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SPECTRAL ANALYSIS OF INTEGRO-DIFFERENTIAL OPERATORS APPLIED IN LINEAR ANTENNA MODELLING

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Abstract The current on a linear strip or wire solves an equation governed by a linear integro-differential operator that is the composition of the Helmholtz operator and an integral operator with a logarithmically singular displacement kernel. Investigating the spectral behaviour of this classical operator, we first consider the composition of the second-order differentiation operator and the integral operator with logarithmic displacement kernel. Employing methods of an earlier work by J. B. Reade, in particular the Weyl–Courant minimax principle and properties of the Chebyshev polynomials of the first and second kind, we derive index-dependent bounds for the ordered sequence of eigenvalues of this operator and specify their ranges of validity. Additionally, we derive bounds for the eigenvalues of the integral operator with logarithmic kernel. With slight modification our result extends to kernels that are the sum of the logarithmic displacement kernel and a real displacement kernel whose second derivative is square integrable. Employing this extension, we derive bounds for the eigenvalues of the integro-differential operator of a linear strip with the complex kernel replaced by its real part. Finally, for specific geometry and frequency settings, we present numerical results for the eigenvalues of the considered operators using Ritz's methods with respect to finite bases.

Keywords: eigenvalue problems; integro-differential operators; logarithmic kernel; linear antennas

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1. Introduction

Spectral analysis is one of the tools used to obtain insight into the electromagnetic behaviour of antennas and microwave components. The analysis of the eigenmodes of a rectangular waveguide that are obtained from Maxwell's equations by applying Sturm–Liouville theory to electric and magnetic scalar potentials is an example [**20**]. The eigenvalues corresponding to the eigenmodes are directly related to their cut-off frequencies, i.e. frequencies above which the modes propagate. A second example concerns the analysis of antenna arrays, where the eigenfunctions are standing waves that represent specific scan and resonant behaviours of the array. The corresponding eigenvalues are characteristic impedances; they predict resonance phenomena, which are related to the occurrence of surface waves supported by the truncated periodic structure [**1**, **3**, **18**]. For overviews with other examples we refer the reader to [**15**, **26**].

Apart from the electromagnetic insight, existing calculational methods rely on pre-knowledge of the spectrum [**1**, **2**, **9**, **11**, **12**, **17**, **22**]. Generally, a sufficiently fast decay of the eigenvalues or their reciprocals is assumed to limit the numbers of eigenfunctions in the spectral transformations for single scatterers and arrays. In [**1**, **2**, **17**] these numbers are chosen on the basis of physical insight or empirical rules derived from numerical results. In this respect we emphasize that the spectral transformation is analytically known only for relatively simple shapes, such as the rectangular waveguide or a loop antenna [**28**]. For problems that require numerical techniques to obtain this transformation, spectral analysis can provide a basis for its approximation by empirical and physical insight. In this paper we concentrate on properties of the spectrum of a linear antenna, where such techniques are required [**1**, **2**].

One of the most common linear antennas is a straight, good conducting wire, or strip, of approximately half a wavelength, referred to as a dipole. The wire diameter and the strip thickness and strip width are small with respect to the wavelength and the dipole length. For the indicated lengths, the linear antenna carries a sinusoidal current distribution of half a period. For larger lengths the dipole turns into a multipole that carries currents of more periods, while for much smaller lengths it turns into a monopole. Focusing first on a linear strip, we outline the derivation of an equation for the current, where we apply the classical assumptions that the electromagnetic field is time harmonic and that the metal is perfectly conducting. Since the strip thickness is much smaller than its width and the wavelength, we model the strip as an infinitely thin sheet. Then, introducing a magnetic vector potential, we express the scattered electric field in terms of the current by Maxwell's equations. Invoking the condition that the total tangential electric field vanishes at the strip surface, we obtain an integro-differential equation, the electric field integral equation (EFIE), that relates the current to the tangential excitation field. Since the strip width is much smaller than the wavelength, we average the current and the tangential excitation field over the strip width. Thus, we link the averaged current to the averaged tangential excitation field by the operator [**1**, § 2.3.2]

$$\mathcal{Z}w = \tfrac{1}{2}\mathrm{i}Z_0k^2\ell b\biggl(1+\frac{1}{k^2\ell^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\biggr)\mathcal{G}w, \tag{1.1}$$

where 2ℓ and $2b$ are the dipole length and width, k is the wavenumber, $Z_0=\sqrt{\mu_0/\varepsilon_0}$ is the characteristic impedance of free space, x is the length coordinate normalized on ℓ, w is the width-averaged current, and $\mathcal{G}$ is the integral operator defined by

$$(\mathcal{G}w)(x) = \int_{-1}^{1} w(\xi)G(x-\xi)\,\mathrm{d}\xi. \tag{1.2}$$

The displacement kernel G of this operator is defined by

$$G(x) = \frac{1}{2\pi k\ell}\int_0^2 (2-y)\frac{\exp(\mathrm{i}k\ell\sqrt{x^2+\beta^2y^2})}{\sqrt{x^2+\beta^2y^2}}\,\mathrm{d}y, \quad \beta=\frac{b}{\ell}. \tag{1.3}$$

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and decomposes as

$$G(x) = -\frac{1}{\pi k \ell \beta} \log|x| + G_{\text{reg}}(x), \tag{1.4}$$

where G_{reg} is even and once differentiable with a square integrable derivative [**1**, § 2.3.2, Appendix A.1]. For linear wires, a similar expression for the integro-differential operator is obtained in the literature, where the current is averaged with respect to the wire circumference. The corresponding equation is called Pocklington's equation with exact kernel, which has a similar decomposition as (1.4) [**13**, **21**]. Recently, the decomposition $F_1(z)\log|z| + F_2(z)$ has been proposed with F_1 and F_2 analytic functions on the real line [**4**]. In other modelling approaches for linear wire antennas the result is an equation with a continuous kernel, which is called the reduced kernel. Contrary to the equation with the exact kernel [**25**], the equation with the reduced kernel is driven by a compact operator and is therefore ill-posed [**6**, **30**]. For justifications of the approximations made in the derivations of both kernels we refer the reader to [**5**, **29**].

Several investigations of spectra of integral operators related to the integral operator $\mathcal{G}$ can be found in the literature. Reade [**23**] derived upper and lower bounds for the integral operators generated by the kernels $\log|x-\xi|$ and $|x-\xi|^{-\alpha}$ with $(x,\xi) \in [-1,1] \times [1,1]$ and $0 < \alpha < 1$. These kernels were considered earlier by Richter, who characterized the singularities in the solutions of the corresponding integral equations [**24**]. Asymptotic expressions for the eigenvalues of the slightly modified kernel $V(y)|x-\xi|^{-\alpha}$ were derived by Kac [**14**]. Estrada and Kanwal [**8**, Lemmas A.1 and A.2] proved two results for eigenvalue bounds of positive compact operators and applied them to kernels considered by Reade. Dostaníc [**7**] derived asymptotic expressions for the eigenvalues related to the kernel $|x-\xi|^{-\alpha}$ and put them in correspondence with the Riemann zeta function. Simíc [**27**] followed similar lines as Dostaníc to derive asymptotic upper and lower bounds for the singular values of the integral operator with kernel $\log^{\beta}|x-\xi|^{-1}$ with $0 < \xi < x < 1$ and $\beta > 0$.

No investigations seem to exist of spectra of integro-differential operators related to $\mathcal{Z}$, in particular the composition of the second-order differentiation operator and an integral operator with displacement kernel $\log|x-\xi|$. Given the aforementioned approximations of the spectral transformation in several methods of analysis and solution, particularly [**1**, **2**, **17**], the objective of our paper is the asymptotics of the eigenvalues of the integro-differential operator $\mathcal{Z}$ and related operators. In our approach we consider the integral operator $\mathcal{K}$ on the Hilbert space $\mathfrak{L}_2([-1,1])$,

$$(\mathcal{K}f)(x) = \int_{-1}^{1} \log|x-\xi| f(\xi)\,\mathrm{d}\xi, \tag{1.5}$$

and the integro-differential operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ on the domain

$$\mathfrak{W} = \{f \in \mathfrak{H}_{2,1}([-1,1]) \mid f(-1) = f(1) = 0\}. \tag{1.6}$$

According to Reade [**23**], $\mathcal{K}$ is a compact self-adjoint operator with negative eigenvalues $\lambda_n(\mathcal{K})$ that satisfy the inequalities

$$\frac{\pi}{4n} \leqslant |\lambda_n(\mathcal{K})| \leqslant \frac{\pi}{n-1}, \quad n \geqslant n_0, \tag{1.7}$$

where the eigenvalues are indexed according to decreasing magnitude starting from $n = 1$. The upper bound is valid for $n_0 = 3$, while the validity of the lower bound is not specified by Reade. In this paper we put Reade's result in a more general perspective. By this generalization we prove that $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$, with domain $\mathfrak{W}$, extends to a positive self-adjoint operator with compact inverse and that the ordered sequence of eigenvalues $\lambda_n((\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K})$, $n = 1, 2, \dots$, satisfies $\lambda_n \geqslant \pi n$ for $n \geqslant 1$ and $\lambda_n \leqslant \pi(4n - 2)$ for $n \geqslant 2$. These results extend with a slight modification to integral operators $\tilde{\mathcal{K}}$ with displacement kernels of the form

$$\tilde{k}(x - \xi) = \log|x - \xi| + h(x - \xi), \tag{1.8}$$

where h is real, even and twice differentiable. Employing this modified result, we derive bounds for the eigenvalues of the integro-differential operator $\mathcal{Z}$ with kernel G_{reg} replaced by its real part. As an additional result of our generalization we find values for n_0 in (1.7) for which the upper and lower bounds are valid. In the last section of this paper we compare our analytic approach with numerical results for the eigenvalues of the considered operators.

