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Conjugacies between linear and nonlinear non-uniform contractions

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Abstract. We construct conjugacies between linear and nonlinear *non-uniform* exponential contractions with discrete time. We also consider the general case of a *non-autonomous* dynamics defined by a sequence of maps. The results are obtained by considering both linear and nonlinear perturbations of the dynamics $x_{m+1} = A_m x_m$ defined by a sequence of linear operators A_m. In the case of conjugacies between *linear* contractions we describe them explicitly. All the conjugacies are locally Hölder, and in fact are locally Lipschitz outside the origin. We also construct conjugacies between linear and nonlinear non-uniform exponential dichotomies, building on the arguments for contractions. All the results are obtained in Banach spaces.

1. *Introduction*

In the stability theory of dynamical systems one wants to understand what properties of a given dynamics persist under sufficiently small perturbations. This leads to the introduction of the notion of conjugacy that provides a faithful correspondence between the orbits of two given dynamics. Namely, given homeomorphisms $f, g\colon X \to X$ of the topological space X, we say that a homeomorphism $h\colon X \to X$ is a *topological conjugacy* if

$$h \circ f = g \circ h.$$

This implies that

$$h \circ f^m = g^m \circ h \quad \text{for every } m \in \mathbb{Z},$$

and thus the conjugacy h transforms the orbit $\{f^m(x) : m \in \mathbb{Z}\}$ of f into the orbit $\{g^m(h(x)) : m \in \mathbb{Z}\}$ of g. Roughly speaking, this provides a dictionary between the two dynamics.

Our main objective is to construct conjugacies, as explicitly as possible, between linear and nonlinear dynamics. Here we concentrate on the case of contractions with discrete time, and we consider arbitrary *non-uniform* exponential contractions, for which the Lyapunov stability is not necessarily uniform. We also consider the general case of a *non-autonomous* dynamics defined by a sequence of maps. This means that, given

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sequences of homeomorphisms $f_m, g_m : X \to X$, $m \in \mathbb{Z}$ (that define two non-autonomous dynamics), we look for homeomorphisms $h_m : X \to X$ such that

$$h_{m+1} \circ f_m = g_m \circ h_m, \quad m \in \mathbb{Z}. \tag{1}$$

It should be noted that there is an obvious solution of (1). Namely, we can take $h_0 = \mathrm{Id}$, and for each $m > 0$ set

$$h_m = g_{m-1} \circ \cdots \circ g_0 \circ f_0^{-1} \circ \cdots \circ f_{m-1}^{-1} \tag{2}$$

and

$$h_{-m} = g_{-m}^{-1} \circ \cdots \circ g_{-1}^{-1} \circ f_{-1} \circ \cdots \circ f_{-m}. \tag{3}$$

However, this solution does not satisfy the following property.

(A) Given homeomorphisms $f, g : X \to X$ with $f_m = f$ and $g_m = g$ for $m \in \mathbb{Z}$, there exists $h : X \to X$ such that $h_m = h$ for every $m \in \mathbb{Z}$.

The conjugacies that we construct in this paper have this additional property. Furthermore, they are locally Hölder, and even locally Lipschitz outside the origin, provided that the maps f_m and g_m are sufficiently regular (again this is obvious for the construction in (2) and (3)).

A related fundamental problem is whether the linearization $g = d_x f$ of a diffeomorphism f along the orbit of a fixed point x is conjugated to the original dynamics. For hyperbolic trajectories the solution of this problem is given by the Grobman–Hartman theorem. Strictly speaking, this only provides conjugacies between dynamics with the same linear part, although some approaches to the proof of the Grobman–Hartman theorem lead to conjugacies between maps with different linear parts. In the case of conjugacies between *linear* contractions we describe the conjugacies explicitly (see §5). To the best of our knowledge, explicit conjugacies satisfying Property A are given here for the first time even in the particular case of non-autonomous *uniform* exponential contractions.

We also consider the general hyperbolic situation in which there are simultaneously contraction and expansion. This case is treated by first obtaining separately conjugacies between contractions and between expansions, and then putting the two together.

The content of the paper is the following. In §2 we consider the problem of robustness of a non-uniform exponential contraction, which asks whether any of its linear perturbations that is sufficiently small is still a non-uniform exponential contraction. Section 3 presents our results on the existence of Hölder conjugacies between linear and nonlinear contractions. In §4 we obtain corresponding results for non-uniform exponential dichotomies. Section 5 contains the proof of Theorem 2, which establishes the existence of locally Hölder conjugacies between linear contractions.

