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Triangulations with few vertices of manifolds with non-free fundamental group

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We study lower bounds for the number of vertices in a PL-triangulation of a given manifold M. While most of the previous estimates are based on the dimension and the connectivity of M, we show that further information can be extracted by studying the structure of the fundamental group of M and applying techniques from the Lusternik-Schnirelmann category theory. In particular, we prove that every PL-triangulation of a d-dimensional manifold ($d \geqslant 3$) whose fundamental group is not free has at least $3d + 1$ vertices. As a corollary, every d-dimensional homology sphere that admits a combinatorial triangulation with less than $3d$ vertices is PL-homeomorphic to S^d. Another important consequence is that every triangulation with small links of M is combinatorial.

Keywords: minimal triangulation; PL-manifold; homology sphere; good cover; Lusternik-Schnirelmann category

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1. Introduction and results

A *triangulation* of a topological space M is a simplicial complex K together with a homeomorphism $M \approx |K|$ between M and the geometric realization of K. If M is a closed d-dimensional manifold then we are particularly interested in *combinatorial* triangulations, where we require that the link of every simplex in K is PL-homeomorphic to a sphere (cf. [**4**]). A manifold admitting a combinatorial triangulation is called a *PL-manifold*.

Given a PL-manifold M, what is the minimal number of vertices in a combinatorial triangulation of M? This is a difficult question because there are no standard constructions for triangulations with few vertices of a given manifold, nor there are sufficiently general methods to prove that some specific triangulation is in fact minimal. Apart from classical results on minimal triangulations of spheres and closed surfaces, and a special family of minimal triangulations for certain sphere bundles over a circle (so-called Császár tori - see [**13**]), there exists only a handful of examples for which the minimal triangulations are known. An exhaustive survey of the results and the existing literature on this problem can be found in [**14**]. See also the recent article [**12**] which discusses a more general question of the number of faces in triangulations of manifolds and polytopes.

Generally speaking, one may expect that the minimal number of vertices in a triangulation of a space increases with its complexity. Most results that can be found in the literature use dimension, connectivity or Betti numbers of M to express lower bounds for the number of vertices in a triangulation of M. In this paper, we have been able to exploit the fundamental group and the Lusternik-Schnirelmann category to improve several estimates of the minimal number of vertices in a triangulation of a manifold.

In the rest of this section, we state our main results. In § 2, we introduce and explain prerequisites on triangulations, Lusternik-Schnirelmann category and the covering type. Finally, in § 3, we give the proofs of the theorems presented below.

Let us begin with a slight improvement of the theorem first proved by Brehm and Kühnel [**3**]. Our approach is based on the notion of covering type [**8**] and is much simpler than the original one. Recall that Poincaré duality together with the positive answer to the Poincaré conjecture imply that every simply-connected closed d-manifold, whose reduced homology is trivial in dimensions less or equal to $d/2$ is homeomorphic to the d-sphere. For the remaining cases, the minimal number of vertices in a triangulation can be estimated as follows.

Theorem 1.1. *Let M be a simply-connected d-dimensional closed PL-manifold, and let i be the minimal index for which $\widetilde{H}_i(M) \neq 0$.*

(a) *If $i = d/2$ then every combinatorial triangulation of M has at least $3d/2 + k + 2$ vertices, where k is the minimal integer for which $\binom{i+k}{i+1} \geqslant \operatorname{rank} H_i(M)$. Moreover, k can be equal to 1 only if $d \in \{4, 8, 16\}$.*

(b) *If $i < d/2$ then every combinatorial triangulation of M has at least $2d - i + 4$ vertices.*

In particular, every combinatorial triangulation of a closed, simply-connected d-manifold with at most $3d/2 + 2$ vertices represents the d-dimensional sphere.

