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THE MOUSE SET CONJECTURE FOR SETS OF REALS

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Abstract. We show that the *Mouse Set Conjecture* for sets of reals is true in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular". As a consequence, we get that below $AD_{\mathbb{R}}$ + "Θ is regular", models of $AD^+ + \neg AD_{\mathbb{R}}$ are hybrid mice over $\mathbb{R}$. Such a representation of models of AD^+ is important in core model induction applications.

One of the central open problems in descriptive inner model theory is the conjecture known as the *Mouse Set Conjecture* (MSC). It conjectures that under AD^+ ordinal, definable reals are exactly those that appear in ω_1-iterable mice. The counterpart of this conjecture for sets of reals conjectures that under AD^+, the sets of reals which are ordinal definable from a real are exactly those that appear in countably iterable mice over $\mathbb{R}$. In [3], the first author proved that MSC holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular", but MSC for sets of reals was left open. The goal of this paper is to establish that MSC for sets of reals holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular".

We will establish a stronger form of MSC known as the *Strong Mouse Set Conjecture* $(SMSC)$. We say $\mathcal{M}$ is countably κ-iterable if all of its sufficiently elementary countable substructures are κ-iterable. We say $\mathcal{M}$ is countably iterable if $\mathcal{M}$ is countably ω_1-iterable. Thus, under AD, if $\mathcal{M}$ is countably iterable then $\mathcal{M}$ is countably $\omega_1 + 1$-iterable.

In what follows, we will let "hod pair" stand for a hod pair below $AD_{\mathbb{R}}$ + "Θ is regular", i.e., the corresponding hod mouse cannot have inaccessible limit of Woodin cardinals (see Definition 1.34 of [3]). Given an iteration strategy Σ for a countable structure, we let $Code(\Sigma)$ be the set of reals coding Σ for trees of length ω_1. Given a hod pair $(\mathcal{P}, \Sigma)$ we let

$$Lp^{\Sigma}(\mathbb{R}) = \cup \{\mathcal{M} : \mathcal{M} \text{ is a sound countably iterable } \Sigma\text{-mouse over } \mathbb{R} \text{ projecting to } \mathbb{R}\}.$$

The following is the statement of $SMSC$ for sets of reals. Recall the notions of branch condensation and fullness preservation from [3] (see Definition 2.14 and

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Definition 2.27 of [3]). Recall that OD_X stands for the class of sets ordinal definable from a finite sequence consisting of members of X.

The Strong Mouse Set Conjecture for sets of reals, $SMSC(\mathbb{R})$: Assume AD^+. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is fullness preserving. Then

$$\{A \subseteq \mathbb{R} : \exists x \in \mathbb{R}(A \text{ is } OD_{\{\Sigma, x\}})\} = Lp^{\Sigma}(\mathbb{R}).$$

Following is the main theorem of this paper.

THEOREM 0.1. *Assume $AD^+ + V = L(\wp(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that the following holds.*

1. *$\mathcal{P}$ does not have inaccessible limit of Woodin cardinals.*
2. *Σ has branch condensation and is fullness preserving.*
3. *MSC for Σ holds, i.e., for every $x, y \in \mathbb{R}$, $x \in OD(\Sigma, y)$ iff x is in a Σ-mouse over y.*
4. *Every set of reals A is $OD(\Sigma, x)$ for some real x.*

Then

$$\wp(\mathbb{R}) = \wp(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R}).$$

In particular, $V = L(Lp^{\Sigma}(\mathbb{R}))$.

COROLLARY 0.2. *Suppose $V = L(\wp(\mathbb{R}))$ and AD^+ holds. Suppose further that for any α such that $\theta_\alpha < \Theta$, letting $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$, $L(\Gamma, \mathbb{R}) \vDash \neg AD_{\mathbb{R}}$. Then $SMSC(\mathbb{R})$ holds. In particular, $SMSC(\mathbb{R})$ is true in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular".*

PROOF. It is shown in [3] that if $(\mathcal{P}, \Sigma)$ is as in the hypothesis of Theorem 0.1 then clause 3 holds in $L(\Gamma_{\alpha+1})$, where α is such that $\theta_\alpha = w(Code(\Sigma))$ and $\Gamma_{\alpha+1} = \{A \subseteq \mathbb{R} : w(A) < \theta_{\alpha+1}\}$. It then follows from Theorem 0.1 that $\Gamma_\alpha = \wp(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})$ implying that $SMSC(\mathbb{R})$ holds. ⊣

All the background material that we will need in this paper is spelled out in [3]. We assume that our reader is familiar with some aspects of it. One important comment is that, in general, hybrid mice over $\mathbb{R}$ or any nonself-wellordered set are not defined (recall that a set X is self-wellordered if there is a wellordering of it in $\mathcal{J}_\omega(X)$). Given an iteration strategy Σ with hull condensation, the Σ-mice over self-wellordered sets are defined according to the following principle. At a typical stage where we would like to add more of Σ to the model, we choose the least tree $\mathcal{T}$ for which $\Sigma(\mathcal{T})$ hasn't been defined. However, $\mathbb{R}$ isn't self-wellordered and hence, we cannot choose the least such $\mathcal{T}$.

In [3], the first author gave a definition of premice over any nonself-wellordered sets under the hypothesis that $\mathcal{M}_1^{\#,\Sigma}$ exists, i.e., there is a minimal active Σ-mouse with one Woodin cardinal (see Definition 3.37 of [3]). This extra assumption is benign as under AD^+ whenever $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is fullness preserving, $\mathcal{M}_1^{\#,\Sigma}$ exists and is Θ-iterable. The proof is the same as the proof that shows that $AD^{L(\mathbb{R})}$ implies that $\mathcal{M}_1^{\#}$ exists and is Θ-iterable in $L(\mathbb{R})$ (see [11]). One consequence of the indexing of the strategy introduced in Definition 3.37 of [3] is that it allows us to perform S-constructions, which we will use in this paper (see Chapter 3 of [3]).

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Corollary 0.2 has been used in core model induction applications. See, for instance, [4], [2], [5], or Chapter 7 of [7]. Before we begin the proof of Theorem 0.1, we introduce Prikry tree forcing associated with Martin's measure on degrees.

§1. Prikry tree forcing on degrees. We develop the notion of Prikry forcing that we need in a general context. Assume $ZF - Replacement + AD$. Let $\mathcal{D}$ be the set of Turing degrees. Let $f : \mathcal{D}^{<\omega} \to HC$ be some function. We would like to define *Prikry tree forcing* on degrees associated to f. Let μ be Martin's measure. We let $(p, A) \in \mathbb{P}^f$ if

1. $p \in \mathcal{D}^{<\omega}$,
2. for any $n < lh(p)$, $(f(p \restriction n), p \restriction n) \in L[p(n)]$,
3. $A \subseteq \cup_{n<\omega} \mathcal{D}^{<n}$ is a tree with stem p such that for every $q \in A$ (in particular, $p \subseteq q$),
$$\{d : q^\frown d \in A\} \in \mu.$$

Given $(p, A), (q, B) \in \mathbb{P}^f$ we let

$$(p, A) \preccurlyeq (q, B) \text{ iff } p \text{ end-extends } q, A \subseteq B \text{ and } p \in B.$$

We say $p \in \mathcal{D}^{<\omega}$ is a *precondition* if it satisfies 1 and 2 above. Given a precondition p and $d \in \mathcal{D}$, we say d is *valid* at p if $p^\frown d$ is a precondition.

