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A variational principle of scaled entropy for amenable group actions

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Abstract. We study scaled topological entropy, scaled measure entropy, and scaled local entropy in the context of amenable group actions. In particular, a variational principle is established.

1 Introduction

Entropy is one of the most widely used notions in the characterization of the complexity of topological dynamical systems. In 1965, Adler et al. [1] defined topological entropy. In 1973, Bowen [4] introduced the topological entropy of any subsets resembling Hausdorff dimension. Later in 1984, inspired by Bowen's approach, Pesin and Pitskel [25, 26] extended the notion of topological pressure to arbitrary subsets and established a variational principle which is a generalization of classical variational principle established by Goodwyn, Dinaburg, and Goodman [7, 13, 14].

In 1983, Brin and Katok [2] gave the topological version of the Shannon–McMillan–Breiman theorem with a local decomposition of the metric entropy. Further, Feng and Huang [11] introduced the notion of measure-theoretical entropy for any (not necessarily invariant) Borel measure which is a modification of Brin and Katok's local metric entropy and established the variational principle between Bowen's topological entropy and measure-theoretical entropy for any non-empty compact subset.

In 1975, Kieffer [22] firstly introduced entropy for amenable group actions. In 2000, Rudolph and Weiss [28], the properties of entropy for amenable group actions was elaborately explored. In particular, the classical variational principle for sofic group actions was established by Kerr and Li [19–21]. The variational principle of topological entropy and measure-theoretical entropy for amenable group actions was obtained by Huang et al. [17]. The variational principle related to the Bowen entropy and the Brin-Katok local entropy with respect to invariant measures for amenable group actions was established by Zheng and Chen [31]. The variational principle of packing entropy and measure-theoretical entropy for amenable group actions was obtained by Dou et al. [9]. For other recent related work, we refer to [4–6, 8, 15, 16–18, 20, 27, 30, 31, 32]. One may refer to Ornstein and Weiss [24], or Kerr and Li [21] for more details on dynamics for amenable group actions.

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In 2015, Zhao and Pesin [33] defined the scaled topological entropy and scaled measure entropy. In this paper, we will work in the frame of countable discrete amenable group actions and introduce scaled measure entropy and scaled local entropy for any Borel measure. In particular, we prove a variational principle between scaled topological entropy and scaled local entropy:

$$E_K(\{F_n\},\mathbf{a}) = \sup\{\underline{h}_\mu(\{F_n\},\mathbf{a}) : \mu \in \mathcal{M}(X), \mu(K) = 1\},$$

where K is any non-empty compact subset of X. It is a scaled version of the variational principles obtained by Feng and Huang for continuous maps [11] and by Huang et al. [17] for amenable group actions. It worth to point out that the variational principle in [25, 26] is not true for scaled entropies with respect to nontrivial sequences, we refer to [33, Example 4.3] for the counter examples. Meanwhile, our result also holds for continuous maps, we would like to directly present this result in a more general setting.

The paper is organized as follows. In Sections 2 and 3, we introduce the notions of scaled topological entropy, scaled weighted topological entropy, scaled measure entropy E_μ and scaled local entropy h_μ for amenable group actions and investigate their properties. We also give the definitions of equivalent scaled sequences and equivalent Følner sequences. In Section 4, a variational principle is established, i.e., the scaled topological entropy is the supremum over all Borel probability measure of the scaled local entropy.

2 Scaled topological entropy

In this section, we give the definitions of scaled topological entropy, lower and upper scaled topological entropies on an arbitrary subset and some related properties. Let (X,G) be a topological dynamical system, where X is a compact metric space and G is a discrete countable amenable group. A group G is amenable if it admits a left invariant mean. This is equivalent to the existence of a sequence of finite subsets $\{F_n\}$ of G which are asymptotically invariant, i.e.,

$$\lim_{n\to+\infty} \frac{|F_n \triangle gF_n|}{|F_n|}, \text{ for all } g \in G.$$

Such sequences are called Følner sequence. One may refer to Ornstein and Weiss [24], or Kerr and Li [21] for more details on dynamics for amenable group actions.

Let $\{F_n\}$ be a Følner sequence in G. Throughout this paper, we always let the Følner sequence $\{F_n\}$ be fixed.

2.1 Scaled topological entropy

We follow the approach described in [25]. Let $\mathcal{U}$ be an open cover of X. Denote by $\mathcal{W}_{F_n}(\mathcal{U})$ the collection of families $\mathbf{U} = \{U_g\}_{g\in F_n}$ of length $m(\mathbf{U}) = |F_n|$ with $U_g \in \mathcal{U}$ and by $\mathcal{W}(\mathcal{U}) = \bigcup_{n\geq 1} \mathcal{W}_{F_n}(\mathcal{U})$. For $\mathbf{U} \in \mathcal{W}_{F_n}(\mathcal{U})$ define

$$\begin{aligned} X(\mathbf{U}) &= \bigcap_{g\in F_n} g^{-1}U_g \\ &= \{x \in X : gx \in U_g \text{ for } g \in F_n\}. \end{aligned}$$

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Let $Z \subset X$ be a subset of X. We say that a collection of strings $\Gamma \subset \mathcal{W}(\mathcal{U})$ covers Z if $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$.

We call a sequence of positive numbers $\mathbf{a} = \{a(n)\}_{n\geq 1}$ a scaled sequence if it is positive and monotonically increasing to infinity. We denote by $\mathcal{SS}$ the set of all scaled sequences.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$. Given a subset $Z \subset X$, $s \geq 0$, $N \in \mathbb{N}$ and a scaled sequence $\mathbf{a} \in \mathcal{SS}$, let

$$M(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}) = \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-sa(m(\mathbf{U}))),$$

where the infimum is taken over all covers $\Gamma \subset \bigcup_{n\geq N} \mathcal{W}_{F_n}(\mathcal{U})$ of Z. It is easy to see that $M(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a})$ is monotone in N. Define

$$M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = \lim_{N\to\infty} M(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}).$$

By the construction of Carathéodory dimension characteristics one can show that when s goes from $-\infty$ to $+\infty$, $M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a})$ jump from $+\infty$ to 0 at a unique critical value. Hence, let

$$\begin{aligned} E_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) &= \inf\{s : M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = 0\} \\ &= \sup\{s : M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = +\infty\}. \end{aligned}$$

Definition 2.1 Let (X, G) be a topological dynamical system. For $\mathbf{a} \in \mathcal{SS}$, $Z \subset X$,

$$E_Z(\{F_n\}, \mathbf{a}) = \sup\{E_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) |\, \mathcal{U} \text{ is a finite open cover of } X\}$$

is called the scaled topological entropy of (X, G) on the set Z (with respect to the sequence $\mathbf{a} \in \mathcal{SS}$ and the Følner sequence $\{F_n\}$).

Given a subset $Z \subset X$, $s \geq 0$, $N \in \mathbb{N}$ and a scaled sequence $\mathbf{a} \in \mathcal{SS}$, define

$$R(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}) = \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-sa(m(\mathbf{U}))),$$

where the infimum is taken over all covers $\Gamma \subset \mathcal{W}_{F_N}(\mathcal{U})$ of Z. We set

$$\begin{aligned} \underline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) &= \liminf_{N\to\infty} R(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}), \\ \overline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) &= \limsup_{N\to\infty} R(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}), \end{aligned}$$

and define the critical values of $\underline{r}(Z, s, \mathcal{U}, \{F_n\}, \mathbf{a})$ and $\overline{r}(Z, s, \mathcal{U}, \{F_n\}, \mathbf{a})$ as

$$\begin{aligned} \underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) &= \inf\{s : \underline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = 0\} \\ &= \sup\{s : \underline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = +\infty\}, \\ \overline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) &= \inf\{s : \overline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = 0\} \\ &= \sup\{s : \overline{r}(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = +\infty\}, \end{aligned}$$

respectively. For existences of the above critical values, we refer to [25, page 16].

Definition 2.2

$$\underline{E}_Z(\{F_n\}, \mathbf{a}) = \sup\{\underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \ : \ \mathcal{U} \ \text{ is a finite open cover of } \ X\}$$

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and

$$\overline{E}_Z(\{F_n\},\mathbf{a}) = \sup\{\overline{E}_Z(\{F_n\},\mathcal{U},\mathbf{a}) \ : \ \mathcal{U} \text{ is a finite open cover of } X\},$$

are called the lower and upper scaled topological entropy of (X,G) on the set Z.

Let $\mathcal{U}$ be an open cover of X and $|\mathcal{U}| = \max\{\operatorname{diam}(U) : U \in \mathcal{U}\}$ denote the diameter of the cover $\mathcal{U}$. In what follows we use the notation $\mathcal{E}$ for either E, $\underline{E}$, or $\overline{E}$.

Proposition 2.1 *Let* $\mathbf{a} \in \mathcal{SS}$ *and* $\mathcal{U}$ *be an open cover of* X. *Then* $\lim_{|\mathcal{U}|\to 0} \mathcal{E}_Z(\{F_n\},\mathcal{U},\mathbf{a})$ *exists and is equal to* $\mathcal{E}_Z(\{F_n\},\mathbf{a})$.

Proof Let $\mathcal{V}$ be a finite open cover of X with diameter smaller than the Lebesgue number of $\mathcal{U}$. Each element $V \in \mathcal{V}$ is contained in some element $U(V) \in \mathcal{U}$. For any $n \in \mathbb{N}$ and any string $\mathbf{V} = \{V_g\}_{g\in F_n} \in \mathcal{W}_{F_n}(\mathcal{V})$, there exists a corresponding string $\mathbf{U}(\mathbf{V}) = \{U(V_g)\}_{g\in F_n} \in W_{F_n}(\mathcal{U})$. If $\Gamma \subset \mathcal{W}_{F_n}(\mathcal{V})$ covers a set $Z \subset X$, then $\mathbf{U}(\Gamma) = \{\mathbf{U}(\mathbf{V}) : \mathbf{V} \in \Gamma\}$ also covers Z. By definition of the scaled topological entropy, $M(Z,s,N,\{F_n\},\mathcal{U},\mathbf{a}) \le M(Z,s,N,\{F_n\},\mathcal{V},\mathbf{a})$. Then, $E_Z(\{F_n\},\mathcal{U},\mathbf{a}) \le E_Z(\{F_n\},\mathcal{V},\mathbf{a})$. Therefore, $E_Z(\{F_n\},\mathcal{U},\mathbf{a}) \le \liminf_{|\mathcal{V}|\to 0} E_Z(\{F_n\},\mathcal{V},\mathbf{a})$, hence

$$\limsup_{|\mathcal{U}|\to 0} E_Z(\{F_n\},\mathcal{U},\mathbf{a}) \le \liminf_{|\mathcal{V}|\to 0} E_Z(\{F_n\},\mathcal{V},\mathbf{a}).$$

This implies that

$$E_Z(\{F_n\},\mathbf{a}) = \lim_{|\mathcal{U}|\to 0} E_Z(\{F_n\},\mathcal{U},\mathbf{a}).$$

$\underline{E}_Z(\{F_n\},\mathbf{a}) = \lim_{|\mathcal{U}|\to 0} \underline{E}_Z(\{F_n\},\mathcal{U},\mathbf{a})$ and $\overline{E}_Z(\{F_n\},\mathbf{a}) = \lim_{|\mathcal{U}|\to 0} \overline{E}_Z(\{F_n\},\mathcal{U},\mathbf{a})$ can be proved in a similar manner. ∎

Now, we describe the second equivalent definition of scaled topological entropy. Let X be a compact metric space. Given $\varepsilon > 0$, $n \in \mathbb{N}$ and $x, y \in X$, denote by $d_{F_n}(x,y) = \max_{g\in F_n} d(gx,gy)$ and $B_{F_n}(x,\varepsilon)$ the open Bowen ball of radius $\varepsilon > 0$ in the metric d_{F_n} around x, i.e., $B_{F_n}(x,\varepsilon) = \{y \in X : d_{F_n}(x,y) < \varepsilon\}$. We follow the approach described in [25], for each subset $Z \subset X$, $\mathbf{a} \in \mathcal{SS}$, $N \in \mathbb{N}$ and $\varepsilon, s > 0$, set

$$\begin{aligned} M(Z,s,N,\{F_n\},\varepsilon,\mathbf{a}) = \inf\{\sum_i \exp(-sa(|F_{n_i}|)) :\\ \bigcup_i B_{F_{n_i}}(x_i,\varepsilon) \supset Z, x_i \in X \ and \ n_i \ge N \ for \ \ all \ i\}. \end{aligned}$$

We note that $M(\cdot,s,N,\{F_n\},\varepsilon,\mathbf{a})$ is an outer measure on X.

