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Non-realizability of the pure braid group as area-preserving homeomorphisms

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Abstract. Let $\mathrm{Homeo}_+(D^2_n)$ be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. The Nielsen realization problem for the braid group asks whether the natural projection $p_n : \mathrm{Homeo}_+(D^2_n) \to B_n := \pi_0(\mathrm{Homeo}_+(D^2_n))$ has a section over subgroups of B_n. All of the previous methods use either torsion or Thurston stability, which do not apply to the pure braid group PB_n, the subgroup of B_n that fixes n marked points pointwise. In this paper, we show that the pure braid group has no realization inside the area-preserving homeomorphisms using rotation numbers.

Key words: group actions, low-dimensional dynamics, topological dynamics
2020 Mathematics Subject Classification: 57K20 (Primary); 37E30, 37C25 (Secondary)

1. *Introduction*

Denote by D^2 the two-dimensional disk. Let $\mathrm{Homeo}_+(D^2_n)$ be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. Denote $B_n := \pi_0(\mathrm{Homeo}_+(D^2_n))$. The Nielsen realization problem for B_n asks whether the natural projection

$$p_n : \mathrm{Homeo}_+(D^2_n) \to B_n$$

has a section over subgroups of B_n. For the whole group B_n, this question has several previous results. Salter and Tshishiku [**12**] used Thurston stability to show that B_n has no realization in $\mathrm{Diff}_+(D^2_n)$ and the author [**1**] used 'hidden torsion' and Markovic's machinery [**9**] to show that B_n has no realization in $\mathrm{Homeo}_+(D^2_n)$. Let $PB_n < B_n$ be the subgroup that preserves n marked points pointwise. The Nielsen realization problem for PB_n is widely open since the two methods in [**12**] and [**1**] fail to work and have no hope to repair. The following question is asked by [**8**, Question 3.12] and [**12**, Remark 1.4].

PROBLEM 1.1. (Realization of pure braid group) *Does PB_n have realization as diffeomorphisms or homeomorphisms? In other words, does p_n have sections over PB_n?*

Denote by $\mathrm{Homeo}^a_+(D^2_n)$ the group of orientation-preserving, area-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. In this paper, we make progress by proving the following result.

THEOREM 1.2. *The pure braid group cannot be realized as area-preserving homeomorphisms on D^2_n for $n \geq 9$. In other words, the natural projection $p^a_n : \mathrm{Homeo}^a_+(D^2_n) \to B_n$ has no sections over PB_n.*

We remark that the Nielsen realization problem is closely related to the existence of flat structures on a surface bundle. We refer the reader to [**8**] for more history and background.

Comparing with the method in [**2**], the novelty of this paper is to provide a different proof towards the final contradiction of the result in [**2**]. The original contradiction is to use the fact that a certain Dehn twist is a product of commutators in its centralizer. However, such structure does not hold in PB_n. Instead, we prove a stronger dynamical property about Dehn twists about non-separating curves. In the beginning of §4, we present an outline of the proof. Since this paper has a lot of overlap with [**2**], we omit or sketch many proofs to reduce redundancy.

This paper is organized as follows.

- In §2, we discuss rotation numbers.
- In §3, we discuss the pure braid group and the minimal decomposition theory.
- In §4, we give an outline of the proof and finish the argument.

2. *Rotation numbers of annulus homeomorphisms*

In this section, we discuss the properties of rotation numbers on annuli.

2.1. *Rotation number of an area-preserving homeomorphism of an annulus.* Firstly, we define the rotation number for geometric annuli. Let

$$N = N(r) = \left\{ w \in \mathbb{C} : \frac{1}{r} < |w| < r \right\}$$

be the geometric annulus in the complex plane $\mathbb{C}$. Denote the geometric strip in $\mathbb{C}$ by

$$P = P(r) = \left\{ x + iy = z \in \mathbb{C} : |y| < \frac{\log r}{2\pi} \right\}.$$

The map $\pi(z) = e^{2\pi i z}$ is a holomorphic covering map $\pi : P \to N$. The deck transformation on P is $T(x, y) = (x + 1, y)$.