2. Prerequisites

The Weyl–Courant minimax principle for positive, or non-negative, compact operators is formulated in [**10**, Chapter 2, § 1].

Theorem 2.1. *Let $\mathcal{C}$ be a positive self-adjoint compact operator with eigenvalues $\lambda_1(\mathcal{C}) \geqslant \lambda_2(\mathcal{C}) \geqslant \cdots \geqslant 0$. Then,*

$$\lambda_n(\mathcal{C}) = \min_{\mathcal{F}} \|\mathcal{C} - \mathcal{F}\|, \tag{2.1}$$

where the minimum is taken over all finite rank operators $\mathcal{F}$ with rank less than or equal to $n - 1$.

The principle has the following two consequences.

Corollary 2.2. *Let $\mathcal{C}$ be a compact positive self-adjoint operator and let $\mathcal{B}$ be a bounded operator. Then $\lambda_n(\mathcal{B}\mathcal{C}\mathcal{B}^*) \leqslant \|\mathcal{B}\|^2 \lambda_n(\mathcal{C})$.*

Proof. The chain of inequalities

$$\lambda_n(\mathcal{B}\mathcal{C}\mathcal{B}^*) = \min_{\mathcal{F}} \|\mathcal{B}\mathcal{C}\mathcal{B}^* - \mathcal{F}\| \leqslant \inf_{\mathcal{F}} \|\mathcal{B}\mathcal{C}\mathcal{B}^* - \mathcal{B}\mathcal{F}\mathcal{B}^*\| \leqslant \|\mathcal{B}\|^2 \min_{\mathcal{F}} \|\mathcal{C} - \mathcal{F}\| \tag{2.2}$$

proves the statement. □

Corollary 2.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be compact positive self-adjoint operators such that $\mathcal{C}_1 \geqslant \mathcal{C}_2$, i.e.*

$$\text{for all } f\colon \langle \mathcal{C}_1 f, f\rangle \geqslant \langle \mathcal{C}_2 f, f\rangle. \tag{2.3}$$

Let the eigenvalues of both operators be indexed according to decreasing magnitude as in Theorem 2.1. Then, $\lambda_n(\mathcal{C}_1) \geqslant \lambda_n(\mathcal{C}_2)$ for all n.

Proof. Let $\mathcal{P}_n$ be the orthogonal projection onto the linear span of the eigenvectors corresponding to $\lambda_1(\mathcal{C}_1), \dots, \lambda_{n-1}(\mathcal{C}_1)$. Since $\mathcal{C}_1$ and $\mathcal{P}_n$ commute, $\mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)$ is self-adjoint. Then, its spectral radius equals its norm [**16**, pp. 391, 394] and its norm equals its numerical radius [**16**, p. 466]. Consequently,

$$\lambda_n(\mathcal{C}_1) = \|\mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)\| = \max_{f,\|f\|=1} \langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)f, f\rangle, \tag{2.4}$$

where we write the maximum instead of the supremum, because $\mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)$ is compact. By decomposing $f = \mathcal{P}_n f + (\mathcal{I} - \mathcal{P}_n)f$, we readily observe that $\langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)f, f\rangle = \langle \mathcal{C}_1(\mathcal{I}-\mathcal{P}_n)f, (\mathcal{I}-\mathcal{P}_n)f\rangle$. Then, substituting this result in (2.4) and subsequently applying the assumption $\mathcal{C}_1 \geqslant \mathcal{C}_2$, we derive

$$\begin{aligned}
\lambda_n(\mathcal{C}_1) &= \max_{f,\|f\|=1} \langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)f, (\mathcal{I} - \mathcal{P}_n)f\rangle && (2.5)\\
&\geqslant \max_{f,\|f\|=1} \langle \mathcal{C}_2(\mathcal{I} - \mathcal{P}_n)f, (\mathcal{I} - \mathcal{P}_n)f\rangle \\
&= \|(\mathcal{I} - \mathcal{P}_n)\mathcal{C}_2(\mathcal{I} - \mathcal{P}_n)\| && (2.6)\\
&= \|\mathcal{C}_2 - (\mathcal{P}_n\mathcal{C}_2 + \mathcal{C}_2\mathcal{P}_n - \mathcal{P}_n\mathcal{C}_2\mathcal{P}_n)\|. && (2.7)
\end{aligned}$$

Since $\mathcal{P}_n\mathcal{C}_2 + \mathcal{C}_2\mathcal{P}_n - \mathcal{P}_n\mathcal{C}_2\mathcal{P}_n$ has finite rank n, it follows from Theorem 2.1 that $\lambda_n(\mathcal{C}_1) \geqslant \lambda_n(\mathcal{C}_2)$. □

As we noted in § 1, our techniques are closely related to the ones used by Reade, who employs properties of the Chebyshev polynomials. Since we want to keep the paper self-contained, we introduce these properties, starting with the definition of the Chebyshev polynomials $\{T_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$:

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad n = 0, 1, 2, \dots. \tag{2.8}$$

The polynomials satisfy the orthogonality relations

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \begin{cases} \pi, & (n,m) = (0,0), \\ \tfrac{1}{2}\pi\delta_{nm}, & (n,m) \neq (0,0), \end{cases} \tag{2.9}$$

and

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,\mathrm{d}x = \tfrac{1}{2}\pi\delta_{nm}. \tag{2.10}$$

Correspondingly, we introduce two complete orthogonal sequences in $\mathfrak{L}_2([-1,1])$:

$$\hat{T}_n(x) = (1-x^2)^{-1/4}T_n(x), \quad \hat{U}_n(x) = (1-x^2)^{1/4}U_n(x), \quad n = 0, 1, \dots. \tag{2.11}$$

Furthermore, for $\nu \in \mathbb{R}$ we introduce the self-adjoint multiplication operator $\mathcal{M}_\nu$ in $\mathfrak{L}_2([-1,1])$ by $\mathcal{M}_\nu f = (1-x^2)^\nu f$. For $\nu \geqslant 0$, the operator is bounded with $\|\mathcal{M}_\nu\| = 1$, i.e. the essential supremum of the function $(1-x^2)^\nu$ on the interval $[-1,1]$. From the goniometric formulae

$$\cos n\theta \sin\theta = \tfrac{1}{2}(\sin(n+1)\theta - \sin(n-1)\theta), \tag{2.12}$$
$$\sin\theta\sin(n+1)\theta = \tfrac{1}{2}(\cos n\theta - \cos(n+2)\theta), \tag{2.13}$$

we derive the relations

$$\mathcal{M}_{1/2}\hat{T}_0 = \hat{U}_0, \quad \mathcal{M}_{1/2}\hat{T}_1 = \tfrac{1}{2}\hat{U}_1, \quad \mathcal{M}_{1/2}\hat{T}_n = \tfrac{1}{2}(\hat{U}_n - \hat{U}_{n-2}), \quad n = 2,3,\dots, \tag{2.14}$$
$$\mathcal{M}_{1/2}\hat{U}_n = \tfrac{1}{2}(\hat{T}_n - \hat{T}_{n+2}), \quad n = 0,1,\dots. \tag{2.15}$$

3. Asymptotic behaviour of eigenvalues of integral operators described by Chebyshev polynomial expansions

In this section we study the asymptotics of the eigenvalues of the integral operators on the Hilbert space $\mathfrak{L}_2([-1,1])$ related to the following two types of kernels:

$$k_1(x,\xi) = \sum_{n=0}^{\infty} \alpha_n T_n(x) T_n(\xi) \tag{3.1}$$

and

$$k_2(x,\xi) = \sum_{n=0}^{\infty} \alpha_n U_n(x) U_n(\xi)\sqrt{1-x^2}\sqrt{1-\xi^2}. \tag{3.2}$$

Here the sequence (α_n) satisfies $\alpha_n \downarrow 0$ as $n\to\infty$, $\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant 0$. The symmetric kernels k_1 and k_2 correspond to the integral operators $\mathcal{K}_1$ and $\mathcal{K}_2$

$$(\mathcal{K}_{1,2}f)(x) = \int_{-1}^{1} k_{1,2}(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{3.3}$$

We define the positive self-adjoint compact operators $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$ on $\mathfrak{L}_2([-1,1])$ by

$$\hat{\mathcal{K}}_1 f = \sum_{n=0}^{\infty} \alpha_n \langle f, \hat{T}_n\rangle_{\mathfrak{L}_2} \hat{T}_n, \qquad \hat{\mathcal{K}}_2 f = \sum_{n=0}^{\infty} \alpha_n \langle f, \hat{U}_n\rangle_{\mathfrak{L}_2} \hat{U}_n. \tag{3.4}$$

Employing (2.9), (2.10) and (2.11), we find

$$\hat{\mathcal{K}}_1\hat{T}_0 = \pi\alpha_0\hat{T}_0, \quad \hat{\mathcal{K}}_1\hat{T}_n = \tfrac{1}{2}\pi\alpha_n\hat{T}_n, \quad n=1,2,\dots, \tag{3.5}$$
$$\hat{\mathcal{K}}_2\hat{U}_n = \tfrac{1}{2}\pi\alpha_n\hat{U}_n, \quad n=0,1,\dots. \tag{3.6}$$

The operators $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$ are related to the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ according to

$$\mathcal{K}_1 = \mathcal{M}_{1/4}\hat{\mathcal{K}}_1\mathcal{M}_{1/4}, \qquad \mathcal{K}_2 = \mathcal{M}_{1/4}\hat{\mathcal{K}}_2\mathcal{M}_{1/4}. \tag{3.7}$$

Since $\mathcal{M}_{1/4}$ is self-adjoint and bounded, $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact, positive and self-adjoint. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the bounded operators on $\mathfrak{L}_2([-1,1])$ defined by $\mathcal{S}_1\hat{T}_n = \hat{T}_{n+2}$ and

