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THE METRIC DIMENSION OF METRIC MANIFOLDS

MAJID HEYDARPOUR✉ and SAEID MAGHSOUDI

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Abstract

In this paper we determine the metric dimension of n-dimensional metric (X, G)-manifolds. This category includes all Euclidean, hyperbolic and spherical manifolds as special cases.

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1. Introduction

The concept of the metric dimension of a metric space was defined for the first time in 1953 in [2]. Around 1975, because of applications involving the set of vertices of a graph, the concept attracted more attention (see, for example, [6, 12]). Since then it has found many other applications (see, for example, [3–5, 8, 10]). In 2013, returning to the original idea of the metric dimension of a metric space, Bau and Beardon in [1], among other things, computed the metric dimension for n-dimensional Euclidean space, spherical space, hyperbolic space and Riemann surfaces. Recently, the authors in [7] presented some generalisations of [1] and computed the metric dimensions of n-dimensional geometric spaces. Note that geometric spaces are in fact equivalent to the connected homogeneous Riemannian manifolds.

We recall from [1] that for a metric space (X, d) a resolving set is a nonempty subset A of X such that if $d(x, a) = d(y, a)$ for all $a \in A$ then $x = y$. The *metric dimension* $\beta(X)$ of (X, d) is the smallest cardinality κ such that there is a resolving subset of X with the cardinality κ. A subset of (X, d) with cardinality $\beta(X)$ that resolves X is called a *metric basis* for X. As X resolves X every metric space X has a metric dimension which is at most the cardinality $|X|$ of X.

Our aim in this note is to determine the metric dimension of a class of metric spaces called metric (X, G)-manifolds. Euclidean, hyperbolic and spherical manifolds and geometric spaces are special cases of this class.

2. Preliminaries

To state our results we need to recall some definitions. Our definitions and notation concerning manifolds are standard (see, for example, [9]).

Let (X, d) be a metric space. The open ball with centre x and radius r will be denoted by $B_X(x, r)$ or simply $B(x, r)$.

As usual we define Euclidean space, hyperbolic space and spherical space, respectively, by

$\mathbb{E}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ with the metric $d(x, y) = \|x - y\|$;

$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with path metric derived from $|dx|/x_n$;

$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ with path metric induced by the Euclidean metric on $\mathbb{R}^{n+1}$.

Note that the metric dimension of any open subset of $\mathbb{E}^n, \mathbb{H}^n$, or $\mathbb{S}^n$ is $n + 1$ (see [1] or [7]). We remark that in a metric space X, the relation $A \subseteq B \subseteq X$ implies neither $\beta(A) \leq \beta(B)$ nor $\beta(B) \leq \beta(A)$ (see [7]).

In a metric space (X, d), a geodesic arc is a distance-preserving function $\alpha : [a, b] \to X$, with $a < b$ in $\mathbb{R}$. A geodesic segment joining a point p to a point q in X is the image of a geodesic arc whose initial point is p and terminal point q. A geodesic line is a function $\lambda : \mathbb{R} \to X$ that preserves distance locally (see [11, page 25]). The image of a geodesic line is called a geodesic.

An n-dimensional geometric space is a metric space (X, d) such that:

(1) X is geodesically connected; that is, each pair of distinct points of X is joined by a geodesic segment in X;
(2) X is geodesically complete; that is, each geodesic arc $\alpha : [a, b] \to X$ can be extended to a unique geodesic line $\bar{\alpha} : \mathbb{R} \to X$;
(3) there exist a continuous function $\varepsilon : \mathbb{E}^n \to X$ and a real number $k > 0$ such that ε maps $B_{\mathbb{E}^n}(0, k)$ homeomorphically onto $B_X(\varepsilon(0), k)$ and, for each point u of $\mathbb{S}^{n-1}$, the map $\lambda : \mathbb{R} \to X$ defined by $\lambda(t) = \varepsilon(tu)$ is a geodesic line such that λ restricts to a geodesic arc on the interval $[-k, k]$;
(4) the metric space (X, d) is homogenous; that is, for each pair of distinct points p and q in X there is an isometry $\phi : X \to X$ such that $\phi(p) = q$.

Note that (3) and (4) imply that X is an n-manifold. It is known that a Riemannian manifold M is a geometric space if and only if M is homogeneous [11]. We know from [7] that for an n-dimensional geometric space X, $\beta(X) = n + 1$. The spaces $\mathbb{E}^n, \mathbb{H}^n, \mathbb{S}^n, \mathbb{T}^n$ (the n-dimensional torus), $\mathbb{RP}^n$ (the real n-dimensional projective space), and $\mathbb{CP}^n$ (the complex n-dimensional projective space) are elementary examples of geometric spaces. In fact all connected homogeneous Riemannian manifolds are geometric spaces (see [11, page 371]). As a special case, real Grassmannian $O(n)/(O(r) \times O(n - r))$ and complex Grassmannian $U(n)/(U(r) \times U(n - r))$ manifolds are geometric spaces.