The main contribution of this paper is the following theorem and its corollaries. In particular, we obtain considerable improvements of estimates by Brehm-Kühnel [**3**] and Bagchi-Datta [**1**] of the number of vertices in PL-triangulations of homology spheres. By [**3**, corollary 2] every PL-triangulation of a non simply-connected d-manifold ($d \geqslant 3$) has at least $2d + 3$ vertices. That the value cannot be improved in general is shown by Kühnel who constructed a family of S^{d-1}-bundles over the circle S^1 that admit PL-triangulations with $2d + 3$ vertices. However, if the fundamental group of M is not free, then we obtained a better estimate:

Theorem 1.2. *If M is a d-dimensional ($d \geqslant 3$) closed manifold whose fundamental group is not free, then every combinatorial triangulation of M has at least $3d + 1$ vertices.*

It is worth noting that closed 3-manifolds whose fundamental group is free are quite special, being either the 3-sphere or connected sums of tori $S^2 \times S^1$ and twisted tori $S^2 \underline{\times} S^1$. All the other closed 3-manifolds (in particular, all closed hyperbolic 3-manifolds) satisfy the assumptions of the above theorem.

An important family of examples whose fundamental group is not free are the *homology spheres*, that is, manifolds, whose homology groups vanish except in the top dimension, where the homology group is $\mathbb{Z}$. A simply-connected homology sphere is homeomorphic to a sphere by the positive answer to the Poincaré conjecture but for every $d \geqslant 3$ there exist d-dimensional homology spheres that are not homeomorphic to S^d. As the fundamental group of a homology sphere must be a perfect group, it cannot be free (unless it is trivial), therefore theorem 1.2 implies the following improvement of the estimate in [**3**, corollary 4].

Corollary 1.3. *Every d-dimensional homology sphere that admits a combinatorial triangulation with at most $3d$ vertices is a PL-sphere.*

Bagchi and Datta [**1**] obtained an estimate of the minimal number of vertices in PL-triangulations of $\mathbb{Z}_2$-homology spheres, that is, manifolds whose $\mathbb{Z}_2$-homology is isomorphic to that of a sphere (non-trivial examples include 3-dimensional odd lens spaces). Their results are improved in some cases by the following:

Corollary 1.4. *Let M be a d-dimensional $\mathbb{Z}_p$-homology sphere that admits a combinatorial triangulation with at most $3d$ vertices. Then M is simply-connected and its homology groups $H_i(M)$ for $0 < i < d$ are torsion and prime to p. In particular, if $d \leqslant 4$, then M is a PL-sphere.*

We conclude with a useful recognition criterion for combinatorial triangulations. By definition, in order to prove that a given triangulation is combinatorial, we must show that the link of each simplex in the triangulation is a PL-sphere. But the question whether a given finite simplicial complex is homeomorphic to a sphere is in general undecidable because a recognition algorithm would have to be able to determine whether a group given by generators and relators (namely the fundamental group of the link) is trivial. Since this is not possible, there is a proliferation of heuristic methods that can give an answer under favourable conditions (see [**10**] for a report on the current status of the field). Typically, one begins by checking if the homology of the simplicial complex is that of a sphere. If this test is passed, then one tries to reduce the size of the triangulation and simplify the presentation of the fundamental group (e.g. by randomized use of so-called *bistellar flips*). corollaries 1.3 and 1.4 allow to bypass the computation of the fundamental group if the simplified triangulation is sufficiently small.

Our recognition criterion comes in two flavours, one for arbitrary triangulations, and another for triangulations of manifolds. Since the conclusion is the same in both cases we combine them in a single theorem.

Theorem 1.5. *Let K be a d-dimensional simplicial complex, such that the link of every simplex of codimension $k+1$ in K has the homology of a k-dimensional sphere. This is true in particular if K is a triangulation of a d-dimensional manifold.*

If for every $k \geqslant 3$ the link of each simplex of codimension $k+1$ in K has at most $3k$ vertices, then the triangulation K is combinatorial and $|K|$ is a d-dimensional PL-manifold.

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2. Preliminaries

In this section, we recollect concepts and results that are needed in the proofs of the above theorems.

2.1. Simplicial complexes and PL-triangulations

Here we describe two special constructions and refer the reader to the article of J. Bryant [4] for the definitions of triangulations, skeleta, open and closed stars, links, joins, combinatorial triangulations and other standard concepts of PL-topology.

