Autocorrelation functions for quantum particles in supersymmetric Pöschl-Teller potentials

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Abstract. We consider autocorrelation functions for supersymmetric quantum mechanical systems (consisting of a fermion and a boson) confined in trigonometric Pöschl–Teller partner potentials. We study the limit of rescaled autocorrelation functions (at random time) as the localization of the initial state goes to infinity. The limiting distribution can be described using pairs of Jacobi theta functions on a suitably defined homogeneous space, as a corollary of the work of Cellarosi and Marklof. A construction by Contreras-Astorga and Fernández provides large classes of Pöschl-Teller partner potentials to which our analysis applies.

1 Introduction

1.1 Supersymmetric Quantum Mechanics

Supersymmetric (SUSY) quantum mechanics is the study of a pair of Hamiltonians (in units where $\hbar$ = mass = 1)

$$(1.1) \qquad H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \qquad H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x)$$

that are intertwined by a differential operator A and its adjoint $A^\dagger$ as

$$(1.2) \qquad H_0 A = A H_1, \qquad H_1 A^\dagger = A^\dagger H_0.$$

The term "supersymmetric" is motivated as follows. If we define the operator matrices

$$(1.3) \qquad \boldsymbol{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \qquad \boldsymbol{Q}^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{H}_{\mathrm{ss}} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & A A^\dagger \end{pmatrix},$$

$$(1.4) \qquad \boldsymbol{Q}_1 = \frac{\boldsymbol{Q}^\dagger + \boldsymbol{Q}}{\sqrt{2}}, \qquad \boldsymbol{Q}_2 = \frac{\boldsymbol{Q}^\dagger - \boldsymbol{Q}}{i\sqrt{2}},$$

then we have the commutator and anticommutator relations of the *supersimmetry algebra with N = 2 generators*

(1.5)
$$[\boldsymbol{Q}_1, \boldsymbol{H}_{ss}] = \boldsymbol{0}, \quad [\boldsymbol{Q}_2, \boldsymbol{H}_{ss}] = \boldsymbol{0}, \quad \{\boldsymbol{Q}_1, \boldsymbol{Q}_1\} = \{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = \boldsymbol{H}_{ss}, \quad \{\boldsymbol{Q}_1, \boldsymbol{Q}_2\} = \boldsymbol{0},$$

where $[X, Y] = XY - YX$ and $\{X, Y\} = XY + YX$ denote the commutator and the anticommutator of X and Y, respectively. The algebra (1.5) corresponds to the simplest supersymmetric quantum system (see [9], as well as [2, §5]). The operators $\boldsymbol{Q}, \boldsymbol{Q}^\dagger$ are called *supercharges* or *supersymmetry generators*, and the Hamiltonians H_0, H_1 (as well as the potentials V_0, V_1) are called *supersymmetric partners*.

Given a general 1-dimensional Hamiltonian H_0 whose eigenfunctions and eigenvalues are known, there is an intertwining method due to Sukumar (see [5] for an historical account, tracing back to the work of Dirac) to construct various partners H_1 using various differential operators $A, A^\dagger$. The advantage of the intertwining relations (1.2) is to generate eigenvalues and eigenfunctions of H_1 from those of H_0. In general, if A is a differential operator of order k, then the spectra of H_0 and H_1 differ by at most k values. Moreover, if we denote by

(1.6)
$$\boldsymbol{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}$$

the *physical Hamiltonian*, then the *supersymmetric Hamiltonian* $\boldsymbol{H}_{ss}$ in (1.3) and (1.5) can be expressed as a polynomial of degree k in $\boldsymbol{H}$. It is therefore enough to study the time-independent Schrödinger equation

(1.7)
$$\boldsymbol{H}\boldsymbol{\psi} = E\boldsymbol{\psi},$$

where $\boldsymbol{\psi}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_0(x) \end{pmatrix}$. From the physical point of view, SUSY predicts that each particle of the Standard Model has a partner particle with a spin that differs by a half unit. Therefore, we can think of $\boldsymbol{H}$ as describing the joint action of H_0 on fermions (particles with half-integer spin) and of H_1 on bosons (particles with integer spin).

1.2 Autocorrelation Functions

The time-dependent Schrödinger equation corresponding to (1.7) is

(1.8)
$$i\frac{\partial}{\partial t}\boldsymbol{\Psi} = \boldsymbol{H}\boldsymbol{\Psi},$$

where $\boldsymbol{\Psi}(x, t) = \begin{pmatrix} \Psi_1(x, t) \\ \Psi_0(x, t) \end{pmatrix}$. We consider the 1-dimensional case in which $x \in I$, where $I \subseteq \mathbb{R}$ denotes an interval. Moreover, we restrict our analysis to the class of physical Hamiltonians $\boldsymbol{H}$ acting on $L^2(I) \oplus L^2(I)$ with purely discrete spectrum. This means that there exists an orthonormal basis of $L^2(I) \oplus L^2(I)$ consisting of eigenfunctions of $\boldsymbol{H}$, say $\boldsymbol{\psi}_n(x) = \begin{pmatrix} \psi_{1,n}(x) \\ \psi_{0,n}(x) \end{pmatrix}$ with eigenvalues E_n, $n \geq 0$. In this case, the solution of (1.8) with initial condition $\boldsymbol{\Psi}(x, 0)$ is given in terms of the evolution

operator $U(t) = e^{-iHt}$ as

(1.9)
$$\Psi(x,t) = U(t)\Psi(x,0) = \sum_{n=0}^{\infty} c_n e^{-iE_n t}\psi_n(x),$$

where

(1.10)
$$c_n = \begin{pmatrix} c_{1,n} \\ c_{0,n} \end{pmatrix}, \quad c_{\ell,n} = \int_I \Psi_\ell(x,0), \overline{\psi_{\ell,n}(x)}\,dx, \quad \ell = 0,1.$$