2. *Robustness of non-uniform contractions*

Let $\mathcal{B}(X)$ be the space of bounded linear operators in the Banach space X. Consider a sequence of invertible operators $A_m \in \mathcal{B}(X)$ for $m \in \mathbb{Z}$, and set

$$\mathcal{A}(m, n) = \begin{cases} A_{m-1} \cdots A_n, & m > n, \\ \mathrm{Id}, & m = n, \\ A_m^{-1} \cdots A_{n-1}^{-1}, & m < n. \end{cases}$$

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We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *strong non-uniform exponential contraction* if there exist constants $b \geq a > 0$, $D > 0$, and $\varepsilon \geq 0$ such that for each $m \geq n$ we have

$$\|\mathcal{A}(m, n)\| \leq De^{-a(m-n)+\varepsilon|n|}, \qquad \|\mathcal{A}(n, m)\| \leq De^{b(m-n)+\varepsilon|m|}. \tag{4}$$

The following is our robustness result.

THEOREM 1. *Let $A_m, B_m \in \mathcal{B}(X)$, $m \in \mathbb{Z}$ be invertible operators such that:*
(1) *$(A_m)_{m\in\mathbb{Z}}$ admits a strong non-uniform exponential contraction; and*
(2) *$\|B_m\| \leq \delta e^{-\varepsilon|m+1|}$ and $A_m + B_m$ is invertible for every $m \in \mathbb{Z}$.*
If $\delta D < \min\{1 - e^{-a}, e^{-b}\}$, then $(A_m + B_m)_{m\in\mathbb{Z}}$ admits a strong non-uniform exponential contraction, with the constants a and b replaced respectively by $a - \log(1 + \delta De^a)$ and $b - \log(1 - \delta De^b)$ (and with D and ε unchanged).

Proof. One can easily verify that, for each $m \geq n$,

$$\mathcal{C}(m, n) = \mathcal{A}(m, n) + \sum_{l=n}^{m-1} \mathcal{A}(m, l + 1)B_l\mathcal{C}(l, n), \tag{5}$$

where

$$\mathcal{C}(m, n) = \begin{cases} (A_{m-1} + B_{m-1}) \cdots (A_n + B_n), & m > n, \\ \mathrm{Id}, & m = n, \\ (A_m + B_m)^{-1} \cdots (A_{n-1} + B_{n-1})^{-1}, & m < n. \end{cases}$$

Setting $x_m = \|\mathcal{C}(m, n)\|$, it follows from (5) that

$$x_m \leq \|\mathcal{A}(m, n)\| + \sum_{l=n}^{m-1} \|\mathcal{A}(m, l + 1)\| \cdot \|B_l\|x_l,$$

and hence

$$x_m \leq De^{-a(m-n)+\varepsilon|n|} + \delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}x_l.$$

Consider the sequence Φ_m defined recursively by

$$\Phi_m = De^{-a(m-n)+\varepsilon|n|} + \delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l. \tag{6}$$

Clearly, $x_m \leq \Phi_m$. Setting $\Gamma_m = e^{a(m-n)}\Phi_m$, we can rewrite (6) in the form

$$\Gamma_m = De^{\varepsilon|n|} + \delta De^a \sum_{l=n}^{m-1} \Gamma_l.$$

Clearly, $\Gamma_{m+1} - \Gamma_m = \delta De^a\Gamma_m$, i.e.

$$\Gamma_{m+1} = (1 + \delta De^a)\Gamma_m.$$

Furthermore, again by (6), $\Gamma_n = \Phi_n = De^{\varepsilon|n|}$ and, for each $m \geq n$,

$$\begin{aligned} \Gamma_m &= (1 + \delta De^a)^{m-n}\Gamma_n \\ &= De^{\varepsilon|n|}(1 + \delta De^a)^{m-n} \\ &= De^{\log(1+\delta De^a)(m-n)+\varepsilon|n|}. \end{aligned}$$

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Hence,

$$\|\mathcal{C}(m,n)\| = x_m \le \Phi_m = De^{-(a-\log(1+\delta De^a))(m-n)+\varepsilon|n|}.$$

Similarly, for each $m \ge n$ we have

$$\mathcal{C}(n,m) = \mathcal{A}(n,m) - \sum_{l=n}^{m-1} \mathcal{A}(n,l+1)B_l\mathcal{C}(l,m).$$

Setting $y_n = \|\mathcal{C}(n,m)\|$ we obtain

$$y_n \le De^{b(m-n)+\varepsilon|m|} + D\delta \sum_{l=n}^{m-1} e^{b(l+1-n)+\varepsilon|l+1|}e^{-\varepsilon|l+1|}y_l. \tag{7}$$

Set now $\Gamma_l = y_l e^{-b(m-l)-\varepsilon|m|}$. It follows from (7) that

$$\Gamma_n \le D + D\delta e^b \sum_{l=n}^{m-1} e^{b(l-n)-b(m-n)-\varepsilon|m|}y_l = D + D\delta e^b \sum_{l=n}^{m-1} \Gamma_l. \tag{8}$$