Given a $\mathbb{P}^f$-generic G we let $g = \cup\{p : \exists X (p, X) \in G\}$. We then let

$$G^i =_{def} f(g \restriction i + 1) \text{ and } f(G) =_{def} \cup_{i<\omega} G^i.$$

The following is proved by a standard fusion argument.

LEMMA 1.1. *$\mathbb{P}^f$ has the Prikry property. More precisely, suppose Z is a countable set of $\mathbb{P}^f$-terms, ϕ is a formula, and $(p, A) \in \mathbb{P}^f$. Then there is a condition $(p, W) \in \mathbb{P}^f$ deciding $\phi[\tau]$ for all $\tau \in Z$ such that $W \in OD_{Z,\{f,p,A\}}$.*

PROOF. We will show that there is a condition (p, T_τ) deciding $\phi[\tau]$ such that $\langle T_\tau : \tau \in Z\rangle \in OD_{Z,\{f,p,A\}}$. It then follows that $(p, \cap_{\tau\in Z} T_\tau)$ is as desired. We say q is *positive* if $(\exists Y)\ ((q, Y) \Vdash \phi[\tau])$, *negative* if $(\exists Y)((q, Y) \Vdash \neg\phi[\tau])$, and *ambiguous* if it is neither positive nor negative. Notice that q cannot be both positive and negative. Fixing τ, we shrink A to some tree T such that given any $r \in T$ and any one step extensions $q_1, 1_2 \in T$ of r, both q_1 and q_2 are simultaneously ambiguous, positive, or negative.

We define a sequence of functions $\langle H^i : i < \omega\rangle$ such that

$$dom(H^i) = \{q : p \trianglelefteq q \text{ and } q \text{ is a precondition}\}$$

and $rng(H^i) \subseteq \{0, 1, 2\}$. First define H on $\{(q, d) : q^\frown d \text{ is a precondition}\}$ by

$$H(q, d) = \begin{cases} 0: & q^\frown d \text{ is positive,} \\ 1: & q^\frown d \text{ is negative,} \\ 2: & q^\frown d \text{ is ambiguous.} \end{cases}$$

Now, let $H^0(q) = i$ if for μ-a.e. d is such that $H(q, d) = i$. Given $\langle H^i : i \leq k\rangle$ define H^{k+1} by setting $H^{k+1}(q) = i$ if for μ-a.e. d is such that $H^k(q^\frown d) = i$.

We then define a decreasing sequence of conditions (p, T^i) by induction as follows. We will have that $(p, T^0) \preceq (p, A)$. We define T^0 by induction on the length

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of conditions. We let $T^0 \restriction m$ be T^0 restricted to sequences of length m. Suppose we have defined $T^0 \restriction m+1$ for $m+1 \geq lh(p)$. Given $q \in T^0 \restriction m$ such that $lh(q) = m$ we let

$$\{q^\frown d \in A : H(q,d) = H^0(q)\}$$

be the one step extensions of q in T^0. This finishes our description of T^0.

Suppose now we have defined $\langle (p, T^i) : i \leq k\rangle$ and $T^{k+1} \restriction m+1$. Given $q \in T^{k+1} \restriction m$ such that $lh(q) = m$, we let

$$\{q^\frown d \in T^k : H^k(q^\frown d) = H^{k+1}(q)\}$$

be the one step extensions of q in T^{k+1}. This finishes our description of $\langle (p, T^i) : i \leq \omega\rangle$. Let $T_\tau = \cap_{i<\omega} T^i$.

We claim that (p, T_τ) decides τ. Suppose not. We then have two conditions (q, X) and (r, Y) such that both are below (p, T_τ) and

1. $lh(q) = lh(r)$,
2. $(q, X) \Vdash \phi[\tau]$,
3. $(r, Y) \Vdash \neg\phi[\tau]$.

Let now s be the common initial segment of q and r. Let $s = (d_i : i \leq m)$, $q = s^\frown(q_i : i < n)$, and $r = s^\frown(r_i : i < n)$. It follows from our construction that

$$\begin{aligned} H(s^\frown(q_i : i < n-1), q_{n-1}) &= H^0(s^\frown(q_i : i \leq n-1)) = H^1(s^\frown(q_i : i < n-2)) \\ &= \cdots = H^{n-1}(s), \\ H(s^\frown(r_i : i < n-1), r_{n-1}) &= H^0(s^\frown(r_i : i \leq n-1)) = H^1(s^\frown(r_i : i < n-2)) \\ &= \cdots = H^{n-1}(s). \end{aligned}$$

It then follows that $H(s^\frown(q_i : i < n-1), q_{n-1}) = H(s^\frown(r_i : i < n-1), r_{n-1})$, which is a contradiction. ⊣

We now turn to prove Theorem 0.1.

§2. The proof. We assume $AD^+ + V = L(\wp(\mathbb{R}))$ and let $(\mathcal{P}, \Sigma)$ be as in the hypothesis of Theorem 0.1. Given a good pointclass Γ[1] and $a \in HC$, we let $Lp^{\Gamma,\Sigma}(a)$ be the union of sound Σ-mice over a projecting to a whose iteration strategy is coded by a set in Γ. Our first lemma is an easy lemma. Below, $MC(\Sigma)$ (mouse capturing relative to Σ) is the statement that for every $x, y \in \mathbb{R}$, $x \in OD_{\Sigma,y}$ if and only if there is an ω_1-iterable sound Σ-mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$.