Denote by $p_1 : P \to \mathbb{R}$ the projection to the x-coordinate, and by $\mathrm{Homeo}_+(N)$ the group of homeomorphisms of N that preserves orientation and the two ends. Fix $f \in \mathrm{Homeo}_+(N)$ and $x \in N$, and let $\widetilde{x} \in P$ and $\widetilde{f} \in \mathrm{Homeo}_+(P)$ denote lifts of x and f, respectively. We define the translation number of the lift $\widetilde{f}$ at $\widetilde{x}$ by

$$\rho(\widetilde{f}, \widetilde{x}, P) = \lim_{n\to\infty} (p_1(\widetilde{f}^n(\widetilde{x})) - p_1(\widetilde{x}))/n. \tag{1}$$

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The rotation number of f at x is then defined as

$$\rho(f, x, N) = \rho(\widetilde{f}, \widetilde{x}, P) \pmod 1. \tag{2}$$

The rotation number is not defined everywhere (see, e.g., [**4**] for more background on rotation numbers). The closed annulus N_c is

$$N_c = \left\{\omega \in \mathbb{C} : \frac{1}{r} \leq |\omega| \leq r\right\},$$

For $f \in \mathrm{Homeo}_+(N_c)$, the rotation and translation numbers are defined analogously.

Let A be an open annulus embedded in a Riemann surface (in particular this endows A with the complex structure). By the Riemann mapping theorem [**7**, Ch. 3.2], there is a unique r such that there is a biholomorphic map $u_A : A \to N(r) =: N$. For any $f \in \mathrm{Homeo}_+(A)$ (the group of end-preserving homeomorphisms), we define the rotation number of f on A by

$$\rho(f, x, A) := \rho(g, u_A(x), N),$$

where $g = u_A \circ f \circ u_A^{-1}$.

We have the following theorems of Poincaré–Birkhoff and Handel about rotation numbers [**6**] (See also Franks [**4**].)

THEOREM 2.1. (Properties of rotation numbers) *If $f : N_c \to N_c$ is an orientation-preserving, boundary component preserving, area-preserving homeomorphism and $\widetilde{f} : P_c \to P_c$ is any lift, then:*

- *(Handel) the translation set*

$$R(\widetilde{f}) = \bigcup_{\widetilde{x} \in P_c} \rho(\widetilde{f}, \widetilde{x}, P_c)$$

  is a closed interval;
- *(Poincaré–Birkhoff) if $r \in R(\widetilde{f})$ is rational, then there exists a periodic orbit of f realizing the rotation number r mod 1.*

2.2. *Separators and their properties.* We let A continue to denote an open annulus embedded in a Riemann surface. Then A has two ends and we choose one of them to be the left end and the other one to be the right end. We call a subset $X \subset A$ *separating* (or essential) if every arc $\gamma \subset A$ which connects the two ends of A must intersect X.

Definition 2.2. (Separator) We call a subset $M \subset A$ a *separator* if M is compact, connected and separating.

The complement of M in A is a disjoint union of open sets. We have the following lemma.

LEMMA 2.3. *Let M be a separator. Then there are exactly two connected components $A_L(M)$ and $A_R(M)$ of $A - M$ which are open annuli homotopic to A and with the property that $A_L(M)$ contains the left end of A and $A_R(M)$ contains the right end of A. All other components of $A - M$ are simply connected.*

Proof. We compactify the annulus A by adding points p_L and p_R to the corresponding ends of A. The compactifications is a two-sphere S^2. Moreover, M is a compact and connected subset of $S^2 - \{p_L, p_R\}$.

Now we observe that every component of $S^2 - M$ is simply connected. Denote by Ω_L and Ω_R the connected components of $S^2 - M$ containing p_L and p_R, respectively. Since M is separating, we conclude that these are two different components. We define $A_L(M) = \Omega_L - p_L$ and $A_R(M) = \Omega_R - p_R$. It is easy to verify that these are required annuli. □

We now prove another property of a separator. Let $\pi : \widetilde{A} \to A$ be the universal cover.

PROPOSITION 2.4. *Let $M \subset A$ be a compact domain with smooth boundary. Then $\pi^{-1}(M)$ is connected; i.e., M is a separator.*

Proof. Since M is a compact domain with boundary which separates the two ends of A, we can find a circle $\gamma \subset M$ which is essential in A (i.e., γ is a separator itself) (note that M has only finitely many boundary components). Denote by T the deck transformation of $\widetilde{A}$. Thus, the lift $\pi^{-1}(\gamma)$ is a T-invariant, connected subset of $\widetilde{A}$. Let C be the component of $\pi^{-1}(M)$ which contains $\pi^{-1}(\gamma)$. Then C is T invariant. We show that $\pi^{-1}(M) = C$.