We proceed by backwards induction on n to show that

$$\Gamma_n = y_n e^{-b(m-n)-\varepsilon|m|} \le \frac{D}{(1-D\delta e^b)^{m-n}}. \tag{9}$$

For $n = m-1$ this follows readily from (8). We now assume that (9) holds for $n > m-k$. Setting $c = D\delta e^b$, it follows from (8) that, for $n = m-k$,

$$\Gamma_n \le D + c\sum_{l=n}^{m-1} \Gamma_l \le D + c\sum_{l=n+1}^{m-1} \frac{D}{(1-c)^{m-l}} + c\Gamma_n.$$

This yields

$$\begin{aligned}(1-c)\Gamma_n &\le D + c\sum_{l=n+1}^{m-1} \frac{D}{(1-c)^{m-l}}\\ &= D + \frac{(1-c)^{-(m-n-1)}-1}{(1-c)^{-1}-1}\cdot\frac{Dc}{1-c}\\ &= D + \frac{D}{(1-c)^{m-n-1}} - D,\end{aligned}$$

and hence (9) holds for $n = m-k$. It now follows from (9) that

$$\|\mathcal{C}(n,m)\| = y_n \le De^{(b-\log(1-D\delta e^b))(m-n)+\varepsilon|m|}.$$

This completes the proof of the theorem. □

3. *Conjugacies between linear and nonlinear contractions*

This section is dedicated to the construction of conjugacies between two given dynamics, either linear or nonlinear. We emphasize that we always obtain Hölder continuous conjugacies.

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3.1. *Main results.* We consider here two sequences $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ of linear operators admitting strong non-uniform exponential contractions. Without loss of generality we always take for the two contractions the same constants a, b, D and ε in (4). We continue to denote by $\mathcal{B}(X)$ the space of bounded linear operators in the Banach space X. We say that *Property P* holds for the pair $(A_m, \hat{A}_m)$ if there is a Lipschitz curve $\gamma_m\colon [0, 1] \to \mathcal{B}(X)$ in the set of invertible operators with bounded inverse, such that $\gamma_m(0) = \hat{A}_m$ and $\gamma_m(1) = A_m$.

The following is our main result on conjugacies between linear systems.

THEOREM 2. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators admitting strong non-uniform exponential contractions, such that Property P holds for the pair $(A_m, \hat{A}_m)$ for every $m \in \mathbb{Z}$. If $a > 2\varepsilon$, then there exist homeomorphisms $h_m\colon X \to X$, $m \in \mathbb{Z}$, satisfying*

$$h_{m+1} \circ A_m = \hat{A}_m \circ h_m, \quad m \in \mathbb{Z}, \tag{10}$$

and Property A. The maps h_m are locally Hölder with Hölder exponent

$$\alpha = \frac{a - 2\varepsilon}{b + 2\varepsilon}, \tag{11}$$

and are locally Lipschitz outside zero. The same happens with the maps h_m^{-1}.

The proof of Theorem 2 is given in §5.

We now consider the particular case of non-uniform exponential contractions in the finite-dimensional space $X = \mathbb{R}^k$. We denote by $\operatorname{sgn}(\sigma)$ the sign of the number σ.

THEOREM 3. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ be $k \times k$ matrices admitting strong non-uniform exponential contractions, such that $\operatorname{sgn}(\det A_m) = \operatorname{sgn}(\det \hat{A}_m)$ for all $m \in \mathbb{Z}$. If $a > 2\varepsilon$, then there exist homeomorphisms $h_m\colon X \to X$, $m \in \mathbb{Z}$, with the properties in Theorem 2.*

Proof. It is sufficient to verify that, for each $m \in \mathbb{Z}$, Property P holds for the pair $(A_m, \hat{A}_m)$ if and only if $\operatorname{sgn}(\det A_m) = \operatorname{sgn}(\det \hat{A}_m)$. But this is an immediate consequence of the fact that the set of $k \times k$ invertible matrices has two connected components, namely those with positive determinant and those with negative determinant. □

It follows from Theorem 1 that for sufficiently small perturbations of a strong non-uniform exponential contraction, when looking for a conjugacy as in (10) we can in fact assume that only one of the sequences $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ admits a strong non-uniform exponential contraction (in both Theorems 2 and 3), since the other one will then automatically admit such a contraction.

THEOREM 4. *Let $A_m, \hat{A}_m \in \mathcal{B}(X)$, $m \in \mathbb{Z}$, be invertible operators such that:*

(1) *$(A_m)_{m\in\mathbb{Z}}$ admits a strong non-uniform exponential contraction;*
(2) *Property P holds for the pair $(A_m, \hat{A}_m)$ for every $m \in \mathbb{Z}$; and*
(3) *$\|A_m - \hat{A}_m\| \le \delta e^{-\varepsilon|m|}$ for every $m \in \mathbb{Z}$.*

If $a > 2\varepsilon$, and δ is sufficiently small, then there exist homeomorphisms $h_m\colon X \to X$, $m \in \mathbb{Z}$, with the properties in Theorem 2.