Lemma 2.1. *For any good pointclass $\Gamma \neq \Sigma^2_1(Code(\Sigma))$ there is a good pointclass $\Gamma_1 \neq \Sigma^2_1(Code(\Sigma))$ such that $\Gamma \cup \{Code(\Sigma)\} \subseteq \Gamma_1$ and for any $a \in HC$,*

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1,\Sigma}(a).$$

Proof. Fix a good pointclass $\Gamma \neq \Sigma^2_1(Code(\Sigma))$. Because $MC(\Sigma)$ holds, using $\Sigma_1(Code(\Sigma))$-reflection, we can find Γ_1 and α such that $\Gamma_1 \neq \Sigma^2_1(Code(\Sigma))$, $\Gamma \cup \{Code(\Sigma)\} \subseteq \Delta_{\Gamma_1}$, $\mathcal{J}_\alpha(\Gamma_1, \mathbb{R}) \vDash ZF - Replacement$, $\Gamma_1 = (\Sigma^2_1(Code(\Sigma)))^{\mathcal{J}_\alpha(\Gamma_1,\mathbb{R})}$ and

$$\mathcal{J}_\alpha(\Gamma_1, \mathbb{R}) \vDash MC(\Sigma).$$ ⊣

[1] i.e., a point class closed under $\exists^{\mathbb{R}}$, continuous preimages and images, and having the scale property

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Suppose now that $\wp(\mathbb{R}) \neq Lp^{\Sigma}(\mathbb{R})$. Using Σ_1-reflection we get $\Gamma \subset \underset{\sim}{\Delta}^2_1(Code(\Sigma))$ and $\alpha < \underset{\sim}{\delta}^2_1(Code(\Sigma))$ such that

1. $\Gamma = \wp(\mathbb{R}) \cap \mathcal{J}_\alpha(\Gamma, \mathbb{R})$ and α ends a Σ_1-gap,
2. $\mathcal{J}_\alpha(\Gamma, \mathbb{R}) \vDash \phi$ where ϕ is the conjunction of the following statements:
   (a) $ZF - Replacement + DC_{\mathbb{R}} + MC(\Sigma)$,
   (b) there is an OD set of reals A such that $A \notin Lp^{\Sigma}(\mathbb{R})$.

We let $N = \mathcal{J}_\alpha(\Gamma, \mathbb{R})$. Let U be the set of pairs $(x, y) \in \mathbb{R}^2$ such that y codes a sound Σ-mouse $\mathcal{M}$ over x that projects to x and has an ω_1-iteration strategy in N.

Since $MC(\Sigma)$ holds in N, U is a universal $(\Sigma^2_1(Code(\Sigma)))^N$-set. Let $A \in N$ be an OD set of reals witnessing clause (b) of ϕ. We assume that A has the minimal Wadge rank among the sets witnessing clause b of ϕ. Using the results of Chapter 3 of [1], we can get $\vec{B} = \langle B_i : i < \omega \rangle$ which is a semiscale on U^c such that each $B_i \in (OD_\Sigma)^N$. The following fact is a well known consequence of $MC(\Sigma)$.

PROPOSITION 2.2. *The following statements are true.*

1. *There is a cone of x such that there is $\mathcal{M} \trianglelefteq Lp^{\Sigma}(x)$ such that $\rho_\omega(\mathcal{M}) = x$ and $\mathcal{M}$ doesn't have an iteration strategy in N.*
2. *Let x be a base of the above cone. Then for every $a \in HC$ such that $x \in \mathcal{J}_\omega(a)$, there is $\mathcal{M} \trianglelefteq Lp^{\Sigma}(a)$ such that $\rho(\mathcal{M}) = a$ and $\mathcal{M}$ doesn't have an iteration strategy in N.*

PROOF. Clause 2 follows from clause 1. To see this, fix a real x such that it is base for the cone of clause 1. Then whenever $a \in HC$ is such that $x \in a$ and y is a real coding a generically over $Lp^{\Sigma}(a)$, then $Lp^{\Sigma}(a)[y] = Lp^{\Sigma}(y)$. Indeed, this follows from S-constructions (see Section 2.11 of [3]).

Clause 1 is an easy consequence of $MC(\Sigma)$. Indeed, suppose clause 1 fails. Then

$$(1)\ \text{there is an } x \in \mathbb{R} \text{ such that for all } y \in \mathbb{R} \text{ such that } x \leq_T y,\ Lp^{\Sigma}(y) = (Lp^{\Sigma}(y))^N.$$

Because $MC(\Sigma)$ holds, letting C be the set of pairs $(y, z) \in \mathbb{R}^2$ such that $x \leq_T y$ and z codes an ω_1-iterable sound Σ-mouse $\mathcal{M}$ over y projecting to y, C is a universal $\Sigma^2_1(Code(\Sigma))$ set. Because $\Gamma \subset \underset{\sim}{\Delta}^2_1(Code(\Sigma))$, we cannot have that C is the universal $(\Sigma^2_1(Code(\Sigma)))^N$ set. It follows from (1), however, that $C \in N$ and $N \vDash$ "C is the universal $\Sigma^2_1(Code(\Sigma))$-set", contradiction. ⊣

Let now x be a base of the cone from clause 1 of Proposition 2.2. We say a is *good* if $a \in HC$ and $x \in \mathcal{J}_\omega(a)$. For each good a let $\mathcal{M}(a)$ be the least Σ-mouse with no iteration strategy in N. Let F be the set of pairs $(a, \mathcal{M}(a))$. It follows that if F^* is the set of reals coding F then $F^* \in \underset{\sim}{\Delta}^2_1(Code(\Sigma))$. Furthermore, there is a set $C \in \underset{\sim}{\Delta}^2_1(Code(\Sigma))$ such that for every good a, the set of reals coding the unique iteration strategy of $\mathcal{M}(a)$ is Wadge reducible to C. Let then

$$D = \{(y, \sigma) \in \mathbb{R}^2 : y \text{ codes a good } a \text{ and } \sigma \text{ codes a continuous function } f \text{ such that } f^{-1}[C] \text{ is the iteration strategy of } \mathcal{M}(a)\}.$$

We have that $D \in \underset{\sim}{\Delta}^2_1(Code(\Sigma))$.

Let Γ_1 be a good pointclass such that $F^*, Code(\Sigma), \vec{B}, U, C, D \in \underset{\sim}{\Delta}_{\Gamma_1}$. Moreover, it follows from Lemma 2.1 that we can require that for any $a \in HC$

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1, \Sigma}(a).$$

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Let now $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ be as in Theorem 1.2.9 of [3] with the property that $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Code(\Sigma), \vec{B}, U, C, D$ (where Suslin capturing is defined on page 36 of [3], also see the next paragraph)[2]. We have that for any $\eta < \delta_z$, $C_{\Gamma_1}(\mathcal{N}_z^*|\eta) \in \mathcal{N}_z^*$.

Let $\Phi = (\underset{\sim}{\Sigma}^2_1)^N$. We have that Φ is a good pointclass. Because $\vec{B}$ is Suslin captured by $\mathcal{N}_z^*$, we have $(\delta_z^+)^{\mathcal{N}_z^*}$-complementing trees $T, S \in \mathcal{N}_z^*$ which capture $\vec{B}$ in the sense that whenever $i : \mathcal{N}_z^* \to \mathcal{N}$ is an iteration embedding according to Σ_z and g is a generic over $\mathcal{N}$ for a poset of size $\leq i(\delta_z)$ then

$$p[i(T)] \cap \mathcal{N}[g] = B \cap \mathcal{N}[g].$$

Let κ be the least cardinal of $\mathcal{N}_z^*$ which is $< \delta_z$-strong in $\mathcal{N}_z^*$.