Let $p \in M$. Since M is a compact domain with smooth boundary, we can find an embedded closed arc $\alpha \subset M$ which connects p and γ. Let $\widetilde{p}$ be a lift of p and let $\widetilde{\alpha}$ be the corresponding lift of α such that $\widetilde{p}$ is one of its end points. Then the other end point of $\widetilde{\alpha}$ is in $\pi^{-1}(\gamma)$ and this shows that $\widetilde{p} \in C$. This concludes the proof. □

Now we discuss an ordering on the set of separators.

PROPOSITION 2.5. *Suppose that $M_1, M_2 \subset A$ are two disjoint separators. Then either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$. Moreover, $M_1 \subset A_L(M_2)$ implies that $M_2 \subset A_R(M_1)$.*

Proof. Since M_1 is connected, it follows that M_1 is a subset of a connected component C of $A - M_2$. Since C is open, we know that there is a neighborhood N_1 of M_1 with smooth boundary such that $N_1 \subset C$ (It is elementary to construct such N_1.) If C is simply connected, the cover $\pi^{-1}(C) \to C$ is a trivial cover. Let $\widetilde{C}$ be a connected component of $\pi^{-1}(C)$. By Proposition 2.4, the set $\pi^{-1}(N_1)$ is connected, so it is contained in a single connected component of $\pi^{-1}(C)$. However, this contradicts the fact that $\pi^{-1}(N_1)$ is also translation invariant. Thus, either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$.

Suppose that $M_1 \subset A_L(M_2)$. Then $A_L(M_1) \subset A_L(M_2)$ as well. On the other hand, by the first part of the proposition, we already know that either $M_2 \subset A_L(M_1)$ or $M_2 \subset A_R(M_1)$. If $M_2 \subset A_L(M_1)$, then $A_L(M_2) \subset A_L(M_1)$. This shows that $A_L(M_1) \subset A_L(M_2)$, which implies that $M_2 \subset A_L(M_2)$. This is absurd, so we must have $M_2 \subset A_R(M_1)$. □

Definition 2.6. The inclusion $M_1 \subset A_L(M_2)$ is denoted as $M_1 < M_2$.

2.3. *The rotation interval of an annular continuum and prime ends.* Let $K \subset A$ be a separator (in the literature, also known as an *essential continuum*). We call K an essential *annular continuum* if $A - K$ has exactly two components. Observe that an essential annular continuum can be expressed as a decreasing intersection of essential closed topological annuli in A.

It is possible to turn any separator $M \subset A$ into an essential annular continuum. Let M be a separating connected set. By Lemma 2.3, we know that $A - M$ has exactly two connected annular components $A_L(M)$ and $A_R(M)$, and all other components of $A - M$ are simply connected. We call a simply connected component of $A - M$ a *bubble component*. Then the *annular completion* $K(M)$ of M is defined as the union of M and the corresponding bubble components of $A - M$.

PROPOSITION 2.7. *Let $M \subset A$ be a separator. Then the annular completion $K(M)$ is an annular continuum.*

Proof. We can again compactify A by adding the points p_L and p_R, one at each end. The compactification is the two-sphere S^2. Then $A_L(M)$ and $A_R(M)$ are two disjoint open disks in S^2, and $K(M) = S^2 - (A_L(M) \cup A_R(M))$. But the complement of two disjoint open disks in S^2 is connected. This proves the proposition. □

Now let f be a homeomorphism of A that leaves an annular continuum K invariant. If μ is an invariant Borel probability measure supported on K, we define the μ-rotation number

$$\sigma(f, \mu) = \int_A \phi \, d\mu,$$

where $\phi : A \to \mathbb{R}$ is the function which lifts to the function $p_1 \circ f - p_1$ on $\widetilde{A}$ (recall that $p_1 : \widetilde{A} \to \mathbb{R}$ is the projection onto the first coordinate).