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3.2. *Linear and nonlinear dynamics.* To obtain conjugacies between two arbitrary systems, either linear or nonlinear, we require a non-uniform version of the classical Grobman–Hartman theorem. We recall an appropriate version established in [**1**]. Consider:

(1) invertible operators $A_m \in \mathcal{B}(X)$, $m \in \mathbb{Z}$; and
(2) maps $f_m \colon X \to X$, $m \in \mathbb{Z}$, such that, for some $\delta > 0$ and each $m \in \mathbb{Z}$,

$$\sup\{\|f_m(x)\| : x \in X\} \leq \delta e^{-\varepsilon|m|}, \tag{12}$$

$$\|f_m(x) - f_m(y)\| \leq \delta e^{-4\varepsilon|m|}\|x - y\|, \quad x, y \in X, \tag{13}$$

with the constant $\varepsilon \geq 0$ as in (4).

We also introduce new norms. Choose $\varrho \in (0, a)$ with a as in (4), and for each $m \in \mathbb{Z}$ set

$$\|x\|_m = \sum_{k \geq m} \|\mathcal{A}(k, m)\| e^{-(a+\varrho)(k-m)}.$$

The following statement is a particular case of more general results established in [**1**] (see also Proposition 2).

PROPOSITION 1. *If the sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a strong non-uniform exponential contraction and δ is sufficiently small, then:*

(1) *there exist unique homeomorphisms $h_m \colon X \to X$, $m \in \mathbb{Z}$, satisfying*

$$h_{m+1} \circ A_m = (A_m + f_m) \circ h_m, \quad m \in \mathbb{Z},$$

and

$$\sup_{m\in\mathbb{Z}} \sup_{x\in X} \|h_m(x) - x\|_m < \infty; \tag{14}$$

(2) *if there exist maps $A \in \mathcal{B}(X)$ and $f \colon X \to X$ such that*

$$A_m = A \quad \textit{and} \quad f_m = f \quad \textit{for each } m \in \mathbb{Z}, \tag{15}$$

then there is a homeomorphism $h \colon X \to X$ with $h_m = h$ for $m \in \mathbb{Z}$; and

(3) *for each $\alpha \in (0, a/b)$, there exists $K > 0$ such that*

$$\|h_m(x) - h_m(y)\| \leq K e^{2\varepsilon\alpha|m|}\|x - y\|^\alpha, \tag{16}$$

$$\|h_m^{-1}(x) - h_m^{-1}(y)\| \leq K e^{2\varepsilon\alpha|m|}\|x - y\|^\alpha, \tag{17}$$

for every $m \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\| \leq e^{-2\varepsilon|m|}$.

Proof of statement (2). We note that, although statement (2) is not formulated explicitly in [**1**], it is a simple consequence of the proofs of Theorems 1 and 3 in that paper. The argument is the following. Let $\mathfrak{X}$ be the space of sequences $(u_m)_{m\in\mathbb{Z}}$ of continuous functions $u_m \colon X \to X$ such that

$$\|(u_m)_{m\in\mathbb{Z}}\| := \sup\{\|u_m(x)\|_m : x \in X,\, m \in \mathbb{Z}\} < \infty.$$

One can easily verify that $\mathfrak{X}$ is a Banach space with this norm. We now observe that

$$p_u := \{(u_m)_{m\in\mathbb{Z}} : u_m = u \text{ for every } m \in \mathbb{Z}\}$$

is in $\mathfrak{X}$ for each continuous function $u \colon X \to X$. Furthermore, when (15) holds, the contraction maps $S \colon \mathfrak{X} \to \mathfrak{X}$ and $T \colon \mathfrak{X} \to \mathfrak{X}$ in the proofs of Theorems 1 and 3 take the set

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$\mathcal{D} = \{p_u : u \text{ is continuous}\}$ into itself. Finally, since $\mathcal{D}$ is a closed non-empty subset of the Banach space $\mathcal{X}$, the unique fixed points of the maps S and T, namely $(h_m^{-1} - \mathrm{Id})_{m\in\mathbb{Z}}$ and $(h_m - \mathrm{Id})_{m\in\mathbb{Z}}$, are also in $\mathcal{D}$. This yields statement (2). □