The *autocorrelation function* for the initial state $\Psi(x,0)$ is the function $A : \mathbb{R} \to \mathbb{C}^2$,

(1.11)
$$A(t) = \int_I \Psi(x,0)\overline{\Psi(x,t)}\,dx = \begin{pmatrix} \sum\limits_{n\geq 0} |c_{1,n}|^2 e^{iE_n t} \\ \sum\limits_{n\geq 0} |c_{0,n}|^2 e^{iE_n t} \end{pmatrix}.$$

We further restrict our attention to the case of autocorrelation functions $A(t)$ as in (1.11) where the coefficients $c_{\ell,n}$ have a special form, namely,

(1.12)
$$|c_{\ell,n}|^2 = \frac{1}{N} f_\ell\left(\frac{n}{N}\right),$$

for some nonnegative functions f_1, f_0 supported on $\mathbb{R}_{\geq 0}$ and some $N \geq 1$. This is a natural choice, which appears in several interesting cases. Define

(1.13)
$$A_N(f_1, f_0; t) = \begin{pmatrix} \frac{1}{N} \sum\limits_{n\geq 0} f_1\left(\frac{n}{N}\right) e^{iE_n t} \\ \frac{1}{N} \sum\limits_{n\geq 0} f_0\left(\frac{n}{N}\right) e^{iE_n t} \end{pmatrix}.$$

We will consider the *large N* regime, which physically represents the case of initial states $\Psi(x,0)$ that are highly localized in space. We will focus on the autocorrelation functions for a class of Hamiltonians H as in (1.6), where the spectra of H_0 and of H_1 differ by at most finitely many values. For random t, we will treat $A_N(f_1, f_0; t)$ as a random variable at the scale $N^{-1/2}$. In this case, as $N \to \infty$, the contribution of finitely many eigenvalues to $A_N(f_1, f_0, t)$ is $O(N^{-1})$ and hence negligible in our analysis. Therefore, without loss of generality, we can assume that H_0 and H_1 have the same spectrum, and thus, $\mathrm{sp}(H) = \mathrm{sp}(H_0) = \mathrm{sp}(H_1)$.

1.3 Trigonometric Pöschl–Teller Potentials

Let us discuss the case when H_0 is the Hamiltonian corresponding to the so-called *trigonometric Pöschl–Teller* potential with parameters (α, β), i.e.,

(1.14)
$$V_0(x) = \frac{(\alpha-1)\alpha}{2\sin^2(x)} + \frac{(\beta-1)\beta}{2\cos^2(x)}, \quad \alpha, \beta > 1, \quad 0 \leq x \leq \frac{\pi}{2}.$$

The potential V_0 can be interpreted as an infinite well, confining the particle to the 1-dimensional box $[0, \frac{\pi}{2}]$, with "soft walls", where the the parameters α and β represent the strength of the reflection of the particle off the two walls. Denote $\gamma = \alpha + \beta$. The
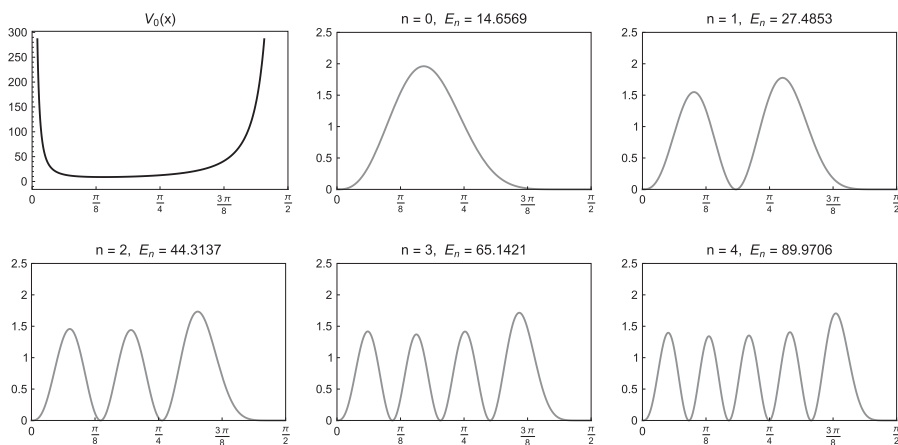
Figure 1: The potential $V_0(x)$ for $\alpha = \sqrt{2}$ and $\beta = 4$ and the probability densities $x \mapsto |\psi_{0,n}(x)|^2$ on $\left[0, \frac{\pi}{2}\right]$ for $0 \le n \le 4$.