Next, we fix a notation. For each $a \in HC$, we let $\mathcal{W}(a) = Lp^{\Gamma,\Sigma}(a)$. Using the results of Section 2.11 of [3], we have that if $g \subseteq Coll(\omega, a)$ is $\mathcal{W}(a)$-generic then

$$\mathcal{W}(a)[g] = \mathcal{W}(a, g) = \mathcal{W}(x_g), \tag{1}$$

where x_g is the generic real coding a. The following claim is standard.

LEMMA 2.4. *$\mathcal{N}_z^* \vDash$ "κ is a limit of cardinals η such that η is a Woodin cardinal in $\mathcal{W}(\mathcal{N}_z^*|\eta)$".*

PROOF. Working in $\mathcal{N}_z^*$, let $\lambda = \delta_z^{++}$ and let $\pi : M \to \mathcal{N}_z^*|\lambda$ be an elementary substructure such that

1. $T, S \in ran(\pi)$ and
2. letting $crit(\pi) = \eta$, $V_\eta^{\mathcal{N}_z^*} \subseteq M$, $\pi(\eta) = \delta_z$ and $\eta > \kappa$.

By elementarity, we have that $M \vDash$ "η is Woodin". Letting $\pi^{-1}(\langle T, S\rangle) = \langle \bar{T}, \bar{S}\rangle$, we have that $(\bar{T}, \bar{S})$ Suslin captures $\vec{B}$ over M at $(\eta^+)^M$. This implies that whenever $a \in M|(\eta^+)^M$, $\mathcal{W}(a) \in M$. To see this, first note that we have that whenever $g \subseteq Coll(\omega, a)$ is M-generic and x_g is the generic real then $\mathcal{W}(x_g) \in M$. But using (1) above, we have that $\mathcal{W}(x_g) = \mathcal{W}(a)[g]$. Therefore, $\mathcal{W}(a) \in M[g]$. Since g was arbitrary, we have that $\mathcal{W}(a) \in M$.

We now have that $\mathcal{W}(\mathcal{N}^*|\eta) \in M$ and since $M \vDash$ "η is Woodin", we have that $\mathcal{W}(\mathcal{N}^*|\eta) \vDash$ "η is Woodin". Because κ is $< \delta_z$-strong in $\mathcal{N}_z^*$ and because $a \to \mathcal{W}(a)$ is definable over $\mathcal{N}_z^*$, we have that for unboundedly many $\nu < \eta$, $\mathcal{W}(\mathcal{N}_z^*|\nu) \vDash$ "ν is Woodin". ⊣

[2] For convenience, we restate Theorem 1.2.9 of [3].

THEOREM 2.3 (Woodin, Theorem 10.3 of [9]). *Assume AD^+. Suppose Γ is a good pointclasses and there is a good pointclass Γ^* such that $\Gamma \subseteq \Delta_{\Gamma^*}$. Suppose (N, Ψ) Suslin, co-Suslin capture Γ. There is then a function F defined on $\mathbb{R}$ such that for a Turing cone of x, $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ such that*

1. *$N \in L_1[x]$,*
2. *$\mathcal{N}_x^*|\delta_x = \mathcal{M}_x|\delta_x$,*
3. *$\mathcal{M}_x$ is a Ψ-mouse: in fact, $\mathcal{M}_x = \mathcal{M}_1^{\Psi,\#}(x)|\kappa_x$ where κ_x is the least inaccessible cardinal of $\mathcal{M}_1^{\Psi,\#}$,*[3]
4. *$\mathcal{N}_x^* \vDash$ "δ_x is the only Woodin cardinal",*
5. *Σ_x is the unique iteration strategy of $\mathcal{M}_x$,*
6. *$\mathcal{N}_x^* = L(\mathcal{M}_x, \Lambda)$ where Λ is the restriction of Σ_x to stacks $\vec{\mathcal{T}} \in \mathcal{M}_x$ that have finite length and are based on $\mathcal{M}_x \restriction \delta_x$,*
7. *$(\mathcal{N}_x^*, \Sigma_x)$ Suslin, co-Suslin captures $Code(\Psi)$ and hence, $(\mathcal{N}_x^*, \Sigma_x)$ Suslin, co-Suslin captures Γ,*
8. *$(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ is a self-capturing background triple.*

[3] $\mathcal{M}_1^{\Psi,\#}$ is the minimal Ψ-mouse having a Woodin cardinal and a last extender.

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2.1. A Σ-mouse beyond N. In this section, we prove the following important lemma.

LEMMA 2.5. *There is a Σ-mouse $\mathcal{N}$ such that there is a sequence $(\gamma_i : i < \omega)$ with the property that*

1. *$(\gamma_i : i < \omega)$ is the sequence of Woodin cardinals of $\mathcal{N}$,*
2. *letting $\gamma = \sup_{i<\omega} \gamma_i$, $\rho_\omega(\mathcal{N}) < \gamma$,*
3. *for some $k < \omega$, $\mathcal{N}$ is a sound Σ-mouse over $\mathcal{N}|\gamma_k$,*
4. *for every cutpoint cardinal $\eta < \gamma$, $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathcal{W}(\mathcal{N}|\eta)$, and*
5. *letting Λ be the $(\omega, \omega_1, \omega_1)$-strategy of $\mathcal{N}$, $Code(\Lambda) \notin \Gamma$ and Λ is Γ-fullness preserving (i.e., clause 4 holds for any Λ-iterate of $\mathcal{N}$).*

We now begin the proof of Lemma 2.5. We continue with previous subsection's notation and start working in $\mathcal{N}_z^*$. Our aim is to use fully backgrounded constructions of $\mathcal{N}_z^*$ to produce a mouse $\mathcal{N}^*$ such that for some l, $\mathcal{N} = \mathcal{C}_l(\mathcal{N}^*)$ has the desired properties. Let $\langle \eta_i : i < \omega \rangle$ be the first ω cardinals below κ such that for every $i < \omega$, $\mathcal{W}(\mathcal{N}_z^*|\eta_i) \vDash$ "η_i is a Woodin cardinal" (it follows from Lemma 2.4 that there are such cardinals). Let now $\langle \mathcal{N}_i : i < \omega \rangle$ be a sequence constructed according to the following rules:

1. $\mathcal{N}_0 = (\mathcal{J}^{\vec{E},\Sigma})^{\mathcal{N}_z^*|\eta_0}$,
2. $\mathcal{N}_{i+1} = (\mathcal{J}^{\vec{E},\Sigma}[\mathcal{N}_i])^{\mathcal{N}_z^*|\eta_{i+1}}$.

Let $\mathcal{N}_\omega = \cup_{i<\omega} \mathcal{N}_i$.