The set of f invariant Borel probability measures on K is a non-empty, convex and compact set (with respect to the weak topology on the space of measures), which is denoted by $M(K)$. We define the *rotation interval* of K

$$\sigma(f, K) = \{\sigma(f, \mu) | \mu \in M(K)\},$$

which is a non-empty segment $[\alpha, \beta]$ of $\mathbb{R}$. The interval is non-empty because there exists at least one f invariant measure, and it is an interval because the set of f invariant measures is convex.

The following is a classical result of Franks and Le Calvez [**5**, Corollary 3.1].

PROPOSITION 2.8. *If $\sigma(f, K) = \{\alpha\}$, the sequence*

$$\frac{p_1 \circ f^n(x) - p_1(x)}{n}$$

converges uniformly for $x \in \pi^{-1}(K)$ to the constant function α.

We remark that this implies that points in K all have the rotation number α.

The following theorem of Franks and Le Calvez [**5**, Proposition 5.4] is a generalization of the Poincaré–Birkhoff theorem.

THEOREM 2.9. *If f is area-preserving and K is an annular continuum, then every rational number in $\sigma(f, K)$ is realized by a periodic point in K.*

The theory of prime ends is an important tool in the study of two-dimensional dynamics which can be used to transform a two-dimensional problem into a one-dimensional problem. Recall that we assume that A is an open annulus embedded in a Riemann surface S. Suppose that f is a homeomorphism of S which leaves A invariant. Furthermore, let $K \subset A$ be an annular continuum and suppose that f leaves K invariant. Then both $A_L(K)$ and $A_R(K)$ are f invariant.

Since A is embedded in S, we can define the frontiers of A, $A_L(K)$ and $A_R(K)$. By Carathéodory's theory of prime ends (see, e.g., [**11**, Ch. 15]), the homeomorphism f yields an action on the frontiers of $A_L(K)$ and $A_R(K)$. Consider the right-hand frontier of $A_L(K)$ (the one which is contained in A). Then the set of prime ends on this frontier is homeomorphic to the circle, and we denote by f_L the induced homeomorphism of this circle. Likewise, the set of prime ends on the left-hand frontier of $A_R(K)$ is homeomorphic to the circle, and we denote by f_R the induced homeomorphism of this circle.

The rotation number of a circle homeomorphism (defined by equation (2)) is well defined everywhere and is the same number for any point on the circle. The rotation numbers of f_L and f_R are called r_L and r_R. We refer to them as the left and right prime end rotation numbers of f. We have the following theorem of Matsumoto [**10**].

THEOREM 2.10. (Matsumoto's theorem) *If K is an annular continuum, then its left and right prime end rotation numbers r_L, r_R belong to the rotation interval $\sigma(f, K)$.*

3. *Minimal decompositions and characteristic annuli*

3.1. *Minimal decompositions.* We recall the theory of minimal decompositions of surface homeomorphisms. This is established in [**9**]. Firstly we recall the definition of the upper semi-continuous decomposition of a surface and the minimal decomposition theory; see also Markovic [**9**, Definition 2.1]. Let M be a surface.

Definition 3.1. (Upper semi-continuous decomposition) Let $\mathbf{S}$ be a collection of closed, compact, connected subsets of M. We say that $\mathbf{S}$ is an upper semi-continuous decomposition of M if the following holds.

- If $S_1, S_2 \in \mathbf{S}$, then $S_1 \cap S_2 = \emptyset$.
- If $S \in \mathbf{S}$, then S does not separate M; i.e., $M - S$ is connected.
- We have $M = \bigcup_{S \in \mathbf{S}} S$.
- If $S_n \in \mathbf{S}$, $n \in \mathbb{N}$ is a sequence that has the Hausdorff limit equal to S_0, then there exists $S \in \mathbf{S}$ such that $S_0 \subset S$.

Now we define acyclic sets on a surface.

Definition 3.2. (Acyclic sets) Let $S \subset M$ be a closed, connected subset of M which does not separate M. We say that S is *acyclic* if there is a simply connected open set $U \subset M$ such that $S \subset U$ and $U - S$ is homeomorphic to an annulus.

The simplest examples of acyclic sets are points, embedded closed arcs and embedded closed disks in M. Let $S \subset M$ be a closed, connected set that does not separate M. Then S is acyclic if and only if there is a lift of S to the universal cover $\widetilde{M}$ of M, which is a compact subset of $\widetilde{M}$. The following theorem is a classical result called Moore's theorem; see, e.g., [**9**, Theorem 2.1].