Since $M(Z,s,N,\{F_n\},\varepsilon,\mathbf{a})$ is monotonically increasing with respect to N,

$$M(Z,s,\{F_n\},\varepsilon,\mathbf{a}) = \lim_{N\to\infty} M(Z,s,N,\{F_n\},\varepsilon,\mathbf{a}),$$

It is easy to show that there is a jump-up value

$$\begin{aligned} E_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) &= \inf\{s : M(Z,s,\{F_n\},\varepsilon,\mathbf{a}) = 0\} \\ &= \sup\{s : M(Z,s,\{F_n\},\varepsilon,\mathbf{a}) = +\infty\}. \end{aligned}$$

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Let

$$E_Z^{B_2}(\{F_n\},\mathbf{a})=\lim_{\varepsilon\to 0}E_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}).$$

For any subset $Z\subset X$ and $\mathbf{a}\in\mathcal{SS}$, let $\aleph(Z,n,\varepsilon)$ denote the smallest number of Bowen's balls $\{B_{F_n}(x,\varepsilon)\}$ whose union covers the set Z,

$$\underline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a})=\liminf_{n\to\infty}\frac{1}{a(|F_n|)}\log\aleph(Z,n,\varepsilon),$$

and

$$\overline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a})=\limsup_{n\to\infty}\frac{1}{a(|F_n|)}\log\aleph(Z,n,\varepsilon).$$

Set

$$\underline{E}_Z^{B_2}(\{F_n\},\mathbf{a})=\lim_{\varepsilon\to 0}\underline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}),$$

and

$$\overline{E}_Z^{B_2}(\{F_n\},\mathbf{a})=\lim_{\varepsilon\to 0}\overline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}).$$

We have the following result.

Proposition 2.2 *For any subset $Z\subset X$ and $\mathbf{a}\in\mathcal{SS}$, we have*

(a) $E_Z(\{F_n\},\mathbf{a})=E_Z^{B_1}(\{F_n\},\mathbf{a})=E_Z^{B_2}(\{F_n\},\mathbf{a})$;

(b) $\underline{E}_Z(\{F_n\},\mathbf{a})=\underline{E}_Z^{B_2}(\{F_n\},\mathbf{a})$; $\overline{E}_Z(\{F_n\},\mathbf{a})=\overline{E}_Z^{B_2}(\{F_n\},\mathbf{a})$.

Proof We only give the proof of (a). (b) can be proved in a similar manner as Proposition 2.3.

Firstly, we prove $E_Z^{B^1}(\{F_n\},\mathbf{a})\le E_Z(\{F_n\},\mathbf{a})$. Let $\mathcal{U}$ be a finite open cover of X. If s satisfies $M(Z,s,\{F_n\},\mathcal{U},\mathbf{a})=0$, then by definition for every $\varepsilon>0$, there exists $N'\in\mathbb{N}$ such that for each $N\in\mathbb{N}$ with $N>N'$,

$$\inf_{\Gamma}\sum_{\mathbf{U}\in\Gamma}\exp(-sa(m(\mathbf{U})))<\frac{\varepsilon}{2},$$

where the infimum is taken over all covers $\Gamma\subset\bigcup_{n\ge N}\mathcal{W}_{F_n}(\mathcal{U})$ of Z. So there exists Γ_N that satisfies $\Gamma_N\subset\bigcup_{n\ge N}\mathcal{W}_{F_n}(\mathcal{U})$ and Γ_N covers Z, such that

$$\sum_{\mathbf{U}\in\Gamma_N}\exp(-sa(m(\mathbf{U})))<\varepsilon.$$

For any $n\in\mathbb{N}$ and $\mathbf{U}\in\mathcal{W}_{F_n}(\mathcal{U})$, $n_{\mathcal{U}}^{\{F_n\}}(X(\mathbf{U}))\ge|F_n|$. Then

$$\sum_{\mathbf{U}\in\Gamma_N}\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(X(\mathbf{U}))))\le\sum_{\mathbf{U}\in\Gamma_N}\exp(-sa(m(\mathbf{U})))<\varepsilon$$

and

$$\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(X(\mathbf{U}))))\le\sum_{\mathbf{U}\in\Gamma_N}\exp(-sa(m(\mathbf{U})))<\varepsilon.$$

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This implies that

$$M_{\{F_n\},\mathcal{U}}(Z,s,\mathbf{a})=0,$$

and hence

$$E_Z^{B_1}(\{F_n\},\mathbf{a})\le E_Z(\{F_n\},\mathbf{a}).$$

Next, we prove $E_Z(\{F_n\},\mathbf{a})\le E_Z^{B_1}(\{F_n\},\mathbf{a})$. Let $\mathcal{U}$ be a finite open cover of X. If s satisfies $M_{\{F_n\},\mathcal{U}}(Z,s,\mathbf{a})=0$, then by definition for every $\varepsilon>0$, there exists $k'\in\mathbb{N}$ such that for each $k\in\mathbb{N}$ with $k>k'$,

$$\inf_{\mathcal{D}\in\mathcal{G}(G,\mathcal{U},Z,k)}\sum_{D\in\mathcal{D}}\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(D)))<\frac{\varepsilon}{4}.$$

Thus there exists $\mathcal{D}=\{D_i\}_{i=1}^{\infty}\in\mathcal{G}(G,\mathcal{U},Z,k)$ such that

$$\sum_{D\in\mathcal{D}}\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(D)))<\frac{\varepsilon}{2}.$$

Without loss of generality, we can assume that each $D_i\in\mathcal{D}$ is open with

$$\sum_{D\in\mathcal{D}}\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(D)))<\varepsilon.$$

Indeed, if $n_{\mathcal{U}}^{\{F_n\}}(D_i)<\infty$, we can take $\widehat{D}_i=X(\mathbf{U})$, where $X(\mathbf{U})=\bigcap_{g\in F_n}g^{-1}U_g$ and for each $g\in F_n$, $gD_i\subset U_g$. It is easy to see that $n_{\mathcal{U}}^{\{F_n\}}(D_i)=n_{\mathcal{U}}^{\{F_n\}}(\widehat{D}_i)$ and $D_i\subset\widehat{D}_i$. If $n_{\mathcal{U}}^{\{F_n\}}(D_i)=\infty$, we can take $n\in\mathbb{N}$ sufficiently large and $\widehat{D}_i=X(\mathbf{U})$, where $X(\mathbf{U})=\bigcap_{g\in F_n}g^{-1}U_g$ and for each $g\in F_n$, $gD_i\subset U_g$, so that $\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(\widehat{D}_i)))<\frac{\varepsilon}{2^{i+1}}$.

Then we can assume that there exists an open cover $\mathcal{D}=\{D_i\}_{i=1}^{\infty}$ of Y, such that for every $D_i\in\mathcal{D}$, there exists $\mathbf{U}_i\in\mathcal{W}_{F_n}(\mathcal{U})$ with $D_i=X(\mathbf{U}_i)$ and

$$\sum_{D\in\mathcal{D}}\exp(-sa(n_{\mathcal{U}}^{\{F_n\}}(D)))<\varepsilon.$$

Then $Z\subset\bigcup_{i=1}^{\infty}D_i=\bigcup_{i=1}^{\infty}X(\mathbf{U}_i)$ and $\sum_{i=1}^{\infty}\exp(-sa(m(\mathbf{U})))<\varepsilon$. By the arbitrariness of ε, we get $M(Z,s,\{F_n\},\mathcal{U},\mathbf{a})=0$ and $E_Z(\{F_n\},\mathbf{a})\le E_Z^{B_1}(\{F_n\},\mathbf{a})$. So

$$E_Z(\{F_n\},\mathbf{a})=E_Z^{B_1}(\{F_n\},\mathbf{a}).$$

Let $\delta(\mathcal{U})$ be the Lebesgue number of $\mathcal{U}$. Clearly, for every $x\in X$ and $n\in\mathbb{N}$, if $x\in X(\mathbf{U})$ for some string $\mathbf{U}\in\mathcal{W}_{F_n}(\mathcal{U})$, then $B_{F_n}(x,\delta(\mathcal{U}))\subset X(\mathbf{U})\subset B_{F_n}(x,|\mathcal{U}|)$. By Proposition 2.1, this implies that

$$E_Z(\{F_n\},\mathbf{a})=\lim_{\delta\to0}E_Z^{B_2}(\{F_n\},\delta,\mathbf{a})=E_Z^{B_2}(\{F_n\},\mathbf{a}).$$

Therefore, $E_Z(\{F_n\},\mathbf{a})=E_Z^{B_1}(\{F_n\},\mathbf{a})=E_Z^{B_2}(\{F_n\},\mathbf{a})$. ∎

2.2 Properties of scaled topological entropy

For any subset $Z\subset X$, any open cover $\mathcal{U}$ of X and $\mathbf{a}\in\mathcal{SS}$, let $\aleph(\mathcal{U},Z)$ denote the number of sets in a finite subcover of $\mathcal{U}$ with the smallest cardinality. We have the following equivalent definition of the lower and upper scaled topological entropy.

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Proposition 2.3 *Let $Z \subset X$, $\mathbf{a} \in \mathcal{SS}$ and $\mathcal{U}$ be an open cover of X, then*

$$\underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) = \liminf_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right),$$

$$\overline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) = \limsup_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right).$$

Proof We will prove the first equality, the second one can be proved in a similar fashion. Let us put

$$\underline{\beta} = \underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}), \gamma = \liminf_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right).$$

Given $\eta > 0$, one can choose a subsequence $\{n_i\}$ such that

$$0 = \underline{r}(Z, \underline{\beta} + \eta, \{F_n\}, \mathcal{U}, \mathbf{a}) = \lim_{i\to\infty} R(Z, \underline{\beta} + \eta, n_i, \{F_n\}, \mathcal{U}, \mathbf{a}).$$

When $i \in \mathbb{N}$ is sufficiently large, we have

$$\aleph\left(\bigvee_{g\in F_{n_i}} g^{-1}\mathcal{U}, Z\right) \exp(-(\underline{\beta} + \eta) a(|F_{n_i}|)) \le 1,$$

and hence

$$\frac{1}{a(|F_{n_i}|)} \log \aleph\left(\bigvee_{g\in F_{n_i}} g^{-1}\mathcal{U}, Z\right) \le \underline{\beta} + \eta.$$

Let $i \to \infty$, we have $\gamma \le \underline{\beta} + \eta$, therefore, $\gamma \le \underline{\beta}$.

Let us choose a subsequence $\{n_i\}$ such that

$$+\infty = \underline{r}(Z, \underline{\beta} - \eta, \{F_n\}, \mathcal{U}, \mathbf{a}) = \lim_{i\to\infty} R(Z, \underline{\beta} - \eta, n_i, \{F_n\}, \mathcal{U}, \mathbf{a}).$$

When $i \in \mathbb{N}$ is sufficiently large, we have

$$\aleph\left(\bigvee_{g\in F_{n_i}} g^{-1}\mathcal{U}, Z\right) \exp(-(\underline{\beta} - \eta) a(|F_{n_i}|)) \ge 1,$$

and hence

$$\frac{1}{a(|F_{n_i}|)} \log \aleph\left(\bigvee_{g\in F_{n_i}} g^{-1}\mathcal{U}, Z\right) \ge \underline{\beta} - \eta.$$

Let $i \to \infty$, we have $\gamma \ge \underline{\beta} - \eta$, therefore, $\gamma \ge \underline{\beta}$.

So $\gamma = \underline{\beta}$, i.e.,

$$\underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) = \liminf_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right).$$

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Given two open covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{U}$ is finer than $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is an element $V \in \mathcal{V}$ such that $U \subset V$. We denote by $\mathcal{U} \geq \mathcal{V}$. Set

$$\mathcal{U} \vee \mathcal{V}_o = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}, g^{-1}\mathcal{U} := \{g^{-1}U : U \in \mathcal{U}\}.$$

In what follows, we use the notation $\mathcal{E}$ for either E or $\underline{E}$ or $\overline{E}$. The following Propositions describe some basic properties of scaled topological entropy and lower (upper) scaled topological entropies.

Proposition 2.4 *Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of X, $Z \subset X$ and $\mathbf{a} \in \mathcal{SS}$, the following properties hold:*

(1) *If $\mathcal{U} \leq \mathcal{V}$, then $\mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \leq \mathcal{E}_Z(\{F_n\}, \mathcal{V}, \mathbf{a})$;*
(2)

$$E_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \leq \underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \leq \overline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a})$$

and

$$E_Z(\{F_n\}, \mathbf{a}) \leq \underline{E}_Z(\{F_n\}, \mathbf{a}) \leq \overline{E}_Z(\{F_n\}, \mathbf{a}).$$

Proof (1) Since $\mathcal{U} \leq \mathcal{V}$, each element $V \in \mathcal{V}$ is contained in some element in $\mathcal{U}$ which we denote by $U(V)$. Therefore, for each string $\mathbf{V} \in \mathcal{W}_{F_n}(\mathcal{V})$ there exists a corresponding string $\mathbf{U}(\mathbf{V}) \in W_{F_n}(\mathcal{U})$. This yields that

$$\aleph\left(\bigvee_{g \in F_n} g^{-1}\mathcal{U}, Z\right) \leq \aleph\left(\bigvee_{g \in F_n} g^{-1}\mathcal{V}, Z\right),$$

and hence

$$\underline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \leq \underline{E}_Z(\{F_n\}, \mathcal{V}, \mathbf{a})$$

and

$$\overline{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \leq \overline{E}_Z(\{F_n\}, \mathcal{V}, \mathbf{a}).$$

Let $\Gamma \subset \mathcal{W}(\mathcal{V})$ be a collection of strings that covers Z. The corresponding collection of strings $\{\mathbf{U}(\mathbf{V}) : \mathbf{V} \in \Gamma\} \subset \mathcal{W}(\mathcal{U})$ also covers Z. This implies that $M(Z, s, N, \{F_n\}, \mathcal{U}, \mathbf{a}) \leq M(Z, s, N, \{F_n\}, \mathcal{V}, \mathbf{a})$ for each $s \geq 0$ and $N > 0$. Thus, $M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a}) \leq M(Z, s, \{F_n\}, \mathcal{V}, \mathbf{a})$. The first statement follows.

