbol{x} = 0 \tag{4.3}$$

to be satisfied for any $\varphi \in \mathcal{D}(\mathcal{T}^2 \times (0, \infty))$. As a matter of fact, (4.3) holds for all vector fields $\boldsymbol{u}_\varepsilon$ integrable on $\bar{\Omega}_\varepsilon$ together with $\operatorname{div} \boldsymbol{u}_\varepsilon$.

We get

$$0 = \int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, d\boldsymbol{x} \\ = \int_{\Omega_\varepsilon \setminus \Omega} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, d\boldsymbol{x} + \int_{\Omega} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, d\boldsymbol{x},$$

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where, up to a suitable subsequence,

$$\int_{\Omega_\varepsilon \setminus \Omega} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, \mathrm{d}\boldsymbol{x} \to 0 \quad \text{as } \varepsilon \to 0,$$

$$\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u}_\varepsilon \varphi) \, \mathrm{d}\boldsymbol{x} \to \int_{\Omega} (\boldsymbol{u} \cdot \nabla_x \varphi - \operatorname{div} \boldsymbol{u} \varphi) \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for } \varepsilon \to 0$$

provided that

$$\Omega \subset \Omega_\varepsilon \quad \text{for all } \varepsilon > 0, \qquad |\Omega_\varepsilon \setminus \Omega| \to 0 \quad \text{for } \varepsilon \to 0. \tag{4.4}$$

Thus, we have shown the following assertion.

Lemma 4.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains satisfying the hypotheses of theorem 3.1. Moreover, let $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ be vector fields on Ω_ε satisfying (4.1), with $p > 1$, and (4.3) for any $\varphi \in \mathcal{D}(\mathcal{T}^2 \times (0, \infty))$.*

Then, passing to a subsequence as the case may be, we have

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \textit{weakly in } W^{1,p}(\Omega; R^3),$$

where $\boldsymbol{u}$ satisfies (4.3) on Ω for any $\varphi \in \mathcal{D}(\mathcal{T}^2 \times (0, \infty))$.

As the target domain $\Omega = \mathcal{T}^2 \times (0, 1)$ is smooth, the conclusion of lemma 4.1 reads

$$u_3|_{\Gamma_{\text{top}}} = 0, \tag{4.5}$$

provided that (4.1) holds.

Let us try to formulate, first intuitively, the meaning of *rugosity* of the surface Γ_ε in a certain (tangent) direction $\boldsymbol{w}$. Very roughly indeed, one may say that such a quantity should be proportional to *probability* that a normal vector to Γ_ε is parallel to $\boldsymbol{w}$. Given a measurable set $D \subset \mathcal{T}^2$ this can be expressed as

$$\mathrm{pr}_D^\varepsilon \approx \frac{\text{meas}\{\boldsymbol{y} \in D \mid \text{the normal vector at } (\boldsymbol{y}, \Phi_\varepsilon(\boldsymbol{y})) \text{ parallel to } \boldsymbol{w}\}}{\text{meas}(D)}.$$

However, the set of boundary points at which the normal is parallel to a single vector $\boldsymbol{w}$ may be very small; in particular, its two-dimensional Hausdorff measure could be zero. For this reason, it seems more convenient to replace $\boldsymbol{w}$ by a *cone* C_w^δ, given by

$$C_w^\delta = \{\boldsymbol{v} \mid \boldsymbol{v} \cdot \boldsymbol{w} \geqslant (1 - \delta) |\boldsymbol{v}| |\boldsymbol{w}|\}.$$

Accordingly, we take

$$\mathrm{pr}_D^\varepsilon \approx \frac{\text{meas}\{\boldsymbol{y} \in D \mid \text{the normal vector at } (\boldsymbol{y}, \Phi_\varepsilon(\boldsymbol{y})) \text{ belongs to } C_w^\delta\}}{\text{meas}(D)}$$

and require this quantity to be positive uniformly with respect to $\varepsilon \to 0$:

$$\liminf_{\varepsilon \to 0} \mathrm{pr}_D^\varepsilon > 0 \quad \text{for any } D \subset \mathcal{T}^2. \tag{4.6}$$

Formula (4.6) is reminiscent of the definition of the Young measure associated to the family of normal vectors on Γ_ε (see [18, ch. 1, § 2]). Motivated by the above