eigenvalues of H_0 on $\mathrm{L}^2\left(\left[0, \frac{\pi}{2}\right]\right)$ are

$$(1.15) \qquad E_n = \frac{1}{2}(2n + \gamma)^2, \qquad n \ge 0.$$

The corresponding normalized eigenfunctions are

$$(1.16) \qquad \psi_{0,n}(x) = \sqrt{\frac{2(2n+\gamma)\Gamma(n+\gamma)(\alpha + \frac{1}{2})_n}{n!\,\Gamma(\alpha + \frac{1}{2})\Gamma(\beta + \frac{1}{2})(\beta + \frac{1}{2})_n}}\, \sin^\alpha(x)\cos^\beta(x)$$
$$\times\ {}_2F_1\!\left(-n, n+\gamma; \alpha + \tfrac{1}{2}; \sin^2(x)\right),$$

where ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}$ is the hypergeometric function, and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol; see [3, 4].

Using various differential operators $A, A^\dagger$ as in (1.2), Contreras-Astorga and Fernández [3] were able to construct various families of partner potentials V_1 such that H_0 and H_1 are isospectral. We discuss two such families in Section 4.

1.4   The Main Theorem

Let $H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0$ be the Hamiltonian with with Pöschl-Teller potential (1.14) with parameters (α, β). Let $\gamma = \alpha + \beta$. Let H_1 be any supersymmetric partner of H_0 such that $\mathrm{sp}(H_1)$ and $\mathrm{sp}(H_0)$ differ by at most finitely many eigenvalues. Fix two compactly supported, Riemann integrable functions $f_0, f_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}$. Let λ be a probability measure on $\mathbb{R}$, absolutely continuous with respect to the Lebesgue measure. Denote by ρ the density of λ, *i.e.*, $\mathrm{d}\lambda(t) = \rho(t)\mathrm{d}t$. Rescaled autocorrelation functions are viewed as random variables, *i.e.*, as $\mathbb{C}^2$-valued measurable functions of $t \in (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

More precisely, consider the random variables

(1.17)
$$X_N : \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda\big) \longrightarrow \big(\mathbb{C}^2, \mathscr{B}(\mathbb{C}^2)\big), \quad X_N(t) = \sqrt{N} A_N(f_0, f_1; t), \quad N \geq 1.$$

Throughout the paper, $\mathscr{B}(\cdot)$ denotes the Borel σ-algebra. Our main result is the following theorem.

Theorem 1.1   *Assume $\gamma \notin \mathbb{Q}$. Then the random variables X_N have a limiting distribution as $N \to \infty$. That is, there exists a (non-degenerate) random variable X on $\mathbb{C}^2$ such that X_N converge in law to X as $N \to \infty$. Moreover, the law of X (which does not depend on ρ nor γ) can be realized as the push forward onto $\mathbb{C}^2$ of the Haar measure on a homogeneous space $\Gamma\backslash G$ via an explicit function $\Gamma\backslash G \to \mathbb{C}^2$.*

Remark 1.2   In this case, $\Gamma < G$ is a lattice in the Lie group G, and $\Gamma\backslash G$ has finite volume. We describe $\Gamma\backslash G$, its normalized Haar measure, and the function X on $\Gamma\backslash G$ explicitly in Section 2. The proof of Theorem 1.1 is provided in Section 3.

In particular, we show that the law of X is not the product of two measures on $\mathbb{C}$. This means that the two random variables

(1.18)
$$\frac{1}{\sqrt{N}} \sum_{n \geq 0} f_0\Big(\frac{n}{N}\Big) e^{iE_n t} \quad \text{and} \quad \frac{1}{\sqrt{N}} \sum_{n \geq 0} f_1\Big(\frac{n}{N}\Big) e^{iE_n t}$$

do not become independent as $N \to \infty$. This can be interpreted physically as follows. The probability distribution of the autocorrelation function (at a random time) of an individual particle is altered if we condition on the event that the autocorrelation function of the partner particle lies in a certain set in $\mathbb{C}$, at least for highly localized particles. This means, in particular, that if we observe—at a random time t—a localized quantum particle undergoing a quantum revival (*i.e.,* we observe a large value of its correlation function), then the probability that the partner particle (which we may have no access to) is also undergoing a quantum revival at time t is not the same as the unconditional probability of the same event; see Remark 3.2.

As an illustration, in Figure 2, we plot the real and imaginary parts of both components of $X_N(t)$ when f_1 and f_0 are indicator functions for different values of N.

2 The Limiting Random Variable X

In this section, we describe explicitly the random variable X featured in Theorem 1.1. We refer the reader to [1, §2] for more details.