(2) The last statement follows immediately from the definitions. ■

The following proposition shows that the scaled topological entropy as well as lower and upper scaled topological entropies for amenable group actions are invariant under a topological conjugacy. Its proof is similar to the proofs of [25, Theorems 1.3 and 2.5].

Definition 2.3 [21, Definition 1.3] Two continuous actions $G \curvearrowright X_1$ and $G \curvearrowright X_2$ of the same group on compact metric spaces are said to be topologically conjugate if there is a homeomorphism $\phi : X_1 \to X_2$ such that $\phi(gx) = g\phi(x)$ for all $x \in X_1$ and $g \in G$.

Proposition 2.5 *Given two topologically conjugate actions $G \curvearrowright X_1$ and $G \curvearrowright X_2$ and $\mathbf{a} \in \mathcal{SS}$, then for each $Z \subset X_1$ and each open cover $\mathcal{U}$ of X_2 we have*

$$\mathcal{E}_Z(\{F_n\}, \phi^{-1}\mathcal{U}, \mathbf{a}) = \mathcal{E}_{\phi(Z)}(\{F_n\}, \mathcal{U}, \mathbf{a}).$$

In particular, $\mathcal{E}_Z(\{F_n\}, \mathbf{a}) = \mathcal{E}_{\phi(Z)}(\{F_n\}, \mathbf{a})$.

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Proposition 2.6 *The following statements hold:*

(1) *If* $Z_1 \subset Z_2$, *then* $\mathcal{E}_{Z_1}(\{F_n\}, \mathcal{U}, \mathbf{a}) \le \mathcal{E}_{Z_2}(\{F_n\}, \mathcal{U}, \mathbf{a})$, *hence* $\mathcal{E}_{Z_1}(\{F_n\}, \mathbf{a}) \le \mathcal{E}_{Z_2}(\{F_n\}, \mathbf{a})$;

(2) *If* $Z_i \subset X$, $i \ge 1$ *and* $Z = \bigcup_{i \ge 1} Z_i$, *then* $E_Z(\{F_n\}, \mathbf{a}) = \sup_{i \ge 1} E_{Z_i}(\{F_n\}, \mathbf{a})$, $\underline{E}_Z(\{F_n\}, \mathbf{a}) \ge \sup_{i \ge 1} \underline{E}_{Z_i}(\{F_n\}, \mathbf{a})$ *and* $\overline{E}_Z(\{F_n\}, \mathbf{a}) \ge \sup_{i \ge 1} \overline{E}_{Z_i}(\{F_n\}, \mathbf{a})$.

Proof (1) The statements follow directly from the definitions.

(2) By (1), $\mathcal{E}_{Z_i}(\{F_n\}, \mathbf{a}) \le \mathcal{E}_Z(\{F_n\}, \mathbf{a})$ and hence $\sup_{i \ge 1} \mathcal{E}_{Z_i}(\{F_n\}, \mathbf{a}) \le \mathcal{E}_Z(\{F_n\}, \mathbf{a})$.

Now we are left to show that $E_Z(\{F_n\}, \mathbf{a}) \ge \sup_{i \ge 1} E_{Z_i}(\{F_n\}, \mathbf{a})$. In fact, suppose $E_{Z_i}(\{F_n\}, \mathbf{a}) < s (i = 1, 2, \cdots)$, it follows that $M(Z_i, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = 0$, and hence $M(\bigcup_{i \ge 1} Z_i, s, \{F_n\}, \mathcal{U}, \mathbf{a}) = 0$. Then $E_{Z = \bigcup_{i \ge 1} Z_i}(\{F_n\}, \mathbf{a}) < s$. This implies that, $E_Z(\{F_n\}, \mathbf{a}) \le \sup_{i \ge 1} E_{Z_i}(\{F_n\}, \mathbf{a})$. ∎

2.3 Equivalent scaled sequences and equivalent Følner sequences

We call two scaled sequences $\mathbf{a}, \mathbf{b} \in \mathcal{SS}$ equivalent and we write $\mathbf{a} \sim \mathbf{b}$ if the following condition holds

$$0 < \liminf_{n \to \infty} \frac{b(|F_n|)}{a(|F_n|)} \le \limsup_{n \to \infty} \frac{b(|F_n|)}{a(|F_n|)} < \infty.$$

Obviously, $\sim$ defines an equivalence relation on $\mathcal{SS}$. Let $\mathbf{a} \in \mathcal{SS}$, we denote its equivalence class by $[\mathbf{a}] := \{\mathbf{b} \in \mathcal{SS} : \mathbf{b} \sim \mathbf{a}\}$ and we let $\mathcal{A} := \mathcal{SS}/\sim$. Given two equivalence classes $[\mathbf{a}], [\mathbf{b}] \in \mathcal{A}$, we say that $[\mathbf{a}] \preceq [\mathbf{b}]$ if for each $\mathbf{a} \in [\mathbf{a}]$ and $\mathbf{b} \in [\mathbf{b}]$ the following holds

$$\limsup_{n \to \infty} \frac{a(|F_n|)}{b(|F_n|)} = 0.$$

The following result is immediate.

Proposition 2.7 *For* $\mathbf{a}, \mathbf{b} \in \mathcal{SS}$, $Z \subset X$ *and each open cover* $\mathcal{U}$ *of* X, *the following properties hold:*

(1) *If* $a(|F_n|) \le b(|F_n|)$ *for all sufficiently large* $n \in \mathbb{N}$, *then* $\mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \ge \mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{b})$ *and* $\mathcal{E}_Z(\{F_n\}, \mathbf{a}) \ge \mathcal{E}_Z(\{F_n\}, \mathbf{b})$;

(2) *For each* $K > 0$ *we have that*

$$K \cdot \mathcal{E}_Z(\{F_n\}, \mathcal{U}, K\mathbf{a}) = \mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}), K \cdot \mathcal{E}_Z(\{F_n\}, K\mathbf{a}) = \mathcal{E}_Z(\{F_n\}, \mathbf{a}),$$

where $K\mathbf{a} = \{K \cdot a(|F_n|)\}$;

(3) *If there exists a constant* $C > 0$ *such that* $\frac{1}{C} \cdot b(|F_n|) \le a(|F_n|) \le C \cdot b(|F_n|)$ *for all sufficiently large* $n \in \mathbb{N}$, *then*

$$\frac{1}{C} \cdot \mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{b}) \le \mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) \le C \cdot \mathcal{E}_Z(\{F_n\}, \mathcal{U}, \mathbf{b})$$

and

$$\frac{1}{C} \cdot \mathcal{E}_Z(\{F_n\}, \mathbf{b}) \le \mathcal{E}_Z(\{F_n\}, \mathbf{a}) \le C \cdot \mathcal{E}_Z(\{F_n\}, \mathbf{b}).$$

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Remark 2.1 By Statement (3) of Proposition 2.7, for each equivalence class $[\mathbf{a}] \in \mathcal{A}$ and for each $\mathbf{a_1}, \mathbf{a_2} \in [\mathbf{a}]$ we have that $\mathcal{E}_Z(\{F_n\}, \mathbf{a_1}) = \mathcal{E}_Z(\{F_n\}, \mathbf{a_2}) = 0$, or $\mathcal{E}_Z(\{F_n\}, \mathbf{a_1}) = \mathcal{E}_Z(\{F_n\}, \mathbf{a_2}) = \infty$ or both $\mathcal{E}_Z(\{F_n\}, \mathbf{a_1})$ and $\mathcal{E}_Z(\{F_n\}, \mathbf{a_2})$ are positive and finite. In the first two cases, we write $\mathcal{E}_Z(\{F_n\}, [\mathbf{a}]) = 0$ and $\mathcal{E}_Z(\{F_n\}, [\mathbf{a}]) = \infty$ respectively and in the third case, we say that $\mathcal{E}_Z(\{F_n\}, [\mathbf{a}])$ is positive and finite. In this sense, entropy depends not on the scaled sequence but on its class of equivalence.

By Statement (1) of Proposition 2.7, we have $\mathcal{E}_Z(\{F_n\}, [\mathbf{a}]) \le \mathcal{E}_Z(\{F_n\}, [\mathbf{b}])$ whenever $[\mathbf{a}] \ge [\mathbf{b}]$.

Theorem 2.1 *If there is $[\mathbf{a}] \in \mathcal{A}$ such that $E_Z(\{F_n\}, [\mathbf{a}])$ is positive and finite, then*

$$E_Z(\{F_n\}, [\mathbf{b}]) = \begin{cases} 0, & \text{if } [\mathbf{a}] \preceq [\mathbf{b}], \\ \infty, & \text{if } [\mathbf{b}] \preceq [\mathbf{a}]. \end{cases}$$

In particular, there may exist at most one element in $(\mathcal{A}, \preceq)$ such that the corresponding scaled topological entropy is positive and finite. Similar results hold for lower and upper scaled topological entropy.

Proof We shall prove the result for the scaled topological entropy $E_Z(\{F_n\}, [\mathbf{a}])$, the arguments for the lower and upper scaled topological entropies are similar.

Suppose there is $[\mathbf{a}] \in \mathcal{A}$ such that $E_Z(\{F_n\}, [\mathbf{a}])$ is positive and finite. Then for each $[\mathbf{b}] \succeq [\mathbf{a}]$,

$$\limsup_{n\to\infty} \frac{a_1(|F_n|)}{b_1(|F_n|)} = 0,$$

for arbitrary $\mathbf{a_1} = \{a_1(|F_n|)\} \in [\mathbf{a}]$ and $\mathbf{b_1} = \{b_1(|F_n|)\} \in [\mathbf{b}]$. Let us fix such two scaled sequences $\mathbf{a_1}$ and $\mathbf{b_1}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $a_1(|F_n|) < \beta b_1(|F_n|)$ and hence, $M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{a_1}) \ge M(Z, s, \{F_n\}, \mathcal{U}, \beta\mathbf{b_1})$. This implies that

$$E_Z(\{F_n\}, \mathbf{a_1}) \ge E_Z(\{F_n\}, \beta\mathbf{b_1}) = \frac{1}{\beta} E_Z(\{F_n\}, \mathbf{b_1}),$$

i.e., $\beta E_Z(\{F_n\}, \mathbf{a_1}) \ge E_Z(\{F_n\}, \mathbf{b_1})$. Since β is arbitrary, we conclude that

$$E_Z(\{F_n\}, \mathbf{b_1}) = 0,$$

and hence

$$E_Z(\{F_n\}, [\mathbf{b}]) = 0.$$

On the other hand, if $[\mathbf{b}] \preceq [\mathbf{a}]$, then

$$\limsup_{n\to\infty} \frac{b_2(|F_n|)}{a_2(|F_n|)} = 0,$$

for arbitrary $\mathbf{a_2} = \{a_2(|F_n|)\} \in [\mathbf{a}]$ and $\mathbf{b_2} = \{b_2(|F_n|)\} \in [\mathbf{b}]$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $b_2(|F_n|) < \beta a_2(|F_n|)$ and hence, $M(Z, s, \{F_n\}, \mathcal{U}, \mathbf{b_2}) \ge M(Z, s, \{F_n\}, \mathcal{U}, \beta\mathbf{a_2})$. It follows that $E_Z(\{F_n\}, \mathbf{b_2}) > \frac{1}{\beta} E_Z(\{F_n\}, \mathbf{a_2})$. Again since β is arbitrary,

$$E_Z(\{F_n\}, \mathbf{b_2}) = \infty,$$

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then

$$E_Z(\{F_n\},[\mathbf{b}]) = \infty.$$ ■

Next we discuss the equivalence of Følner sequence. Let (X,G) be a topological dynamical system, we denote by $\mathcal{SF}(G)$ the set of all Følner sequences in G. We call two Følner sequences $\{F_n\},\{Q_n\} \in \mathcal{SF}(G)$ equivalent with respect to $\mathbf{a} \in \mathcal{SS}$ and we write $\{F_n\} \overset{\mathbf{a}}{\sim} \{Q_n\}$ if the following condition holds:

$$0 < \liminf_{n\to\infty} \frac{a(|Q_n|)}{a(|F_n|)} \le \limsup_{n\to\infty} \frac{a(|Q_n|)}{a(|F_n|)} < \infty.$$

Obviously, $\{F_n\} \overset{\mathbf{a}}{\sim} \{Q_n\}$ defines an equivalence relation on $\mathcal{SF}(G)$. Let $\{F_n\} \in \mathcal{SF}(G)$, we denote its equivalence class by $\Big[\{F_n\}\Big]_{\mathbf{a}} := \Big\{\{Q_n\} \in \mathcal{SF}(G) : \{Q_n\}\{F_n\} \overset{\mathbf{a}}{\sim} \{Q_n\}\{F_n\}\Big\}$ and we let $\mathcal{F}(G)_{\mathbf{a}} := \mathcal{SF}(G)/\overset{\mathbf{a}}{\sim}$. Given two equivalence classes $\Big[\{F_n\}\Big]_{\mathbf{a}}, \Big[\{Q_n\}\Big]_{\mathbf{a}} \in \mathcal{F}(G)_{\mathbf{a}}$, we say that $\Big[\{F_n\}\Big]_{\mathbf{a}} \preceq \Big[\{Q_n\}\Big]_{\mathbf{a}}$ if for each $\{F_n\} \in \Big[\{F_n\}\Big]_{\mathbf{a}}$ and $\{Q_n\} \in \Big[\{Q_n\}\Big]_{\mathbf{a}}$ the following holds:

$$\limsup_{n\to\infty} \frac{a(|F_n|)}{a(|Q_n|)} = 0.$$

The following result is immediate.