CLAIM 2.6. *For every $i < \omega$, $\mathcal{N}_\omega \vDash$ "η_i is a Woodin cardinal" and $\mathcal{N}_\omega|(\eta_i^+)^{\mathcal{N}_\omega} = \mathcal{W}(\mathcal{N}_i)$.*

PROOF. It is enough to show that

1. $\mathcal{N}_{i+1} \vDash$ "η_i is a Woodin cardinal",
2. no level of $\mathcal{N}_{i+1}$ projects across η_i, and
3. $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = \mathcal{W}(\mathcal{N}_i)$.

To show 1–3, it is enough to show that if $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(\mathcal{Q}) \leq \eta_i$ then the fragment of the iteration strategy of $\mathcal{Q}$ that acts on trees above η_i is coded by a set in Γ (this is simply because $\mathcal{N}_{i+1}$ is Γ-full). Fix then i and let $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$ be such that $\rho_\omega(\mathcal{Q}) \leq \eta_i$. Let ξ be such that if $\mathcal{S}$ is the ξth model of the fully backgrounded construction producing $\mathcal{N}_{i+1}$ then $\mathcal{Q}$ is the core of $\mathcal{S}$. Let $\pi : \mathcal{Q} \to \mathcal{S}$ be the uncollapse map. It is a fine structural map but that is irrelevant and we suppress this point.

Let $\nu < \eta_{i+1}$ be a cardinal such that $\mathcal{S}$ is the ξth model of the full background construction of $\mathcal{N}_z^*|\nu$. Let Ψ be the fragment of Σ_z that acts on nondropping trees that are based on $\mathcal{N}_z^*|(\nu^+)^{\mathcal{N}_z^*}$ and are above η_i. We have that Ψ induces an iteration strategy Ψ^* for $\mathcal{S}$ and that π-pullback of Ψ^* is an iteration strategy for $\mathcal{Q}$. It is then enough to show that $Code(\Psi) \in \Gamma$.

Notice that whenever $\mathcal{T}$ is a tree on $\mathcal{N}_z^*|(\nu^+)^{\mathcal{N}_z^*}$ according to Ψ and $b = \Psi(\mathcal{T})$ then $\mathcal{Q}(b, \mathcal{T})$ is defined. Also, notice that because of our choice of η_{i+1}, for any such $\mathcal{T}$ and b, $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$. Because the function $a \to \mathcal{W}(a)$ is coded by a set in Γ, we have that $Code(\Psi) \in \Gamma$. ⊣

CLAIM 2.7. *There is $\mathcal{Q} \trianglelefteq (\mathcal{J}^{\vec{E},\Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$ such that $\rho_\omega(\mathcal{Q}) < \eta_\omega$.*

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PROOF. To see this suppose not. Let $\mathcal{R} = (\mathcal{J}^{\vec{E},\Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$. It follows from universality of $\mathcal{R}$ (with respect to Σ-mice that have iteration strategies in Γ), we have that

$$Lp^{\Gamma_1,\Sigma}(\mathcal{N}_\omega) \trianglelefteq \mathcal{R}.$$

It follows from our choice of Γ_1 and from our hypothesis that $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$. Notice that if $\mathcal{M}(a)$ is defined for some a then because of our choice of Γ_1, $\mathcal{M}(a) \trianglelefteq Lp^{\Gamma_1,\Sigma}(a)$.

We claim that $\mathcal{M}(\mathcal{N}_0)$ is defined. To see this, notice that x is generic over $\mathcal{J}[\mathcal{N}_0]$ for the extender algebra at η_0. Hence, if $g \subseteq Coll(\omega, \eta_0)$ is $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$-generic such that $x \in Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)[g]$, then by the results of Section 2.11 of [3], we have that

$$Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)[g] = Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0[g]).$$

But now, because $x \in \mathcal{J}[\mathcal{N}_0][g]$, we have that $\mathcal{M}(\mathcal{N}_0[g])$ is defined, and by our choice of Γ_1, we have that $\mathcal{M}(\mathcal{N}_0[g]) \trianglelefteq Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0[g])$. Again using the results of Section 2.11 of [3], we have that some initial segment of $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$ has an iteration strategy which is not coded by a set of reals in Γ. Hence, $\mathcal{M}(\mathcal{N}_0)$ is defined.

Because $\mathcal{M}(\mathcal{N}_0)$ is defined, we have that $\mathcal{M}(\mathcal{N}_0) \trianglelefteq Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$ and therefore, $\mathcal{M}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$. However, it follows from the proof of Claim 1 that all initial segments of $\mathcal{N}_\omega$ projecting to η_0 have an iteration strategy coded by a set in Γ. This implies that $\mathcal{M}(\mathcal{N}_0)$ has an iteration strategy coded by a set in Γ, contradiction! ⊣

Let now $\mathcal{N}^* \trianglelefteq Lp(\mathcal{N}_\omega)$ be least such that $\rho_\omega(\mathcal{N}^*) < \eta_\omega$. Let l be least such that $\rho_l(\mathcal{N}^*) < \eta_\omega$ and let k be least such that $\rho_l(\mathcal{N}^*) < \eta_k$. In what follows, we will regard $\mathcal{N}^*$ as a Σ-mouse over $\mathcal{N}^*|\eta_k$. We let $\mathcal{N} = \mathbb{C}_l(\mathcal{N}^*)$. Thus, $\mathcal{N}$ is sound (as a Σ-mouse over $\mathcal{N}|\eta_k$). We let $\langle \gamma_i : i < \omega \rangle$ be the Woodin cardinals of $\mathcal{N}$ and $\gamma = \sup_{i<\omega} \gamma_i$. Let Λ be the $(\omega, \omega_1, \omega_1)$-strategy of $\mathcal{N}$ induced by Σ_z. Notice that $Code(\Lambda) \notin \Gamma$ because otherwise, since $\mathcal{N}_\omega$ is Γ-full, $\mathcal{N} \trianglelefteq \mathcal{N}_\omega \trianglelefteq \mathcal{N}^*$.

CLAIM 2.8. *Λ is Γ-fullness preserving.*

PROOF. To see this fix $\mathcal{N}_1$ which is a Λ-iterate of $\mathcal{N}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $i : \mathcal{N} \to \mathcal{N}_1$ exists. If $\mathcal{N}_1$ isn't Γ-full then there is a cutpoint ν of $\mathcal{N}_1$ and a sound Σ-mouse $\mathcal{Q}$ over $\mathcal{N}_1|\nu$ with (ω, ω_1)-iteration strategy Ψ such that $Code(\Psi) \in \Gamma$, $\rho_\omega(\mathcal{Q}) = \nu$ and $\mathcal{Q} \not\trianglelefteq \mathcal{N}_1$.