A similarity from (X, d_X) to (Y, d_Y) is a bijective map $\Phi : X \to Y$ for which there is a real number $k > 0$ such that

$$d_Y(\Phi(x), \Phi(y)) = k d_X(x, y)$$

for all x, y in X. In this case we say that (X, d_X) is similar to (Y, d_Y). It is obvious that, like the isometries, the similarities between metric spaces preserve the metric dimension. Also a similarity preserves the geodesics.

Now we can define the (X, G)-manifolds which are the main focus of this note.

Definition 2.1.

(i) Let G be a subgroup of $S(X)$, the similarity group of an n-dimensional geometric space X and let M be an n-manifold. An (X, G)-atlas for M is defined as a family of charts

$$\Phi = \{\phi_i : U_i \to X \mid i \in I\}$$

covering M such that the coordinate changes

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

agree in a neighbourhood of each point with an element of G. There is a unique maximal (X, G)-atlas for M containing Φ. An (X, G)-structure for M is a maximal (X, G)-atlas for M and an (X, G)-*manifold* is an n-manifold M together with an (X, G)-structure for M.

(ii) A *metric* (X, G)-*manifold* is a connected (X, G)-manifold M such that G is a subgroup of $I(X)$, the group of isometries of X.

It can be shown that on a metric (X, G)-manifold M there exists a metric induced by X which is the path metric on M and generates a topology on M equivalent to the manifold topology. Throughout, we consider this metric on a metric (X, G)-manifold M. We refer the reader to [11] for more details of the X-metric on a metric (X, G)-manifold M and other related material.

Example 2.2.

(i) Let $X = \mathbb{E}^n, \mathbb{H}^n$, or $\mathbb{S}^n$. Then an $(X, I(X))$-structure on a manifold is called, respectively, a Euclidean, hyperbolic, or spherical structure and we call X, respectively, a Euclidean, hyperbolic, or spherical n-manifold.

(ii) Let X be a geometric space and suppose that the subgroup $\Gamma \leq I(X)$ acts freely and properly discontinuously on X. Then the quotient space X/Γ has a metric $(X, I(X))$-manifold structure.

A function $\varphi : X \to Y$ is called a local isometry if, for each $x \in X$, there is a real number $r > 0$ such that φ maps $B_X(x, r)$ isometrically onto $B_Y(\varphi(x), r)$. Local isometries preserve the length of curves and geodesics. For a metric (X, G)-manifold the charts are locally isometric [11, page 352], so we can consider them (by restricting the domains) as isometries, and also the change of coordinates maps are isometries on each component of their domains.

Finally, let us recall a well-known result on metric (X, G)-manifolds; for a proof we refer to [11, page 367].

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Theorem 2.3. *Let M be a metric (X, G)-manifold. Then the following statements are equivalent:*

(i) *M is a complete metric space;*
(ii) *M is geodesically complete;*
(iii) *M is complete, that is, the universal covering space of each connected component of M is a complete metric space.*

All the metric (X, G)-manifolds we consider in this note are complete metric spaces. Hence the manifolds considered in this paper are geodesically complete and, by [11, page 359], geodesically connected. Recall that M is called geodesically connected whenever each pair of distinct points in M can be joined by a geodesic segment in M.

3. Results

In this section we determine the metric dimension of n-dimensional metric (X, G)-manifolds. To begin with, we compute the metric dimension for a special case of this class of manifolds.

Proposition 3.1. *The metric dimension of a unique geodesic n-dimensional metric (X, G)-manifold is $n + 1$.*

Proof. A unique geodesic manifold is simply connected. Thus, if M is a unique geodesic manifold, there exists a developing map $\phi : M \to X$ that is locally isometric (see [11]). But a bijective local isometry is an isometry, so $\phi : M \to \phi(M)$ is an isometry. From this, we infer that $\beta(M) = \beta\phi(M)$. By [7], the metric dimension of an open subset of n-dimensional geometric space X is $n + 1$, hence $\beta(M) = n + 1$. □

We recall that for any distinct points p, q in a metric space (X, d), the bisector $B(p \mid q)$ is defined by

$$B(p \mid q) = \{x \in X \mid d(x, p) = d(x, q)\}.$$

From the definition we see that a subset $A \subseteq X$ resolves X if and only if it is not contained in any bisector.

Now we are ready to prove our main result. In [1] it is proved that for a Riemann surface M with constant sectional curvature we have $\beta(M) = 3$. The following theorem not only includes this result for Riemann surfaces with constant sectional curvature but also determines the metric dimension of all Riemann surfaces (orientable and nonorientable cases).

Theorem 3.2. *Let M be a complete n-dimensional metric (X, G)-manifold. Then for each open subset A of M, $\beta(A) = n + 1$.*

Proof. Let $x_0 \in A$ be an arbitrary point. Since A is open, there exist an open neighbourhood $U \subseteq A$ of x_0 and a chart $\phi : U \to X$ such that, for some $r > 0$, the mapping $\phi : B_M(x_0, r) \to B_X(y_0, r)$ is an isometry, where $y_0 = \phi(x_0)$ (see [11]). Also, since X is an n-dimensional geometric space, we know from [7] that $\beta(X) = n + 1$.