Proposition 2.8 *For $\{F_n\},\{Q_n\} \in \mathcal{SF}(G)$, $\mathbf{a} \in \mathcal{SS}$, $Z \subset X$, and each open cover $\mathcal{U}$ of X, the following properties hold:*

(1) *If $a(|F_n|) \le a(|Q_n|)$ for all sufficiently large $n \in \mathbb{N}$, then $\mathcal{E}_Z(\{F_n\},\mathcal{U},\mathbf{a}) \ge \mathcal{E}_Z(\{Q_n\},\mathcal{U},\mathbf{a})$ and $\mathcal{E}_Z(\{F_n\},\mathbf{a}) \ge \mathcal{E}_Z(\{Q_n\},\mathbf{a})$;*
(2) *If there exists a constant $C > 0$ such that $\frac{1}{C} \cdot a(|Q_n|) \le a(|F_n|) \le C \cdot a(|Q_n|)$ for all sufficiently large $n \in \mathbb{N}$, then*

$$\frac{1}{C} \cdot \mathcal{E}_Z(\{Q_n\},\mathcal{U},\mathbf{a}) \le \mathcal{E}_Z(\{F_n\},\mathcal{U},\mathbf{a}) \le C \cdot \mathcal{E}_Z(\{Q_n\},\mathcal{U},\mathbf{a})$$

and

$$\frac{1}{C} \cdot \mathcal{E}_Z(\{Q_n\},\mathbf{a}) \le \mathcal{E}_Z(\{F_n\},\mathbf{a}) \le C \cdot \mathcal{E}_Z(\{Q_n\},\mathbf{a}).$$

Remark 2.2 By Statement (2) of Proposition 2.8, for each $\mathbf{a} \in \mathcal{SS}$, equivalence class $\Big[\{F_n\}\Big]_{\mathbf{a}} \in \mathcal{F}(G)_{\mathbf{a}}$ and $\{F_n^*\},\{F_n^{**}\} \in \Big[\{F_n\}\Big]_{\mathbf{a}}$, we have that $\mathcal{E}_Z(\{F_n^*\},\mathbf{a}) = \mathcal{E}_Z(\{F_n^{**}\},\mathbf{a}) = 0$, or $\mathcal{E}_Z(\{F_n^*\},\mathbf{a}) = \mathcal{E}_Z(\{F_n^{**}\},\mathbf{a}) = \infty$ or both $\mathcal{E}_Z(\{F_n^*\},\mathbf{a})$ and $\mathcal{E}_Z(\{F_n^{**}\},\mathbf{a})$ are positive and finite. In the first two cases, we write $\mathcal{E}_Z(\Big[\{F_n\}\Big]_{\mathbf{a}},\mathbf{a}) = 0$ and $\mathcal{E}_Z(\Big[\{F_n\}\Big]_{\mathbf{a}},\mathbf{a}) = \infty$, respectively, and in the third case, we say that $\mathcal{E}_Z(\Big[\{F_n\}\Big]_{\mathbf{a}},\mathbf{a})$ is positive and finite. In this sense, entropy depends not on the Følner sequence but on its class of equivalence.

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By Statement (1) of Proposition 2.8, we have $\mathcal{E}_Z(\left[\{F_n\}\right]_{\mathbf{a}}, \mathbf{a}) \le \mathcal{E}_Z(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a})$ whenever $\left[\{F_n\}\right]_{\mathbf{a}} \ge \left[\{Q_n\}\right]_{\mathbf{a}}$.

Theorem 2.2 *Let $\mathbf{a} \in \mathcal{SS}$. If there is $\left[\{F_n\}\right]_{\mathbf{a}} \in \mathcal{F}(G)_{\mathbf{a}}$ such that $E_Z\left(\left[\{F_n\}\right]_{\mathbf{a}}, \mathbf{a}\right)$ is positive and finite, then*

$$E_Z\left(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a}\right) = \begin{cases} 0, & \text{if} \left[\{F_n\}\right]_{\mathbf{a}} \le \left[\{Q_n\}\right]_{\mathbf{a}}, \\ \infty, & \text{if} \left[\{Q_n\}\right]_{\mathbf{a}} \le \left[\{F_n\}\right]_{\mathbf{a}}. \end{cases}$$

In particular, there may exist at most one element in $(\mathcal{F}(G)_{\mathbf{a}}, \le)$ such that the corresponding scaled topological entropy is positive and finite. Similar results hold for lower and upper scaled topological entropy.

Proof We shall prove the result for the scaled topological entropy $E_Z\left(\left[\{F_n\}\right]_{\mathbf{a}}, \mathbf{a}\right)$, the arguments for the lower and upper scaled topological entropies are similar.

Suppose there is $\left[\{F_n\}\right]_{\mathbf{a}} \in \mathcal{F}(G)_{\mathbf{a}}$ such that $E_Z\left(\left[\{F_n\}\right]_{\mathbf{a}}, \mathbf{a}\right)$ is positive and finite. Then for each $\left[\{Q_n\}\right]_{\mathbf{a}} \ge \left[\{F_n\}\right]_{\mathbf{a}}$,

$$\limsup_{n\to\infty} \frac{a(|F_n^*|)}{a(|Q_n^*|)} = 0,$$

for arbitrary $\{F_n^*\} \in \left[\{F_n\}\right]_{\mathbf{a}}$ and $\{Q_n^*\} \in \left[\{Q_n\}\right]_{\mathbf{a}}$. Let us fix such two Følner sequences $\{F_n^*\}$ and $\{Q_n^*\}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $a(|F_n^*|) < \beta a(|Q_n^*|)$ and hence, $M(Z, s, \{F_n^*\}, \mathcal{U}, \mathbf{a}) \ge M(Z, s, \{Q_n^*\}, \mathcal{U}, \beta\mathbf{a})$. By (2) of Proposition 2.7,

$$E_Z(\{F_n^*\}, \mathbf{a}) \ge E_Z(\{Q_n^*\}, \beta\mathbf{a}) = \frac{1}{\beta} E_Z(\{Q_n^*\}, \mathbf{a}),$$

i.e., $\beta E_Z(\{F_n^*\}, \mathbf{a}) \ge E_Z(\{Q_n^*\}, \mathbf{a})$. Since β is arbitrary, we conclude that

$$E_Z(\{Q_n^*\}, \mathbf{a}) = 0,$$

and hence

$$E_Z\left(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a}\right) = 0.$$

On the other hand, if $\left[\{F_n\}\right]_{\mathbf{a}} \ge \left[\{Q_n\}\right]_{\mathbf{a}}$, then

$$\limsup_{n\to\infty} \frac{a(|Q_n^{**}|)}{a(|F_n^{**}|)} = 0$$

for arbitrary $\{F_n^{**}\} \in \left[\{F_n\}\right]_{\mathbf{a}}$ and $\{Q_n^{**}\} \in \left[\{Q_n\}\right]_{\mathbf{a}}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $a(|Q_n^{**}|) < \beta a(|F_n^{**}|)$ and hence, $M(Z, s, \{Q_n^{**}\}, \mathcal{U}, \mathbf{a}) \ge M(Z, s, \{F_n^{**}\}, \mathcal{U}, \beta\mathbf{a})$. It follows that $E_Z(\{Q_n^{**}\}, \mathbf{a}) >$

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$\frac{1}{\beta}E_Z(\{F_n^{**}\}, \mathbf{a})$. Again since β is arbitrary,

$$E_Z(\{Q_n^{**}\}, \mathbf{a}) = \infty,$$

then

$$E_Z\left(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a}\right) = \infty. \qquad \blacksquare$$

Remark 2.3 As is shown in Theorem 2.2, the scaled entropy depends on the choice of Følner sequence.

2.4 Scaled weighted topological entropy

For any positive function $f : X \to [0, \infty)$, $N \in \mathbb{N}$, $\varepsilon > 0$ and $\mathbf{a} \in \mathcal{SS}$, we define

$$W(f, s, N, \varepsilon, \{F_n\}, \mathbf{a}) = \inf \sum_i c_i \exp(-sa(|F_{n_i}|)),$$

where the infimum is taken over all finite or countable families $\{(B_{F_{n_i}}(x_i, \varepsilon), c_i)\}$ such that $x_i \in X, n_i \geq N, 0 < c_i < \infty$ and $\sum_i c_i \chi_{B_i} \geq f$, where $B_i = B_{F_{n_i}}(x_i, \varepsilon)$. We note that $W(\cdot, s, N, \varepsilon, \{F_n\}, \mathbf{a})$ is an outer measure on X.

For $Z \subset X, f = \chi_Z$, set $W(Z, s, N, \varepsilon, \{F_n\}, \mathbf{a}) = W(\chi_Z, s, N, \varepsilon, \{F_n\}, \mathbf{a})$. Clearly, the function $W(Z, s, N, \varepsilon, \{F_n\}, \mathbf{a})$ does not decrease as N increases and ε decreases. So the following limits exist:

$$W(Z, s, \varepsilon, \{F_n\}, \mathbf{a}) = \lim_{N\to\infty} W(Z, s, N, \varepsilon, \{F_n\}, \mathbf{a}),$$
$$W(Z, s, \{F_n\}, \mathbf{a}) = \lim_{\varepsilon\to 0} W(Z, s, \varepsilon, \{F_n\}, \mathbf{a}).$$

It's not difficult to prove that there exists a critical value of parameter s, which we will denote by $h_{top}^W(Z, \{F_n\}, \mathbf{a})$, such that

$$W(Z, s, \{F_n\}, \mathbf{a}) = \begin{cases} 0, & s > h_{top}^W(Z, \{F_n\}, \mathbf{a}), \\ \infty, & s < h_{top}^W(Z, \{F_n\}, \mathbf{a}). \end{cases}$$

We call $h_{top}^W(Z, \{F_n\}, \mathbf{a})$ scaled weighted topological entropy of (X, G) on the set Z (with respect to the sequence $\mathbf{a} \in \mathcal{SS}$ and the Følner sequence $\{F_n\}$).

2.5 Examples

Example 2.1 [33, Example 4.1] Suppose X is a compact metric space and $G = \mathbb{Z}$. Let (X, G) is a topological dynamical system induced by an expansive homeomorphism $f : X \to X$. Let $F_n = [0, n-1] \cap \mathbb{Z}$ for each $n \in \mathbb{N}$ and $\mathcal{V}$ be a generating open cover of X. Then $\{F_n\}$ is a Følner sequence. For $\mathbf{a} \in \mathcal{SS}$, let $Z \subset X$ with

(1) $\overline{E}_Z(\{F_n\}, \mathbf{a}) = 0$;
(2) There is an open cover $\mathcal{U}$ of X such that $\mathcal{U}$ is finer than $\mathcal{V}$ and $\aleph(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z) \to \infty$ as $n \to \infty$. By [33, Proposition 2.2], if $\lim\limits_{n\to\infty} \frac{a(n)}{a(n+1)} = 1$, then

$$\underline{E}_Z(\{F_n\}, \mathbf{a}) = \liminf_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right)$$

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and

$$\overline{E}_Z(\{F_n\}, \mathbf{a}) = \limsup_{n\to\infty} \frac{1}{a(|F_n|)} \log \aleph\left(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z\right).$$

If $a(|F_n|) = \aleph(\bigvee_{g\in F_n} g^{-1}\mathcal{U}, Z)$ increases monotonically, then

$$\underline{E}_Z(\{F_n\}, \mathbf{a}) = \overline{E}_Z(\{F_n\}, \mathbf{a}) = 1.$$

Example 2.2 For $k, d, n \in \mathbb{N}$, let $X = \{0, 1, \ldots, k\}^G$, $G = \mathbb{Z}^d$, $F_n = [-n+1, n-1]^d \cap \mathbb{Z}^d$ and $\sigma_g : X \to X$ be the natural shift action for $g \in G$, then $\{F_n\}$ is a Følner sequence. Suppose $x = (x_g)_{g\in G}, y = (y_g)_{g\in G} \in X$ and $\mathbf{a} \in \mathcal{SS}$ with $a(n) = \log n$, set $n(x, y) = \min\{n \in \mathbb{N} : x_g = y_g$ for all $g \in F_n$ and $x_g \neq y_g$ for some $g \in F_{n+1} \backslash F_n\}$, then $d(x, y) = \exp(-a(|F_{n(x,y)}|))$ is a compatible metric and (X, G) is a topological dynamical system.