Given a triangulation $M \approx |K|$ we identify the set of vertices of the triangulation with the 0-skeleton K^0 of the simplicial complex K. For a subset $V \subseteq K^0$, let $K(V)$ denote the full subcomplex of K spanned by V, that is, the maximal subcomplex of K whose 0-skeleton is V. It is easy to check that for every vertex $v \in K^0$ the subcomplex $K(V \cup \{v\})$ can be obtained as the union of $K(V)$ and the join of v with the part of the link of v contained in $K(V)$, which can be expressed by the following formula:

$$K(V \cup \{v\}) = K(V) \cup v * (\mathrm{lk}(v) \cap K(V)) \tag{2.1}$$

Furthermore, let us denote by $N(V) \subseteq |K|$ the union of open stars (with respect to K) of vertices in V. Clearly, the geometric realization $|K(V)|$ is a subspace of $N(V)$, and we have the following standard fact (cf. proof of [15, corollary 3.3.11]).

Lemma 2.1. *$N(V) = |K| - |K(K^0 - V)|$, therefore $N(V)$ is an open neighbourhood of $|K(V)|$ in $|K|$. Moreover, $|K(V)|$ is a deformation retract of $N(V)$.*

Proof. The first statement is obvious. In order to obtain a deformation retraction observe that every point $x \in |K|$ can be written uniquely in terms of barycentric coordinates

$$x = \sum_{v \in K^0} \lambda_v(x) \cdot v.$$

By definition, for every $x \in N(V)$ there is at least one $v \in V$ for which $\lambda_v(x) > 0$, so we may define the retraction

$$r\colon N(V) \to |K(V)| \quad \text{as} \quad r(x) := \left(\sum_{v \in V} \lambda_v(x) \cdot v\right) \Big/ \left(\sum_{v \in V} \lambda_v(x)\right),$$

which can be extended to a deformation retraction of $N(V)$ to $|K(V)|$ through a straight-line homotopy. □

In particular, if K^0 is partitioned into two disjoint subsets V, V' then $N(V)$ and $N(V')$ form an open cover of $|K|$ and

$$N(V) \cap N(V') = N(V) - |K(V)| = N(V') - |K(V')|.$$

LEMMA 2.2. *Let K be a triangulation of a closed d-dimensional manifold. If $V \subseteq K^0$ spans a d-dimensional simplex in K then for every $i < d$*

$$H_i(K(K^0 - V)) \cong H_i(K) \quad \text{and} \quad H^i(K(K^0 - V)) \cong H^i(K)$$

((co)homology is with integer coefficients unless $|K|$ is non-orientable and $i = d-1$, in which case one should use $\mathbb{Z}_2$-coefficients).

Proof. First observe that by lemma 2.1 $N(V) \simeq |K(V)|$ is contractible, therefore

$$\widetilde{H}_i(N(V) - |K(V)|) \cong H_{i+1}(N(V), N(V) - |K(V)|) .$$

On the other hand, by a version of Poincaré-Lefshetz duality (see [**9**, proposition 3.46]) we have

$$H_{i+1}(N(V), N(V) - |K(V)|) \cong H^{d-i-1}(K(V)) .$$

By combining the two isomorphisms we conclude that $H_i(N(V) - |K(V)|) \cong H_i(S^{d-1})$ for all i.

Denote $V' = K^0 - V$ and consider the following portion of the Mayer-Vietoris sequence

$$H_i(N(V) \cap N(V')) \to H_i(N(V)) \oplus H_i(N(V')) \to H_i(K) \to H_{i-1}(N(V) \cap N(V'))$$