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discussion, we can define *parametrized rugosity measure* $\mathcal{R}_y$, $y \in \mathcal{T}^2$, as a Young measure associated to the family $\{\nabla_y \Phi_\varepsilon\}_{\varepsilon>0}$, i.e. $\mathcal{R}_y$ is a probability measure on R^2 defined as

$$\langle \mathcal{R}_y, G\rangle = \overline{G(\nabla_y \Phi)}(\boldsymbol{y}) \quad \text{for all } G \in C(R^2) \text{ and for a.a. } \boldsymbol{y} \in \mathcal{T}^2,$$

where $\overline{G(\nabla_y \Phi)}$ denotes a weak limit of $\{G(\nabla_y \Phi_\varepsilon)\}_{\varepsilon>0}$ in $L^1(\mathcal{T}^2)$ (see [5, § 3]). As a direct consequence of (3.7), the centre of gravity associated to the parametrized rugosity measure $\mathcal{R}_y$ is always located at the origin. Note that the vectors

$$(\nabla_y \Phi_\varepsilon, 0) = (\nabla_y \Phi_\varepsilon, -1) - (0,0,1) = \boldsymbol{n}_\varepsilon - \boldsymbol{n}$$

express the deviations of the normal vector fields on $\Gamma^\varepsilon_{\text{top}}$ from the vertical direction.

A remarkable property of the parametrized rugosity measure is the following identity:

$$\boldsymbol{u}(\boldsymbol{y},1)\cdot\biggl(\int_{R^2} G(\boldsymbol{Z})\boldsymbol{Z}\,\mathrm{d}\mathcal{R}_y(\boldsymbol{Z}),0\biggr) = 0 \quad \text{for all } G \in C(R^2) \text{ and a.a. } y \in \mathcal{R}^2, \tag{4.7}$$

where $\boldsymbol{u}$ is the weak limit appearing in (4.2). Indeed, the impermeability condition (1.1) written in terms of Φ_ε reads

$$\boldsymbol{u}_\varepsilon(y, 1+\Phi_\varepsilon(y))\cdot(\nabla_x \Phi_\varepsilon(y), -1) = 0 \quad \text{for a.a. } y \in \mathcal{T}^2;$$

in particular,

$$\int_{\mathcal{T}^2} \psi(y) G(\nabla_x \Phi_\varepsilon(y)) \boldsymbol{u}_\varepsilon(y, 1+\Phi_\varepsilon(y))\cdot(\nabla_x \Phi_\varepsilon(y), -1)\,\mathrm{d}y = 0 \tag{4.8}$$

for any $G \in C(R^2)$ and all $\psi \in \mathcal{D}(\mathcal{T}^2)$. On the other hand,

$$\int_{\mathcal{T}^2} |\boldsymbol{u}_\varepsilon(y, 1+\Phi_\varepsilon(y)) - \boldsymbol{u}_\varepsilon(y,1)|\,\mathrm{d}y \leqslant \int_{\mathcal{T}^2}\int_1^{1+\Phi_\varepsilon(y)} |\nabla_x \boldsymbol{u}_\varepsilon(y,z)|\,\mathrm{d}z\,\mathrm{d}y,$$

where the right-hand side tends to zero for $\varepsilon \to 0$ as a consequence of (4.2). Consequently, it follows from (4.8) that

$$\lim_{\varepsilon\to 0}\int_{\mathcal{T}^2} \psi(y) G(\nabla_x \Phi_\varepsilon(y)) \boldsymbol{u}_\varepsilon(y,1)\cdot(\nabla_x \Phi_\varepsilon(y), -1)\,\mathrm{d}y = 0 \tag{4.9}$$

for any $G \in C(R^2)$ and all $\psi \in \mathcal{D}(\mathcal{T}^2)$. As the trace operator $\boldsymbol{u}_\varepsilon \in W^{1,p}(\Omega; R^3) \mapsto \boldsymbol{u}_\varepsilon|_{\{x_3=1\}}$ is absolutely continuous with respect to the topology $L^p(\{x_3 = 1\})$, relation (4.9) yields (4.7) (see also [5, lemma 7.1] for the case in which $p = 2$).