We claim that for any $Z \subset X$, $E_Z(\{F_n\}, \mathbf{a}) = dim_H(Z)$, where $dim_H(Z)$ is the Hausdorff dimension in (X, d). Moreover, by [23, Theorem 8.19], for any $0 \le t \le dim_H(X)$, there exists a compact subset $Z_t \subset X$ such that $E_{Z_t}(\{F_n\}, \mathbf{a}) = t$.

In fact, for any $n \in \mathbb{N}$ and $x \in X$, let $C_n(x) = \{y \in X : \ x_g = y_g, \ g \in F_n\}$ be the cylinder set. For any $s \ge 0$, one can show that

$$\mathcal{H}^s(Z) := \lim_{\varepsilon\to 0} \inf_{\mathcal{C}} \operatorname{diam}(C_{k_i}(x_i))^s,$$

where $\mathcal{H}^s(Z)$ is the s-Hausdorff outer measure of Z and the infimum is taken over all finite or countable family $\mathcal{C} := \{C_{k_i}(x_i)\}$ which covers Z with $\sup_i \operatorname{diam}(C_{k_i}(x_i)) < \varepsilon$. For any sufficiently small $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\exp(-a(|F_{n+1}|)) \le \varepsilon < \exp(-a(|F_n|))$. By the choice of the metric d, we have $B_{F_k}(x, \varepsilon) = C_{k+n-1}(x)$ and $\operatorname{diam}(C_k(x)) = \exp(-a(|F_{k+1}|))$ for all $k \in \mathbb{N}$ and $x \in X$. The desired conclusion follows exactly by the definitions of $E_Z(\{F_n\}, \mathbf{a})$ and $dim_H(Z)$, we refer to [29, Theorem 4.2] for a similar and detailed proof.

3 Scaled measure entropy

In this Section, we introduce different types of scaled measure entropy and study their properties .

3.1 Scaled measure entropy

Let X be a compact Hausdorff space, (X, G) be a measurable dynamical system. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X. Denote by $\mathcal{M}(X, G)$ (respectively, $\mathcal{M}^e(X, G)$) the set of all G-invariant (respectively, ergodic G-invariant) Borel probability measure on X. We follow the approach described in [25] and introduce the notion of scaled measure entropy using the inverse variational principle. Given $\mu \in \mathcal{M}(X, G)$ and $\mathbf{a} \in \mathcal{SS}$, let

$$\begin{aligned} E_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}) &= \inf\{E_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) : \mu(Z) = 1\} \\ &= \lim_{\delta\to 0} \inf\{E_Z(\{F_n\}, \mathcal{U}, \mathbf{a}) : \mu(Z) \ge 1 - \delta\}. \end{aligned}$$

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The fact that the second equality holds can be proven in the same way as [25, p. 22], and for that reason we shall omit its proof.

Let

$$E_\mu(\{F_n\},\mathbf{a}) = \sup\{E_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\}.$$

We call the quantity $E_\mu(\{F_n\},\mathbf{a})$ the scaled measure entropy of (X,G) with respect to μ and $\mathbf{a} \in \mathcal{SS}$. Let further

$$\underline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) = \lim_{\delta\to 0}\inf\{\underline{E}_Z(\{F_n\},\mathcal{U},\mathbf{a}) : \mu(Z) \geq 1-\delta\},$$
$$\overline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) = \lim_{\delta\to 0}\inf\{\overline{E}_Z(\{F_n\},\mathcal{U},\mathbf{a}) : \mu(Z) \geq 1-\delta\}.$$

We call the quantities

$$\underline{E}_\mu(\{F_n\},\mathbf{a}) = \sup\{\underline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\},$$

$$\overline{E}_\mu(\{F_n\},\mathbf{a}) = \sup\{\overline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \mid \mathcal{U} \text{ is a finite open cover of } X\},$$

respectively the lower and upper scaled measure entropy of (X,G) with respect to μ and $\mathbf{a} \in \mathcal{SS}$.

We describe another equivalent definition of scaled measure entropy. Given $\mu \in \mathcal{M}(X,G)$ and $\mathbf{a} \in \mathcal{SS}$, let

$$\begin{aligned} E_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) &= \inf\{E_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) : \mu(Z) = 1\} \\ &= \lim_{\delta\to 0}\inf\{E_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) : \mu(Z) \geq 1-\delta\} \end{aligned}$$

and then let

$$E_\mu^{B_2}(\{F_n\},\mathbf{a}) = \lim_{\varepsilon\to 0} E_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}).$$

Let further

$$\underline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) = \lim_{\delta\to 0}\inf\{\underline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) : \mu(Z) \geq 1-\delta\},$$
$$\overline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) = \lim_{\delta\to 0}\inf\{\overline{E}_Z^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) : \mu(Z) \geq 1-\delta\}.$$

Set

$$\underline{E}_\mu^{B_2}(\{F_n\},\mathbf{a}) = \lim_{\varepsilon\to 0} \underline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}),$$

and

$$\overline{E}_\mu^{B_2}(\{F_n\},\mathbf{a}) = \lim_{\varepsilon\to 0} \overline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}).$$

It is easy to see that

$$E_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) \leq \underline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a}) \leq \overline{E}_\mu^{B_2}(\{F_n\},\varepsilon,\mathbf{a})$$

and

$$E_\mu^{B_2}(\{F_n\},\mathbf{a}) \leq \underline{E}_\mu^{B_2}(\{F_n\},\mathbf{a}) \leq \overline{E}_\mu^{B_2}(\{F_n\},\mathbf{a}).$$

We have the following result.

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Proposition 3.1 *For any $\mu \in \mathcal{M}(X,G)$ and $\mathbf{a} \in \mathcal{SS}$, we have*
(a) $E_\mu(\{F_n\},\mathbf{a}) = E_\mu^{B_1}(\{F_n\},\mathbf{a}) = E_\mu^{B_2}(\{F_n\},\mathbf{a})$;
(b) $\underline{E}_\mu(\{F_n\},\mathbf{a}) = \underline{E}_\mu^{B_2}(\{F_n\},\mathbf{a})$; $\overline{E}_\mu(\{F_n\},\mathbf{a}) = \overline{E}_\mu^{B_2}(\{F_n\},\mathbf{a})$.

This proof is similar to the proof of Proposition 2.2, so we omit it.

3.2 Properties of the scaled measure entropy

In what follows, we use the notation $\mathcal{E}_\mu$ for either E_μ, $\underline{E}_\mu$ or $\overline{E}_\mu$. The following Propositions describe some basic properties of scaled measure entropy and lower and upper scaled measure entropies.

The following Proposition is a direct consequence of the definition of scaled measure entropy and Proposition 2.4.

Proposition 3.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of X, $\mu \in \mathcal{M}(X,G)$ and $\mathbf{a} \in \mathcal{SS}$, the following properties hold:*

(1) *If $\mathcal{U} \le \mathcal{V}$, then $\mathcal{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \le \mathcal{E}_\mu(\{F_n\},\mathcal{V},\mathbf{a})$;*
(2)

$$E_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \le \underline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a}) \le \overline{E}_\mu(\{F_n\},\mathcal{U},\mathbf{a})$$

and

$$E_\mu(\{F_n\},\mathbf{a}) \le \underline{E}_\mu(\{F_n\},\mathbf{a}) \le \overline{E}_\mu(\{F_n\},\mathbf{a}).$$

The following Proposition shows that the scaled measure entropy as well as lower and supper scaled measure entropies for amenable group actions are invariant under a measure conjugacy.

Definition 3.1 [21, Definition 1.4] Two probability measure preserving actions $G \curvearrowright X_1$, $G \curvearrowright X_2$ of the same group are said to be measure conjugate if there are conull sets $X_1^{'} \subset X_1$ and $X_2^{'} \subset X_2$ with $GX_1^{'} \subset X_1$ and $GX_2^{'} \subset X_2$ and an equivariant measure isomorphism $\varphi : X_1^{'} \to X_2^{'}$.

Proposition 3.3 *Given two measure conjugate actions $G \curvearrowright X_1$ and $G \curvearrowright X_2$. For any $\mathbf{a} \in \mathcal{SS}$ and $\mu \in \mathcal{M}(X,G)$, we have*

$$\mathcal{E}_\mu(\{F_n\},\mathbf{a}) = \mathcal{E}_{\varphi_*\mu}(\{F_n\},\mathbf{a}),$$

where φ is the equivariant measure isomorphism and $\varphi_*\mu = \mu \circ \varphi^{-1}$.

3.3 Scaled measure entropy for equivalent sequences and equivalent Følner sequences

Following the discussion on scaled topological entropy for equivalent scaled sequence and equivalent Følner sequencesin Section 2.3, we introduce a similar notion of equivalence for the scaled measure entropy.

The following Proposition is a direct consequence of Proposition 2.7.

Proposition 3.4 *Let $\mathbf{a},\mathbf{b} \in \mathcal{SS}$, for every G-invariant measure $\mu \in \mathcal{M}(X,G)$, the following properties hold:*

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(1) *If* $a(|F_n|) \le b(|F_n|)$ *for all sufficiently large* $n \in \mathbb{N}$, *then* $\mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}) \ge \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{b})$ *and* $\mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \ge \mathcal{E}_\mu(\{F_n\}, \mathbf{b})$;

(2) *For each* $K > 0$ *we have that*

$$K \cdot \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, K\mathbf{a}) = \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}),\ K \cdot \mathcal{E}_\mu(\{F_n\}, K\mathbf{a}) = \mathcal{E}_\mu(\{F_n\}, \mathbf{a}),$$

where $K\mathbf{a} = \{K \cdot a(|F_n|)\}$;

(3) *If there exists a constant* $C > 0$ *such that* $\frac{1}{C} \cdot b(|F_n|) \le a(|F_n|) \le C \cdot b(|F_n|)$ *for all sufficiently large n, then*

$$\frac{1}{C} \cdot \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{b}) \le \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}) \le C \cdot \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{b})$$

and

$$\frac{1}{C} \cdot \mathcal{E}_\mu(\{F_n\}, \mathbf{b}) \le \mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \le C \cdot \mathcal{E}_\mu(\{F_n\}, \mathbf{b}).$$

Remark 3.1 By Statement (1) of Proposition 3.4, we have that $\mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \ge \mathcal{E}_\mu(\{F_n\}, \mathbf{b})$ whenever $[\mathbf{a}] \le [\mathbf{b}]$ and by Statement (3) of Proposition 3.4, for each equivalence class $[\mathbf{a}] \in \mathcal{A}$ and for each $\mathbf{a_1}, \mathbf{a_2} \in [\mathbf{a}]$ we have that $\mathcal{E}_\mu(\{F_n\}, \mathbf{a_1}) = \mathcal{E}_\mu(\{F_n\}, \mathbf{a_2}) = 0$ or $\mathcal{E}_\mu(\{F_n\}, \mathbf{a_1}) = \mathcal{E}_\mu(\{F_n\}, \mathbf{a_2}) = \infty$ or both $\mathcal{E}_\mu(\{F_n\}, \mathbf{a_1})$ and $\mathcal{E}_\mu(\{F_n\}, \mathbf{a_2})$ are positive and finite.

Theorem 3.1 *For every* $\mu \in \mathcal{M}(X, G)$, *if there is* $[\mathbf{a}] \in \mathcal{A}$ *such that* $E_\mu(\{F_n\}, [\mathbf{a}])$ *is positive and finite, then*

$$E_\mu(\{F_n\}, [\mathbf{b}]) = \begin{cases} 0, & \text{if } [\mathbf{a}] \le [\mathbf{b}], \\ \infty, & \text{if } [\mathbf{b}] \le [\mathbf{a}]. \end{cases}$$

Similar results holds for lower and upper scaled measure entropy.