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THEOREM 3.3. (Moore's theorem) *Let M be a surface and* $\mathbf{S}$ *be an upper semi-continuous decomposition of M so that every element of* $\mathbf{S}$ *is acyclic. Then there is a continuous map $\phi : M \to M$ that is homotopic to the identity map on M and such that for every $p \in M$, we have $\phi^{-1}(p) \in \mathbf{S}$. Moreover,* $\mathbf{S} = \{\phi^{-1}(p) | p \in M\}$.

We call the map $M \to M/\sim$ the *Moore map*, where $x \sim y$ if and only if $x, y \in S$ for some $S \in \mathbf{S}$. The following definition appears in [**9**, Definition 3.1]

Definition 3.4. (Admissible decomposition) Let $\mathbf{S}$ be an upper semi-continuous decomposition of M. Let G be a subgroup of $\text{Homeo}(M)$. We say that $\mathbf{S}$ is *admissible* for the group G if the following holds.

- Each $f \in G$ preserves setwise every element of $\mathbf{S}$.
- Let $S \in \mathbf{S}$. Then every point in every frontier component of the surface $M - S$ is a limit of points from $M - S$ which belong to acyclic elements of $\mathbf{S}$.

If G is a cyclic group generated by a homeomorphism $f : M \to M$, we say that $\mathbf{S}$ is an admissible decomposition of f.

An admissible decomposition for $G < \text{Homeo}(M)$ is called *minimal* if it is contained in every admissible decomposition for G. We have the following theorem [**9**, Theorem 3.1].

THEOREM 3.5. (Existence of minimal decompositions) *Every group $G < \text{Homeo}(M)$ has a unique minimal decomposition.*

Denote by $\mathbf{A}(G)$ the subcollection of acyclic sets from $\mathbf{S}(G)$. By a mild abuse of notation, we occasionally refer to $\mathbf{A}(G)$ as a subset of M (the union of all sets from $\mathbf{A}(G)$). To distinguish the two notions, we do as follows. When we refer to $\mathbf{A}(G)$ as a collection, then we consider it as the collection of acyclic sets. When we refer to as a set (or a subsurface of M), we have in mind the other meaning.

We have the following result [**9**, Proposition 2.1].

PROPOSITION 3.6. *Every connected component of* $\mathbf{A}(G)$ *(as a subset of S_g) is a subsurface of M with finitely many ends.*

LEMMA 3.7. *For $H < G < \text{Homeo}(M)$, we have that* $\mathbf{A}(G) \subset \mathbf{A}(H)$.

Proof. The inclusion $\mathbf{A}(G) \subset \mathbf{A}(H)$ is because the minimal decomposition of G is also an admissible decomposition of H and the minimal decomposition of H is finer than that of G. □

3.2. *Lifting through hyperelliptic branched cover.* Denote by $S_{g;n,b}$ the surface of genus g with b boundary components and n marked points. To make the analysis easier, we take the following hyperelliptic $\mathbb{Z}/2$ branched cover:

$$\pi_n : S = S_{(n-1)/2;n,1} \to S_{0;n,1} \text{ for } n \text{ odd or } \pi_n : S = S_{(n/2)-1;n,2} \to S_{0;n} \text{ for } n \text{ even.}$$

The cover is shown by Figures 1 and 2. The hyperelliptic involution on S is denoted by τ.

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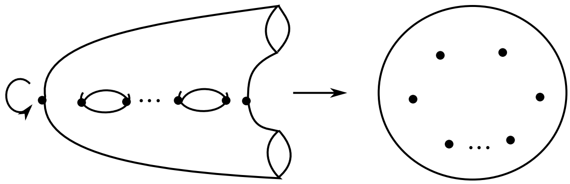

FIGURE 1. n even.

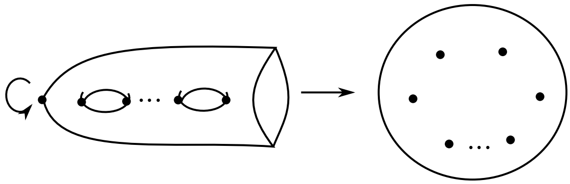

FIGURE 2. n odd.