$\mathcal{S}_2\hat{U}_n = \hat{U}_{n+2}$, with adjoints $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ that satisfy

$$\mathcal{S}_1^*\hat{T}_0 = \mathcal{S}_1^*\hat{T}_1 = 0, \quad \mathcal{S}_1^*\hat{T}_2 = \tfrac{1}{2}\hat{T}_0, \quad \mathcal{S}_1^*\hat{T}_n = \hat{T}_{n-2}, \quad n \geqslant 3, \tag{3.8}$$

$$\mathcal{S}_2^*\hat{U}_0 = \mathcal{S}_2^*\hat{U}_1 = 0, \quad \mathcal{S}_2^*\hat{U}_n = \hat{U}_{n-2}, \quad n \geqslant 2. \tag{3.9}$$

Employing the relations (2.14)–(2.15), we obtain

$$\begin{aligned}\mathcal{M}_{1/2}&\hat{\mathcal{K}}_1\mathcal{M}_{1/2}f \\ &= \alpha_0\langle f,\hat{U}_0\rangle_{\mathfrak{L}_2}\hat{U}_0 + \tfrac{1}{4}\alpha_1\langle f,\hat{U}_1\rangle_{\mathfrak{L}_2}\hat{U}_1 + \frac{1}{4}\sum_{n=2}^{\infty}\alpha_n\langle f,\hat{U}_n-\hat{U}_{n-2}\rangle_{\mathfrak{L}_2}(\hat{U}_n-\hat{U}_{n-2}) \\ &= \tfrac{3}{4}\alpha_0\langle f,\hat{U}_0\rangle_{\mathfrak{L}_2}\hat{U}_0 + \tfrac{1}{4}(\mathcal{I}-\mathcal{S}_2{}^*)\hat{\mathcal{K}}_2(\mathcal{I}-\mathcal{S}_2)f\end{aligned} \tag{3.10}$$

and, similarly,

$$\mathcal{M}_{1/2}\hat{\mathcal{K}}_2\mathcal{M}_{1/2} = \tfrac{1}{4}(\mathcal{I}-\mathcal{S}_1)\hat{\mathcal{K}}_1(\mathcal{I}-\mathcal{S}_1{}^*). \tag{3.11}$$

Let $\mathcal{P}_{1,N}$ and $\mathcal{P}_{2,N}$ denote the orthogonal projections onto the linear spans of $\{\hat{T}_n \mid n = 0,\dots,2N\}$ and $\{\hat{U}_n \mid n = 0,\dots,2N\}$, respectively. Then $\hat{\mathcal{K}}_2 \geqslant \frac{1}{2}\pi\alpha_{2N}\mathcal{P}_{2,N}$ and $\hat{\mathcal{K}}_1 \geqslant \frac{1}{2}\pi\alpha_{2N}\mathcal{Q}_{1,N}$, where

$$\mathcal{Q}_{1,N}f = \mathcal{P}_{1,N}f + \frac{1}{\pi}(f,\hat{T}_0)_{\mathfrak{L}_2}\hat{T}_0,$$

so that

$$\mathcal{M}_{1/2}\hat{\mathcal{K}}_1\mathcal{M}_{1/2} \geqslant \tfrac{1}{8}\pi\alpha_{2N}\mathcal{A}_N, \qquad \mathcal{M}_{1/2}\hat{\mathcal{K}}_2\mathcal{M}_{1/2} \geqslant \tfrac{1}{8}\pi\alpha_{2N}\mathcal{B}_N. \tag{3.12}$$

where the operators $\mathcal{A}_N$ and $\mathcal{B}_N$ are defined as

$$\mathcal{A}_N = (\mathcal{I}-\mathcal{S}_2{}^*)\mathcal{P}_{2,N}(\mathcal{I}-\mathcal{S}_2), \qquad \mathcal{B}_N = (\mathcal{I}-\mathcal{S}_1)\mathcal{Q}_{1,N}(\mathcal{I}-\mathcal{S}_1{}^*). \tag{3.13}$$

By Corollary 2.3 it then follows that

$$\lambda_n(\mathcal{M}_{1/2}\hat{\mathcal{K}}_1\mathcal{M}_{1/2}) \geqslant \tfrac{1}{8}\pi\alpha_{2N}\lambda_n(\mathcal{A}_N), \qquad \lambda_n(\mathcal{M}_{1/2}\hat{\mathcal{K}}_2\mathcal{M}_{1/2}) \geqslant \tfrac{1}{8}\pi\alpha_{2N}\lambda_n(\mathcal{B}_N), \tag{3.14}$$

where all eigenvalues are indexed according to decreasing magnitude.

Since $\mathcal{A}_N\hat{U}_n = 0$ for $n > 2N$ and

$$\mathcal{A}_N\hat{U}_n = \begin{cases} 2\hat{U}_n - \hat{U}_{n+2}, & n = 0,1, \\ 2\hat{U}_n - \hat{U}_{n+2} - \hat{U}_{n-2}, & n = 2,\dots,2N-2, \\ \hat{U}_n - \hat{U}_{n-2}, & n = 2N-1, 2N, \end{cases} \tag{3.15}$$

its matrix A_N $(N \geqslant 2)$ with respect to the orthonormal basis

$$\{\sqrt{2/\pi}\hat{U}_{2n+1} \mid n = 0,\dots,N-1\} \cup \{\sqrt{2/\pi}\hat{U}_{2n} \mid n = 0,\dots,N\} \tag{3.16}$$

has the block structure

$$A_N = \begin{pmatrix} C_N & 0 \\ 0 & C_{N+1}\end{pmatrix}, \tag{3.17}$$

where C_N is the symmetric tridiagonal $N \times N$ matrix with main diagonal $(2, \dots, 2, 1)$ and codiagonal $(-1, \dots, -1)$. Reade [**23**] derives the eigenvalues of C_N by expressing its characteristic polynomial q_N as

$$q_N(\lambda) = p_N(\lambda) - p_{N-1}(\lambda), \tag{3.18}$$

where p_N is the characteristic polynomial of the symmetric tridiagonal $N \times N$ matrix with main diagonal $(2, \dots, 2)$ and codiagonal $(-1, \dots, -1)$. The polynomials p_N can be expressed as $p_N(\lambda) = U_N(1 - \frac{1}{2}\lambda)$, since they satisfy the recurrence relation of the Chebyshev polynomials U_N with argument $1 - \frac{1}{2}\lambda$,

$$p_{N+1}(\lambda) = (2 - \lambda)p_N(\lambda) - p_{N-1}(\lambda), \tag{3.19}$$

with initial conditions $p_0 = 1$ and $p_1(\lambda) = 2 - \lambda$. The eigenvalues of C_N follow by substitution of this expression in (3.18), by which [**23**, p. 143]

$$q_N(\lambda) = \frac{\cos(\frac{1}{2}(2N+1)\theta)}{\cos(\frac{1}{2}\theta)}, \quad \lambda = 2(1 - \cos\theta). \tag{3.20}$$

Finally, the eigenvalues of A_N with $N \geqslant 2$ are those of C_N and C_{N+1},

$$\nu^{(1)}_{A_N,m} = 4\cos^2 \frac{\pi m}{2N+1}, \quad m = 1, 2, \dots, N, \tag{3.21}$$

$$\nu^{(2)}_{A_N,m} = 4\cos^2 \frac{\pi m}{2N+3}, \quad m = 1, 2, \dots, N+1. \tag{3.22}$$

To calculate the eigenvalues of $\mathcal{B}_N$, we employ a basis decomposition similar to (3.16) and properties of the characteristic polynomials q_N. Since $\mathcal{B}_N \hat{T}_n = 0$ for $n > 2N + 2$ and

$$\mathcal{B}_N \hat{T}_n = \begin{cases} 2(\hat{T}_0 - \hat{T}_2), & n = 0, \\ \hat{T}_1 - \hat{T}_3, & n = 1, \\ 2\hat{T}_n - \hat{T}_{n+2} - \hat{T}_{n-2}, & n = 2, \dots, 2N, \\ \hat{T}_n - \hat{T}_{n-2}, & n = 2N+1, 2N+2, \end{cases} \tag{3.23}$$

its matrix B_N $(N \geqslant 1)$ with respect to the orthonormal basis

$$\{\sqrt{1/\pi}\hat{T}_0\} \cup \{\sqrt{2/\pi}\hat{T}_{2n} \mid n = 1, \dots, N+1\} \cup \{\sqrt{2/\pi}\hat{T}_{2n+1} \mid n = 0, \dots, N\} \tag{3.24}$$

has the block structure

$$B_N = \begin{pmatrix} D_{N+2} & 0 \\ 0 & E_{N+1} \end{pmatrix}, \tag{3.25}$$

where the symmetric tridiagonal matrices D_{N+2} and E_{N+1} have main diagonals $(2, \dots, 2, 1)$ and $(1, 2, \dots, 2, 1)$, and codiagonals $(-\sqrt{2}, -1, \dots, -1)$ and $(-1, \dots, -1)$, respectively. In terms of the polynomials q_N, their characteristic polynomials satisfy

$$\chi_{D_{N+2}}(\lambda) = (2 - \lambda)q_{N+1}(\lambda) - 2q_N(\lambda), \qquad \chi_{E_{N+1}}(\lambda) = (1 - \lambda)q_N(\lambda) - q_{N-1}(\lambda). \tag{3.26}$$

Since the polynomials q_N satisfy the same recurrence relation (3.19) as the polynomials p_N (but with different initial conditions), we obtain the relations

$$\chi_{D_{N+2}}(\lambda) = q_{N+2}(\lambda) - q_N(\lambda), \qquad \chi_{E_{N+1}}(\lambda) = q_{N+1}(\lambda) - q_N(\lambda). \tag{3.27}$$