It is the uniqueness property of the sequence of homeomorphisms $(h_m)_{m\in\mathbb{Z}}$ among those satisfying (14) that makes Proposition 1 non-trivial. Otherwise, proceeding as in the introduction, we could choose $h_0 = \mathrm{Id}$, and for each $m > 0$ set

$$h_m = (A_{m-1} + f_{m-1}) \circ \cdots \circ (A_0 + f_0) \circ A_0^{-1} \circ \cdots \circ A_{m-1}^{-1} \tag{18}$$

and

$$h_{-m} = (A_{-m} + f_{-m})^{-1} \circ \cdots \circ (A_{-1} + f_{-1})^{-1} \circ A_{-1} \circ \cdots \circ A_{-m} \tag{19}$$

(one can show that for δ sufficiently small the transformations $A_m + f_m$ are invertible). Then the maps h_m defined by (18) and (19) are automatically Lipschitz (or even more regular if this happens with the maps f_m) and not only locally Hölder as in (16) and (17). In the particular case when (15) holds, the maps h_m in (18) and (19) are given by $h_m = (A + f)^m \circ A^{-m}$ and thus depend on m. On the other hand, in this situation Proposition 1 provides a homeomorphism h such that $h_m = h$ for every $m \in \mathbb{Z}$. This is the crucial difference in Proposition 1 with respect the construction in (18) and (19).

Notice that when $\varepsilon = 0$ and (15) holds we recover the classical Grobman–Hartman theorem, in fact with the additional Hölder regularity property in (16) and (17). Thus, Proposition 1 provides new information even when $\varepsilon = 0$ and in the autonomous case (when (15) holds). The original references for the Grobman–Hartman theorem are Grobman [**2, 3**] and Hartman [**4, 5**]. It was extended to Banach spaces by Palis [**6**] and Pugh [**7**]. We emphasize that in general the conjugacies are not more than Hölder continuous. In particular, the work of Sternberg [**8, 9**] showed that there are algebraic obstructions, expressed in terms of resonances between the eigenvalues of the linear approximation, that prevent the existence of conjugacies with a prescribed high regularity. See [**1**] for more details.

The following statement is an immediate consequence of Theorem 2 and Proposition 1.

THEOREM 5. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators admitting strong non-uniform exponential contractions, such that Property P holds for the pair $(A_m, \hat{A}_m)$ for every $m \in \mathbb{Z}$. Given maps $f_m, \hat{f}_m : X \to X$, $m \in \mathbb{Z}$, such that (12) and (13) hold, also with f_m replaced by $\hat{f}_m$, if $a > 2\varepsilon$, and δ is sufficiently small, then there exist locally Hölder homeomorphisms $h_m : X \to X$, $m \in \mathbb{Z}$, with locally Hölder inverse, satisfying*

$$h_{m+1} \circ (A_m + f_m) = (\hat{A}_m + \hat{f}_m) \circ h_m, \quad m \in \mathbb{Z}, \tag{20}$$

and Property A.

4. *The case of non-uniform exponential dichotomies*

4.1. *Conjugacies between linear systems.* Here we construct conjugacies between two linear dynamics given by sequences of linear operators $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ admitting strong non-uniform exponential dichotomies.

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We continue to denote by $\mathcal{B}(X)$ the space of bounded linear operators in the Banach space X. We say that a sequence of invertible linear operators $(A_m)_{m\in\mathbb{Z}}$ admits a *strong non-uniform exponential dichotomy* if there exist projections $P_m \in \mathcal{B}(X)$ for $m \in \mathbb{Z}$ satisfying

$$P_m \mathcal{A}(m, n) = \mathcal{A}(m, n) P_n \quad \text{for every } m, n \in \mathbb{Z},$$

and there exist constants

$$\underline{a} \le \overline{a} < 0 < \underline{b} \le \overline{b}, \quad D > 0, \quad \text{and} \quad \varepsilon \ge 0$$

such that for each $m \ge n$ we have

$$\|\mathcal{A}(m, n) P_n\| \le D e^{\overline{a}(m-n)+\varepsilon|n|}, \quad \|\mathcal{A}(n, m) Q_m\| \le D e^{-\underline{b}(m-n)+\varepsilon|m|}, \tag{21}$$

and for each $m \le n$ we have

$$\|\mathcal{A}(m, n) P_n\| \le D e^{-\underline{a}(m-n)+\varepsilon|n|}, \quad \|\mathcal{A}(n, m) Q_m\| \le D e^{\overline{b}(m-n)+\varepsilon|m|}, \tag{22}$$

where $Q_m = \mathrm{Id} - P_m$ is the complementary projection of P_m for each m.

Now let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators admitting strong non-uniform exponential dichotomies, with projections respectively P_m and $\hat{P}_m$ for $m \in \mathbb{Z}$. For simplicity of the exposition we assume that there exists a decomposition $X = E \oplus F$ (independent of m) with

$$A_m = \begin{pmatrix} B_m & 0 \\ 0 & C_m \end{pmatrix} \quad \text{and} \quad \hat{A}_m = \begin{pmatrix} \hat{B}_m & 0 \\ 0 & \hat{C}_m \end{pmatrix},$$

such that, for each $m \in \mathbb{Z}$,

$$A_m P_m = B_m, \quad A_m Q_m = C_m, \quad \hat{A}_m \hat{P}_m = \hat{B}_m, \quad \hat{A}_m \hat{Q}_m = \hat{C}_m.$$

The following is our main result.