In particular, for the sequence of domains considered in theorem 3.1, hypothesis (3.3) gives rise to

$$\operatorname{supp}\mathcal{R}_y \subset \{(y_1,0), y_1 \in R\},\ \operatorname{supp}\mathcal{R}_y \neq (0,0) \text{ for a.a. } \boldsymbol{y} \in \mathcal{T}^2\}. \tag{4.10}$$

Thus, combining (4.7), (4.10), together with (4.5), we conclude that, under the hypotheses of theorem 3.1,

$$u_1|_{\Gamma_{\text{top}}} = u_3|_{\Gamma_{\text{top}}} = 0 \tag{4.11}$$

in accordance with (1.8), (2.2). However, the validity of (4.11) is conditioned by the uniform bound anticipated in (4.1). After some preliminary material presented in § 5, a rigorous justification of (4.1) will be given in § 6.

5. Equation div $\boldsymbol{v} = g$

5.1. Bogovskii's operator

For the purposes of this section, we adopt a slightly more general situation than in theorem 3.1, assuming that Φ_ε are effective functions of both variables $(y_1, y_2) \in \mathcal{T}^2$ and replacing hypothesis (3.1) by

$$\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}_2), \quad 0 \leqslant \Phi_\varepsilon \leqslant \varepsilon, \quad |\nabla_x \Phi_\varepsilon| \leqslant L, \text{ with } L \text{ independent of } \varepsilon > 0. \tag{5.1}$$

By virtue of (5.1), there exists $\omega > 0$ independent of ε such that the interior of the cone

$$(x_1, x_2, 1 + \Phi_\varepsilon(x_1, x_2)) + \mathcal{K}, \quad \mathcal{K} = \{(x_1, x_2, x_3) \mid x_3 \in (-1, 0),\ |(x_1, x_2)| < \omega |x_3|\},$$

is contained in Ω_ε for any $(x_1, x_2) \in \mathcal{T}^2$. Consequently, there is a finite number of domains Ω_ε^k, $k = 1, \dots, m$, such that

$$\Omega_\varepsilon = \bigcup_{k=1}^m \Omega_\varepsilon^k,$$

and each Ω_ε^k is star shaped with respect to any point of a ball of a radius $r > 0$ contained in Ω_ε^k, where both m and r can be chosen independent of ε (for the relevant definition of a star-shaped domain see [9, ch. III.3]).

Consider an auxiliary problem.

Problem 5.1. Given

$$g \in L^q(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} g \, \mathrm{d}\boldsymbol{x} = 0, \quad 1 < q < \infty, \tag{5.2}$$

find a vector field $\boldsymbol{v} = \mathcal{B}_\varepsilon[g]$ such that

$$\boldsymbol{v} \in W_0^{1,q}(\Omega_\varepsilon; R^3), \qquad \operatorname{div} \boldsymbol{v} = g \quad \text{a.a. in } \Omega_\varepsilon. \tag{5.3}$$

We report the following result (see [9, ch. III.3, theorem 3.1]).

Proposition 5.2. *For each $\varepsilon > 0$ there is a solution operator $\mathcal{B}_\varepsilon$ associated to problem (5.2), (5.3) such that*

$$\|\mathcal{B}_\varepsilon[g]\|_{W_0^{1,q}(\Omega_\varepsilon; R^3)} \leqslant c(r, m, q) \|g\|_{L^q(\Omega_\varepsilon)}; \tag{5.4}$$

in particular, the norm of $\mathcal{B}_\varepsilon$ is independent of ε.

Remark 5.3. The construction of the operator $\mathcal{B}$ used in [9] is due to Bogovskii [3]. Clearly, the parameters r and m depend solely on the value of the Lipschitz constant L in (5.1).

5.2. Korn's inequality

The so-called Korn inequality yields a bound on the full velocity gradient $\nabla_x \boldsymbol{u}$ in terms of its symmetric part $\mathbb{D}[\boldsymbol{u}]$ introduced in (1.10). The uniform estimates stated in proposition 5.2 can be used in order to establish a version of Korn's inequality

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depending solely on the value of the Lipschitz constant L in (5.1). To this end, we adapt the approach in [16].

Initially, we apply the standard Korn inequality on the domain Ω to obtain

$$\|\boldsymbol{v}\|_{W^{1,q}(\Omega;R^3)} \leqslant c(q)\|\mathbb{D}[\boldsymbol{v}]\|_{L^q(\Omega;R^{3\times3}_{\mathrm{sym}})} \leqslant c(q)\|\mathbb{D}[\boldsymbol{v}]\|_{L^q(\Omega_\varepsilon;R^{3\times3}_{\mathrm{sym}})} \tag{5.5}$$

for any $\boldsymbol{v} \in W^{1,q}(\Omega_\varepsilon;R^3)$, $q > 1$, such that $\boldsymbol{v}|_{\Gamma_{\mathrm{bot}}} = 0$.