Observe that $H_i(N(V)) = 0$, that $H_i(N(V) \cap N(V')) = H_i(S^{d-1}) = 0$ for $i < d-1$, and that $H_d(|K|) \to H_{d-1}(N(V) \cap N(V'))$ is surjective (with $\mathbb{Z}_2$-coefficients if $|K|$ is non-orientable). By exactness of the above sequence

$$H_i(K) \cong H_i(N(V')) \cong H_i(K(V'))$$

for $i < d$. The proof for cohomology groups is similar. □

2.2. Lusternik-Schnirelmann category

A subset $A \subseteq X$ of a topological space X is said to be *categorical* if the inclusion map $A \hookrightarrow X$ is null-homotopic (i.e., if there exists a homotopy between the inclusion and the constant map). The minimal cardinality of an open categorical cover of X is denoted $\mathrm{cat}(X)$ and is called the *Lusternik-Schnirelmann category* of X. For example, the category of a space is 1 if, and only if, it is contractible, and the category of a (non-contractible) suspension is 2, because every suspension has a natural cover by two contractible cones. See [**5**] for a comprehensive survey of the results and the vast literature about Lusternik-Schnirelmann category and related topics. (Keep in mind when comparing the results, the survey [**5**] and the article [**6**] use the normalized value of $\mathrm{cat}(X)$ which is one less than in our definition so that contractible spaces have category 0 and non-contractible suspensions have category 1.) Lusternik-Schnirelmann category is tightly related to other homotopy invariants, for example, a well-known result states that if $\mathrm{cat}(X) \leqslant 2$ then the fundamental group of X is free (see [**5**, p. 44]).

We will base our results on a similar but much deeper theorem proved by Dranishnikov, Katz and Rudyak [**6**, corollary 1.2]: if M is a closed d-dimensional manifold

($d \geqslant 3$) and if $\mathrm{cat}(M) \leqslant 3$ then the fundamental group of M is free. Their proof is based on the notion of category weight which we briefly recall. Roughly speaking, a non-zero class $u \in \widetilde{H}^*(M)$ (here we omit the coefficients for cohomology from the notation) has *category weight* at least k if the restriction of u to any union of k categorical subsets of M is trivial. A recise definition is slightly more technical - see [5, § 2.7] or [6, § 3]. Clearly, if we can find classes $u, v \in \widetilde{H}^*(M)$ of weight k and l respectively, and such that $0 \neq u \cdot v \in \widetilde{H}^*(M)$, then $\mathrm{cat}(M) > k + l$. We can summarize the main result of [6, § 4] as follows:

THEOREM 2.3. *Let M be a closed d-dimensional ($d \geqslant 3$) manifold M whose fundamental group is not free. Then there exist suitable systems of coefficients on M and cohomology classes $u \in H^2(M)$ of weight 2 and $v \in H^{d-2}(M)$ of weight 1, such that $0 \neq u \cdot v \in H^d(M)$. As a consequence,* $\mathrm{cat}(M) \geqslant 4$.

2.3. Homotopy triangulations and covering type

Let us denote by $\Delta(X)$ the minimal number of vertices in a triangulation of a compact polyhedron. Clearly, $\Delta(X)$ is a topological invariant of compact polyhedra but it is in general very far from being a homotopy invariant. As an easy example let $X_1 := S^1 \vee S^1 \vee S^1$, the one-point union of three circles, let X_2 be the graph with two vertices and four parallel edges between them and let $X_3 := \Delta_3^{(1)}$, the 1-skeleton of the tetrahedron. All three spaces have the same homotopy type and yet easy geometric reasoning shows that $\Delta(X_1) = 7$, $\Delta(X_2) = 5$, $\Delta(X_3) = 4$. To obtain a homotopy invariant notion recall that a *homotopy triangulation* of X is a simplicial complex K together with a homotopy equivalence $X \simeq |K|$. Then the minimal number of vertices among all possible homotopy triangulations of X is not only a homotopy invariant of X but it also provides a link to the concept of covering type that was recently introduced by M. Karoubi and C. Weibel [11].

Recall that a cover $\mathcal{U}$ of a space X is said to be *good* if all finite non-empty intersections of elements of $\mathcal{U}$ are contractible. Standard examples are covers by convex sets, covers of polyhedra by open stars of vertices and covers of Riemannian manifolds by geodesic balls. One of the main facts about good covers is the Nerve theorem (see [9, corollary 4.G3]): if $\mathcal{U}$ is a good open cover of a paracompact space X, then $X \simeq |N(\mathcal{U})|$, where $|N(\mathcal{U})|$ is the geometric realization of the nerve of $\mathcal{U}$. Karoubi and Weibel defined the *covering type* of X as the minimum cardinality of a good open cover of a space that is homotopy equivalent to X.