THEOREM 6. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ admit strong non-uniform exponential dichotomies, such that Property P holds for the pairs $(B_m, \hat{B}_m)$ in E and $(C_m, \hat{C}_m)$ in F for each $m \in \mathbb{Z}$. If $\min\{-\overline{a}, \underline{b}\} > 2\varepsilon$, then there exist homeomorphisms $h_m \colon X \to X$, $m \in \mathbb{Z}$, satisfying (10) and Property A. The maps h_m are locally Hölder, and are locally Lipschitz outside zero. The same happens with the maps h_m^{-1}.*

The proof of Theorem 6 is given in §5.

We now consider the particular case of non-uniform exponential dichotomies in the finite-dimensional space $X = \mathbb{R}^k$. The following is an immediate consequence of Theorem 6.

THEOREM 7. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ be $k \times k$ matrices admitting strong non-uniform exponential dichotomies, such that*

$$\operatorname{sgn}(\det B_m) = \operatorname{sgn}(\det \hat{B}_m) \quad \text{and} \quad \operatorname{sgn}(\det C_m) = \operatorname{sgn}(\det \hat{C}_m),$$

for each $m \in \mathbb{Z}$. If $\min\{-\overline{a}, \underline{b}\} > 2\varepsilon$, then there exist homeomorphisms $h_m \colon X \to X$, $m \in \mathbb{Z}$, with the properties in Theorem 6.

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4.2. *Linear and nonlinear dynamics.* Now we obtain conjugacies between a strong non-uniform exponential dichotomy and its sufficiently small nonlinear perturbations.

We first recall the following statement established in [**1**].

PROPOSITION 2. *If the sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators admits a strong non-uniform exponential dichotomy and δ is sufficiently small, then statements (1)–(3) in Proposition 1 hold.*

The following result is now an immediate consequence of Theorem 6 and Proposition 2.

THEOREM 8. *Assume that:*
(1) *$(A_m)_{m\in\mathbb{Z}}$ and $(\hat{A}_m)_{m\in\mathbb{Z}}$ admit strong non-uniform exponential contractions, such that Property P holds for the pairs $(B_m, \hat{B}_m)$ in E and $(C_m, \hat{C}_m)$ in F for each $m \in \mathbb{Z}$; and*
(2) *the maps $f_m, \hat{f}_m \colon X \to X$, $m \in \mathbb{Z}$, satisfy (12) and (13), also with f_m replaced by $\hat{f}_m$.*

If $\min\{-\overline{a}, \underline{b}\} > 2\varepsilon$ and δ is sufficiently small, then there exist locally Hölder homeomorphisms $h_m \colon X \to X$, $m \in \mathbb{Z}$, with locally Hölder inverse, satisfying (20) and Property A.

5. *Proofs of Theorems 2 and 6*

We separate the proof of Theorem 2 into several steps.

5.1. *Preliminaries.*

For each $m \in \mathbb{Z}$ and $x \in X$ we set

$$q_m(x) = \sum_{k=0}^{\infty} \|\mathcal{A}(k+m, m)x\|. \tag{23}$$

It follows from (4) that

$$q_m(x) \le D\|x\| \sum_{k=0}^{\infty} e^{-ak+\varepsilon|m|} = \frac{De^{\varepsilon|m|}}{1-e^{-a}}\|x\| < \infty, \tag{24}$$

and that

$$\begin{aligned} q_m(x) &\ge \|x\| \sum_{k=0}^{\infty} \frac{1}{\|\mathcal{A}(m, k+m)\|} \\ &\ge \frac{1}{D}\|x\| \sum_{k=0}^{\infty} \frac{1}{e^{bk+\varepsilon|k+m|}} \\ &\ge \frac{1}{D}\|x\| \sum_{k=0}^{\infty} \frac{1}{e^{(b+\varepsilon)k+\varepsilon|m|}} = \frac{1}{D(1-e^{-b-\varepsilon})}\|x\|e^{-\varepsilon|m|}. \end{aligned} \tag{25}$$

Setting $\alpha = 1/(D(1-e^{-b-\varepsilon}))$ and $\beta = D/(1-e^{-a})$, we thus obtain

$$\alpha e^{-\varepsilon|m|}\|x\| \le q_m(x) \le \beta e^{\varepsilon|m|}\|x\|. \tag{26}$$