Proof Suppose there is $[\mathbf{a}] \in \mathcal{A}$ such that $E_\mu(\{F_n\}, [\mathbf{a}])$ is positive and finite. Then for each $[\mathbf{b}] \ge [\mathbf{a}]$,

$$\limsup_{n \to \infty} \frac{a_1(|F_n|)}{b_1(|F_n|)} = 0$$

for arbitrary $\mathbf{a_1} = \{a_1(|F_n|)\} \in [\mathbf{a}]$ and $\mathbf{b_1} = \{b_1(|F_n|)\} \in [\mathbf{b}]$. Let us fix such two scaled sequences $\mathbf{a_1}$ and $\mathbf{b_1}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $a_1(|F_n|) < \beta b_1(|F_n|)$. By Proposition 3.4, we have that

$$E_\mu(\{F_n\}, \mathbf{a_1}) \ge E_\mu(\{F_n\}, \beta\mathbf{b_1}) = \frac{1}{\beta} E_\mu(\{F_n\}, \mathbf{b_1}),$$

i.e., $\beta E_\mu(\{F_n\}, \mathbf{a_1}) \ge E_\mu(\{F_n\}, \mathbf{b_1})$. Since β is arbitrary, we conclude that

$$E_\mu(\{F_n\}, \mathbf{b_1}) = 0,$$

and hence

$$E_\mu(\{F_n\}, [\mathbf{b}]) = 0.$$

On the other hand, if $[\mathbf{b}] \preceq [\mathbf{a}]$ then

$$\limsup_{n\to\infty} \frac{b_2(|F_n|)}{a_2(|F_n|)} = 0$$

for arbitrary $\mathbf{a_2} = \{a_2(|F_n|)\} \in [\mathbf{a}]$ and $\mathbf{b_2} = \{b_2(|F_n|)\} \in [\mathbf{b}]$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $b_2(|F_n|) < \beta a_2(|F_n|)$. It follows that $E_\mu(\{F_n\}, \mathbf{b_2}) > \frac{1}{\beta} E_\mu(\{F_n\}, \mathbf{a_2})$. Again since β is arbitrary,

$$E_\mu(\{F_n\}, \mathbf{b_2}) = \infty,$$

and hence

$$E_\mu(\{F_n\}, [\mathbf{b}]) = \infty.$$

■

And the following proposition is a direct consequence of Proposition 2.8.

Proposition 3.5 *Let $\{F_n\}, \{Q_n\} \in \mathcal{SF}(G)$. For $\mathbf{a} \in \mathcal{SS}$ and every G-invariant measure $\mu \in \mathcal{M}(X, G)$, the following properties hold:*

(1) *If $a(|F_n|) \le a(|Q_n|)$ for all sufficiently large $n \in \mathbb{N}$, then $\mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}) \ge \mathcal{E}_\mu(\{Q_n\}, \mathcal{U}, \mathbf{a})$ and $\mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \ge \mathcal{E}_\mu(\{Q_n\}, \mathbf{a})$;*
(2) *If there exists a constant $C > 0$ such that $\frac{1}{C} \cdot a(|Q_n|) \le a(|F_n|) \le C \cdot a(|Q_n|)$ for all sufficiently large $n \in \mathbb{N}$, then*

$$\frac{1}{C} \cdot \mathcal{E}_\mu(\{Q_n\}, \mathcal{U}, \mathbf{a}) \le \mathcal{E}_\mu(\{F_n\}, \mathcal{U}, \mathbf{a}) \le C \cdot \mathcal{E}_\mu(\{Q_n\}, \mathcal{U}, \mathbf{a})$$

and

$$\frac{1}{C} \cdot \mathcal{E}_\mu(\{Q_n\}, \mathbf{a}) \le \mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \le C \cdot \mathcal{E}_\mu(\{Q_n\}, \mathbf{a}).$$

Remark 3.2 By Statement (1) of Proposition 3.5, we have that $\mathcal{E}_\mu(\{F_n\}, \mathbf{a}) \ge \mathcal{E}_\mu(\{Q_n\}, \mathbf{a})$ whenever $\left[\{F_n\}\right]_\mathbf{a} \preceq \left[\{Q_n\}\right]_\mathbf{a}$ and by Statement (2) of Proposition 3.5, for each equivalence class $\left[\{F_n\}\right]_\mathbf{a} \in \mathcal{F}(G)_\mathbf{a}$ and for each $\{F_n^*\}, \{F_n^{**}\} \in \left[\{F_n\}\right]$ we have that $\mathcal{E}_\mu(\{F_n\}, \mathbf{a}) = \mathcal{E}_\mu(\{F_n\}, \mathbf{a}) = 0$ or $\mathcal{E}_\mu(\{F_n^*\}, \mathbf{a}) = \mathcal{E}_\mu(\{F_n^{**}\}, \mathbf{a}) = \infty$ or both $\mathcal{E}_\mu(\{F_n^*\}, \mathbf{a})$ and $\mathcal{E}_\mu(\{F_n^{**}\}, \mathbf{a})$ are positive and finite.

Theorem 3.2 *Let $\mu \in \mathcal{M}(X, G)$ and $\mathbf{a} \in \mathcal{SS}$. If there is $\left[\{F_n\}\right]_\mathbf{a} \in \mathcal{F}(G)_\mathbf{a}$ such that $E_\mu\left(\left[\{F_n\}\right]_\mathbf{a}, \mathbf{a}\right)$ is positive and finite, then*

$$E_\mu\left(\left[\{Q_n\}\right]_\mathbf{a}, \mathbf{a}\right) = \begin{cases} 0, & \text{if} \left[\{F_n\}\right]_\mathbf{a} \preceq \left[\{Q_n\}\right]_\mathbf{a}, \\ \infty, & \text{if} \left[\{Q_n\}\right]_\mathbf{a} \preceq \left[\{F_n\}\right]_\mathbf{a}. \end{cases}$$

Similar results hold for lower and upper scaled measure entropy.

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Proof Suppose there is $\left[\{F_n\}\right]_{\mathbf{a}} \in \mathcal{F}(G)_{\mathbf{a}}$ such that $E_\mu\left(\left[\{F_n\}\right]_{\mathbf{a}}, \mathbf{a}\right)$ is positive and finite. Then for each $\left[\{Q_n\}\right]_{\mathbf{a}} \geq \left[\{F_n\}\right]_{\mathbf{a}}$,

$$\limsup_{n\to\infty} \frac{a(|F_n^*|)}{a(|Q_n^*|)} = 0$$

for arbitrary $\{F_n^*\} \in \left[\{F_n\}\right]_{\mathbf{a}}$ and $\{Q_n^*\} \in \left[\{Q_n\}\right]_{\mathbf{a}}$. Let us fix such two Følner sequences $\{F_n^*\}$ and $\{Q_n^*\}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$, we have that $a(|F_n^*|) < \beta a(|Q_n^*|)$. By (2) of Proposition 3.4, we have that

$$E_\mu(\{F_n^*\}, \mathbf{a}) \geq E_\mu(\{Q_n^*\}, \beta\mathbf{a}) = \frac{1}{\beta} E_\mu(\{Q_n^*\}, \mathbf{a}),$$

i.e., $\beta E_\mu(\{F_n^*\}, \mathbf{a}) \geq E_\mu(\{Q_n^*\}, \mathbf{a})$. Since β is arbitrary, we conclude that

$$E_\mu(\{Q_n^*\}, \mathbf{a}) = 0,$$

and hence

$$E_\mu\left(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a}\right) = 0.$$

On the other hand, if $\left[\{F_n\}\right]_{\mathbf{a}} \geq \left[\{Q_n\}\right]_{\mathbf{a}}$ then

$$\limsup_{n\to\infty} \frac{a(|Q_n^{**}|)}{a(|F_n^{**}|)} = 0$$

for arbitrary $\{F_n^{**}\} \in \left[\{F_n\}\right]_{\mathbf{a}}$ and $\{Q_n^{**}\} \in \left[\{Q_n\}\right]_{\mathbf{a}}$. Given a small number $\beta > 0$, for all sufficiently large $n \in \mathbb{N}$ we have that $a(|Q_n^{**}|) < \beta a(|F_n^{**}|)$. It follows that $E_\mu(\{Q_n^{**}\}, \mathbf{a}) > \frac{1}{\beta} E_\mu(\{F_n^{**}\}, \mathbf{a})$. Again since β is arbitrary,

$$E_\mu(\{Q_n^{**}\}, \mathbf{a}) = \infty,$$

then

$$E_\mu\left(\left[\{Q_n\}\right]_{\mathbf{a}}, \mathbf{a}\right) = \infty.$$

■

3.4 Scaled local entropy

In this Section, we introduce the scaled local entropy following the approach of Brin and Katok defined for amenable group actions as follows:

Definition 3.2 For any $\mathbf{a} \in \mathcal{SS}$, $\varepsilon > 0$ and $\mu \in \mathcal{M}(X)$, define

$$\underline{h}_\mu(\{F_n\}, \mathbf{a}) = \int \underline{h}_\mu(\{F_n\}, \mathbf{a}, x)\, d\mu(x),$$
$$\overline{h}_\mu(\{F_n\}, \mathbf{a}) = \int \overline{h}_\mu(\{F_n\}, \mathbf{a}, x)\, d\mu(x),$$

where

$$\underline{h}_\mu(\{F_n\}, \mathbf{a}, x) = \lim_{\varepsilon\to 0} \liminf_{n\to+\infty} -\frac{1}{a(|F_n|)} \log \mu(B_{F_n}(x, \varepsilon)),$$

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$$\overline{h}_\mu(\{F_n\},\mathbf{a},x)=\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}-\frac{1}{a(|F_n|)}\log\mu(B_{F_n}(x,\varepsilon)).$$

We call $\underline{h}_\mu(\{F_n\},\mathbf{a})$, $\overline{h}_\mu(\{F_n\},\mathbf{a})$ the lower and upper scaled local entropy of (X,G) (with respect to the sequence $\mathbf{a}\in\mathcal{SS}$, the Følner sequence $\{F_n\}$ and $\mu\in\mathcal{M}(X)$).

Remark 3.3 It is pointed out in [12] that Dan Rudolph showed that for an amenable group G, the generic measure-preserving action of G on a Lebesgue space has zero entropy. Indeed, this is extended to nonamenable groups by Lewis Bowen in [3] in which the proof shows that every action is a factor of a zero entropy action. In this sense, for generic measure-preserving actions, if we can find certain sub-exponential scaled sequences, then the scaled measure entropy could be positive. Thus the scaled measure entropy is a possible candidate to classify generic measure-preserving actions. For more examples of scaled measure entropies for $\mathbb{Z}$ or $\mathbb{N}$ actions, we refer to [33, 4 Examples].

4 Variational principle

The notion of scaled weighted topological entropy is introduced, which is important to prove the variational principle.

4.1 Equivalence of $E_Z^{B_2}(\{F_n\},\mathbf{a})$ and $h^W_{top}(Z,\{F_n\},\mathbf{a})$.

Lemma 4.1 [23, Theorem 2.1] *Let (X,d) be a compact metric space and $\mathcal{B}=\{B(x_i,r_i)\}_{i\in\mathcal{I}}$ be a family open of (or closed) balls in X. Then there exists a finite or countable subfamily $\mathcal{B}'=\{B(x_i,r_i)\}_{i\in\mathcal{I}'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}}B\subseteq\bigcup_{i\in\mathcal{I}'}B(x_i,5r_i).$$

Theorem 4.1 *For any $\mathbf{a}\in\mathcal{SS}$, Borel set $L\subset X$, $\mu\in\mathcal{M}(X)$ and $s\ge 0$, we have*

(1) *If $\underline{h}_\mu(\{F_n\},\mathbf{a},x)\le s$ for all $x\in L$, then $E_L(\{F_n\},\mathbf{a})\le s$;*
(2) *If $\underline{h}_\mu(\{F_n\},\mathbf{a},x)\ge s$ for all $x\in L$ and $\mu(L)>0$, then $E_L(\{F_n\},\mathbf{a})\ge s$.*

Proof (1) For a fixed $r>0$ and $k\in\mathbb{N}$, let

$$L_k=\left\{x\in L:\liminf_{n\to\infty}\frac{-\log\mu(B_{F_n}(x,\varepsilon))}{a(|F_n|)}<s+r\text{ for all }\varepsilon\in(0,\frac{1}{k})\right\}.$$

Then we have $L=\bigcup_{k=1}^\infty L_k$, since $\underline{h}_\mu(\{F_n\},\mathbf{a},x)\le s$ for all $x\in L$.