2.1 The Universal Jacobi Group G

Let $\mathfrak{H} := \{w \in \mathbb{C} : \mathfrak{I}(w) > 0\}$ denote the upper half plane. The group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathfrak{H}$ by Möbius transformations $z \mapsto gz := \frac{az+b}{cz+d}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Every
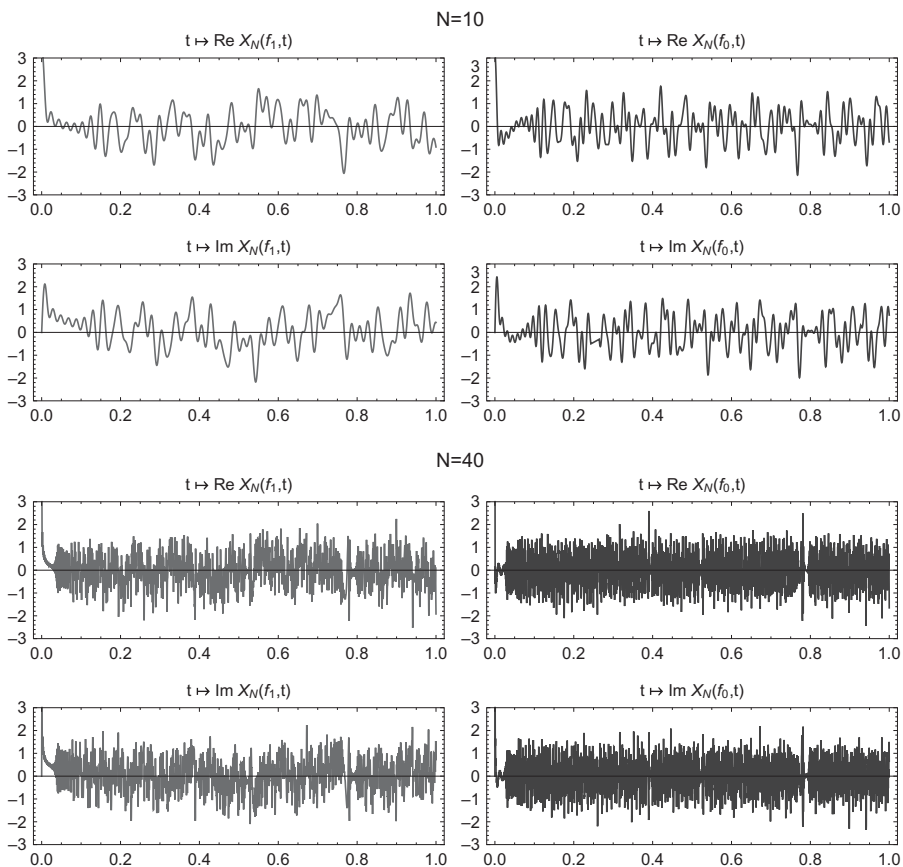
Figure 2: The real and imaginary parts of the two components of the rescaled autocorrelation function $t \mapsto \boldsymbol{X}_N(t) = (X_N(f_1, t), X_N(f_0, t))$ for $f_1 = \boldsymbol{1}_{[0,1]}$ (left panels) and $f_0 = \boldsymbol{1}_{[1/3, 4/3]}$ (right panels) and $N = 10$ (top four panels) and $N = 40$ (bottom four panels). In all the panels, $\alpha = \sqrt{2}$ and $\beta = 3$.

$g \in \mathrm{SL}(2, \mathbb{R})$ can be written uniquely via Iwasawa decomposition as

$$(2.1) \qquad g = n_x a_y k_\phi,$$

where

$$(2.2) \qquad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \quad k_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

and $z = x + iy \in \mathfrak{H}$, $\phi \in [0, 2\pi)$. Set $\mathrm{e}(t) := e^{2\pi i t}$ and $\varepsilon_g(z) = (cz + d)/|cz + d|$. The universal cover of $\mathrm{SL}(2, \mathbb{R})$ is defined as

$$(2.3) \qquad \widetilde{\mathrm{SL}}(2, \mathbb{R}) := \big\{ [g, \beta_g] : g \in \mathrm{SL}(2, \mathbb{R}),\ \beta_g \text{ a continuous function on } \mathfrak{H}$$
$$\text{such that } \mathrm{e}(i\beta_g(z)) = \varepsilon_g(z) \big\},$$

and has the group structure given by

$$(2.4) \qquad [g, \beta_g^1][h, \beta_h^2] = [gh, \beta_{gh}^3], \qquad\qquad \beta_{gh}^3(z) = \beta_g^1(hz) + \beta_h^2(z),$$

$$(2.5) \qquad [g, \beta_g]^{-1} = [g^{-1}, \beta'_{g^{-1}}], \qquad\qquad \beta'_{g^{-1}}(z) = -\beta_g(g^{-1}z).$$

The group $\widetilde{SL}(2, \mathbb{R})$ is identified with $\mathfrak{H} \times \mathbb{R}$ via $[g, \beta_g] \mapsto (z, \phi) = (gi, \beta_g(i))$ and acts on $\mathfrak{H} \times \mathbb{R}$ via

$$(2.6) \qquad [g, \beta_g](z, \phi) = (gz, \phi + \beta_g(z)).$$

The Iwasawa decomposition (2.1) of $SL(2, \mathbb{R})$ extends to a decomposition of $\widetilde{SL}(2, \mathbb{R})$: for every $\widetilde{g} = [g, \beta_g] \in \widetilde{SL}(2, \mathbb{R})$, we have

$$(2.7) \qquad \tilde{g} = [g, \beta_g] = \widetilde{n}_x \tilde{a}_y \widetilde{k}_\phi = [n_x, 0][a_y, 0][k_\phi, \beta_{k_\phi}].$$