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On the other hand,

$$q_m(\mathcal{A}(m,n)x) = \sum_{k=0}^{\infty} \|\mathcal{A}(k+m,m)\mathcal{A}(m,n)x\| \\ = \sum_{k=0}^{\infty} \|\mathcal{A}(k+m,n)x\| = \sum_{j=m}^{\infty} \|\mathcal{A}(j,n)x\|, \tag{27}$$

and hence

$$q_m(\mathcal{A}(m,n)x) - q_{m-1}(\mathcal{A}(m-1,n)x) \\ = \sum_{j=m}^{\infty} \|\mathcal{A}(j,n)x\| - \sum_{j=m-1}^{\infty} \|\mathcal{A}(j,n)x\| = -\|\mathcal{A}(m-1,n)x\|. \tag{28}$$

In particular, whenever $x \neq 0$ the sequence $m \mapsto q_m(\mathcal{A}(m,n)x)$ is strictly decreasing. By (24) and (27) we have that

$$q_m(\mathcal{A}(m,n)x) \to 0 \quad \text{as } m \to +\infty. \tag{29}$$

Furthermore, using (26) and (4) we find that, for $m \leq n$,

$$q_m(\mathcal{A}(m,n)x) \geq \alpha\|\mathcal{A}(m,n)x\|e^{-\varepsilon|m|} \\ \geq \frac{\alpha\|x\|e^{-\varepsilon|m|}}{\|\mathcal{A}(n,m)\|} \geq \frac{\alpha\|x\|}{D}e^{a(n-m)-2\varepsilon|m|}.$$

Thus, since $a > 2\varepsilon$, we have that

$$q_m(\mathcal{A}(m,n)x) \to +\infty \quad \text{as } m \to -\infty. \tag{30}$$

We now define

$$\mathcal{D}_m = \{x \in X : q_m(x) \leq 1\} \quad \text{and} \quad \mathcal{R}_m = \mathcal{D}_m \setminus A_{m-1}(\mathcal{D}_{m-1}). \tag{31}$$

By (28), (29), and (30), for each $n \in \mathbb{Z}$ and $x \in X \setminus \{0\}$ there is a unique integer $m \in \mathbb{Z}$ such that $\mathcal{A}(m,n)x \in \mathcal{R}_m$. We denote it by $\tau_{n,x}$.

We also consider the function

$$\hat{q}_m(x) = \sum_{k=0}^{\infty} \|\hat{\mathcal{A}}(k+m,m)x\|,$$

and the sets

$$\hat{\mathcal{D}}_m = \{x \in X : \hat{q}_m(x) \leq 1\} \quad \text{and} \quad \hat{\mathcal{R}}_m = \hat{\mathcal{D}}_m \setminus \hat{A}_{m-1}(\hat{\mathcal{D}}_{m-1}).$$

The strategy of the proof of Theorem 2 is to construct maps $z_m \colon \mathcal{R}_m \to \hat{\mathcal{R}}_m$ such that $z_{m+1}(A_m x) = \hat{A}_m z_m(x)$ for every $x \in X$ with $q_m(x) = 1$. They will be given in the form

$$z_m = \hat{u}_m \circ H_m \circ u_m^{-1} \colon \mathcal{R}_m \to \hat{\mathcal{R}}_m, \tag{32}$$

with the transformations u_m, H_m and $\hat{u}_m$ constructed in the following sections. The conjugacies h_m will then be obtained from the maps z_m by using the evolution given by the two dynamics $\mathcal{A}(m,n)$ and $\hat{\mathcal{A}}(m,n)$.

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5.2. *Construction of the maps u_m.* Set $\mathcal{S} = \{x \in X : \|x\| = 1\}$. For each $m \in \mathbb{Z}$ we define the map

$$u_m : [0, 1] \times \mathcal{S} \to \overline{\mathcal{R}_m} \quad \text{by} \quad u_m(t, x) = \tau_m(t, x)x,$$

where

$$\tau_m(t, x) = \frac{t}{r(x)} + \frac{1-t}{s(x)},$$

having set for simplicity

$$r(x) = q_m(x) \quad \text{and} \quad s(x) = q_{m-1}(A_{m-1}^{-1}x).$$

By (28) we have $r(x) < s(x)$, and hence

$$q_m(u_m(t, x)) = t + (1-t)\frac{r(x)}{s(x)} \leq t + (1-t) = 1.$$

Analogously,

$$q_{m-1}(A_{m-1}^{-1}u_m(t, x)) = t\frac{s(x)}{r(x)} + 1 - t \geq 1.$$

This shows that indeed $u_m(t, x) \in \overline{\mathcal{R}_m}$ for every $t \in [0, 1]$ and $x \in \mathcal{S}$. We also define the map $v_m : \overline{\mathcal{R}_m} \to [0, 1] \times \mathcal{S}$ by

$$v_m(y) = \left(\frac{(s(y) - 1)r(y)}{s(y) - r(y)}, \frac{y}{\|y\|}\right). \tag{33}$$