Substituting the expression (3.20) for q_N in these expressions we obtain

$$\chi_{D_{N+2}}(\lambda) = -4\sin\tfrac{1}{2}(2N+3)\theta\sin\tfrac{1}{2}\theta, \qquad \chi_{E_{N+1}}(\lambda) = -\frac{2\sin(N+1)\theta\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}. \tag{3.28}$$

Then, the eigenvalues of D_{N+2} and E_{N+1} follow straightforwardly from the expression for λ in (3.20). Therewith we obtain the eigenvalues of B_N for $N \geqslant 1$,

$$\nu^{(1)}_{B_N,m} = 4\sin^2\frac{m\pi}{2N+3}, \qquad m = 0,\dots,N+1, \tag{3.29}$$

$$\nu^{(2)}_{B_N,m} = 4\sin^2\frac{m\pi}{2(N+1)}, \quad m = 0,\dots,N. \tag{3.30}$$

Theorem 3.1. *Let $\mathcal{K}_1$ be the integral operator defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ by (3.3) with kernel k_1 defined by (3.1), where the sequence (α_r) satisfies $\alpha_r \downarrow 0$ as $r \to \infty$, $\alpha_r \geqslant 0$ for all r, $\alpha_r \geqslant \alpha_{r+1}$ for $r \geqslant N_0 \geqslant 1$ and $\alpha_r \geqslant \alpha_{N_0}$ for $r < N_0$. The operator $\mathcal{K}_1$ is compact and positive with eigenvalues $\lambda_n(\mathcal{K}_1)$, $n = 1,2,\dots$, that satisfy*

$$\lambda_n(\mathcal{K}_1) \leqslant \tfrac{1}{2}\pi\alpha_{n-1}, \quad n \geqslant \max(N_0, 2), \tag{3.31}$$

$$\lambda_n(\mathcal{K}_1) \geqslant \tfrac{1}{4}\pi\alpha_{2n}, \qquad n \geqslant \max(\lceil\tfrac{1}{2}N_0\rceil, 2), \tag{3.32}$$

where the eigenvalues are indexed according to decreasing magnitude.

Proof. Since $\mathcal{K}_1 = \mathcal{M}_{1/4}\hat{\mathcal{K}}_1\mathcal{M}_{1/4}$, we obtain by Corollary 2.2 and by (3.5)

$$\lambda_n(\mathcal{K}_1) \leqslant \lambda_n(\hat{\mathcal{K}}_1) = \tfrac{1}{2}\pi\alpha_{n-1}, \quad n \geqslant \max(N_0, 2). \tag{3.33}$$

To derive a lower bound we recall that the inequalities in (3.14) are derived for monotonically decreasing sequences (α_n). It straightforwardly follows that the first inequality is also valid for the sequences (α_n) in this theorem if the requirement $N \geqslant \lceil\frac{1}{2}N_0\rceil$ is added. Then, since $\mathcal{A}_N$ is defined for $N \geqslant 2$, it follows from (3.14) and Corollary 2.2 that

$$\lambda_n(\mathcal{K}_1) \geqslant \lambda_n(\mathcal{M}_{1/2}\hat{\mathcal{K}}_1\mathcal{M}_{1/2}) \geqslant \tfrac{1}{8}\pi\alpha_{2N}\lambda_n(\mathcal{A}_N), \tag{3.34}$$

where we can select an appropriate $N \geqslant \max(\lceil\frac{1}{2}N_0\rceil, 2)$. Indexing the eigenvalues $\lambda_n(\mathcal{A}_N)$, $n = 1,2,\dots,2N+1$, given by (3.21) and (3.22) according to decreasing magnitude, we find that $\lambda_1(\mathcal{A}_N)$ is $\nu^{(2)}_{A_N,1}$. To determine $\lambda_n(\mathcal{A}_N)$ we invoke the property

$$\nu^{(2)}_{A_N,m} > \nu^{(1)}_{A_N,m} > \nu^{(2)}_{A_N,m+1}. \tag{3.35}$$

Then,

$$\lambda_n(\mathcal{A}_N) = \begin{cases} \nu^{(1)}_{A_N,n/2} = 4\cos^2\left(\dfrac{\pi}{4}\dfrac{n}{N+\frac{1}{2}}\right), & n \text{ even}, \\[2ex] \nu^{(2)}_{A_N,(n+1)/2} = 4\cos^2\left(\dfrac{\pi}{4}\dfrac{n+1}{N+\frac{3}{2}}\right), & n \text{ odd}. \end{cases} \tag{3.36}$$

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Considering the eigenvalues with index $n \geqslant \max(\lceil \frac{1}{2}N_0 \rceil, 2)$ and choosing $N = n$, we obtain from (3.36) that $\lambda_n(\mathcal{A}_n) \geqslant 4\cos^2(\pi/4) = 2$. Consequently, (3.34) with $N = n$ yields (3.32). We note that Reade chooses $N = n+1$ in his analysis of the eigenvalues of the integral operator with logarithmic kernel [**23**, p. 143], since he indexes the eigenvalues starting from $n = 0$. We index the eigenvalues starting from $n = 1$ because of the application of the Weyl–Courant minimax principle in the form (2.1). □

Theorem 3.2. *Let $\mathcal{K}_2$ be the integral operator defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ by (3.3) with kernel k_2 defined by (3.2), where the sequence (α_r) satisfies $\alpha_r \downarrow 0$ as $r \to \infty$, $\alpha_r \geqslant 0$ for all r, $\alpha_r \geqslant \alpha_{r+1}$ for $r \geqslant N_1 \geqslant 1$ and $\alpha_r \geqslant \alpha_{N_1}$ for $r < N_1$. The operator $\mathcal{K}_2$ is compact and positive with eigenvalues $\lambda_n(\mathcal{K}_2)$, $n = 1, 2, \dots$, that satisfy*

$$\lambda_n(\mathcal{K}_2) \leqslant \tfrac{1}{2}\pi\alpha_{n-1}, \qquad n \geqslant N_1, \tag{3.37}$$

$$\lambda_n(\mathcal{K}_2) \geqslant \tfrac{1}{4}\pi\alpha_{2(n-1)}, \quad n \geqslant \lceil \tfrac{1}{2}N_1 \rceil + 1, \tag{3.38}$$

where the eigenvalues $\lambda_n(\mathcal{K}_2)$ are indexed according to decreasing magnitude.

Proof. Analogously to (3.33), the upper bound follows from $\mathcal{K}_2 = \mathcal{M}_{1/4}\hat{\mathcal{K}}_2\mathcal{M}_{1/4}$ by application of Corollary 2.2 and (3.6). To derive the lower bound we follow similar arguments as in the previous proof; the difference is mainly in the way in which the eigenvalues are counted. Analogously to (3.34), we obtain from (3.14) and Corollary 2.2

$$\lambda_n(\mathcal{K}_2) \geqslant \tfrac{1}{8}\pi\alpha_{2N}\lambda_n(\mathcal{B}_N), \tag{3.39}$$

where we can select an appropriate $N \geqslant \lceil \frac{1}{2}N_1 \rceil$. Indexing the eigenvalues $\lambda_n(\mathcal{B}_N)$, $n = 1, 2, \dots, 2N+3$, given by (3.29) and (3.30) according to decreasing magnitude, we find that $\lambda_1(\mathcal{B}_N)$ is $\nu^{(1)}_{B_N,N+1}$. The eigenvalues of B_N satisfy (3.35) with A_N replaced by B_N and with the superindices interchanged. Hence,

$$\lambda_n(\mathcal{B}_N) = \begin{cases} \nu^{(2)}_{B_N,N-(n-2)/2} = 4\sin^2\left(\dfrac{\pi}{4}\dfrac{2N+2-n}{N+1}\right), & n \text{ even}, \\[2ex] \nu^{(1)}_{B_N,N+1-(n-1)/2} = 4\sin^2\left(\dfrac{\pi}{4}\dfrac{2N+3-n}{N+\frac{3}{2}}\right), & n \text{ odd}. \end{cases} \tag{3.40}$$

Then, (3.38) follows from (3.39) and (3.40) with $N = n-1$ and $n \geqslant \lceil \frac{1}{2}N_1 \rceil + 1$. □

4. Application: asymptotic behaviour of an integro-differential operator with logarithmically singular kernel

Since

$$\log|x-\xi| = -\sum_{n=0}^{\infty} \gamma_n T_n(\xi) T_n(x) \tag{4.1}$$

with $\gamma_0 = \log 2$ and $\gamma_n = 2/n$ [**23**, Lemma 1], the operator $\mathcal{K}$ on $\mathfrak{L}_2([-1,1])$ defined by (1.5) is compact, negative and self-adjoint. Applying Theorem 3.1 with $N_0 = 3$ (since $\gamma_2 > \gamma_0 > \gamma_3$), we obtain the inequalities (1.7), as follows.

Corollary 4.1. *The (negative) eigenvalues $\lambda_n(\mathcal{K})$, $n = 1, 2, \dots,$ of the compact self-adjoint operator $\mathcal{K}$ defined by (1.5) on the Hilbert space $\mathfrak{L}_2([-1, 1])$ satisfy*

$$\frac{\pi}{4n} \leqslant |\lambda_n(\mathcal{K})|, \quad n \geqslant 2, \qquad |\lambda_n(\mathcal{K})| \leqslant \frac{\pi}{n-1}, \quad n \geqslant 3, \tag{4.2}$$

where the eigenvalues are indexed according to decreasing magnitude.