Denote by $\widetilde{PB}_n$ the lifts of mapping classes under π_n, where they satisfy the following exact sequence:

$$1 \to \mathbb{Z}/2 \to \widetilde{PB}_n \xrightarrow{L} PB_n \to 1.$$

Let c be a simple closed curve on $S_{0;n,1}$ and denote by T_c the Dehn twist about c. For every simple closed curve c on $S_{0;n,1}$, we have the following easy fact about its preimage under π_n.

Fact 3.8.

(1) If c bounds an odd number of points, then the lift is a single curve c'. The preimages of T_c^2 under L are $T_{c'}$ and $T_{c'}\tau$.

(2) If c bounds an even number of points, then the lift is two curves c_1, c_2. The preimages of T_c under L are $T_{c_1}T_{c_2}$ and $T_{c_1}T_{c_2}\tau$. In particular, if c bounds 2 points, then $c_1 = c_2$.

From the above fact, we know that if c bounds two points and $c_1 = c_2$ are the lifts, we have that $T_{c_1}^2 \in \widetilde{PB}_n$. We have the following fact.

Fact 3.9. If α is a non-separating simple closed curve that is invariant under τ, then a square of the Dehn twist about c is in $\widetilde{PB}_n$. We call such element an *invariant Dehn twist square*.

Let b be the curve in D_n^2 bounding five points $P_1, \ldots, P_5$. The lift of b under the cover π_n is a curve c bounding a genus-two subsurface as Figure 3.

If a curve α is on the genus-two subsurface of S that is cut out by c, then we call the invariant Dehn twist square about α a *left invariant Dehn twist square*. We have the following important relation in $\widetilde{PB}_n$.

PROPOSITION 3.10. *The element $T_c \in \widetilde{PB}_n$ is a product of left invariant Dehn twist squares in $\widetilde{PB}_n$.*

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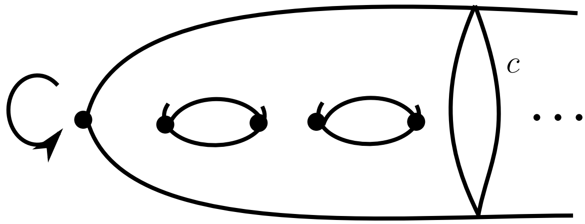

FIGURE 3. The curve c bounding a genus-two surface is the lift of a curve bounding five points.

Proof. We have the basic fact that PB_n is generated by Dehn twists about curves in the interior of b bounding two points; see, e.g., [**3**, Ch. 9]. Take a lift of all of the elements; we obtain a product of squares of Dehn twists about non-separating curves that are disjoint from c and on the left of c in $\widetilde{PB}_n$. After taking a square of the equation, we obtain the proposition. □

3.3. *Characteristic annuli.* From now on, we work with the assumption that there exists a realization of the pure braid group

$$\mathcal{E}' : PB_n \to \mathrm{Homeo}^a_+(D^2_n).$$

Lifting by the hyperelliptic involution, we obtain a new realization

$$\mathcal{E} : \widetilde{PB}_n \to \mathrm{Homeo}^a_+(S_g)^\tau,$$

where the image lies in the centralizer of the hyperelliptic involution τ. We now only work with the new realization $\mathcal{E}$.

For an element $f \in \widetilde{PB}_n$ or a subgroup $F < \widetilde{PB}_n$, we shorten $\mathbf{A}(\mathcal{E}(f))$ to $\mathbf{A}(f)$, and $\mathbf{A}(\mathcal{E}(F))$ to $\mathbf{A}(F)$, denoting the corresponding collections of acyclic components. Denote by S the hyperelliptic cover we defined in §3.2. Recall that $c \subset S$ is a separating curve that is invariant under τ and divides S into subsurfaces S_L of genus two and $S_R = S - S_L$ (see more about c in the previous section). We know that $T_c \in \widetilde{PB}_n$. We have the following theorem about the minimal decomposition of $\mathcal{E}(T_c)$.

THEOREM 3.11. *The set* $\mathbf{A}(T_c)$ *has a component* $\mathbf{L}(c)$ *which is homotopic to* S_L *and a component* $\mathbf{R}(c)$ *homotopic to* S_R.