Let ω be the standard symplectic form on $\mathbb{R}^2$, $\omega(\xi, \xi') = xy' - yx'$, where $\xi = \begin{pmatrix} x \\ y \end{pmatrix}, \xi' = \begin{pmatrix} x' \\ y' \end{pmatrix}$. The Heisenberg group $\mathbb{H}(\mathbb{R})$ is defined as $\mathbb{R}^2 \times \mathbb{R}$ with the multiplication law

$$(2.8) \qquad (\xi, t)(\xi', t') = \left(\xi + \xi', t + t' + \tfrac{1}{2}\omega(\xi, \xi')\right).$$

We consider universal Jacobi group

$$(2.9) \qquad G = \widetilde{SL}(2, \mathbb{R}) \ltimes \mathbb{H}(\mathbb{R}),$$

having the multiplication law

$$(2.10) \quad \left([g, \beta_g]; \xi, \zeta\right)\left([g', \beta'_{g'}]; \xi', \zeta'\right) = \left([gg', \beta''_{gg'}]; \xi + g\xi', \zeta + \zeta' + \tfrac{1}{2}\omega(\xi, g\xi')\right),$$

where $\beta''_{gg'}(z) = \beta_g(g'z) + \beta'_{g'}(z)$. The Haar measure on G is given in coordinates $(x + iy, \phi; \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \zeta)$ by

$$(2.11) \qquad \mathrm{d}\mu(g) = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\phi\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\zeta}{y^2}.$$

2.2 Hermite Expansion

Let H_k be the k-th Hermite polynomial

$$(2.12) \qquad H_k(t) = (-1)^k e^{t^2} \frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{-t^2} = k! \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m (2t)^{k-2m}}{m!(k-2m)!}.$$

We define the Hermite functions

$$(2.13) \qquad \psi_k(t) = (2\pi)^{\frac{1}{4}} h_k(\sqrt{2\pi}t) = (2^{k-\frac{1}{2}}k!)^{-1/2} H_k(\sqrt{2\pi}\,t)e^{-\pi t^2},$$

which form an orthonormal basis for $L^2(\mathbb{R}, dx)$. Let $f \in L^2(\mathbb{R})$. Following [6], we set

$$(2.14) \qquad f_\phi(t) = \sum_{k=0}^{\infty} \widehat{f}(k) e^{-i(2k+1)\phi/2} \psi_k(t),$$

where $\widehat{f}(k) = \langle f, \psi_k \rangle_{L^2(\mathbb{R})}$. For an equivalent definition of f_ϕ, see [1, §2.3]. The space of functions $f : \mathbb{R} \to \mathbb{R}$ for which f_ϕ has a prescribed decay at infinity, uniformly in ϕ, is denoted by $\mathcal{S}_\eta(\mathbb{R})$. More precisely,

$$(2.15) \qquad \mathcal{S}_\eta(\mathbb{R}) := \left\{ f : \mathbb{R} \longrightarrow \mathbb{R} : \sup_{t,\phi} |f_\phi(t)| (1 + |t|)^\eta < \infty \right\};$$

see *e.g.,* [8].

2.3   Jacobi Theta Functions on *G*

For $g = (z, \phi; \boldsymbol{\xi}, \zeta) \in G$ and $f \in \mathcal{S}_\eta(\mathbb{R})$, $\eta > 1$, define the *Jacobi theta function* as

$$(2.16) \quad \Theta_f(z, \phi; \boldsymbol{\xi}, \zeta) = y^{1/4} e\left(\zeta - \tfrac{1}{2}\xi_1\xi_2\right) \sum_{n \in \mathbb{Z}} f_\phi\left((n - \xi_2) y^{1/2}\right) e\left(\tfrac{1}{2}(n - \xi_2)^2 x + n\xi_1\right),$$

where $z = x + iy$, $\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and f_ϕ is as in (2.14).

2.4   A Lattice $\Gamma < G$ such that Θ_f is Γ-invariant

Consider the following elements of G, each written in two ways using the identification described in Section 2.1:

$$(2.17) \qquad \gamma_1 = \left(\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \arg \right]; \mathbf{0}, \tfrac{1}{8} \right) = \left(i, \tfrac{\pi}{2}; \mathbf{0}, \tfrac{1}{8} \right),$$

$$(2.18) \qquad \gamma_2 = \left(\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 0 \right]; \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, 0 \right) = \left(1 + i, 0; \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, 0 \right),$$

$$(2.19) \qquad \gamma_3 = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right]; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right) = \left(i, 0; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right),$$

$$(2.20) \qquad \gamma_4 = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right]; \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0 \right) = \left(i, 0; \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0 \right),$$

$$(2.21) \qquad \gamma_5 = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right]; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1 \right) = \left(i, 0; \mathbf{0}, 1 \right).$$