If X admits a homotopy triangulation $X \simeq |K|$, where the simplicial complex K has n vertices, then the open stars of the vertices form a good cover for $|K|$, therefore $\mathrm{ct}(X) \leqslant n$. Conversely, if there exists a homotopy equivalence $X \simeq Y$ where Y has a good open cover $\mathcal{U}$ with n elements, then $X \simeq Y \simeq |N(\mathcal{U})|$ is a homotopy triangulation of X with n vertices. Thus we have proved the following result (cf. [8, theorem 1.2]):

PROPOSITION 2.4. *If X has the homotopy type of a compact polyhedron, then* $\mathrm{ct}(X)$ *equals the minimal number of vertices in a homotopy triangulation of X.*

For every compact polyhedron X there is the obvious relation $\Delta(X) \geqslant \mathrm{ct}(X)$ and we have seen previously that $\Delta(X)$ can be in fact much bigger that $\mathrm{ct}(X)$. However, if M is a closed triangulable manifold then there is some evidence that $\Delta(M)$ are ct(M) close and often equal. Notably, Borghini and Minian [**2**] showed that for closed surfaces $\Delta(M)$ and $\mathrm{ct}(M)$ coincide, with the sole exception of the orientable surface of genus 2, where the two quantities differ by one.

There are several useful estimates of $\mathrm{ct}(X)$ based on other homotopy invariants of X. For example, let $\mathrm{hdim}(X)$ denote the homotopy dimension of X, that is, the minimal dimension of a homotopy triangulation of X. Then we have the following estimate (cf. [**11**, proposition 3.1]):

Proposition 2.5. *If* $\mathrm{ct}(X) = \mathrm{hdim}(X) + 2$*, then* X *is homotopy equivalent to the sphere* $S^{\mathrm{hdim}(X)}$*, otherwise* $\mathrm{ct}(X) \geqslant \mathrm{hdim}(X) + 3$*.*

Proof. If $\mathrm{ct}(X) \leqslant n$, then Nerve theorem implies that X admits a homotopy triangulation by a subcomplex of Δ_{n-1}. There is only one subcomplex of Δ_{n-1} whose homotopy dimension equals $n-2$, namely its boundary $|\partial\Delta_{n-1}| \approx S^{n-2}$. Indeed, $|\Delta_{n-1}|$ is contractible, so its homotopy dimension is 0, while all subcomplexes of $\partial\Delta_{n-1}$ have homotopy dimension at most $n-3$. As a consequence $\mathrm{ct}(X) - \mathrm{hdim}(X)$ is at least 3, unless X is homotopy equivalent to a sphere. □

Govc, Marzantowicz and Pavešić [**8**] applied techniques from Lusternik-Schnirelmann category to obtain further estimates of the covering type of a space and proved the following two results. For the sake of completeness, we reproduce here the proof of the first statement.

Theorem 2.6 [**8**, theorem 4.1]. *The covering type of a* r*-fold wedge of spheres of dimension* i *equals the minimal integer* n *for which* $\binom{n-1}{i+1} \geqslant r$*.*

Proof. The case $i = 1$ is covered by [**11**, proposition 4.1], so we may assume $i > 1$.

Let us first compute the homology of $\Delta_{n-1}^{(i)}$, the i-th skeleton of the $(n-1)$-dimensional simplex. The simplicial chain complex of $\Delta_{n-1}^{(i)}$ is obtained by truncating the simplicial chain complex for Δ_{n-1} at degree i:

$$C_i(\Delta_{n-1}) \xrightarrow{\partial_i} C_{i-1}(\Delta_{n-1}) \longrightarrow \cdots\cdots \longrightarrow C_0(\Delta_{n-1}) \xrightarrow{\partial_0} C_{-1} = \mathbb{Z}$$

The homology of Δ_{n-1} is trivial, so the above chain complex is exact, except at the beginning. The rank of each $C_k(\Delta_{n-1})$ is $\binom{n}{k+1}$, and the rank of $H_i(\Delta_{n-1}^{(i)}) = \ker\partial_i$ can be computed by exploiting the exactness:

$$\mathrm{rank}(\ker\partial_i) = \binom{n}{i+1} - \binom{n}{i} + \ldots + (-1)^i\binom{n}{1} + (-1)^{i+1}\binom{n}{0} = \binom{n-1}{i+1}.$$

We conclude that $\Delta_{n-1}^{(i)}$ is the wedge of $\binom{n-1}{i+1}$ spheres of dimension i.