Proof sketch. The proof is the same as the proof of [**2**, Theorem 4.1]. We use the fact that there are pseudo-Anosov elements on the left and on the right of c in $\widetilde{PB}_n$. In this theorem, we need $n \geq 9$. □

For the rest of paper, we write

$$\mathbf{B} := S - \mathbf{L}(c) - \mathbf{R}(c).$$

Let $p_L : \mathbf{L}(c) \to \mathbf{L}(c)/\sim$ and $p_R : \mathbf{R}(c) \to \mathbf{R}(c)/\sim$ be the Moore maps of $\mathbf{L}(c)$ and $\mathbf{R}(c)$ corresponding to the decomposition $\mathbf{S}(c)$. Let $\mathbf{L} \subset \mathbf{L}(c)/\sim$ be an open annulus bounded by the end of $\mathbf{L}(c)'$ on one side, and by a simple closed curve on the other. The open annulus $\mathbf{R} \subset \mathbf{R}(c)/\sim$ is defined similarly. We have the following definition (see [**9**, Ch. 5]).

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Definition 3.12. An annulus of the form $A = p_L^{-1}(\mathbf{L}) \cup \mathbf{B} \cup p_R^{-1}(\mathbf{R})$ is called a *characteristic annulus*.

Denote $f = \mathcal{E}(T_c)$. Every characteristic annulus is invariant under f. We observe that $\mathbf{B}$ is a separator in A, that is, $\mathbf{B}$ is an essential, compact and connected subset of A. Note that a characteristic annulus A is invariant under f, but it may not be invariant under homeomorphisms which are lifts (with respect to $\mathcal{E}$) of other elements from $\widetilde{PB}_n$. However, $\mathbf{B}$ is invariant under these lifts of elements from the image under $\mathcal{E}$ of the centralizer of T_c in $\widetilde{PB}_n$. As we see from the next lemma, the dynamical information about f is contained in $\mathbf{B}$.

LEMMA 3.13. *Fix a characteristic annulus A. Then:*
(1) *every number $0 < r < 1$ appears as the rotation number $\rho(f, x, A)$ for some $x \in A$;*
(2) *if $0 < \rho(f, x, A) < 1$, then $x \in B$.*

The proof of the above lemma can be seen in [**2**, Lemma 4.5], which is a result of two facts. One is that f is homotopic to a Dehn twist and the other is that the realization is area-preserving.

4. *The proof of Theorem 1.2*

In this section, we give the proof of Theorem 1.2. We first discuss the main strategy.

4.1. *Outline of the proof.* Recall that c is a separating simple closed curve that divides the surface S (the hyperelliptic cover of $S_{0;1,n}$) into a genus-two subsurface and its complement. Fix a characteristic annulus A of T_c. Let E_r be the set of points in A that have rotation numbers equal to r under $\mathcal{E}(T_c)$. Lemma 3.13 states that the set E_r is not empty when $0 < r < 1$.

The key observation of the proof lies in the analysis of connected components of E_r. Let E be a component of E_r. We show the following results:
(1) E is $\mathcal{E}(h)$-invariant for h a left invariant Dehn twist square;
(2) $\overline{E}$ is a separator in A;
(3) if E contains a periodic orbit, then E contains a separator.

Denote by $K(\overline{E})$ the annular completion of $\overline{E}$, and let $\rho(\mathcal{E}(T_c), K(\overline{E}))$ be the rotation interval of $K(\overline{E})$. We claim that $\rho(\mathcal{E}(T_c), K(\overline{E})) = \{r\}$. First of all, we know that $r \in \rho(\mathcal{E}(T_c), K(\overline{E}))$. If $\rho(\mathcal{E}(T_c), K(\overline{E})) \neq \{r\}$, then $\rho(\mathcal{E}(T_c), K(\overline{E}))$ contains infinitely many rational numbers. By Theorem 2.9, there exist three periodic points $x_1, x_2, x_3 \in K(\overline{E})$ with different rational rotation numbers r_1, r_2, r_3. Let F_i denote the connected component of E_{r_i} containing r_i, and let $M_i \subset F_i$ be a separator.

By Proposition 2.5, there is an ordering on disjoint separators. Without loss of generality, we assume that $M_1 < M_2 < M_3$. Based on a discussion about the position of E with respect to the M_i, we obtain a contradiction. Thus, $\rho(\mathcal{E}(T_c), K(E))$ is the singleton $\{r\}$. We know from Theorem 2.10 that the left and right prime end rotation numbers of $K(\overline{E})$ are both r. In the group of circle homeomorphisms, the centralizer of an irrational rotation is essentially SO(2).