It was shown in [7] that for $i = 1, \ldots, 5$, we have $\Theta_f(\gamma_i g) = \Theta_f(g)$ for every $g \in G$. The Jacobi theta function Θ_f is therefore invariant under the left action by the group

$$(2.22) \qquad\qquad \Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \rangle < G.$$

This means that Θ_f is well defined on the quotient $\Gamma \backslash G$. The group Γ is a lattice in G, and $\Gamma \backslash G$ is a 4-torus bundle over the modular surface $SL(2, \mathbb{Z}) \backslash \mathfrak{H}$. In particular, $\Gamma \backslash G$ is non-compact. A fundamental domain for the action of Γ on G is

$$(2.23) \qquad \mathcal{F}_\Gamma = \left\{ (z, \phi; \boldsymbol{\xi}, \zeta) \in \mathcal{F}_{SL(2,\mathbb{Z})} \times [0, \pi) \times \left[-\tfrac{1}{2}, \tfrac{1}{2} \right)^2 \times \left[-\tfrac{1}{2}, \tfrac{1}{2} \right) \right\},$$

where $\mathcal{F}_{SL(2,\mathbb{Z})}$ is a fundamental domain of the modular group in $\mathfrak{H}$. The hyperbolic area of $\mathcal{F}_{SL(2,\mathbb{Z})}$ is $\pi/3$, and hence, by (2.11), we have that $\mu(\Gamma \backslash G) = \mu(\mathcal{F}_\Gamma) = \pi^2/3$.

Remark 2.1   Although we defined the Jacobi theta function in (2.16) assuming that f is regular enough ($\eta > 1$), it can be shown Θ_f is a well-defined element of $L^2(\Gamma \backslash G, \mu)$ (in fact, of $L^4(\Gamma \backslash G, \mu)$) provided $f \in L^2(\mathbb{R})$; see [1, §2.9].

We are finally ready to define $\boldsymbol{X}$. Given f_0, f_1 as in the Section 1.4, set $\boldsymbol{X} : \Gamma \backslash G \to \mathbb{C}^2$ as

$$(2.24) \qquad\qquad \boldsymbol{X}(\Gamma g) = \begin{pmatrix} \Theta_{f_1}(\Gamma g) \\ \Theta_{f_0}(\Gamma g) \end{pmatrix}.$$

Observe that $\boldsymbol{X} : (\Gamma \backslash G, \mathscr{B}(\Gamma \backslash G), \frac{3}{\pi^2} \mu) \to (\mathbb{C}^2, \mathscr{B}(\mathbb{C}^2))$ is a random variable whose law is simply the push forward of the normalized Haar measure $\frac{3}{\pi^2} \mu$ onto $\mathbb{C}^2$ via $\boldsymbol{X}$. The properties of the law of each component $\Theta_f(\Gamma g)$ when Γg is Haar-random on $\Gamma \backslash G$ were studied in [1]. In particular, we have the following lemma.

Lemma 2.2 [1]   *Let $\eta > 1$ and $f \in \mathcal{S}_\eta$. Then for all sufficiently large $R > 0$, we have*

$$(2.25) \qquad \mu\big(\{ \Gamma g \in \Gamma \backslash G : |\Theta_f(g)| > R \}\big) = \frac{2}{3} D(f) R^{-6} \big(1 + O_\eta(\kappa_\eta(f)^{2\eta} R^{-2\eta}) \big),$$

where $D(f) := \int_{-\infty}^\infty \int_0^\pi |f_\phi(w)|^6 \, d\phi \, dw$ and $\kappa_\eta(f) := \sup_{w, \phi} |f_\phi(w)|(1 + |w|)^\eta$.

If f is less regular (*e.g.,* when f is the indicator of an interval, in which case $f \in \mathcal{S}_1$ and Lemma 2.2 does not apply) a delicate analysis can be performed to study the tail asymptotic (see [1, §3.6]). In any case, the complex-valued random variable Θ_f has heavy tails and all moments of order $p \geq 6$ are infinite.

3   Proof of Theorem 1.1

Recall (2.7). The following theorem describes how certain "irrational" horocycle lifts of the form $u \mapsto M(u)\Psi^u$, with $M(u) \in G$ and $\Psi^u = (\tilde{n}_u, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right), 0)$, become equidistributed in $\Gamma \backslash G$ under the action of the geodesic flow $\Phi^s = (\tilde{a}_{e^{-s}}, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right), 0)$. It follows immediately from [1, Corollary 4.3].