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Let $B_X = \{y_0, \dots, y_n\} \subset B_X(y_0, r)$ be a metric basis for X and let

$$B_M = \{x_i \mid 0 \le i \le n, x_i = \phi^{-1}(y_i)\} \subset B_M(x_0, r).$$

Since ϕ is an isometry, B_M is a metric basis for $B_M(x_0, r)$ and thus $\beta(B_M(x_0, r)) = n + 1$. We show that B_M is a resolving set for A. To this end let x and x' be two distinct points in A with $d_M(x, x_i) = d_M(x', x_i)$ for all $i = 0, \dots, n$. Then from the definition of X-length [11, page 350] and X-distance [11, page 351] for M and our assumption that M is complete and consequently geodesically complete and geodesically connected, we conclude that there are geodesic segments α_i from x to x_i and α'_i from x' to x_i such that $d_M(x, x_i) = \|\alpha_i\|$ and $d_M(x', x_i) = \|\alpha'_i\|$. Hence there exist lifts $\widetilde{\alpha}_i$ and $\widetilde{\alpha'}_i$ of α_i and α'_i in X such that $\|\alpha_i\| = |\widetilde{\alpha'}_i|$ and $\|\alpha'_i\| = |\widetilde{\alpha}_i|$, where $|\widetilde{\alpha}_i|$ and $|\widetilde{\alpha'}_i|$ are the length of $\widetilde{\alpha}_i$ and $\widetilde{\alpha'}_i$ in X; for more details, see [11]. So there are corresponding points y and y' in X such that $d_X(y, y_i) = d_X(y', y_i)$, for every $i = 0, \dots, n$, which is a contradiction since $\{y_0, \dots, y_n\}$ is a metric basis for X. This shows that $\{x_0, \dots, x_n\}$ is a resolving set for A, and we conclude that $\beta(A) \le n + 1$.

To show that $\beta(A) > n$, let $B = \{x_1, \dots, x_n\}$ be a resolving set for A. Since A is open, each x_i has a ball neighbourhood, say $B(x_i, r_i)$, in A and there are isometric charts $\phi_i : B(x_i, r_i) \to B(\phi_i(x_i), r_i)$. Also, there exists a simply connected open subset U of M such that

$$x_i \in B(x_i, r_i) \subseteq U \quad (i = 1, \dots, n),$$

and, according to the developing theorem (see [11]), there is an extended chart $\phi : U \to X$ that is locally isometric. Now since $\phi(U)$ is open in X, $B_X = \{y_i = \phi(x_i) \mid i = 1, \dots, n\}$ could not be a resolving set for $\phi(U)$. It follows that there are distinct points q and q' such that $d(q, y_i) = d(q', y_i)$ for each $i = 1, \dots, n$. Note that in each ball $B(p, r)$ in X there is a metric basis for X, and around each y_i we have $B(y_i, r_i) = \phi(B(x_i, r_i))$. Now since B_X is not a resolving set for $\phi(U)$, we find that $B_X \subseteq B(q \mid q')$ and $B(q \mid q')$ is an $(n - 1)$-dimensional submanifold (geometric space) of M (see [7]) and $B(y_i, r_i)$ is an n-dimensional submanifold of M. From this we infer that for a fixed $1 \le j \le n$ one may choose y and y' in $\phi(B(x_j, r_j))$ such that $d(y, y_i) = d(y', y_i)$ for each $i = 1, \dots, n$ (in fact, we have $B(q \mid q') = B(y \mid y')$ (see [7]). This implies that

$$x = \phi^{-1}(y) \in B(x_j, r_j) \quad \text{and} \quad x' = \phi^{-1}(y') \in B(x_j, r_j)$$

which means that x and x' are in A. The fact that local isometries preserve the length of curves implies that $d(x, x_i) = d(x', x_i)$ for each $i = 1, \dots, n$. This shows that B is not a metric basis for A. Hence $\beta(A) > n$. Consequently, we conclude that $\beta(A) = n + 1$. □

As an immediate consequence we obtain the following corollary.

Corollary 3.3.

(i) *For a complete n-dimensional metric (X, G)-manifold M we have $\beta(M) = n + 1$.*
(ii) *The metric dimension of every n-dimensional Euclidean, spherical and hyperbolic manifold is $n + 1$.*

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Proof.

(i) Take $M = A$ in Theorem 3.2.

(ii) By part (i), we only need to note that Euclidean, spherical and hyperbolic manifolds are metric $(\mathbb{E}^n, I(\mathbb{E}^n))$-, $(\mathbb{S}^n, I(\mathbb{S}^n))$- and $(\mathbb{H}^n, I(\mathbb{H}^n))$-manifolds, respectively. □

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MAJID HEYDARPOUR, Department of Mathematics,
University of Zanjan, Zanjan 45195-313, Iran
e-mail: Heydarpour@znu.ac.ir

SAEID MAGHSOUDI, Department of Mathematics,
University of Zanjan, Zanjan 45195-313, Iran
e-mail: s_maghsodi@znu.ac.ir