It is obvious from the definition of simplicial homology that the rank of $H_i(\Delta_{n-1}^{(i)})$ is maximal among all sub-complexes of Δ_{n-1}. Therefore, if $r > \binom{n-1}{i+1}$, then the r-fold wedge of i-dimensional spheres cannot be represented by a subcomplex of Δ_{n-1}.

To show the converse, note that $\mathrm{im}(\partial_i)$ is $\binom{n-2}{i+1}$-dimensional, so we may find up to $\binom{n}{i+1} - \binom{n-2}{i+1} = \binom{n-1}{i+1}$ i-simplices in $\Delta_{n-1}^{(i)}$ whose removal does not alter the image of ∂_i. In particular, if $r \leqslant \binom{n-1}{i+1}$, then we may remove $\binom{n-1}{i+1} - r$ simplices of dimension i, so that the remaining simplices form a r-fold wedge of i-dimensional spheres. We conclude that the r-fold wedge of i-dimensional spheres can be represented by a subcomplex of Δ_{n-1} if, and only if $\binom{n-1}{i+1} \geqslant r$, which proves our claim. □

The next theorem relates the covering type (and thus the size of a minimal triangulation) of a closed manifold to its Lusternik-Schnirelmann category.

THEOREM 2.7 [**8**, corollary 2.4]. *Let M be a d-dimensional closed manifold. Then every triangulation of M has at least*

$$1 + d + \frac{1}{2}\,\mathrm{cat}(M)(\mathrm{cat}(M) - 1)$$

vertices.

Observe that by theorem 2.3 the category of a 3-dimensional closed manifold M with non-free fundamental group is at least 4. As a consequence one needs at least $1 + 3 + 4 \cdot 3/2 = 10$ vertices to triangulate M, so we obtain the estimate of theorem 1.2 for $d = 3$.

3. Proofs

In this section, we provide the proofs for the results stated in § 1.

Proof of theorem 1.1. Observe that M is by assumption simply-connected and hence orientable, which implies that Poincaré duality holds with arbitrary coefficients. Moreover, the statement of the theorem is trivial for $d = 2$ since the only simply-connected close 2-manifold is S^2, so we assume henceforth that $d > 2$.

Let K be a combinatorial triangulation of M. Since M is d-dimensional, there exists a $(d + 1)$-element subset $V \subset K^0$ spanning a simplex. Lemma 2.2, together with Seifert-van Kampen theorem imply that $K(K^0 - V)$ is simply connected and that $H_j(K(K^0 - V)) = H_j(K)$ for $j < d$.

Under the assumption (a) $d = 2i$ and $H_i(M)$ is the first non-trivial homology group of M, therefore M is an $(i - 1)$-connected $2i$-dimensional manifold. It is well-known (see [**16**]) that such a manifold is homotopy equivalent to a CW-complex obtained by attaching a $2i$-dimensional cell to a wedge of $r = \mathrm{rank}\, H_i(M)$ spheres of dimension i. By the previous paragraph the homology of $K(K^0 - V)$ is free and concentrated in dimension i, and $H_i(K(K^0 - V)) = H_i(M)$. It follows that $K(K^0 - V)$ is homotopy equivalent to a wedge of r copies of i-dimensional spheres. By theorem 2.6 the covering type of $K(K^0 - V)$ is equal to $i + k + 1$ where k is the minimal integer satisfying $\binom{i+k}{i+1} \geqslant r$. We conclude that K^0 has at least $(d + 1) + (d/2 + k + 1) = 3d/2 + k + 2$ elements.

