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A SHORT PROOF OF AN INTERPOLATION THEOREM

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In this note we give a simple proof of an operator-interpolation theorem (Theorem 2) due originally to Donoghue [6], and Lions-Foias [7].

DEFINITION. Let $\mathscr{C}$ be the complex plane, $\mathscr{C}^+$ the open upper half-plane, $\mathscr{R}$ the real line, $\mathscr{R}^+$ and $\mathscr{R}^-$ the non-negative and non-positive axes. Denote by $\mathscr{K}$ the class of positive functions on $\mathscr{R}^+$ which extend analytically to $\mathscr{C}-\mathscr{R}^-$, and map $\mathscr{C}^+$ into itself. Denote by $\mathscr{K}'$ the class of functions φ such that $\varphi(x^{1/2})^2$ is in $\mathscr{K}$.

Let $\varphi \in \mathscr{K}$. By reflection φ takes the lower half-plane to itself, and is increasing and concave (i.e. $\varphi'' \leq 0$) on $\mathscr{R}^+$. Turner [5] proves the following generalisation of a matrix theorem of Löwner [4] (see also Heinz [2]):

LEMMA 1. *Let φ be a positive function on $\mathscr{R}^+$. Then $\varphi \in \mathscr{K}$ if and only if for any positive selfadjoint operators A and B on a Hilbert space which satisfy $0 \leq A \leq B$, we have $\varphi(A) \leq \varphi(B)$.*

By a simple exercise with harmonic functions one proves:

LEMMA 2. *Let $\varphi \in \mathscr{K}$. Then if $z \in \mathscr{C}^+$,*

$$0 \leq \arg \varphi(z) \leq \arg z,$$

If $\varphi \in \mathscr{K}$ or $\mathscr{K}'$, let $\varphi^c(x)=x/\varphi(x)$. It follows from Lemma 2 that $\varphi \in \mathscr{K}$ (or $\mathscr{K}'$) implies $\varphi^c \in \mathscr{K}$ (or $\mathscr{K}'$). The following theorem was first proved by Heinz [3] for the functions $\varphi(x)=x^\theta$ ($\theta \in [0, 1]$).

THEOREM 1. *Let A and B be positive operators on Hilbert spaces H_1 and H_2. Let Q be a closed, densely defined linear map from H_1 to H_2, such that $D(A) \subseteq D(Q)$, $D(B) \subseteq D(Q^*)$, and for all $f \in D(A)$, $g \in D(B)$, we have: $\|Qf\| \leq \|Af\|$;* and *$\|Q^*g\| \leq \|Bg\|$. Let $\varphi \in \mathscr{K}'$. Then for $f \in D(A)$, $g \in D(B)$,*

$$|(Qf, g)| \leq \|\varphi(A)f\| \, \|\varphi^c(B)g\|.$$

In proving this theorem it suffices to assume that $H_1=H_2$. Then Q has the "polar" representation $Q=PS$, where $S \geq 0$ and P is a partial isometry [1, p. 1249]. The rest of the proof is a straightforward use of Lemma 1.

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We now have the main interpolation result:

THEOREM 2. *Let $A \geq 0$, $B \geq 0$ be positive operators on H_1 and H_2. Let $\varphi \in \mathscr{K}'$. Suppose T is a bounded linear map from H_1 to H_2 which takes $D(A)$ into $D(B)$, and satisfies*

$$\|BTf\| \leq C_1 \|Af\| \qquad (f \in D(A))$$

$$\|Tg\| \leq C_2 \|g\| \qquad (g \in H_1).$$

Then if $f \in D(A)$,

$$\|\varphi(C_2\mathrm{B})Tf\| \leq C_2 \|\varphi(C_1A)f\|.$$

Proof. Assume first that $A \geq \varepsilon > 0$, $B \geq \varepsilon$. Let $Q = BT$. Then Q^* is the closure of T^*B, and hence the hypotheses of Theorem 1 are satisfied. Thus,

$$|(Qf, g)| \leq \|\varphi(C_1A)f\| \, \|\varphi^c(C_2B)g\|, \qquad \text{for } f \in D(A), \qquad g \in D(B).$$

Since $B \geq \varepsilon$, $\varphi^c(C_2B)$ is invertible. Let $h = \varphi^c(C_2B)g$. Since

$$\varphi^c(C_2B)^{-1}Qf = C_2^{-1}\varphi(C_2B)Tf,$$

the above becomes

$$|(\varphi(C_2B)Tf, h)| \leq C_2 \|\varphi(C_2A)f\| \, \|h\|,$$

from which the conclusion follows. A limiting argument removes the assumptions on A and B.

The concavity of φ implies

COROLLARY.

$$\|\varphi(B)Tf\| \leq d_1d_2 \|\varphi(A)f\|,$$

where $d_i = \max(C_i, 1)$.

Theorem 2 has a converse, which is an easy consequence of Lemma 1, and which states that $\mathscr{K}'$ is the largest class of functions enjoying the property stated in Theorem 2 for all A, B, and T.

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