Next we apply the results of § 3 to the derivation of the asymptotics of the eigenvalues of the operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$. For that we do some auxiliary work. The Hilbert transform $\mathcal{V}$ on the Hilbert space $\mathfrak{L}_2(\mathbb{R})$ is defined by the principal-value (PV) integral

$$(\mathcal{V}w)(x) = \mathrm{PV} \int_{-\infty}^{\infty} \frac{w(\xi)}{x - \xi} \, \mathrm{d}\xi. \tag{4.3}$$

The operator is bounded and its adjoint satisfies $\mathcal{V}^* = -\mathcal{V}$. The Fourier transformation $\mathcal{F}$ on $\mathfrak{L}_2(\mathbb{R})$ defined by

$$(\mathcal{F}w)(y) = \int_{-\infty}^{\infty} w(x) \mathrm{e}^{-\mathrm{i}yx} \, \mathrm{d}x \tag{4.4}$$

and the Hilbert transform satisfy the relation

$$((\mathcal{F} \circ \mathcal{V})w)(y) = -\pi \mathrm{i} \operatorname{sgn}(y)(\mathcal{F}w)(y). \tag{4.5}$$

The well-known identity $-\mathcal{V}^2 = \mathcal{V}^*\mathcal{V} = \pi^2 \mathcal{I}$ follows by composing $\mathcal{F}$ and $\mathcal{V}^2$, applying (4.5) twice and taking the inverse Fourier transform of the result.

On $\mathfrak{L}_2([-1, 1])$ we introduce the finite Hilbert transform $\mathcal{H}$ by $\mathcal{H}f = (\mathcal{V}w_f)|_{[-1,1]}$ with w_f the natural extension of $f \in \mathfrak{L}_2([-1, 1])$ to $\mathfrak{L}_2(\mathbb{R})$. We conclude that $\mathcal{H}$ is bounded with $\|\mathcal{H}\| \leqslant \pi$. The Chebyshev polynomials satisfy the relations [**19**, p. 261]

$$\frac{1}{\pi} \mathrm{PV} \int_{-1}^{1} \frac{1}{x - \xi} \frac{1}{\sqrt{1 - \xi^2}} T_n(\xi) \, \mathrm{d}\xi = -U_{n-1}(x), \tag{4.6}$$

$$\frac{1}{\pi} \mathrm{PV} \int_{-1}^{1} \frac{1}{x - \xi} \sqrt{1 - \xi^2} U_{n-1}(\xi) \, \mathrm{d}\xi = T_n(x), \tag{4.7}$$

for $-1 \leqslant x \leqslant 1$ and $n = 1, 2, \dots$. Note that from (4.7) we conclude that $\|\mathcal{H}\| = \pi$. We write the relations (4.6) and (4.7) as

$$\mathcal{H}\mathcal{M}_{-1/4}\hat{T}_n = -\pi \mathcal{M}_{-1/4}\hat{U}_{n-1}, \qquad \mathcal{H}\mathcal{M}_{1/4}\hat{U}_{n-1} = \pi \mathcal{M}_{1/4}\hat{T}_n. \tag{4.8}$$

The first relation inspires us to introduce $\hat{\mathcal{H}}$ on $\mathfrak{L}_2([-1, 1])$ by

$$\hat{\mathcal{H}}g = -2 \sum_{n=0}^{\infty} \langle g, \hat{T}_{n+1} \rangle_{\mathfrak{L}_2} \hat{U}_n, \tag{4.9}$$

such that $\hat{\mathcal{H}}\hat{T}_n = -\pi \hat{U}_{n-1}$ for $n = 1, 2, \dots$ and $\hat{\mathcal{H}}\hat{T}_0 = 0$. Applying $\hat{\mathcal{H}}$ to $\mathcal{M}_{1/4}f$ with $f \in \mathfrak{L}_2([-1, 1])$ and multiplying by $\mathcal{M}_{-1/4}$, we obtain

$$\mathcal{M}_{-1/4}\hat{\mathcal{H}}\mathcal{M}_{1/4}f = -2 \sum_{n=0}^{\infty} \langle f, T_{n+1} \rangle_{\mathfrak{L}_2} U_n \tag{4.10}$$

by straightforwardly employing the definitions of $\mathcal{M}_\nu$, $\hat{T}_n$ and $\hat{U}_n$. Alternatively, the action of the operator $\mathcal{M}_{-1/4}\hat{\mathcal{H}}\mathcal{M}_{1/4}$ can also be calculated as

$$\begin{aligned}\mathcal{M}_{-1/4}\hat{\mathcal{H}}\mathcal{M}_{1/4}f &= -2\sum_{n=0}^{\infty}\langle f,\mathcal{M}_{1/4}\hat{T}_{n+1}\rangle_{\mathfrak{L}_2}U_n \\ &= -\frac{2}{\pi}\sum_{n=0}^{\infty}\langle f,\mathcal{H}\mathcal{M}_{1/4}\hat{U}_n\rangle_{\mathfrak{L}_2}U_n \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\langle \mathcal{H}f,\mathcal{M}_{1/2}U_n\rangle_{\mathfrak{L}_2}U_n = \mathcal{H}f, \end{aligned} \tag{4.11}$$

where the second equality is obtained by invoking the second relation of (4.8) and the third equality is obtained by invoking the adjoint $\mathcal{H}^* = -\mathcal{H}$. Combining (4.10) and (4.11), we conclude that $\mathcal{H} = \mathcal{M}_{-1/4}\hat{\mathcal{H}}\mathcal{M}_{1/4}$ and

$$\mathcal{H}f = -2\sum_{n=0}^{\infty}\langle f,T_{n+1}\rangle_{\mathfrak{L}_2}U_n \tag{4.12}$$

with convergence in $\mathfrak{L}_2([-1,1])$. Since

$$\mathcal{K}f = -\log 2\langle f,T_0\rangle_{\mathfrak{L}_2}T_0 - 2\sum_{n=0}^{\infty}\frac{1}{n+1}\langle f,T_{n+1}\rangle_{\mathfrak{L}_2}T_{n+1} \tag{4.13}$$

and $\mathrm{d}T_{n+1}/\mathrm{d}x = (n+1)U_n$ for $n = 0,1,2,\dots$, we observe that, for all $f \in \mathfrak{L}_2([-1,1])$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{K}f = -2\sum_{n=0}^{\infty}\langle f,T_{n+1}\rangle_{\mathfrak{L}_2}U_n = \mathcal{H}f \tag{4.14}$$

and thus $\mathcal{K}f \in \mathfrak{H}_{2,1}([-1,1])$. By straightforward partial integration, we derive, for $f \in \mathfrak{H}_{2,1}([-1,1])$,

$$\begin{aligned}(\mathcal{K}f)(x) &= \int_{-1}^{1} f(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\biggl(-\int_0^{x-\xi}\log|t|\,\mathrm{d}t\biggr)\,\mathrm{d}\xi \\ &= -f(1)\int_{-1}^{x-1}\log|t|\,\mathrm{d}t + f(-1)\int_0^{x+1}\log|t|\,\mathrm{d}t + \int_{-1}^{1}\int_0^{x-\xi}\log|t|\,\mathrm{d}t\frac{\mathrm{d}f}{\mathrm{d}x}(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{4.15}$$

From this expression we observe that $\mathcal{K}$ satisfies $\mathrm{d}(\mathcal{K}f)/\mathrm{d}x = \mathcal{K}(\mathrm{d}f/\mathrm{d}x)$ for all $f \in \mathfrak{W}$, where $\mathfrak{W}$ is the dense subspace of $\mathfrak{L}_2([-1,1])$ defined by (1.6). Employing this property and (4.14), we derive, for $f \in \mathfrak{W}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathcal{K}f) = \mathcal{H}\biggl(\frac{\mathrm{d}f}{\mathrm{d}x}\biggr) = -2\sum_{n=0}^{\infty}\biggl\langle\frac{\mathrm{d}f}{\mathrm{d}x},T_{n+1}\biggr\rangle_{\mathfrak{L}_2}U_n = 2\sum_{n=0}^{\infty}(n+1)\langle f,U_n\rangle_{\mathfrak{L}_2}U_n. \tag{4.16}$$

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We introduce the compact positive self-adjoint operator $\hat{\mathcal{T}}$ on $\mathfrak{L}_2([-1,1])$ by

$$\hat{\mathcal{T}}f = \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1}\langle f,\hat{U}_n\rangle_{\mathfrak{L}_2}\hat{U}_n \tag{4.17}$$

and, correspondingly, we introduce $\mathcal{T}$ on $\mathfrak{L}_2([-1,1])$ by $\mathcal{T} = \mathcal{M}_{1/4}\hat{\mathcal{T}}\mathcal{M}_{1/4}$. Then, for all $f \in \mathfrak{W}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathcal{K}f) = 2\sum_{n=0}^{\infty}(n+1)\langle \mathcal{M}_{-1/4}f,\hat{U}_n\rangle_{\mathfrak{L}_2}\mathcal{M}_{-1/4}\hat{U}_n = \mathcal{M}_{-1/4}\hat{\mathcal{T}}^{-1}\mathcal{M}_{-1/4}f = \mathcal{T}^{-1}f. \tag{4.18}$$

Thus, we show that the unbounded operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ extends to a positive self-adjoint operator, given by $\mathcal{T}^{-1}$, with domain the range of the compact self-adjoint operator $\mathcal{T}$. From (4.17) and the definition of $\mathcal{T}$, it follows that the kernel of $\mathcal{T}$ is equal to

$$K_{\mathcal{T}}(x,\xi) = \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1}U_n(x)U_n(\xi)\sqrt{1-x^2}\sqrt{1-\xi^2}. \tag{4.19}$$

Applying Theorem 3.2 with $N_1 = 1$, we obtain the following result.