Theorem 3.1   *Let $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a probability density, and let $F : \Gamma\backslash G \to \mathbb{C}^2$ be a bounded continuous function. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and any $\varpi \in \mathbb{R}$, we have*

$$(3.1) \quad \lim_{s \to \infty} \int_{\mathbb{R}} F\left(\Gamma\left(I_2 ; \begin{pmatrix} \theta u \\ 0 \end{pmatrix}\right), \varpi u\right) \Psi^u \Phi^s \right) \sigma(u) du = \frac{1}{\mu(\Gamma\backslash G)} \int_{\Gamma\backslash G} F(\Gamma g) d\mu(\Gamma g).$$

Note that the limit depends neither on σ, nor on θ (provided it is irrational), nor on ϖ. Our assumptions on the Pöschl–Teller potential V_0 and on the functions f_0, f_1, along with (1.15), allow us to write

$$\frac{1}{2\pi} E_n t = \frac{1}{2\pi} \frac{(2n+\gamma)^2}{2} t = \frac{1}{2}(n - \xi_2)^2 u + n\xi_1 + \zeta - \frac{1}{2}\xi_1 \xi_2,$$

where $\xi_1 = \frac{\gamma}{2} u$, $\xi_2 = 0$, $\zeta = \frac{\gamma^2}{8} u$, and $u = \frac{2t}{\pi}$. Therefore, using (2.16) and setting $s = 2\log N$, we have

$$(3.2) \quad \frac{1}{\sqrt{N}} \sum_{n \geq 0} f_\ell\left(\frac{n}{N}\right) e^{iE_n t} = \Theta_{f_\ell}\left(u + ie^{-s}, 0; \begin{pmatrix} \frac{\gamma}{2} u \\ 0 \end{pmatrix}, \frac{\gamma^2}{8} u\right), \quad \ell = 0, 1.$$

Recall that ρ denotes the density of the probability measure λ on $\mathbb{R}$, and that t in (1.17) is distributed according to the law λ. Therefore, $u = \frac{2t}{\pi}$ has density $\sigma = \frac{\pi}{2}\rho(\frac{\pi}{2}\cdot)$.

We now can write

$$X_N(t) = X\left(u + ie^{-s}, 0; \begin{pmatrix} \frac{\gamma}{2} u \\ 0 \end{pmatrix}, \frac{\gamma^2}{8} u\right),$$

and proving Theorem 1.1 is equivalent to showing that for every bounded, continuous function $h : \mathbb{C}^2 \to \mathbb{R}$, we have that

$$\lim_{s \to \infty} \int_{\mathbb{R}} h\left(X\left(u + ie^{-s}, 0; \begin{pmatrix} \frac{\gamma}{2} u \\ 0 \end{pmatrix}, \frac{\gamma^2}{8} u\right)\right) \sigma(u) du$$

$$(3.3) \qquad = \frac{1}{\mu(\Gamma\backslash G)} \int_{\Gamma\backslash G} h\left(X(\Gamma g)\right) d\mu(\Gamma g).$$

Observe that if $f_0, f_1 \in \mathcal{S}_\eta$ with $\eta > 1$, then $h \circ X$ is bounded and continuous, and, since we are assuming that γ is irrational, we can apply Theorem 3.1 to achieve (3.3).

Finally, if f_0 and f_1 are bounded, Riemann-integrable functions, then we can use a standard approximation argument, analogous to the one used in [1, Lemmata 4.5–4.9].

Remark 3.2   Since the two components of X are both functions of the same random variable (Γg distributed in $\Gamma\backslash G$ according to the normalized Haar measure), it is clear that the law on $\mathbb{C}^2$ of X is not the product measure of its two marginals on $\mathbb{C}$. The analogue of Theorem 1.1 for rational γ can also be considered, but the statement of Theorem 3.1 needs to be modified. In fact, the horocycle lifts do not equidistribute on the homogeneous space $\Gamma\backslash G$; instead, they equidistribute on a submanifold of positive codimension of the form $\Gamma_\gamma\backslash G$. In this case, the random variable Γg on the right-hand side of (3.1) would be distributed according the normalized Haar measure on $\Gamma_\gamma\backslash G$.

The extension of Theorem 3.1 for rational γ is the subject of a joint work in progress with Tariq Osman.

4   A Class of SUSY Partner Pöschl–Teller Potentials

In this section, we explore two constructions that, given a Pöschl–Teller potential, allow us to construct infinitely many isospectral supersymmetric partners, thus providing us with a plethora of cases in which our Theorem 1.1 can be applied. Let H_0 be the Hamiltonian with potential V_0 as in (1.14). Recall that $\gamma = \alpha + \beta$. A general solution to the equation $H_0 u = \varepsilon u$ (regardless of boundary conditions) for any $\varepsilon > 0$ is

(4.1)

$$
u_{\varepsilon,a,b}(x) = \sin^\alpha(x)\cos^\beta(x)\left(a\ _2F_1\left(\frac{\gamma}{2}+\sqrt{\frac{\varepsilon}{2}},\frac{\gamma}{2}-\sqrt{\frac{\varepsilon}{2}};\alpha+\frac{1}{2};\sin^2(x)\right)\right.
$$

$$
\left.+\ b\ \sin^{1-2\alpha}(x)\ _2F_1\left(\frac{1+\beta-\alpha}{2}+\sqrt{\frac{\varepsilon}{2}},\frac{1+\beta-\alpha}{2}-\sqrt{\frac{\varepsilon}{2}};\frac{3}{2}-\alpha;\sin^2(x)\right)\right).
$$

Using first-order intertwining operators of the form

(4.2) $\qquad A = \dfrac{1}{\sqrt{2}}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}+\kappa(x)\right), \qquad\qquad A^\dagger = \dfrac{1}{\sqrt{2}}\left(-\dfrac{\mathrm{d}}{\mathrm{d}x}+\kappa(x)\right),$