SUBCLAIM. *Ψ can be extended to an $(\omega, \omega_1, \omega_1)$-iteration strategy.*

PROOF. We can find a good pointclass Γ^* such that $Code(\Psi) \in \underset{\sim}{\Delta}_{\Gamma^*}$. Using Theorem 1.2.9 of [3], we can find $(\mathcal{N}_y^*, \Sigma_y, \delta_y)$ that Suslin captures $Code(\Psi)$. Notice that Σ_y is an $(\omega, \omega_1, \omega_1)$-iteration strategy. It follows from universality that $\mathcal{Q} \trianglelefteq (\mathcal{J}^{\vec{E},\Sigma}[\mathcal{N}_1|\nu])^{\mathcal{N}_y^*|\delta_y}$. Hence, $\mathcal{Q}$ has an $(\omega, \omega_1, \omega_1)$-iteration strategy Ψ^+. Because Ψ is the unique (ω, ω_1)-iteration strategy of $\mathcal{Q}$, we have that Ψ^+ extends Ψ. ⊣

We now compare $\mathcal{Q}$ with $\mathcal{N}_1$. Let $\mathcal{S}$ be the comparison tree on the $\mathcal{Q}$ side with last model $\mathcal{Q}^*$ and $\mathcal{T}$ be the comparison tree on the $\mathcal{N}_1$ side with last model $\mathcal{N}_1^*$. Because $\mathcal{Q} \not\trianglelefteq \mathcal{N}_1$, we must have that $\mathcal{N}_1^* \trianglelefteq \mathcal{Q}^*$ and $\pi^{\mathcal{T}} : \mathcal{N}_1 \to \mathcal{N}_1^*$ exists. Because the (ω, ω_1)-fragment of Λ is the unique (ω, ω_1)-iteration strategy of $\mathcal{N}$, we must have that it is the $\pi^{\mathcal{T}} \circ i$-pullback of $\Psi_{\mathcal{N}_1^*, \vec{\mathcal{T}}^\frown \mathcal{S}}$ (recall that this is the strategy of $\mathcal{N}_1^*$ induced by Ψ). This implies that $\Lambda \in \Gamma$, contradiction. ⊣

It is now clear that $(\mathcal{N}, \Lambda)$ is as desired. This completes the proof of Lemma 2.5.

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2.2. A Prikry generic. In this subsection, while working in N, we define a Prikry forcing with the property that the generic object produces a sound countably iterable Σ-mouse $\mathcal{R}$ over $\mathbb{R}$ such that $\mathcal{R} \in N$ and extends $(Lp^{\Sigma}(\mathbb{R}))^N$. Clearly this is a contradiction.

We now start working in N. We now describe a function $f : \mathcal{D}^{<\omega} \to HC$ such that if $G \subseteq \mathbb{P}^f$ is N-generic then $f(G)$ is a Σ-premouse such that certain $\mathcal{J}^{\vec{E},\Sigma}$-construction of it is an initial segment of some Λ-iterate of $\mathcal{N}$.

Following [3], we say $\mathcal{Q}$ is Σ-*suitable* (in N) if for some ordinal δ

1. δ is the unique Woodin cardinal of $\mathcal{Q}$,
2. $o(\mathcal{Q}) = \sup_{n<\omega}(\delta^{+n})^{\mathcal{Q}}$,
3. $\mathcal{Q}$ is full with respect to Σ-mice, i.e., for any cutpoint η, $Lp^{\Sigma}(\mathcal{Q}|\eta) \trianglelefteq \mathcal{Q}$.

We let $\delta^{\mathcal{Q}}$ be the Woodin cardinal of $\mathcal{Q}$. Similarly we can define the notion of a Σ-suitable $\mathcal{Q}$ over any set a. In particular, if $\mathcal{Q}$ is Σ-suitable and $\mathcal{R}$ is Σ-suitable over $\mathcal{Q}$ then $\mathcal{R} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". Because we will only deal with Σ-suitable structures, we omit Σ and just say suitable instead of Σ-suitable.

A normal iteration tree $\mathcal{U}$ on a suitable $\mathcal{P}$ is *short* if for all limit $\xi \leq lh(\mathcal{U})$,

$$Lp^{\Sigma}(\mathcal{M}(\mathcal{U}|\xi)) \vDash \text{"}\delta(\mathcal{U}|\xi) \text{ is not Woodin"}.$$

Otherwise, we say that $\mathcal{U}$ is *maximal*. We say that a suitable $\mathcal{P}$ is *short tree iterable* if for any short tree $\mathcal{T}$ on $\mathcal{P}$, there is a cofinal wellfounded branch b such that $\mathcal{Q}(b,\mathcal{T})$ exists and if $\pi^{\mathcal{T}}_b : \mathcal{P} \to \mathcal{M}^{\mathcal{T}}_b$ exists then $\mathcal{M}^{\mathcal{T}}_b$ is suitable.

Write $\mathcal{P}_y$ for the premouse coded by the real y. Let a be countable transitive and $d \in \mathcal{D}$ be such that a is coded by a real recursive in d. Put

$$\mathcal{F}^d_a = \{\mathcal{P}_z : z \leq_T d, \mathcal{P}_z \text{ is a short-tree iterable suitable premouse over } a\}.$$

LEMMA 2.9. *For any fixed a, there is a cone of d such that $\mathcal{F}^d_a \neq \emptyset$.*

PROOF. If not, the failure of the statement in the claim is a Σ_1 statement. Call this statement $\phi[a]$. Using $\underset{\sim}{\Sigma}^2_1(Code(\Sigma))$-reflection, we get a transitive model

$$H \vDash ZF^- + \Theta = \Theta_{Code(\Sigma)} + \phi[a],$$

$\mathbb{R} \subseteq H$ and $\wp(\mathbb{R}) \cap H \subsetneq \underset{\sim}{\Delta}^2_1(Code(\Sigma))$.

Let Γ^* be a good pointclass beyond H. Such a Γ^* exists by our assumption on H. We use Theorem 1.2.9 of [3] to get a triple $\langle N^*_w, \delta_w, \Sigma_w\rangle$ (for some real w) that Suslin captures the universal Γ^* set. Using universality of fully backgrounded constructions and the proofs of the claims from the proof of Lemma 2.5 (or the results of Section 3.2.2 of [3]), we conclude that the $(\mathcal{J}^{\vec{E},\Sigma}[a])^{\mathcal{N}^*_w|\delta_w}$ reaches a premouse $\mathcal{Q}_a$ such that in H, $\mathcal{Q}_a$ is short tree iterable and suitable (with respect to H). This contradicts our assumptions on H. ⊣

For each a and for each Turing degree d from the cone of Lemma 2.9, we can simultaneously compare all $\mathcal{Q} \in \mathcal{F}^d_a$ while doing the generic genericity iteration to make d generic over the common part of the final model $\mathcal{Q}^{d,-}_a$. This process (hence $\mathcal{Q}^{d,-}_a$) depends only on d. Set

$$\mathcal{Q}^d_a = Lp^{\Sigma}_{\omega}(\mathcal{Q}^{d,-}_a) \text{ and } \delta^d_a = o(\mathcal{Q}^{d,-}_a).$$

Recall that we are working in N (thus, we really have that $\mathcal{Q}^d_a = Lp^{\Gamma,\Sigma}_{\omega}(\mathcal{Q}^{d,-}_a)$).