Now fix $k\ge 1$ and $0<\varepsilon<\frac{1}{5k}$. For each $x\in L_k$, there exists a strictly increasing sequence $\{n_j\}_{j=1}^\infty$(depending on the point x) such that

$$\mu(B_{F_{n_j}}(x,\varepsilon))\ge\exp(-(s+r)a(|F_{n_j}|))\text{ for all }j\ge 1.$$

For any $N\in\mathbb{N}$, the set L_k is contained in the union of the sets in the family

$$\mathcal{F}=\{B_{F_{n_j}}(x,\varepsilon):x\in L_k,n_j\ge N\}.$$

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By Lemma 4.1, there exists a finite or countable subfamily $\mathcal{B} = \{B_{F_{n_i}}(x_i, \varepsilon)\}_{i\in\mathcal{I}} \subset \mathcal{F}$ of pairwise disjoint balls such that

$$L_k \subset \bigcup_{i\in I} B_{F_{n_i}}(x_i, 5\varepsilon).$$

The subfamily is at most countable since μ is a probability measure and the elements in $\mathcal{B}$ are pairwise disjoint and have positive μ-measure. Note that

$$\mu(B_{F_{n_i}}(x_i, \varepsilon)) \geq \exp(-(s+r)a(|F_{n_i}|)) \text{ for all } i \in \mathcal{I}.$$

The disjointness of $\{B_{F_{n_i}}(x_i, \varepsilon)\}_{i\in\mathcal{I}}$ yields that

$$M(L_k, s+r, N, \{F_n\}, 5\varepsilon, \mathbf{a}) \leq \sum_{i\in\mathcal{I}} \exp(-(s+r)a(|F_{n_i}|)) \leq \sum_{i\in\mathcal{I}} \mu(B_{F_{n_i}}(x_i, \varepsilon)) \leq 1.$$

It follows that

$$M(L_k, s+r, \{F_n\}, 5\varepsilon, \mathbf{a}) = \lim_{N\to\infty} M(L_k, s+r, N, \{F_n\}, 5\varepsilon, \mathbf{a}) \leq 1.$$

Hence,

$$E_{L_k}(\{F_n\}, 5\varepsilon, \mathbf{a}) \leq s + r,$$

which implies that

$$E_{L_k}(\{F_n\}, \mathbf{a}) \leq s + r \text{ for all } k \geq 1.$$

Hence,

$$E_L(\{F_n\}, \mathbf{a}) = E_{\bigcup_{k=1}^{\infty} L_k}(\{F_n\}, \mathbf{a}) = \sup_{k\geq 1} E_{L_k}(\{F_n\}, \mathbf{a}) \leq s + r.$$

Since r can be arbitrary, this implies that $E_L(\{F_n\}, \mathbf{a}) \leq s$.

(2) For a fixed $r > 0$ and $k \in \mathbb{N}$, let

$$L_k = \left\{x \in L : \liminf_{n\to\infty} \frac{-\log \mu(B_{F_n}(x, \varepsilon))}{a(|F_n|)} > s - r \text{ for all } \varepsilon \in (0, \frac{1}{k})\right\}.$$

Since $\underline{h}_\mu(\{F_n\}, \mathbf{a}, x) \geq s$ for all $x \in L$, we have that $L_k \subset L_{k+1}$ and $L = \bigcup_{k=1}^{\infty} L_k$. Fix a sufficiently large $k \geq 1$ with $\mu(L_k) > \frac{1}{2}\mu(L) > 0$. For each $N \in \mathbb{N}$, set

$$L_{k,N} = \left\{x \in L_k : \frac{-\log \mu(B_{F_n}(x, \varepsilon))}{a(|F_n|)} > s - r \text{ for all } n \geq N, \varepsilon \in (0, \frac{1}{k})\right\}.$$

It is easy to see that $L_{k,N} \subset L_{k,N+1}$ and $L_k = \bigcup_{N=1}^{\infty} L_{k,N}$. Thus we can pick $N^* \geq 1$ such that $\mu(L_{k,N^*}) > \frac{1}{2}\mu(L_k) > 0$. For simplicity of notation, let $L^* = L_{k,N^*}$ and $\varepsilon^* = \frac{1}{k}$. By the choice of L^*, we have that

$$\mu(B_{F_n}(x, \varepsilon)) \leq \exp(-(s-r)a(|F_n|)) \text{ for all } x \in L^*, 0 < \varepsilon < \varepsilon^*, n \geq N^*.$$

Fix a sufficiently large $N > N^*$. For each cover $\mathcal{F} = \{B_{F_{n_i}}(y_i, \frac{\varepsilon}{2})\}_{i\geq 1}$ of L^* with $0 < \varepsilon < \varepsilon^*$ and $n_i \geq N \geq N^*$ for each $i \geq 1$. Without loss of generality, assume that $L^* \bigcap B_{F_{n_i}}(y_i, \frac{\varepsilon}{2}) \neq \varnothing$ for all i. Thus, for each $i \geq 1$ pick a point $x_i \in L^* \bigcap B_{F_{n_i}}(y_i, \frac{\varepsilon}{2})$

so that

$$B_{F_{n_i}}\left(y_i,\frac{\varepsilon}{2}\right)\subset B_{F_{n_i}}(x_i,\varepsilon).$$

It follows that

$$\sum_{i\geq 1}\exp(-(s-r)a(|F_{n_i}|))\geq\sum_{i\geq 1}\mu(B_{F_{n_i}}(x_i,\varepsilon))\geq\mu(L^*).$$

Therefore,

$$M(L^*,s-r,N,\{F_n\},\frac{\varepsilon}{2},\mathbf{a})\geq\mu(L^*)>0.$$

Consequently,

$$M(L^*,s-r,\{F_n\},\frac{\varepsilon}{2},\mathbf{a})=\lim_{N\to\infty}M(L^*,s-r,N,\{F_n\},\frac{\varepsilon}{2},\mathbf{a})\geq\mu(L^*)>0,$$

which implies that $E_{L^*}(\{F_n\},\mathbf{a})\geq s-r$.

It follows that

$$E_L(\{F_n\},\mathbf{a})\geq E_{L^*}(\{F_n\},\mathbf{a})\geq s-r.$$

Since r can be arbitrary, this implies that $E_L(\{F_n\},\mathbf{a})\geq s$ completing the proof of the theorem. ∎

We need to emphasize that the following proof uses the similar idea as [11, Proposition 3.2].

Proposition 4.1 *For any* $\mathbf{a}\in\mathbb{SS}$, $s\geq 0,\varepsilon,\delta>0$ *and* $Z\subset X$. *If* $\liminf\limits_{n\to+\infty}\frac{a(|F_n|)}{n}>0$, *we have*

$$M(Z,s+\delta,N,\{F_n\},6\varepsilon,\mathbf{a})\leq W(Z,s,N,\varepsilon,\{F_n\},\mathbf{a})\leq M(Z,s,N,\{F_n\},\varepsilon,\mathbf{a})$$

for large enough $N\in\mathbb{N}$, *and then* $E_Z^{B_2}(\{F_n\},\mathbf{a})=h_{top}^W(Z,\{F_n\},\mathbf{a})$.

Proof Let $Z\subset X, s\geq 0,\varepsilon,\delta>0$, set $f=\chi_Z, c_i\equiv 1$ in the definition of scaled weighted topological entropy, we have

$$W(Z,s,N,\varepsilon,\{F_n\},\mathbf{a})\leq M(Z,s,N,\{F_n\},\varepsilon,\mathbf{a})\ for\ every\ N\in\mathbb{N}.$$

Next, we prove $M(Z,s+\delta,N,\{F_n\},6\varepsilon,\mathbf{a})\leq W(Z,s,N,\varepsilon,\{F_n\},\mathbf{a})$ for large enough $N\in\mathbb{N}$.

There are $\xi>0$ and N_1 such that $a(|F_n|)\geq n\xi$ for all $n\geq N_1$. Let $N>\max\{N_1,2\}$ such that $n^2e^{-n\delta\xi}\leq 1$ for all $n\geq N$. Let $\{(B_{F_{n_i}}(x_i,\varepsilon),c_i)\}_{i\in\mathcal{J}}$ be a family so that $\mathcal{J}\subset\mathbb{N}, x_i\in X, 0\leq c_i<\infty, n_i\geq N$ and

$$\sum_i c_i\chi_{B_i}\geq\chi_Z,$$

where $B_i:=B_{F_{n_i}}(x_i,\varepsilon)$. We claim that

$$M(Z,s+\delta,N,\{F_n\},6\varepsilon,\mathbf{a})\leq\sum_{i\in I}c_i\exp(-sa(|F_{n_i}|))\tag{4.1}$$

and hence $M(Z,s+\delta,N,\{F_n\},6\varepsilon,\mathbf{a})\leq W(Z,s,N,\varepsilon,\{F_n\},\mathbf{a})$.

We denote

$$\mathcal{I}_n = \{i \in \mathcal{I} : n_i = n\},$$

and

$$\mathcal{I}_{n,k} = \{i \in \mathcal{I}_n : i \leq k\}$$

for $n \geq N, k \in \mathbb{N}$. We write $B_i := B_{F_{n_i}}(x_i, \varepsilon), 5B_i := B_{F_{n_i}}(x_i, 5\varepsilon)$ for $i \in \mathcal{I}$. Obviously, we may assume $B_i \neq B_j$ for $i \neq j$. For $t > 0$, set

$$Z_{n,t} = \left\{ x \in Z : \sum_{i \in \mathcal{I}_n} c_i \chi_{B_i}(x) > t \right\}$$

and

$$Z_{n,t,k} = \left\{ x \in Z : \sum_{i \in \mathcal{I}_{n,k}} c_i \chi_{B_i}(x) > t \right\}.$$

We divide the proof of (4.1) into the following three steps.

Step 1. For each $n \geq N, k \in \mathbb{N}$ and $t > 0$, there exists a finite set $\mathcal{J}_{n,k,t} \subset \mathcal{I}_{n,k}$ such that the ball $B_i (i \in \mathcal{J}_{n,k,t})$ are pairwise disjoint, $Z_{n,t,k} \subset \cup_{i \in \mathcal{J}_{n,k,t}} 5B_i$, and

$$\aleph(\mathcal{J}_{n,k,t}) \exp(-sa(|F_{n_i}|)) \leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i \exp(-sa(|F_{n_i}|)).$$

We will use the method of Federer [10, 2.10.24] and Mattila [23, Lemma 8.16] for amenable group actions. Since $\mathcal{I}_{n,k}$ is finite, by approximating the c_i's from above, we may assume that each c_i is a positive rational, and then by multiplying with a common denominator we may assume that each c_i is a positive integer. Let m be the least integer with $m \geq t$. Denote $\mathcal{B} = \{B_i, i \in \mathcal{I}_{n,k}\}$, and define $u : \mathcal{B} \to \mathbb{Z}$, by $u(B_i) = c_i$. Since $B_i \neq B_j$ for $i \neq j$, so u is well defined. We define by introduction integer-valued functions $v_0, v_1, \cdots, v_m$ on $\mathcal{B}$ and sub-families $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ of $\mathcal{B}$ starting with $v_0 = u$. Using Lemma 4.1 repeatedly, we define inductively for $j = 1, \cdots, m$, disjoint subfamilies $\mathcal{B}_i$ of $\mathcal{B}$ such that

$$\mathcal{B}_j \subset \{B \in \mathcal{B} : v_{j-1}(B) \geq 1\},$$

$$Z_{n,k,t} \subset \cup_{B \in \mathcal{B}_j} 5B,$$

and the functions v_j such that

$$v_j(B) = \begin{cases} v_{j-1}(B) - 1, & \text{for } B \in \mathcal{B}_j, \\ v_{j-1}(B), & \text{for } B \in \mathcal{B} \backslash \mathcal{B}_j. \end{cases}$$

This is possible for $j < m$,

$$Z_{n,k,t} \subset \left\{ x : \sum_{B \in \mathcal{B} : B \ni x} v_j(B) \geq m - j \right\},$$

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whence every $x \in Z_{n,k,t}$ belongs to some ball $B \in \mathcal{B}$ with $\nu_j(B) \geq 1$. Thus

$$\begin{aligned}\sum_{j=1}^{m} \aleph(\mathcal{B}_j) \exp(-sa(|F_n|)) &= \sum_{j=1}^{m} \sum_{B \in \mathcal{B}_j} (\nu_{j-1}(B) - \nu_j(B)) \exp(-sa(|F_n|)) \\ &\leq \sum_{B \in \mathcal{B}} \sum_{j=1}^{m} (\nu_{j-1}(B) - \nu_j(B)) \exp(-sa(|F_n|)) \\ &\leq \sum_{B \in \mathcal{B}} u(B) \exp(-sa(|F_n|)) \\ &= \sum_{i \in \mathcal{I}_{n,k}} c_i \exp(-sa(|F_n|)).\end{aligned}$$

Choose $j_0 \in \{1, \cdots, m\}$ so that $\aleph(\mathcal{B}_{j_0})$ is the smallest. Then

$$\aleph(\mathcal{B}_{j_0}) \exp(-sa(|F_n|)) \leq \frac{1}{m} \sum_{i \in \mathcal{I}_{n,k}} c_i \exp(-sa(|F_n|)) \leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i \exp(-sa(|F_n|)).$$

So $\mathcal{J}_{n,k,t} = \{i \in \mathcal{I} : B_i \in \mathcal{B}_{j_0}\}$ is desired.

Step 2. For each $n \in \mathbb{N}$ and $t > 0$, we have

$$M(Z_{n,t}, s+\delta, N, \{F_n\}, 6\varepsilon, \mathbf{a}) \leq \frac{\exp(-\delta a(|F_n|))}{t} \sum_{i \in \mathcal{I}_n} c_i \exp(-sa(|F_n|)). \tag{4.2}$$

Assume $Z_{n,t} \neq \varnothing$, otherwise (4.2) is obvious. Since $Z_{n,k,t} \uparrow Z_{n,t}$, $Z_{n,k,t} \neq \varnothing$ for large enough $k \in \mathbb{N}$. Let $\mathcal{J}_{n,k,t}$ be the sets constructed in *Step 1*. Then $\mathcal{J}_{n,k,t} \neq \varnothing$ for large enough $k \in \mathbb{N}$. Set $E_{n,k,t} = \{x_i : i \in \mathcal{J}_{n,k,t}\}$. Note that the family of all non-empty compact subsets of X is compact with respect to Hausdorff distance(Federer [10, 2.10.21]). It follows that there is a subsequence (k_j) of natural numbers and a non-empty compact set $E_{n,t} \subset X$ such that $E_{n,k_j,t}$ converges to $E_{n,t}$ in the Hausdorff distance as $j \to \infty$. Since any two points in $E_{n,k,t}$ have a distance (with respect to d_{F_n}) not less than ε, so do the points in $E_{n,t}$. Thus $E_{n,t}$ is a finite set, moreover, $\aleph(E_{n,k_j,t}) = \aleph(E_{n,t})$ when $j \in \mathbb{N}$ is large enough.