Contreras–Astorga and Fernández [3] were able to construct a 1-parameter family of partner potentials V_1 such that H_0 and H_1 are isospectral; see also [5]. Specifically,

(4.3) $\qquad V_1(x) = \dfrac{\alpha(\alpha+1)}{2\sin^2(x)}+\dfrac{(\beta-2)(\beta-1)}{2\cos^2(x)}-\big(\log v_\varepsilon(x)\big)'', \qquad \alpha>1, \beta>2,$

where

(4.4) $\qquad v_\varepsilon(x) = \dfrac{u_{\varepsilon,1,0}(x)}{\sin^\alpha(x)\cos^{1-\beta}(x)}$

$$
= \cos^{2\beta-1}(x)\,_2F_1\left(\frac{\gamma}{2}+\sqrt{\frac{\varepsilon}{2}},\frac{\gamma}{2}-\sqrt{\frac{\varepsilon}{2}};\alpha+\frac{1}{2};\sin^2(x)\right),
$$

and $\varepsilon < E_0 = \dfrac{\gamma^2}{2}$ is an arbitrary real parameter. The corresponding eigenfunctions for H_1 are

(4.5) $$\psi_{1,n} = \frac{A^\dagger \psi_{0,n}}{\sqrt{E_n-\varepsilon}},$$

where $\kappa(x) = (\log u_{\varepsilon,1,0}(x))'$ in (4.2). Such a potential V_1 for $\alpha = \sqrt{2}, \beta = 3$, and $\varepsilon = 9$ is shown in Figure 3, along with the supersymmetric partner V_0.

In the same paper [3], the authors also used second-order intertwining operators $B, B^\dagger$ of the form so that $H_1 B^\dagger = B^\dagger H_0$, where

(4.6) $$B^\dagger = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\eta(x)\frac{\mathrm{d}}{\mathrm{d}x}+\theta(x)\right),$$

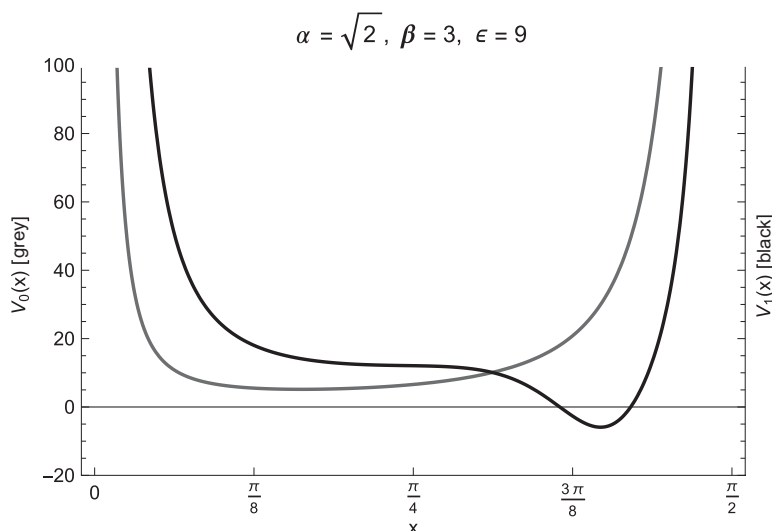$$\alpha = \sqrt{2}\,,\;\; \beta = 3,\;\; \epsilon = 9$$

Figure 3: The potentials V_0 and V_1 of two isospectral supersymmetric partner Hamiltonians, given by (1.14) and (4.3), respectively.

to construct a 2-parameters family of partner potentials V_1 such that H_0 and H_1 are isospectral. Specifically,

$$(4.7) \quad V_1(x) = \frac{(\alpha + 1)(\alpha + 2)}{2\sin^2(x)} + \frac{(\beta - 3)(\beta - 2)}{2\cos^2(x)} - (\log \mathcal{W}(x))'', \quad \alpha > 1, \beta > 3,$$

where

$$(4.8) \qquad\qquad \mathcal{W}(x) = \frac{W(u_{\varepsilon_1,1,0}, u_{\varepsilon_2,1,0})}{\sin^{2\alpha+1}(x)\cos^{3-2\beta}(x)},$$

$W(f, g) = f'g - fg'$ denotes the Wronskian of f and g, $u_{\varepsilon_i,1,0}$ is as in (4.1) for $i = 1, 2$, and $\varepsilon_1, \varepsilon_2$ are real parameters such that $E_l < \varepsilon_2 < \varepsilon_1 < E_{l+1}$ for some $l \geq 0$. In this case, the normalized eigenfunctions of H_1 are

$$(4.9) \qquad\qquad \psi_{1,n} = \frac{B^\dagger \psi_{0,n}}{\sqrt{(E_n - \varepsilon_1)(E_n - \varepsilon_2)}},$$

where in (4.6), we have $\eta = (\log(W(u_{\varepsilon_1,1,0}, u_{\varepsilon_2,1,0})))'$, $\theta = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V_0 + d$, and $d = \varepsilon_1 + \varepsilon_2$.

In all these classes of supersymmetric partner potentials, our Theorem 1.1 applies, provided γ is irrational.

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