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LEMMA 2.10. *The following statements are true* (*in* N).

1. $\mathcal{Q}_a^d$ and δ_a^d depend only on d.
2. $\mathcal{Q}_a^{d,-}$ is Σ-full (no levels of $\mathcal{Q}_a^d$ project strictly below δ_a^d).
3. $\mathcal{Q}_a^d \vDash \delta_a^d$ is Woodin.
4. $\wp(a) \cap \mathcal{Q}_a^d = \wp(a) \cap OD_\Sigma(a \cup \{a\})$ and $\wp(\delta_a^d) \cap \mathcal{Q}_a^d = \wp(\delta_a^d) \cap OD_\Sigma(\mathcal{Q}_a^{d,-} \cup \{\mathcal{Q}_a^{d,-}\})$.
5. $\delta_a^d = \omega_1^{L[S,d]}$.

PROOF. 1–4 just follow from our definitions. We consider 5. Let S be the tree of a $(\Sigma_1^2(Code(\Sigma)))^N$ scale on a universal $(\Sigma_1^2(Code(\Sigma)))^N$ set U. Suppose that in $L[S,d]$, the process producing $\mathcal{Q}_a^d$ stops at stage $\alpha < \omega_1^{L[S,d]}$. We then have that $\mathcal{Q}_a^d$ is countable in $L[S,d]$. The suitability of $\mathcal{Q}_a^d$ then implies that $\mathbb{R} \cap L[S,d] \subseteq \mathcal{Q}_a^d[d]$. It then follows that δ_a^d, the Woodin of $\mathcal{Q}_a^d$, is countable in $L[S,d]$ while it is a cardinal in $\mathcal{Q}_a^d[d]$ (because the extender algebra of $\mathcal{Q}_a^d$ at δ_a^d is δ_a^d-cc). Hence, $\omega_1^{L[S,d]}$ is countable in $L[S,d]$, contradiction! ⊣

We now define $f : \mathcal{D}^{<\omega} \to HC$ by induction on $\mathcal{D}^n$. Fix $(\mathcal{N}, \Lambda)$ as in Lemma 2.5 and let k be as in clause 3. Below we use the notation of Lemma 2.5. We let $f(\emptyset) = \mathcal{N}|\gamma_k$. Suppose we have defined $f \restriction \mathcal{D}^{n+1}$. Given $p \in \mathcal{D}^{n+2}$, we let

$$f(d) = \begin{cases} \mathcal{Q}^d_{f(p\restriction n+1)} & : f(p \restriction n+1) \text{ is countable in } L[d], \\ \emptyset & : \text{otherwise.} \end{cases}$$

Suppose now that $G \subseteq \mathbb{P}^f$ is N-generic. Let $\mathcal{Q}_i = G^i$ and let $\mathcal{Q}_\omega = f(G)$. We let δ_i be the largest Woodin cardinal of $\mathcal{Q}_i$. Without loss of generality, we assume that if $(\langle d \rangle, X) \in G$ then $\mathcal{N}$ is countable in $L[d]$.

Given an increasing function $h : \omega \to \omega$, we define $\langle \mathcal{Q}_i^h, \mathcal{Q}_i^{h,*} : i < \omega \rangle$ according to the following procedure:

1. $\mathcal{Q}_0^{h,*}$ is the output of $\mathcal{J}^{\vec{E},\Sigma}[a]$ construction done in $\mathcal{Q}_{h(0)+1}$ using extenders with critical point $> \delta_{h(0)}$.
2. $\mathcal{Q}_0^h = (Lp^\Sigma(\mathcal{Q}_0^{h,*}))^{\mathcal{Q}_{h(0)+2}}$.
3. $\mathcal{Q}_{i+1}^{h,*}$ is the output of $\mathcal{J}^{\vec{E},\Sigma}[\mathcal{Q}_i^h]$ construction done in $\mathcal{Q}_{h(i+1)+1}$ using extenders with critical point $> \delta_{h(i+1)}$.
4. $\mathcal{Q}_{i+1}^h = (Lp^\Sigma(\mathcal{Q}_{i+1}^{h,*}))^{\mathcal{Q}_{h(i+1)+2}}$.

We let $\mathcal{Q}_\omega^h = \cup_{i<\omega} \mathcal{Q}_i^h$.

LEMMA 2.11. *For some increasing function* $h : \omega \to \omega$ *such that* $h \in V$, $\mathcal{Q}_\omega^h$ *is an initial segment of a* Λ*-iterate of* $\mathcal{N}$.

PROOF. Let $\vec{d} = \langle d_i : i < \omega \rangle$ be the generic sequence of degrees given by G. We define h recursively. It will have the property that $\mathcal{Q}_i^h$ is a Λ-iterate of $\mathcal{N}|(\gamma^+_{k+i+1})^{\mathcal{N}}$. While defining h, we also define a sequence $\vec{H} = \langle \mathcal{N}_i, \mathcal{U}_i, b_i : i \in [-1, \omega) \rangle$ such that

1. $\mathcal{N}_{-1} = \mathcal{N}$,
2. for each i, $\mathcal{U}_i$ is an iteration tree on $\mathcal{N}_i$ and $b_i = \Lambda(\oplus_{m<i+1}\mathcal{U}_m)$,
3. for each i, $\mathcal{N}_{i+1} = \mathcal{M}_{b_i}^{\mathcal{U}_i}$,
4. for each i, $\pi_{b_i}^{\mathcal{U}_i}$-exists and letting $\pi_{i,j} : \mathcal{N}_i \to \mathcal{N}_j$ be the composition of iteration embeddings, $\mathcal{U}_i$ is a tree based on $\mathcal{N}_i|[\pi_{-1,i}(\gamma_{k+i}), \pi_{-1,i}(\gamma_{k+i+1}))$,

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5. $\mathcal{Q}_i^h = \mathcal{N}_i|(\pi_{-1,i}(\gamma^+_{k+i+1})^{\mathcal{N}})$,
6. for each i, $h(i) = m+1$ where m is the least integer such that $\vec{H} \restriction i+1$ is countable in $L[d_m]$.