Moreover, if $k = 1$ then clearly $\mathrm{rank}\, H_i(M) = 1$, therefore M is homotopy equivalent to a CW-complex with three cells in dimensions $0, i$ and $d = 2i$, respectively. The i-dimensional skeleton is the sphere S^i and the d-dimensional cell is attached to S^i by a map with Hopf invariant 1 (see [**7**, §§ 5 and 6] for details). By the celebrated theorem of Adams, this is possible only if $i \in \{2, 4, 8\}$.

Under the assumption (b) $H_i(M) \neq 0$ for some $i < d/2$. If $H_i(M) \cong \mathbb{Z}$, then Poincaré duality and the Universal Coefficients theorem imply that $H_{d-i}(M) \cong \mathbb{Z}$. On the other hand, if $H_i(M) \ncong \mathbb{Z}$, then by Poincaré duality $H^{d-i}(M) \ncong 0$ or $\mathbb{Z}$. Lemma 2.2 yields $H^k(K(K^0 - V)) \cong H^k(M)$ for $k < d$, which in both cases implies that $\mathrm{hdim}(K(K^0 - V)) \geqslant d - i$, and that the cohomology of $K(K^0 - V)$ is not that of a sphere. By proposition 2.5 the covering type of $K(K^0 - V)$ is at least $d - i + 3$. We conclude that K^0 has at least $(d+1) + (d-i+3) = 2d - i + 4$ elements. □

Proof of theorem 1.2. Let K be a combinatorial triangulation of M. Since M is d-dimensional, its triangulation must contain at least one d-simplex, and so there exist vertices $v_1, \ldots, v_{d+1} \in K^0$ that span a d-dimensional simplex in K. Let us enumerate the remaining vertices so that $K^0 = \{v_1, \ldots, v_{d+1}, \ldots, v_n\}$.

By adding one vertex at a time we obtain a sequence of subcomplexes

$$\Delta_d = K_{d+1} < \ldots < K_k < K_n = K,$$

where $K_k = K(v_1, \ldots, v_k) \leqslant K$. Since $\pi_1(M)$ is non-trivial, there exists a minimal l, such that $\pi_1(|K(v_1, \ldots, v_l)|)$ is non-trivial. By expressing K_l as in formula (2.1)

$$K_l = K_{l-1} \cup v_l * (\mathrm{lk}(v_l) \cap K_{l-1}),$$

we see that K_l is a union of two simply-connected subcomplexes. By Seifert-van Kampen theorem its fundamental group can be non-trivial only if (the geometric realization of) the intersection $L := \mathrm{lk}(v_l) \cap K(v_1, \ldots, v_{l-1})$ has at least two components. Let us denote $L' := \mathrm{lk}(v_l) \cap K(v_{l+1}, \ldots, v_n)$. Then L and L' are full subcomplexes of $\mathrm{lk}(v_l)$ and their vertices determine a partition of the vertices of $\mathrm{lk}(v_l)$. By lemma 2.1 $|L'|$ is a deformation retract of $|\mathrm{lk}(v_l)| - |L|$. Since $|\mathrm{lk}(v_l)| \approx S^{d-1}$, we can apply Alexander duality [**9**, theorem 3.44] and obtain that $H^{d-2}(|L'|) \cong \widetilde{H}_0(|L|) \neq 0$. By proposition 2.5 there exist $d-1$ vertices, which we may label as $v_{l+1}, \ldots, v_{l+d-1}$, that span a simplex in L'. Since these vertices are contained in $\mathrm{lk}(v_l)$, they can be joined to v_l in K, therefore vertices $v_l, \ldots, v_{l+d-1}$ span a simplex in K.

Let us denote $A := \{v_1, \ldots, v_{d+1}\}$ and $B := \{v_l, \ldots, v_{l+d-1}\}$. A and B are disjoint and together contain $2d+1$ vertices of K^0. To conclude the proof, we must show that $K^0 - A - B$ contains at least d vertices.