Corollary 4.2. *The eigenvalues $\lambda_n(\mathcal{T})$, $n = 1,2,\dots,$ of the compact positive self-adjoint operator $\mathcal{T}$ defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ satisfy*

$$\frac{1}{\pi(4n-2)} \leqslant \lambda_n(\mathcal{T}), \quad n \geqslant 2, \qquad \lambda_n(\mathcal{T}) \leqslant \frac{1}{\pi n}, \quad n \geqslant 1, \tag{4.20}$$

where the eigenvalues are indexed according to decreasing magnitude.

Theorem 4.3. *Let $\mathcal{K}$ be the compact self-adjoint operator on the Hilbert space $\mathfrak{L}_2([-1,1])$ defined by (1.5). Then, the integro-differential operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ defined on $\mathfrak{W}$ according to*

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{K}f\right)(x) = \mathrm{PV}\int_{-1}^{1}\frac{1}{x-\xi}\frac{\mathrm{d}f}{\mathrm{d}x}(\xi)\,\mathrm{d}\xi \tag{4.21}$$

extends to a positive self-adjoint operator with domain $\mathrm{ran}(\mathcal{T})$, where $\mathcal{T}$ is the integral operator defined by the kernel $K_{\mathcal{T}}$ in (4.19). The eigenvalues $\lambda_n(\mathcal{T}^{-1})$, $n = 1,2,\dots,$ of the self-adjoint extension $\mathcal{T}^{-1}$ of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ satisfy

$$\pi n \leqslant \lambda_n(\mathcal{T}^{-1}), \quad n \geqslant 1, \qquad \lambda_n(\mathcal{T}^{-1}) \leqslant \pi(4n-2), \quad n \geqslant 2, \tag{4.22}$$

where the eigenvalues are indexed according to increasing magnitude.

Theorem 4.4. *Let $\tilde{\mathcal{K}}$ be the integral operator on the Hilbert space $\mathfrak{L}_2([-1,1])$ defined by the displacement kernel*

$$\tilde{k}(x-\xi) = \log|x-\xi| + h(x-\xi), \tag{4.23}$$

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where h is real, even and twice differentiable with square integrable second derivative. Then, the operator

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\mathcal{K}} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{K} + \tilde{\mathcal{H}}$$

extends to a self-adjoint operator with a discrete spectrum of eigenvalues that satisfy

$$\pi n - \|\tilde{\mathcal{H}} - \gamma\mathcal{I}\|_{\mathfrak{L}_2} + \gamma \leqslant \lambda_n\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\mathcal{K}}\right) \leqslant \pi(4n-2) + \|\tilde{\mathcal{H}} - \gamma\mathcal{I}\|_{\mathfrak{L}_2} + \gamma, \tag{4.24}$$

with $\gamma = \inf_{f,\|f\|=1}\langle\tilde{\mathcal{H}}f, f\rangle_{\mathfrak{L}_2}$, $n \geqslant 1$ for the first inequality, and $n \geqslant 2$ for the second inequality.

Proof. We use Corollary A 2 with $\mathcal{A}$ the self-adjoint extension of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ and $\mathcal{D}$ the bounded self-adjoint operator $\tilde{\mathcal{H}} - \gamma\mathcal{I}$, where $\tilde{\mathcal{H}}$ is the integral operator generated by the symmetric kernel $\mathrm{d}^2 h(x-\xi)/\mathrm{d}x^2$. Then, $\mathcal{A}+\mathcal{D}$ has a compact inverse and its eigenvalues $\lambda_n(\mathcal{A}+\mathcal{D})$ satisfy

$$\lambda_n(\mathcal{A}) - \|\mathcal{D}\|_{\mathfrak{L}_2} \leqslant \lambda_n(\mathcal{A}+\mathcal{D}) \leqslant \lambda_n(\mathcal{A}) + \|\mathcal{D}\|_{\mathfrak{L}_2}. \tag{4.25}$$

Moreover, we conclude that $(\mathrm{d}^2/\mathrm{d}x^2)\tilde{\mathcal{K}}$ extends to the self-adjoint operator $\mathcal{A}+\mathcal{D}+\gamma\mathcal{I}$ with a discrete spectrum of eigenvalues that satisfy

$$\lambda_n\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\mathcal{K}}\right) = \lambda_n(\mathcal{A}+\mathcal{D}) + \gamma. \tag{4.26}$$

From (4.22), (4.25) and (4.26) it follows that these eigenvalues satisfy (4.24). □

Theorem 4.4 can be applied to the operator $\mathcal{Z}$ in (1.1) with the integral kernel G replaced by its real part. To this end we first specify the regular part of G by decomposing it as $G_{\mathrm{reg}} = G_1 + G_2 + G_3$, where

$$\begin{aligned} G_1(x) &= \frac{1}{\pi k\ell}\int_0^2 \frac{1}{\sqrt{x^2+\beta^2y^2}}\,\mathrm{d}y + \frac{1}{\pi k\ell\beta}\log|x| \\ &= \frac{1}{\pi k\ell\beta}\log(2\beta + \sqrt{4\beta^2+x^2}), \end{aligned} \tag{4.27}$$

$$\begin{aligned} G_2(x) &= \frac{1}{\pi k\ell}\int_0^2 \frac{\exp(\mathrm{i}k\ell\sqrt{x^2+\beta^2y^2})-1}{\sqrt{x^2+\beta^2y^2}}\,\mathrm{d}y \\ &= \frac{1}{\pi k\ell}\sum_{n=0}^{\infty}\frac{(\mathrm{i}k\ell)^{n+1}}{(n+1)!}Q_n(x), \end{aligned} \tag{4.28}$$

$$Q_n(x) = \int_{y=0}^2 (x^2+\beta^2y^2)^{n/2}\,\mathrm{d}y \tag{4.29}$$

and

$$\begin{aligned} G_3(x) &= -\frac{1}{2\pi k\ell}\int_0^2 \frac{y\exp(\mathrm{i}k\ell\sqrt{x^2+\beta^2y^2})}{\sqrt{x^2+\beta^2y^2}}\,\mathrm{d}y \\ &= -\frac{1}{2\pi\mathrm{i}k^2\ell^2\beta^2}[\exp(\mathrm{i}k\ell\sqrt{x^2+\beta^2y^2}) - \exp(\mathrm{i}k\ell|x|)]. \end{aligned} \tag{4.30}$$

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For decompositions of the thin-wire kernel we refer the reader to [**4**], where Taylor expansions as in G_2 are employed to arrive at the aforementioned kernel decomposition $F_1(z)\log|z| + F_2(z)$. Next, we write the action of $\mathcal{Z}$ as

$$\mathcal{Z}w = -\frac{\mathrm{i}Z_0}{2\pi k\ell}\biggl(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{K}w - \pi k\ell\beta\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{G}_{\mathrm{reg}}w - \pi k^3\ell^3\beta\mathcal{G}w\biggr), \tag{4.31}$$

where $\mathcal{G}_{\mathrm{reg}}$ is the integral operator induced by the kernel G_{reg}. The second derivative of G_1 is square integrable. By termwise differentiation of the series expansion of G_2, it can be shown straightforwardly that the second derivative of G_2 is also square integrable and has a logarithmic singularity. Decomposing G_3 as

$$G_3(x) = -\frac{1}{2\pi\mathrm{i}k^2\ell^2\beta^2}(-\mathrm{i}k\ell|x| + g_3(x)), \tag{4.32}$$

we observe that g_3 is twice continuously differentiable. Moreover, the composition of the second derivative and the integral operator induced by the displacement kernel $|x|$ is equal to twice the identity operator. The action of $\mathcal{Z}$ can thus be written as

$$\frac{1}{Z_1}(\mathcal{Z}w)(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathcal{K}w)(x) - \frac{1}{\beta}w(x) + \int_{-1}^{1}\frac{\mathrm{d}^2}{\mathrm{d}x^2}h(x-\xi)w(\xi)\,\mathrm{d}\xi, \tag{4.33}$$

where $Z_1 = -\mathrm{i}Z_0/2\pi k\ell$ and

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2} = -\pi k\ell\beta\biggl(\frac{\mathrm{d}^2G_1}{\mathrm{d}x^2} + \frac{\mathrm{d}^2G_2}{\mathrm{d}x^2} - \frac{1}{2\pi\mathrm{i}k^2\ell^2\beta^2}\frac{\mathrm{d}^2g_3}{\mathrm{d}x^2}\biggr) - \pi k^3\ell^3\beta G. \tag{4.34}$$

For the second derivative of G_2 and the evaluation of Q_n we refer the reader to Appendix B.