1–6 above tell us how to define the sequence. To see that we can always arrange 6, recall that $\vec{d}$ is cofinal in the set of degrees. To see that $h \in V$, recall that Prikry property implies that $\mathbb{P}^f$ doesn't add new reals. To see 5, notice that by our construction, $\vec{H} \restriction i$ is generic over $\mathcal{Q}_i$ for the extender algebra at δ_i. ⊣

We let h be as in Lemma 2.11. We let $\mathcal{S}_i = \mathcal{Q}_i^h$ and $\mathcal{S}_\omega = \mathcal{Q}_\omega^h$. Also, let $\mathcal{S}$ be the Λ-iterate of $\mathcal{N}$ such that $\mathcal{S}_\omega \trianglelefteq \mathcal{S}$. Because $\rho(\mathcal{N}) \leq \gamma_k$, we have that $\rho(\mathcal{S}) \leq \gamma_k$. Let $\langle \eta_n : n < \omega \rangle$ be the Woodin cardinals of $\mathcal{S}$. Let $\eta_\omega = \sup_{n<\omega} \eta_n$. Notice that in $V[G]$, $\mathcal{S}_\omega$ is (ω, ω_1)-iterable for short trees.

We now have that there is $g \subseteq Col(\omega, < \eta_\omega)$-generic over $\mathcal{S}$ such that

$$\cup_{n<\omega} \mathbb{R}^{\mathcal{S}[g \cap \eta_n]} = \mathbb{R}.$$

Next we perform an S-construction (see Section 2.11 of [3], [6], [8], or [10]) to translate $\mathcal{S}$ to a Σ-mouse over $\mathbb{R}$. To see that the translation procedure works, let $\lambda = \Theta^{\mathcal{J}_\alpha(U,\mathbb{R})}$. Notice that $\mathbb{P}^f \in \mathcal{J}_\lambda(\mathbb{R})$ and that all extenders of $\mathcal{S}$ above η_ω have critical point $> \lambda$. Thus, we can translate Σ-premice over $\mathcal{J}_\lambda[\mathcal{S}_\omega]$ to Σ-premice over $\mathcal{J}_\lambda(\mathbb{R})$. Let then $\mathcal{W}$ be the Σ-premouse over $\mathbb{R}$ that is the result of translating $\mathcal{S}$ into a Σ-premouse over $\mathcal{R}$.

LEMMA 2.12. $(Lp^\Sigma(\mathbb{R}))^N \lhd \mathcal{W}$.

PROOF. Suppose $\mathcal{W} \trianglelefteq (Lp^\Sigma(\mathbb{R}))^N$. Notice that $\mathcal{W}$ is OD^N_Σ. Notice that $\mathcal{N}$ is the δ_k-core of $\mathcal{S}$. Let then τ be a name for a sound Σ-premouse over $\mathcal{N}|\delta_k$ projecting to δ_k such that it is always realized as the δ_k-core of the translation of $\mathcal{W}$ into an extension of $\mathcal{Q}^h_\omega$. Then τ is OD^N_Σ and hence, there is OD^N_Σ condition $(\emptyset, X)$ that decides τ. It then follows that if $\mathcal{N}^*$ is the premouse given by τ and $(\emptyset, X)$ then $\mathcal{N}^* = \mathcal{N}$. But this implies that $\mathcal{N} \in OD^N_{\mathcal{N}|\delta_k,\Sigma}$ and hence, by N-fullness of $\mathcal{N}$, $\mathcal{N} \in \mathcal{N}$, contradiction. ⊣

Let then $\mathcal{R} \trianglelefteq \mathcal{W}$ be the first level of $\mathcal{W}$ such that

$$(Lp^\Sigma(\mathbb{R}))^N \trianglelefteq \mathcal{R} \text{ and } \rho_\omega(\mathcal{R}) = \mathbb{R}.$$

The next two lemmas finishes the proof of Theorem 0.1.

LEMMA 2.13. $\mathcal{R} \in V$.

PROOF. Suppose not. Using DC we can find $\pi : H \to \mathcal{J}_\mu(\wp(\mathbb{R}))$ such that $\mu > \Theta$ and H is countable. We can further assume that $\Sigma, \Lambda, N \in rng(\pi)$. Let then $\bar{N} = \pi^{-1}(N)$. Let $g \subseteq Coll(\omega, \bar{N})$ be H-generic and let $g_1, g_2 \subseteq \pi^{-1}(\mathbb{P}^f) \in H[g]$ be two different $\bar{N}$-generics. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the versions of $\mathcal{R}$ defined for $\bar{N}$ using g_1 and g_2 respectively. Because both are (ω, ω_1)-iterable, we have that $\mathcal{R}_1 = \mathcal{R}_2$. Hence, the version of $\mathcal{R}$ for $\bar{N}$ is $OD^{H[g]}_{\Sigma^H}$ and hence, it is in H. ⊣

It remains to show that $\mathcal{R}$ is in N and countably iterable in N. Granted this, we obtain the desired contradiction, hence complete the proof of Theorem 0.1.

LEMMA 2.14. *$\mathcal{R} \in N$ and $\mathcal{R}$ is countably iterable in N.*

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PROOF. First, we show $\mathcal{R} \in N$. We can assume $o(\mathcal{R})$ is limit and $\rho_1(\mathcal{R}) = \mathbb{R}$ (if not, look at the mastercode structure of $\mathcal{R}$). In V, we can write $\mathcal{R} = \cup_{\xi<o(\mathcal{R})} Th^{\mathcal{R}|\xi}(\mathbb{R})$. Notice that for all $\xi < o(\mathcal{R})$, $|Th^{\mathcal{R}|\xi}(\mathbb{R})|_w < |A|_w$ (where A is the least OD^N_Σ set of reals such that $A \notin Lp^\Sigma(\mathbb{R}))^N$). Since $\mathcal{R}$ is a well-ordered union of sets Wadge reducible to A, it follows from a theorem of Kechris that $\mathcal{R}$ is projective in A. This implies that $\mathcal{R} \in N$.

It remains to show $\mathcal{R}$ is countably iterable in N. Working in N, given $\sigma \in \wp_{\omega_1}(\mathbb{R})$ we say σ is bad if there is a noniterable sound Σ-premouse $\mathcal{W}$ over σ projecting to σ and an embedding $\pi : \mathcal{W} \to \mathcal{R}$. Notice that (in V) $\mathcal{R}$ is countably (ω, ω_1)-iterable and hence, for each $\sigma \in \wp_{\omega_1}(\mathbb{R})$ there is at most one such $\mathcal{W}$. We denote it by $\mathcal{W}(\sigma)$.

To show that $\mathcal{R}$ is countably iterable in N, it is enough to show that for stationary many σ, $\mathcal{W}(\sigma)$ is undefined. Towards a contradiction assume that for a club C of σ, $\mathcal{W}(\sigma)$ is defined. Then the set

$$B = \{(\sigma, \mathcal{W}(\sigma)) : \sigma \in C\}$$

is $OD^N_{\Sigma,u}$ for some real u. It follows that for every $\sigma \in C$ such that $u \in \sigma$, $\mathcal{W}(\sigma) \in (Lp^\Sigma(\sigma))^N$. Because for every σ, $\mathcal{W}(\sigma)$ has an (ω, ω_1)-iteration strategy in V, we get that $\mathcal{W}(\sigma) \trianglelefteq (Lp^\Sigma(\sigma))^N$, which is a contradiction. ⊣

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