On the other hand, it is a routine matter to express

$$\nabla_x(\partial_{x_i} v_j) = \mathcal{A}_{i,j}\mathbb{D}[\boldsymbol{v}] \quad \text{for any } i, j, = 1, \dots, 3, \tag{5.6}$$

where $\mathcal{A}_{i,j}$ are linear differential operators of first order with constant coefficients.

By virtue of proposition 5.2, any function $h \in L^{q'}(\Omega_\varepsilon)$, $1/q' + 1/q = 1$, of zero integral mean can be expressed as a divergence of a vector field $\mathcal{B}_\varepsilon[h]$ belonging to $W_0^{1,q'}(\Omega_\varepsilon, R^3)$. Consequently, we deduce from (5.6) that

$$\|\nabla_x \boldsymbol{v}\|_{L^q(\Omega_\varepsilon;R^{3\times3})} \leqslant c(q,L)(\|\mathbb{D}[\boldsymbol{v}]\|_{L^q(\Omega_\varepsilon;R^{3\times3}_{\mathrm{sym}})} + \|\nabla_x \boldsymbol{v}\|_{L^1(\Omega_\varepsilon;R^{3\times3})}), \quad q > 1, \tag{5.7}$$

for any $\boldsymbol{v}$ as in (5.5).

Finally,

$$\|\nabla_x \boldsymbol{v}\|_{L^1(\Omega_\varepsilon;R^{3\times3})} = \|\nabla_x \boldsymbol{v}\|_{L^1(\Omega;R^{3\times3})} + \|\nabla_x \boldsymbol{v}\|_{L^1(\Omega_\varepsilon\setminus\Omega;R^{3\times3})}, \tag{5.8}$$

where, by means of Hölder's inequality,

$$\|\nabla_x \boldsymbol{v}\|_{L^1(\Omega_\varepsilon\setminus\Omega;R^{3\times3})} \leqslant |\Omega_\varepsilon \setminus \Omega|^{1/q'}\|\nabla_x \boldsymbol{v}\|_{L^q(\Omega_\varepsilon;R^{3\times3})}. \tag{5.9}$$

Combining estimates (5.5) and (5.7)–(5.9) we may infer that

$$\|\nabla_x \boldsymbol{v}\|_{L^q(\Omega_\varepsilon;R^{3\times3})} \leqslant c(q,L)\|\mathbb{D}[\boldsymbol{v}]\|_{L^q(\Omega_\varepsilon;R^{3\times3}_{\mathrm{sym}})}, \quad 1 < q < \infty, \tag{5.10}$$

for any

$$\boldsymbol{v} \in W^{1,q}(\Omega_\varepsilon;R^3), \quad \boldsymbol{v}|_{\Gamma_{\mathrm{bot}}} = 0, \tag{5.11}$$

provided that $0 < \varepsilon < \varepsilon_{q,L}$ is small enough.

Remark 5.4. The fact that the constant in Korn's inequality depends only on L was observed by Nitsche [17] in the case in which $q = 2$.

6. Proof of theorem 3.1

6.1. Uniform estimates

Our first goal is to establish bounds on the solutions $\boldsymbol{u}_\varepsilon$ and P_ε independent of $\varepsilon \to 0$. To this end, we use the quantities $\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}$ as test functions in (2.1), where $\tilde{\boldsymbol{V}}$ is a suitable extension of the boundary velocity field $\boldsymbol{V}$ introduced in (2.4). In accordance with hypothesis (1.11), we get

$$\begin{aligned} d_1\|\mathbb{D}[\boldsymbol{u}_\varepsilon]\|^p_{L^p(\Omega_\varepsilon;R^{3\times3}_{\mathrm{sym}})} &+ \int_{\partial\Omega_\varepsilon} \beta_\varepsilon|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}\sigma \\ &\leqslant \int_{\Omega_\varepsilon} \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \mathbb{D}[\boldsymbol{u}_\varepsilon] \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_\varepsilon} \beta_\varepsilon|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}\sigma \\ &= \int_{\Omega_\varepsilon} (\boldsymbol{f}\cdot(\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}) - (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla_x\tilde{\boldsymbol{V}} + \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \mathbb{D}[\tilde{\boldsymbol{V}}]) \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{6.1}$$