Let $\tilde{\mathcal{Z}}$ be the operator $\mathcal{Z}$ with the kernel G replaced by its real part or, equivalently, with h in (4.33) replaced by $\operatorname{Re} h$. Applying Theorem 4.4 to the kernel $\tilde{k} = \log|\cdot| + \operatorname{Re} h$, we obtain

$$\pi n - \|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2} - \frac{1}{\beta} \leqslant \lambda_n\biggl(\frac{1}{Z_1}\tilde{\mathcal{Z}}\biggr) \leqslant \pi(4n-2) + \|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2} - \frac{1}{\beta}, \tag{4.35}$$

for the same values of n as in Theorem 4.4, where the operator $\tilde{\mathcal{H}}$ is generated by the kernel $\operatorname{Re}(\mathrm{d}^2h/\mathrm{d}x^2)$ and where we employed $\|\tilde{\mathcal{H}} - \gamma\mathcal{I}\|_{\mathfrak{L}_2} \pm \gamma \leqslant \|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2}$ (Theorem 4.4). In our numerical results we replace $\|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2}$ by its upper bound,

$$\|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2} \leqslant \int_{-2}^{2}\biggl|\operatorname{Re}\frac{\mathrm{d}^2h}{\mathrm{d}x^2}(\xi)\biggr|\,\mathrm{d}\xi. \tag{4.36}$$

5. Numerical results

To validate the theorems derived in the previous section, we compute the eigenvalues of the operators $\mathcal{K}$ and $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ by employing a projection method. For a specified set of independent functions that belong to $\mathfrak{W}$, we compute the matrix $G^{-1}Z$, where G is

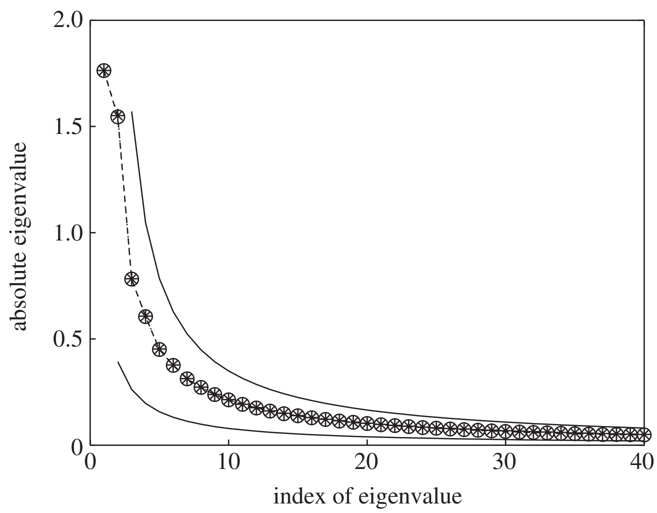

Figure 1. Absolute eigenvalues of $\mathcal{K}$ obtained with piecewise linear splines ($\circ$, $N = 40$) and with the Fourier basis ($*$, $N = 20$). The bounds (4.2) are depicted by solid lines.

the Gram matrix of the set of functions with respect to the classical inner product in $\mathfrak{L}_2([-1,1])$ and Z are the matrices of inner products generated by $\langle \cdot, \mathcal{K}\cdot \rangle_{\mathfrak{L}_2}$ and

$$\left\langle \cdot, \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{K}\cdot \right\rangle_{\mathfrak{L}_2}.$$

On $\mathfrak{W}$, the second inner product can be rewritten as

$$-\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\cdot, \mathcal{K}\frac{\mathrm{d}}{\mathrm{d}x}\cdot \right\rangle_{\mathfrak{L}_2}.$$

We define two sets of functions in $\mathfrak{W}$. The first one is the Fourier basis $\cos(\frac{1}{2}(2n-1)\pi x)$, $\sin n\pi x$, where $n = 1, 2, \dots, N$. The second one is a set of uniformly distributed, piecewise linear splines,

$$\Lambda_n(x) = \Lambda\left(\frac{x - x_n}{\Delta}\right), \tag{5.1}$$

where $\Lambda(x) = (1 - |x|)1_{[1,1]}(x)$, $\Delta = 2/(N+1)$, $x_n = -1 + n\Delta$ and $n = 1, 2, \dots, N$. We calculate the matrix Z by rewriting its entries as the inner product of the kernel and the convolution of the two basis functions. Next we calculate the contribution of the logarithmic part of the integrand analytically and we compute the contribution of the regular part by a composite Simpson rule [**1**, §§ 3.3, 3.4]. For the Fourier basis the Gram matrix G is the identity, and for the splines it is a tridiagonal matrix with $\frac{2}{3}\Delta$ on its diagonal and $\frac{1}{6}\Delta$ on its two codiagonals.

Figure 1 shows the absolute eigenvalues of $\mathcal{K}$ computed with both sets of functions together with the upper and lower bounds of Corollary 4.1. Similarly, Figure 2 shows the eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ together with the upper and lower bounds of Theorem 4.3. For

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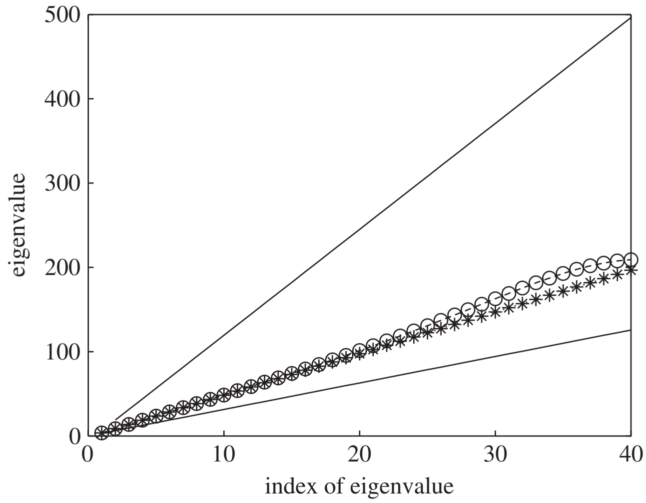

Figure 2. Eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ obtained with piecewise linear splines ($\circ$, $N = 40$) and the Fourier basis ($*$, $N = 20$). The bounds (4.22) are depicted by solid lines.

both results we employed $N = 20$ for the Fourier basis and $N = 40$ for the piecewise linear splines. We may clearly observe that the (absolute) computed eigenvalues satisfy the derived bounds for both operators. Moreover, for the operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$, we observe that the eigenvalues obtained with the two bases start to deviate for eigenvalue indices $n \gtrsim 20$. In this respect, we demonstrated in [**1**, § 5.2] that if the eigenvalues of a dipole are generated by a set of P uniformly distributed linear splines and by the first P functions in the Fourier basis, the first $\lfloor \frac{1}{2}P \rfloor$ eigenvalues match.

Next we consider the operator $\mathcal{Z}$ for the current on a strip. First we choose the frequency such that the dipole is half a wavelength long, $2\ell = \frac{1}{2}\lambda$, and that it is narrow with respect to the wavelength, $\beta = \frac{1}{50}$. Figure 3 shows the real part of the eigenvalues of the operator $\mathcal{Z}/Z_1$, the eigenvalues of $\mathcal{Z}/Z_1$ with the kernel G replaced by its real part (i.e. $\tilde{\mathcal{Z}}/Z_1$), the eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K} - \mathcal{I}/\beta$, and the upper and lower bounds obtained from (4.35) and (4.36). Note that in this numerical example $\|\tilde{\mathcal{H}}\|_{\mathfrak{L}_2} \leqslant 48.5$. For all three operators, the eigenvalues are computed by the Fourier basis with $N = 20$. We observe that the real parts of the eigenvalues of $\mathcal{Z}/Z_1$ and the eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part are the same. We also observe that for $n \gtrsim 20$ the eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K} - \mathcal{I}/\beta$ match the real parts of the eigenvalues of $\mathcal{Z}/Z_1$. The first observation demonstrates that the real parts of the eigenvalues are determined by the real part of the integral kernel and suggests that a similar conclusion is valid for the imaginary parts. The second observation is explained by the boundedness of the integral operator with kernel $\mathrm{d}^2h/\mathrm{d}x^2$ in (4.33) and of its real counterpart $\tilde{\mathcal{H}}$ with kernel $\mathrm{Re}(\mathrm{d}^2h/\mathrm{d}x^2)$. These explanations suggest that the imaginary parts of the eigenvalues of $\mathcal{Z}/Z_1$ are only significant for the lower eigenvalues and that the eigenvalues of $\mathcal{Z}/Z_1$ with complex kernel G also satisfy the bounds in Figure 3, which is confirmed by numerical results. Physically, this observation indicates that only a limited number

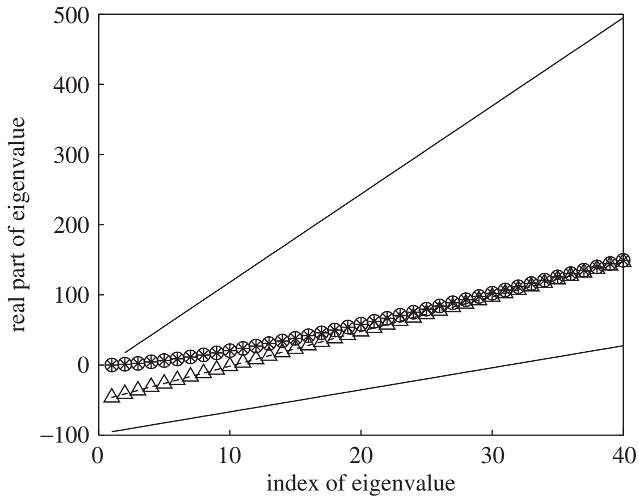

Figure 3. For a dipole of half a wavelength: real part of the eigenvalues of $\mathcal{Z}/Z_1$ ($*$), eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part ($\circ$), and eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}-\mathcal{I}/\beta$ ($\triangle$), computed by the Fourier basis ($N=20$). The bounds given by (4.35), combined with (4.36), are depicted by solid lines. Parameter values: $2\ell = \frac{1}{2}\lambda$, $\beta = \frac{1}{50}$.

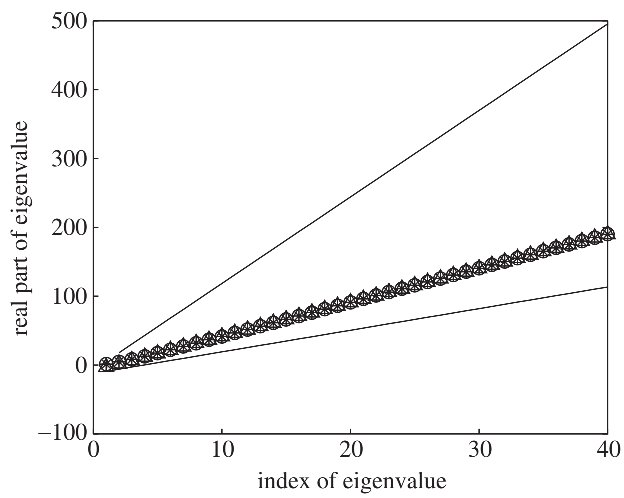

Figure 4. For a dipole of one-fifteenth wavelength: real part of the eigenvalues of $\mathcal{Z}/Z_1$ ($*$), eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part ($\circ$), and eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}-\mathcal{I}/\beta$ ($\triangle$), computed by the Fourier basis ($N=20$). The bounds given by (4.35), combined with (4.36), are depicted by solid lines. Parameter values: $2\ell = \frac{1}{15}\lambda$, $\beta = \frac{3}{20}$.

of eigenfunctions are radiative, since the real parts of the eigenvalues of $\mathcal{Z}/Z_1$, or the imaginary parts of the eigenvalues of $\mathcal{Z}$, correspond to the reactive energy of the eigenfunctions of the dipole, while the imaginary parts correspond to the radiated energy of these eigenfunctions.