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We then show a new ingredient of the proof: the rotation numbers of the realization of a left invariant Dehn twist square on the set of prime ends of $K(\overline{E})$ are all zero. This contradicts the fact that T_c is a product left invariant Dehn twist square as in Proposition 3.10.

4.2. *The set E_r.* Once again we use abbreviation $f = \mathcal{E}(T_c)$. For a characteristic annulus A, we let

$$E_r = \{x \in A : \rho\big(f, x, A\big) = r\}.$$

By Lemma 3.13, if $0 < r < 1$, we know that E_r is non-empty and $E_r \subset \mathbf{B}$.

Next, we have the following key lemmas, which correspond to [**2**, Lemmas 5.1, 5.3 and 5.4].

LEMMA 4.1. *Fix $0 < r < 1$ and let E denote a connected component of E_r. Fix a left invariant Dehn twist square h in $\widetilde{PB}_n$. For $x \in E$, let $C(x) \in \mathbf{A}(h)$ be the corresponding acyclic set. Then $C(x) \subset E$. In particular, E is $\mathcal{E}(C(T_c))$-invariant.*

LEMMA 4.2. *The closed set $\overline{E}$ is a separator (as defined in §2).*

LEMMA 4.3. *Let x be a periodic orbit of f such that $\rho(f, x, A) = p/q$ and $0 < p/q < 1$. Then the connected component E of $E_{p/q}$, which contains x, also contains a separator (as a subset).*

Fix an irrational number $r \in (0, 1)$. By Lemma 3.13, we know that E_r is not empty. Let E be a connected component of E_r. By Lemma 4.1, we know that E is invariant under $\mathcal{E}(C(T_c))$. By Lemma 4.2, we know that $\overline{E}$ is a separator. The annular completion $K(\overline{E})$ of $\overline{E}$ is also $\mathcal{E}(C(T_c))$-invariant since the definition is canonical. The following claim is at the heart of the entire construction.

CLAIM 4.4. *Let r_L and r_R be the left and right prime end rotation numbers of f on $K(\overline{E})$. Then $r_L = r_R = r$.*

Remark. We refer the reader to [**2**, Claim 5.2] for the proof. The only property we use about $\widetilde{PB}_n$ is Proposition 3.10.

4.3. *Finishing the proof.* We need to show a new property of a left invariant Dehn twist square $h \in \widetilde{PB}_n$.

CLAIM 4.5. *The action of $\mathcal{E}(T_b^2)$ on the set of prime ends of $K(\overline{E})$ has rotation number zero.*

Proof. Now we consider the rotation set of $\mathcal{E}(T_b^2)$ on $K(\overline{E})$. We claim that the rotation set satisfies

$$\sigma(\mathcal{E}(T_b), K(\overline{E})) = \{0\}.$$

Since $\overline{E} \subset B$, and the fact that $S - B$ is a union of two open subsurfaces, we know that $K(\overline{E}) \subset B$. This means that for every point $x \in K(\overline{E}) \subset B$, there exists $C(x) \in \mathbf{A}(T_b^2)$ such that $C(x) \subset B$ by Lemma 4.1. However, $C(x)$ is acyclic and fixed by $\mathcal{E}(T_b^2)$. Therefore, we know that the rotation number of $\mathcal{E}(T_b^2)$ on points in $C(x)$ is zero. By

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Theorem 2.10, we know that the rotation number of the action of $\mathcal{E}(T_b^2)$ on the set of prime ends is also zero. □

We now finish the proof.

Proof. Since the rotation number of $\mathcal{E}(T_c)$ on the prime ends of $K(\overline{E})$ is an irrational number r, it is semiconjugate to an irrational rotation. Then, up to the same semiconjugacy, the image of the centralizer of T_c under $\mathcal{E}$ is SO(2). The image of each element is determined by its rotation number. However, $\mathcal{E}(T_c)$ is a product of $\mathcal{E}(T_b^2)$ for b non-separating and invariant under τ by Proposition 3.10. By Lemma 4.5, we know that the rotation number of $\mathcal{E}(T_b^2)$ is zero. Thus, their product should also have zero rotation number. This contradicts the fact that the rotation number of $\mathcal{E}(T_c)$ is r, which is non-zero. □

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