Since $\pi_1(M)$ is not free, theorem 2.3 states that there exist cohomology classes $u \in H^2(M)$ of weight 2 and $v \in H^{d-2}(M)$ of weight 1, such that $u \cdot v \neq 0$. Both $K(A)$ and $K(B)$ are contractible, therefore $N(A \cup B) = N(A) \cup N(B)$ is a union of two categorical sets. It follows that $u|_{N(A\cup B)} = 0$, and so the restriction of v to $N(K^0 - A - B)$ cannot be trivial, as it would contradict $u \cdot v \neq 0$. Therefore $H^{d-2}(N(K^0 - A - B)) \neq 0$, hence proposition 2.5 implies that $K^0 - A - B$ must contain at least d vertices, as claimed. □

Proof of corollary 1.3. Every 1- or 2-dimensional homology sphere is a PL-sphere so we may assume $d \geqslant 3$. If there is a PL-triangulation of M with less than $3d+1$ vertices, then $\pi_1(M)$ is free by theorem 1.2. Therefore, assumption $H_1(M) = 0$ implies that M is simply-connected, and so it is homeomorphic to S^d by the positive answer to the Poincaré conjecture. It is known that for $d \neq 4$ every combinatorial triangulation of a d-dimensional sphere is a PL-sphere (see [**4**, § 9]). The remaining case

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follows from the result of Bagchi and Datta [1, corollary 3] who used combinatorial methods to show that every combinatorial triangulation of a 4-dimensional homology sphere with at most 12 vertices is PL. □

Proof of corollary 1.4. Similarly as in the proof of the previous corollary $H_1(M;\mathbb{Z}_p)=0$ implies that M is simply-connected. Moreover, since $H_i(M;\mathbb{Z}_p)=H_i(S^d;\mathbb{Z}_p)$ the homology of M is torsion and prime to p for $0<i<d$. In particular, as the homology of a simply-connected d-manifold is free if $d\leqslant 4$, it follows that a simply-connected $\mathbb{Z}_p$-homology sphere is actually an integral homology sphere. By corollary 1.3 we conclude that M is a PL-sphere. □

Proof of theorem 1.5. We begin by showing that links of simplices in a triangulation of a manifold are always homology spheres. Assume that K is a triangulation of a d-dimensional manifold and let σ be a simplex in K of codimension $k+1$. If $x\in|K|$ is a point lying in the interior of σ, then we may use excision and homology sequence of the pair to relate the homology of $\mathrm{lk}(\sigma)$ to the local homology of $|K|$ at x:

$$\begin{aligned}\mathbb{Z}&\cong H_d(|K|,|K|-x)\cong H_d(\sigma*\mathrm{lk}(\sigma),\partial\sigma*\mathrm{lk}(\sigma))\\&\cong H_{d-1}(\partial\sigma*\mathrm{lk}(\sigma))=H_{d-1}(\Sigma^{d-k-1}\,\mathrm{lk}(\sigma))\cong\widetilde{H}_k(\mathrm{lk}(\sigma)).\end{aligned}$$

It follows that $\mathrm{lk}(\sigma)$ is a k-dimensional homology sphere.

Let us now continue the proof under the assumption that links of simplices in K are homology spheres of suitable dimensions. If codimension of σ is at most 3, then $\dim\mathrm{lk}(\sigma)\leqslant 2$, therefore $\mathrm{lk}(\sigma)$ is a combinatorial triangulation of a sphere. We will use this as a base for induction.

Let σ be a simplex of codimension $k+1$, and assume that for each $v\in\mathrm{lk}(\sigma)$ the link $\mathrm{lk}(v,\mathrm{lk}(\sigma))=\mathrm{lk}(\{v\}\cup\sigma)$ is combinatorially equivalent to S^{k-1}. It follows that $\mathrm{lk}(\sigma)$ is a combinatorial triangulation of a k-dimensional homology sphere. By assumption $\mathrm{lk}(\sigma)$ has at most $3k$ vertices, so corollary 1.3 implies that $\mathrm{lk}(\sigma)$ is a combinatorial triangulation of S^k. We conclude that links of all simplices in K are homeomorphic to spheres of suitable dimensions, hence the triangulation K is combinatorial. □

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