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As a second example we consider a much shorter dipole, $2\ell = \frac{1}{15}\lambda$, with the same width by which $\beta = \frac{3}{20}$. Analogously to Figure 3, Figure 4 shows the three curves of eigenvalues and the upper and lower bounds. Note that in this numerical example $\|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} \leqslant 5.7$. The real parts of the eigenvalues $\mathcal{Z}/Z_1$ are not only the same as the eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part, but also approximately the same as the eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K} - \mathcal{I}/\beta$, except for significant deviations in the smallest eigenvalues. The imaginary parts of the eigenvalues of $\mathcal{Z}/Z_1$ are a factor 10^3 or more smaller than their real parts, which means physically that all eigenfunctions are reactive. Based on these observations we may consider to compute the smallest eigenvalues of $\mathcal{Z}/Z_1$ from Rayleigh–Ritz quotients applied to the first few eigenfunctions of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ and to approximate the other eigenvalues by the eigenvalues of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K} - \mathcal{I}/\beta$. This approach facilitates a rapid eigenvalue computation, since the eigenvalues and eigenfunctions of $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ do not depend on the geometrical parameters.

6. Conclusion

In our investigation of the spectral behaviour of the integro-differential operator that governs the time-harmonic current on a linear strip or wire, we came across the problem of deriving explicit bounds for the ordered sequence of eigenvalues of the composition of the second-order differentiation operator and the integral operator with logarithmic displacement kernel. To tackle this problem, we used methods of an earlier work by Reade, who employed the Weyl–Courant minimax principle and explicit properties of the Chebyshev polynomials of the first and second kind. By his methods, Reade was able to derive explicit index-dependent bounds for the ordered sequence of eigenvalues of the integral operator $\mathcal{K}$ with logarithmic displacement kernel. In this paper, we modified and extended Reade's result to integral operators with kernels described by arbitrary expansions of $T_m(x)T_m(\xi)$ and $U_m(x)U_m(\xi)$. In particular, we showed that the upper and lower bounds $\pi/(n-1)$ and $\pi/4n$ derived by Reade for the absolute values of the eigenvalues λ_n of $\mathcal{K}$ $(n = 1, 2, \dots)$ are valid for $n \geqslant 4$ and $n \geqslant 2$, respectively. Furthermore, for the integro-differential operator $(\mathrm{d}^2/\mathrm{d}x^2)\mathcal{K}$ we proved that its eigenvalues are bounded from below by πn for $n \geqslant 1$ and from above by $\pi(4n-2)$ for $n \geqslant 2$. We extended this result to kernels that are the sum of the logarithmic displacement kernel and a real displacement kernel whose second derivative is square integrable. Subsequently, we applied this extension to the integro-differential operator corresponding to a linear strip, where we replaced the complex integral kernel by its real part. For this operator we found lower and upper bounds expressed in terms of the bounds πn and $\pi(4n-2)$, a uniform shift, and the norm of the integral operator corresponding to the regular part of the kernel. Numerically, we showed how well the eigenvalues of the considered operators, computed by Ritz's methods, fit the analytically derived bounds. Although our analysis does not provide bounds for the complex kernel corresponding to a linear strip, the absolute values of the computed eigenvalues satisfy the bounds derived for the real part of the kernel. Moreover, for the larger eigenvalue indices, the computed eigenvalues for the complex kernel match those computed for the real kernel.

Appendix A. Additional theorems

Lemma A 1. *Let $\mathcal{A}$ be an invertible positive self-adjoint operator and let the operator $\mathcal{B}$ be such that $\mathcal{B} \geqslant \mathcal{A}$. Then, $\mathcal{B}$ is invertible and $\mathcal{B}^{-1} \leqslant \mathcal{A}^{-1}$.*

Proof. By the Spectral Theorem $A^{1/2}$ exists and is positive and invertible. Define the positive self-adjoint operator $\mathcal{P}$ by

$$\mathcal{P} = \mathcal{A}^{-1/2}(\mathcal{B} - \mathcal{A})\mathcal{A}^{-1/2}.$$

Then,

$$\mathcal{B} = \mathcal{A}^{1/2}(\mathcal{I} + \mathcal{P})\mathcal{A}^{1/2}.$$

It follows from the Spectral Theorem for unbounded self-adjoint operators [**31**] that $\mathcal{I}+\mathcal{P}$ is invertible and $(\mathcal{I} + \mathcal{P})^{-1} \leqslant \mathcal{I}$. We derive

$$\mathcal{B}^{-1} = \mathcal{A}^{-1/2}(\mathcal{I} + \mathcal{P})^{-1}\mathcal{A}^{-1/2} \leqslant \mathcal{A}^{-1/2}\mathcal{I}\mathcal{A}^{-1/2} = \mathcal{A}^{-1}, \tag{A 1}$$

where the inequality follows from $(\mathcal{I} + \mathcal{P})^{-1} \leqslant \mathcal{I}$ and $\mathcal{A}^{-1/2}$ being self-adjoint. □

Corollary A 2. *Let $\mathcal{A}$ be a positive self-adjoint operator with compact inverse and let $\mathcal{D}$ be a bounded self-adjoint operator such that $\mathcal{A} + \mathcal{D}$ is invertible (with bounded inverse). Then $\mathcal{A} + \mathcal{D}$ has a compact inverse and its eigenvalues $\lambda_n(\mathcal{A} + \mathcal{D})$ satisfy*

$$\lambda_n(\mathcal{A}) - \|\mathcal{D}\| \leqslant \lambda_n(\mathcal{A} + \mathcal{D}) \leqslant \lambda_n(\mathcal{A}) + \|\mathcal{D}\|. \tag{A 2}$$

Proof. Since $\mathcal{A} + \mathcal{D} = \mathcal{A}(\mathcal{I} + \mathcal{A}^{-1}\mathcal{D})$ and since $\mathcal{A}+\mathcal{D}$ and $\mathcal{A}$ are invertible, $\mathcal{I} + \mathcal{A}^{-1}\mathcal{D}$ has a bounded inverse. Then, since the operator $(\mathcal{A} + \mathcal{D})^{-1}$ is the product of the bounded operator $(\mathcal{I} + \mathcal{A}^{-1}\mathcal{D})^{-1}$ and the compact operator $\mathcal{A}^{-1}$, it is compact. If $\mathrm{dom}(\mathcal{A})$ is the domain of definition of $\mathcal{A}$, then $\mathcal{A} + \mathcal{D}$ is self-adjoint on $\mathrm{dom}(\mathcal{A})$. By the Cauchy–Schwarz inequality we have $|\langle \mathcal{D}f, f\rangle| \leqslant \|\mathcal{D}\|\langle f, f\rangle$, and thus $\pm\mathcal{D} \leqslant \|\mathcal{D}\|\mathcal{I}$. Consequently, $\mathcal{A} \leqslant \mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I} \leqslant \mathcal{A} + 2\|\mathcal{D}\|\mathcal{I}$. From Corollary A 1 it follows that

$$(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1} \leqslant (\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1} \leqslant \mathcal{A}^{-1}. \tag{A 3}$$

Since $(\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1}$ and $(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1}$ are positive and compact, it follows by Corollary 2.3 that

$$\lambda_n((\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1}) \leqslant \lambda_n((\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1}) \leqslant \lambda_n(\mathcal{A}^{-1}). \tag{A 4}$$

Thus, $\lambda_n(\mathcal{A}) \leqslant \lambda_n(\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I}) \leqslant \lambda_n(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})$, from which (A 2) follows. □

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Appendix B. Derivatives of G_2

The second derivative of G_2 is given by

$$\begin{aligned}\frac{\mathrm{d}^2 G_2}{\mathrm{d}x^2} = \frac{k\ell}{2\pi\beta}\bigg[\log|x| + 1 - \log(2\beta + \sqrt{4\beta^2 + x^2}) - \frac{x^2}{\sqrt{4\beta^2 + x^2}(2\beta + \sqrt{4\beta^2 + x^2})}\bigg] \\ + \frac{k^3\ell^3}{8\pi\beta}x^2[-\log|x| + \log(2\beta + \sqrt{4\beta^2 + x^2})] \\ + \frac{1}{\pi k\ell}\sum_{n=0}^{\infty}\frac{(\mathrm{i}k\ell)^{n+3}}{(n+3)(n+1)!}\bigg(1 - \frac{k^2\ell^2}{n+5}x^2\bigg)Q_n(x). \qquad \text{(B 1)}\end{aligned}$$

The function $Q_n(x)$ given by (4.29) can be evaluated by the recurrence relation

$$Q_n(x) = \frac{2}{n+1}(x^2 + 4\beta^2)^{n/2} + \frac{n}{n+1}x^2 Q_{n-2}(x) \qquad \text{(B 2)}$$

with initial conditions $Q_0(x) = 2$ and

$$Q_1(x) = \sqrt{x^2 + 4\beta^2} + \frac{x^2}{2\beta}[-\log|x| + \log(2\beta + \sqrt{4\beta^2 + x^2})]. \qquad \text{(B 3)}$$

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