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Seeing that

$$\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla_x \tilde{\boldsymbol{V}} \, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \otimes \tilde{\boldsymbol{V}}) : \nabla_x \boldsymbol{u}_\varepsilon \, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \otimes \tilde{\boldsymbol{V}}) : \nabla_x (\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}) \, \mathrm{d}\boldsymbol{x},$$

we obtain

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} (\boldsymbol{f} \cdot (\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}) - (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla_x \tilde{\boldsymbol{V}} + \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \mathbb{D}[\tilde{\boldsymbol{V}}]) \, \mathrm{d}\boldsymbol{x} \right| \\ &\quad \leqslant \|\boldsymbol{f}\|_{L^\infty(\Omega_\varepsilon;R^3)} \|\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}\|_{L^1(\Omega_\varepsilon;R^3)} \\ &\qquad + \|\tilde{\boldsymbol{V}}\|_{L^4(\Omega;R^3)} \|\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}\|_{L^4(\Omega;R^3)} \|\nabla_x (\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}})\|_{L^2(\Omega;R^{3\times 3})} \\ &\qquad + \|\tilde{\boldsymbol{V}}\|^2_{L^4(\Omega;R^3)} \|\nabla_x (\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}})\|_{L^2(\Omega;R^{3\times 3})} \\ &\qquad + + \|\nabla_x \tilde{\boldsymbol{V}}\|_{L^\infty(\Omega;R^3)} \|S(\mathbb{D}[\boldsymbol{u}_\varepsilon])\|_{L^1(\Omega;R^{3\times 3}_{\mathrm{sym}})}, \end{aligned} \tag{6.2}$$

while

$$\begin{aligned} \|\mathbb{D}[\boldsymbol{u}_{\boldsymbol{\varepsilon}}]\|^p_{L^p(\Omega_\varepsilon;R^{3\times 3}_{\mathrm{sym}})} \geqslant \tfrac{1}{2}(\|\mathbb{D}[\boldsymbol{u}_{\boldsymbol{\varepsilon}}]\|^p_{L^p(\Omega_\varepsilon;R^{3\times 3}_{\mathrm{sym}})} + \|\mathbb{D}[\boldsymbol{u}_{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{V}}]\|^p_{L^p(\Omega_\varepsilon;R^{3\times 3}_{\mathrm{sym}})}) \\ - c(p)\|\nabla_x \tilde{\boldsymbol{V}}\|^p_{L^p(\Omega;R^{3\times 3})}. \end{aligned} \tag{6.3}$$

As already pointed out in (2.4), the extension $\tilde{\boldsymbol{V}}$ can be made arbitrarily small in the Lebesgue space $L^4(\Omega; R^3)$. Consequently, it is possible to use estimates (6.1)–(6.3) and hypothesis (1.12), together with Korn's inequality (5.10) applied to $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}$, in order to conclude that

$$\sup_{\varepsilon>0} \int_{\Omega_\varepsilon} |\nabla_x \boldsymbol{u}_\varepsilon|^p + \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} \beta_\varepsilon |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}\sigma < \infty, \tag{6.4}$$

and, by virtue of (1.7),

$$\sup_{\varepsilon>0} \int_{\Omega_\varepsilon} |\boldsymbol{u}_\varepsilon|^p \, \mathrm{d}\boldsymbol{x} < \infty. \tag{6.5}$$

Finally, taking $\boldsymbol{\varphi} = \mathcal{B}_\varepsilon[h]$ for $h \in L^p(\Omega_\varepsilon; R^3)$ as a test function in (2.1) and using (6.4), (6.5), together with proposition 5.2, we obtain

$$\sup_\varepsilon \|P_\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} < \infty, \quad \text{provided that} \int_{\Omega_\varepsilon} P_\varepsilon \, \mathrm{d}\boldsymbol{x} = 0. \tag{6.6}$$