Hence

$$\bigcup_{x \in E_{n,t}} B_{F_n}(x, 5.5\varepsilon) \supset \bigcup_{x \in E_{n,k_j,t}} B_{F_n}(x, 5\varepsilon) = \bigcup_{i \in \mathcal{J}_{n,k_j,t}} 5B_i \supset Z_{n,k_j,t}$$

when $j \in \mathbb{N}$ is large enough, and thus $\bigcup_{x \in E_{n,t}} B_{F_n}(x, 6\varepsilon) \supset Z_{n,t}$. By the way, since $\aleph(E_{n,k_j,t}) = \aleph(E_{n,t})$ when $j \in \mathbb{N}$ is large enough, we have

$$\aleph(E_{n,t}) \exp(-sa(|F_n|)) \leq \frac{1}{t} \sum_{i \in \mathcal{I}_n} c_i \exp(-sa(|F_n|)).$$

Therefore,

$$\begin{aligned}M(Z_{n,t}, s+\delta, N, \{F_n\}, 6\varepsilon, \mathbf{a}) &\leq \aleph(E_{n,t}) \exp(-(s+\delta) a(|F_n|)) \\ &\leq \frac{\exp(-\delta a(|F_n|))}{t} \sum_{i \in \mathcal{I}_n} c_i \exp(-sa(|F_n|))\end{aligned}$$

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$$\leq \frac{\exp(-\delta n\xi)}{t}\sum_{i\in\mathcal{I}_n} c_i \exp(-sa(|F_n|))$$
$$\leq \frac{1}{n^2 t}\sum_{i\in\mathcal{I}_n} c_i \exp(-sa(|F_n|)).$$

Step 3. For any $t \in (0,1)$, we have

$$M(Z, s+\delta, N, \{F_n\}, 6\varepsilon, \mathbf{a}) \leq \frac{1}{t}\sum_{i\in\mathcal{I}} c_i \exp(-sa(|F_{n_i}|)),$$

which implies (4.1). In fact, fix $t \in (0,1)$. Note that $\sum_{n=N}^{\infty} n^{-2} < 1$. Then $Z \subset \bigcup_{n=N}^{\infty} Z_{n,n^{-2}t}$. Also note that $M(Z, s, N, \{F_n\}, \varepsilon, \mathbf{a})$ is an outer measure of X, so we get

$$M(Z, s+\delta, N, \{F_n\}, 6\varepsilon, \mathbf{a}) \leq \sum_{n=N}^{\infty} M(Z_{n,n^{-2}t}, s+\delta, N, \{F_n\}, 6\varepsilon, \mathbf{a})$$
$$\leq \sum_{n=N}^{\infty} \frac{1}{t}\sum_{i\in\mathcal{I}_n} c_i \exp(-sa(|F_n|)) = \frac{1}{t}\sum_{i\in\mathcal{I}} c_i \exp(-sa(|F_{n_i}|)).$$

■

We will give a Frostman's lemma in dynamical system, which is important to our proof.

Lemma 4.2 *Suppose K is a non-empty compact subset of X. Let $s \geq 0, N \in \mathbb{N}, \varepsilon > 0$. If $c := W(K, s, N, \varepsilon, \{F_n\}, \mathbf{a}) > 0$, then there exists a Borel probability measure μ on X such that $\mu(K) = 1$ and*

$$\mu(B_{F_n}(x,\varepsilon)) \leq \frac{1}{c}\exp(-sa(|F_n|)).$$

Proof Clearly $c < \infty$. We define a function p on the space $C(X)$ of continuous real-valued functions on X by

$$p(f) = \frac{1}{c}\mathrm{W}(\chi_K \cdot f, s, N, \varepsilon, \{F_n\}, \mathbf{a}).$$

Let $\mathbf{1} \in C(X)$ denote the constant function $\mathbf{1}(x) \equiv 1$. It is easy to verify that:

(1) $p(tf) = tp(f)$ for $f \in C(X)$ and $t \geq 0$,
(2) $p(f+g) \leq p(f) + p(g)$ for $f, g \in C(X)$,
(3) $p(\mathbf{1}) = 1, 0 \leq P(f) \leq \| f \|_\infty$ for $f \in C(X)$, and $p(g) = 0$ for $g \in C(X), g \leq 0$.

By the Hahn–Banach theorem, we can extend the linear functional $t \to tp(1), t \in \mathbb{R}$, from the subspace of constant functions to a linear functional $L : C(X) \to \mathbb{R}$ satisfying

$$L(\mathbf{1}) = p(\mathbf{1}) = 1 \text{ and } -p(-f) \leq L(f) \leq p(f) \text{ for any } f \in C(X).$$

If $f \in C(X)$ with $f \geq 0$, then $p(-f) = 0$ and so $L(f) \geq 0$. Hence we can use the Riesz representation theorem to find a Borel probability measure μ on X such that $L(f) = \int f d\mu$ for $f \in C(X)$.

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Next, we prove $\mu(K)=1$. For any compact set $E\subset X\backslash K$, by Urysohn lemma there exists $f\in C(X)$ such that $0\le f\le 1, f(x)=1$ for $x\in E$ and $f(x)=0$ for $x\in K$. Then $f\cdot\chi_K=0$ and thus $p(f)=0$. Hence $\mu(E)\le L(f)\le p(f)=0$. This shows $\mu(X\backslash K)=0$, that is $\mu(K)=1$.

In the end, we prove $\mu(B_{F_n}(x,\varepsilon))\le\frac{1}{c}\exp(-sa(|F_n|))$ for any $x\in X, n\ge N$. In fact, for any compact set $E\subset B_{F_n}(x,\varepsilon)$, by Urysohn lemma again, there is $f\in C(X)$, such that $0\le f\le 1, f(y)=1$ for $y\in E$ and $f(y)=0$ for $y\in X\backslash B_{F_n}(x,\varepsilon)$. Then $\mu(E)\le L(f)\le p(f)$. Since $\chi_K\cdot f\le\chi_{B_{F_n}(x,\varepsilon)}$ and $n\ge N$, we get $W(\chi_K\cdot f,s,N,\varepsilon,\{F_n\},\mathbf{a})\le\exp(-sa(|F_n|))$ and hence $p(f)\le\frac{1}{c}\exp(-sa(|F_n|))$. Therefore, we have $\mu(E)\le\frac{1}{c}\exp(-sa(|F_n|))$. It follows that

$$\mu(B_{F_n}(x,\varepsilon))=\sup\{\mu(E): E \text{ is a compact subset of } B_{F_n}(x,\varepsilon)\}\le\frac{1}{c}\exp(-sa(|F_n|)).$$

■

Now we are in a position to present our main result: the variational principle between scaled topological entropy and scaled local entropy.

Theorem 4.2 *Let (X,G) be a topological dynamical system, $\mathbf{a}\in\mathcal{SS}$ and K be a non-empty compact subset of X. If $\liminf_{n\to+\infty}\frac{a(|F_n|)}{n}>0$, then*

$$E_K(\{F_n\},\mathbf{a})=\sup\{\underline{h}_\mu(\{F_n\},\mathbf{a}):\mu\in\mathcal{M}(X),\mu(K)=1\}.$$

Proof Firstly, we prove $E_K^{B_2}(\{F_n\},\mathbf{a})\ge\underline{h}_\mu(\{F_n\},\mathbf{a})$, for any $\mu\in\mathcal{M}(X),\mu(K)=1$. We set

$$\underline{h}_\mu(\{F_n\},\mathbf{a},x,\varepsilon)=\liminf_{n\to\infty}-\frac{1}{a(|F_n|)}\log\mu(B_{F_n}(x,\varepsilon))$$

for $x\in X, n\in\mathbb{N},\varepsilon>0$. It's easy to see that $\underline{h}_\mu(\{F_n\},\mathbf{a},x,\varepsilon)$ is nonnegative and increases as ε decreases. By the monotone convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int\underline{h}_\mu(\{F_n\},\mathbf{a},x,\varepsilon)d\mu(x)=\int\underline{h}_\mu(\{F_n\},\mathbf{a},x)d\mu(x)=\underline{h}_\mu(\{F_n\},\mathbf{a}).$$

Thus to show $E_K^{B_2}(\{F_n\},\mathbf{a})\ge\underline{h}_\mu(\{F_n\},\mathbf{a})$, we only to show

$$E_K^{B_2}(\{F_n\},\mathbf{a})\ge\int\underline{h}_\mu(\{F_n\},\mathbf{a},x,\varepsilon)d\mu(x) \text{ for any } \varepsilon>0.$$

Now we fix $\varepsilon>0, l\in\mathbb{N}$, set $u_l=\min\{l,\int\underline{h}_\mu(\{F_n\},\mathbf{a},x,\varepsilon)d\mu(x)-\frac{1}{l}\}$, then exist a Borel set $A_l\subset X,\mu(A_l)>0, N\in\mathbb{N}$ such that

$$\mu(B_{F_n}(x,\varepsilon))\le\exp(-u_l a(|F_n|)),\ \text{for all } x\in A_l, n\ge N. \tag{4.3}$$

Let $\{B_{F_{n_i}}(x_i,\varepsilon/2)\}$ ba a finite or countable family such that $x_i\in X, n_i\ge N$ and $K\cap A_l\subset\bigcup_i B_{F_{n_i}}(x_i,\varepsilon/2)$. We may as well assume that for each $i\in\mathbb{N}, B_{F_{n_i}}(x_i,\varepsilon/2)\bigcap(K\cap A_l)\ne\varnothing$, and select $y_i\in B_{F_{n_i}}(x_i,\varepsilon/2)\bigcap(K\cap A_l)$. Then by (4.3), we have

$$\begin{aligned}\sum_i\exp(-u_l a(|F_{n_i}|))&\ge\sum_i\mu(B_{F_{n_i}}(y_i,\varepsilon))\\&\ge\sum_i\mu(B_{F_{n_i}}(x_i,\varepsilon/2))\\&\ge\mu(K\cap A_l)=\mu(A_l)>0.\end{aligned}$$

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So, we get

$$\begin{aligned} M\left(K, u_l, \{F_n\}, \frac{\varepsilon}{2}, \mathbf{a}\right) &\geq M(K, u_l, N, \{F_n\}, \frac{\varepsilon}{2}, \mathbf{a}) \\ &\geq M(K \cap A_l, u_l, N, \{F_n\}, \frac{\varepsilon}{2}, \mathbf{a}) \\ &\geq \mu(A_l) > 0. \end{aligned}$$

Therefore, $E_K^{B_2}(\{F_n\}, \mathbf{a}) \geq u_l$. Letting $l \to \infty$, we get

$$E_K^{B_2}(\{F_n\}, \mathbf{a}) \geq \int \underline{h}_\mu(\{F_n\}, \mathbf{a}, x, \varepsilon) d\mu(x).$$

Thus $E_K^{B_2}(\{F_n\}, \mathbf{a}) \geq \underline{h}_\mu(\{F_n\}, \mathbf{a})$.

We next prove $E_K^{B_2}(\{F_n\}, \mathbf{a}) \leq \{\underline{h}_\mu(\{F_n\}, \mathbf{a}) : \mu \in \mathcal{M}(X), \mu(K) = 1\}$. We may as well assume $E_K^{B_2}(\{F_n\}, \mathbf{a}) > 0$, otherwise the conclusion is obvious. By Proposition 4.1, $E_K^{B_2}(\{F_n\}, \mathbf{a}) = h_{top}^W(K, \{F_n\}, \mathbf{a})$. Suppose $0 < s < h_{top}^W(K, \{F_n\}, \mathbf{a})$, then there exists $\varepsilon > 0$ and $N \in \mathbb{N}$, such that $c = W(K, s, N, \varepsilon, \{F_n\}, \mathbf{a}) > 0$. By Lemma 4.2, there exists $\mu \in \mathcal{M}(X), \mu(K) = 1$, such that

$$\mu(B_{F_n}(x, \varepsilon)) \leq \frac{1}{c} \exp(-sa(|F_n|))$$

for any $x \in X, n \geq N$. And then $\underline{h}_\mu(\{F_n\}, \mathbf{a}, x) \geq s$ for each $x \in X$. Therefore, $\underline{h}_\mu(\{F_n\}, \mathbf{a}) \geq \int \underline{h}_\mu(\{F_n\}, \mathbf{a}, x) d\mu(x) \geq s$. By Proposition 2.2, the proof is completed. ∎

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