6.2. Weak convergence

Seeing that the family $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ satisfies (4.1), we may assume that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{weakly in } W^{1,p}(\Omega; R^3), \tag{6.7}$$

where, in accordance with (4.11), the limit vector field $\boldsymbol{u}$ satisfies the boundary conditions

$$\boldsymbol{u}|_{\Gamma_{\mathrm{bot}}} = \boldsymbol{V}, \qquad u_1|_{\Gamma_{\mathrm{top}}} = u_3|_{\Gamma_{\mathrm{top}}} = 0. \tag{6.8}$$

Similarly, in agreement with (6.6),

$$P_\varepsilon \to P \quad \text{weakly in } L^{p'}(\Omega), \int_\Omega P \, \mathrm{d}\boldsymbol{x} = 0. \tag{6.9}$$

In order to perform the limit $\varepsilon \to 0$ in the variational formula (2.1), observe first that any test function for the target problem (2.3) may also be used in (2.1). Indeed, the class of functions

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3), \quad \varphi_1, \varphi_3 \in \mathcal{D}(\Omega), \quad \varphi_2 \in \mathcal{D}(\mathcal{T}^2 \times (0, \infty)), \tag{6.10}$$

form a dense subset of test functions for (2.3) in $W^{1,p}(\Omega; R^3)$.

Plugging a function $\boldsymbol{\varphi}$ satisfying (6.10) in (2.1) and letting $\varepsilon \to 0$, we get

$$\begin{aligned}\int_\Omega ((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla_x \boldsymbol{\varphi} + P \operatorname{div} \boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} \\ = \int_\Omega \overline{\mathbb{S}(\mathbb{D}[\boldsymbol{u}])} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \lim_{\varepsilon \to 0} \int_{\partial \Omega_\varepsilon} \beta_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x},\end{aligned} \tag{6.11}$$

where

$$\mathbb{S}(\mathbb{D}[\boldsymbol{u}]) \to \overline{\mathbb{S}(\mathbb{D}[\boldsymbol{u}])} \quad \text{weakly in } L^{p'}(\Omega; R^{3\times 3}_{\mathrm{sym}}). \tag{6.12}$$

In order to identify the limit of the boundary term, we write

$$\int_{\partial \Omega_\varepsilon} \beta_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma = \int_{\mathcal{T}^2} \beta_\varepsilon(\boldsymbol{y}) \boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1 + \Phi_\varepsilon(\boldsymbol{y})) \cdot \boldsymbol{\varphi}(\boldsymbol{y}, 1 + \Phi_\varepsilon(\boldsymbol{y})) \sqrt{1 + |\Phi'_\varepsilon(y_1)|^2} \, \mathrm{d}\boldsymbol{y}, \tag{6.13}$$

where

$$\boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1 + \Phi_\varepsilon(\boldsymbol{y})) - \boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1) = \int_1^{1+\Phi_\varepsilon(\boldsymbol{y})} \partial_{x_3} \boldsymbol{u}_\varepsilon(\boldsymbol{y}, z) \, \mathrm{d}z \quad \text{for a.a. } \boldsymbol{y} \in \mathcal{T}^2.$$

Thus, by virtue of Jensen's inequality, we get

$$|\boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1 + \Phi_\varepsilon(\boldsymbol{y})) - \boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1)|^2 \leqslant \Phi_\varepsilon(\boldsymbol{y}) \int_1^{1+\Phi_\varepsilon(\boldsymbol{y})} |\partial_{x_3} \boldsymbol{u}_\varepsilon(\boldsymbol{y}, z)|^2 \, \mathrm{d}z,$$

whence we obtain

$$\int_{\mathcal{T}^2} |\boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1 + \Phi_\varepsilon(\boldsymbol{y})) - \boldsymbol{u}_\varepsilon(\boldsymbol{y}, 1)|^2 \, \mathrm{d}\boldsymbol{y} \leqslant \varepsilon \int_{\Omega_\varepsilon} |\nabla_x \boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}\boldsymbol{x}. \tag{6.14}$$

Relations (6.13), (6.14) give rise to

$$\int_{\partial \Omega_\varepsilon} \beta_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma \to \int_{\partial \Omega} \tilde{\beta} \boldsymbol{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{y} \tag{6.15}$$

for any $\boldsymbol{\varphi}$ satisfying (6.10), where

$$\tilde{\beta} = L^1\text{-weak } \lim_{\varepsilon \to 0} \beta_\varepsilon \sqrt{1 + |\Phi'_\varepsilon|^2}, \tag{6.16}$$

in complete agreement with (2.3), (3.5).

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6.3. Strong convergence

In order to complete the proof of theorem 3.1, it is sufficient to establish the strong convergence of the velocity gradients as well as of the pressure as claimed in (3.4). Note that, as soon as this is achieved, we obtain $\overline{\mathbb{S}(\mathbb{D}[\boldsymbol{u}])} = \mathbb{S}(\mathbb{D}[\boldsymbol{u}])$, which converts (6.11) to the desired identity, (2.3).

To this end, we use the classical monotonicity argument. Note that, since $p \geqslant 2 \geqslant \frac{9}{5}$, we are allowed to use the quantity $\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}}$ as a test function in (2.1), which facilitates the analysis considerably.

Accordingly, we get

$$\int_{\Omega_\varepsilon} \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \nabla_x(\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{V}})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_\varepsilon} \beta_\varepsilon |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}\sigma = \int_{\Omega_\varepsilon} \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon \,\mathrm{d}\boldsymbol{x} - \int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla_x \tilde{\boldsymbol{V}}\,\mathrm{d}\boldsymbol{x}.$$

Letting $\varepsilon \to 0$ and using (6.14) we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \nabla_x \boldsymbol{u}_\varepsilon \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \int_{\Omega}(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\tilde{\boldsymbol{V}}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega} \overline{\mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon])} : \nabla_x \tilde{\boldsymbol{V}}\,\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega}\tilde{\beta}|\boldsymbol{u}|^2\,\mathrm{d}\sigma. \tag{6.17}$$

On the other hand, setting $\boldsymbol{\varphi} = \boldsymbol{u} - \tilde{\boldsymbol{V}}$ in (6.11) we can compute the right-hand side in (6.17) in order to conclude that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) : \nabla_x \boldsymbol{u}_\varepsilon \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \overline{S(\mathbb{D}[\boldsymbol{u}])} : \nabla_x \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}; \tag{6.18}$$

in particular,

$$\lim_{\varepsilon\to 0}\int_{\Omega} (\mathbb{S}(\mathbb{D}[\boldsymbol{u}_\varepsilon]) - \mathbb{S}(\mathbb{D}[\boldsymbol{u}])) : (\mathbb{D}(\boldsymbol{u}_\varepsilon) - \mathbb{D}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x} = 0. \tag{6.19}$$

As $\mathbb{S}$ satisfies hypotheses (1.11)–(1.13), relations (6.18), (6.19) imply that

$$\mathbb{D}[\boldsymbol{u}_\varepsilon] \to \mathbb{D}[\boldsymbol{u}] \quad \text{in } L^p(\Omega; R^{3\times 3}_{\mathrm{sym}}),$$

yielding

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } W^{1,p}(\Omega; R^3). \tag{6.20}$$

The last step is to establish the pointwise convergence of the pressure $\{P_\varepsilon\}_{\varepsilon>0}$. To this end, we consider test functions of the form

$$\boldsymbol{\varphi}_\varepsilon = \mathcal{B}\biggl[|P_\varepsilon|^{p'-2}P_\varepsilon - \int_{\Omega}|P_\varepsilon|^{p'-2}P_\varepsilon\,\mathrm{d}\boldsymbol{x}\biggr],$$

where $\mathcal{B}$ is the Bogovskii operator constructed in proposition 5.2 associated to Ω. Note that, by virtue of (6.6),

$$|P_\varepsilon|^{p'-2}P_\varepsilon \quad \text{are bounded in } L^p(\Omega)$$

uniformly for $\varepsilon \to 0$.

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Similarly, we take

$$\varphi = \mathcal{B}\left[\overline{|P|^{p'-2}P} - \int_\Omega \overline{|P|^{p'-2}P}\,\mathrm{d}\boldsymbol{x}\right].$$

Using φ_ε, φ as test functions in (2.1), (2.3, respectively, we conclude that

$$\lim_{\varepsilon\to 0}\int_\Omega |P_\varepsilon|^{p'}\,\mathrm{d}\boldsymbol{x} = \int_\Omega P\overline{|P|^{p'-2}P}\,\mathrm{d}\boldsymbol{x}, \tag{6.21}$$

whence we obtain

$$P_\varepsilon \to P \quad \text{in } L^{p'}(\Omega).$$

Theorem 3.1 has been proved.

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