Using again (28), we can easily verify that the first component of $v_m(y)$ is in $[0, 1]$: given numbers $c > d > 0$ with $c > 1$ we have

$$\frac{(c-1)d}{c-d} \leq 1 \quad \text{if and only if } d \leq 1,$$

and since $s(y) > 1 \geq r(y)$, the first component of $v_m(y)$ is indeed at most one. Observe now that

$$\begin{aligned} u_m(v_m(y)) &= \frac{y(s(y)-1)}{s(y)-r(y)} + \frac{y}{s(y)}\left(1 - \frac{(s(y)-1)r(y)}{s(y)-r(y)}\right) \\ &= \frac{y(s(y)-1)}{s(y)-r(y)} + \frac{y}{s(y)}\frac{s(y)-s(y)r(y)}{s(y)-r(y)} \\ &= \frac{y}{s(y)-r(y)}(s(y)-r(y)) = y. \end{aligned}$$

Thus, v_m is the inverse of u_m.

LEMMA 1. *There exists a continuous function $C_m : (X \setminus \{0\})^2 \to \mathbb{R}$ such that for every $x, y \in \overline{\mathcal{R}_m} \setminus \{0\}$ we have*

$$\|v_m(x) - v_m(y)\| \leq C_m(x, y)\|x - y\|.$$

Proof. We have

$$v_m(x) - v_m(y) = \left(\frac{\mathcal{G}(x, y)}{(s(y) - r(y))(s(x) - r(x))}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\right),$$

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where

$$\begin{aligned}\mathcal{G}(x, y) &= (s(y) - r(y))(s(x) - 1)r(x) \\ &\quad - (s(x) - r(x))(s(y) - 1)r(y) \\ &= (r(x) - r(y))(s(x)s(y) - s(y)) \\ &\quad - (s(x) - s(y))(r(x)r(y) - r(y)).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathcal{G}(x, y)\| &\le |r(x) - r(y)| \cdot |s(x) - 1| s(y) \\ &\quad + |s(x) - s(y)| \cdot |r(x) - 1| r(y).\end{aligned}$$

By (23) and (24),

$$\begin{aligned}r(x) - r(y) &= \sum_{k=0}^{\infty} \|\mathcal{A}(k+m, m)x\| - \sum_{k=0}^{\infty} \|\mathcal{A}(k+m, m)y\| \\ &\le \sum_{k=0}^{\infty} \|\mathcal{A}(k+m, m)(x-y)\| \le \frac{De^{\varepsilon|m|}}{1-e^{-a}} \|x-y\|. \qquad (34)\end{aligned}$$

Since $A_{m-1}^{-1} = \mathcal{A}(m-1, m)$ we have

$$\|A_{m-1}^{-1}\| \le De^{b+\varepsilon|m-1|} \le De^{b+\varepsilon+\varepsilon|m|},$$

and it follows from (34) that

$$\begin{aligned}\|\mathcal{G}(x, y)\| &\le \frac{De^{\varepsilon|m|}}{1-e^{-a}} \|x-y\| \cdot |s(x) - 1|\, s(y) \\ &\quad + \frac{D^2 e^{b+\varepsilon+2\varepsilon|m|}}{1-e^{-a}} \|x-y\| \cdot |r(x) - 1|\, r(y) \\ &\le \frac{D^3 e^{b+\varepsilon+3\varepsilon|m|}}{(1-e^{-a})^2} \|y\| \left(\frac{D^2 e^{b+\varepsilon+2\varepsilon|m|}}{1-e^{-a}} \|x\| + 1 \right) \|x-y\| \\ &\quad + \frac{D^3 e^{b+\varepsilon+3\varepsilon|m|}}{(1-e^{-a})^2} \|y\| \left(\frac{De^{\varepsilon|m|}}{1-e^{-a}} \|x\| + 1 \right) \|x-y\| \\ &\le \frac{2D^3 e^{b+\varepsilon+3\varepsilon|m|}}{(1-e^{-a})^2} \|y\| \\ &\quad \times \left(1 + \frac{De^{b+\varepsilon+2\varepsilon|m|}}{1-e^{-a}} \max\{D, 1\} \|x\| \right) \|x-y\|. \qquad (35)\end{aligned}$$

Furthermore,

$$\begin{aligned}\left| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right| &= \left| \frac{x\|y\| - y\|x\|}{\|x\| \cdot \|y\|} \right| \\ &\le \frac{|x\|y\| - y\|y\| + y\|y\| - y\|x\||}{\|x\| \cdot \|y\|} \\ &\le \frac{1}{\|x\|} \|x-y\| + \frac{1}{\|x\|} |\|y\| - \|x\|| \le \frac{2}{\|x\|} \|x-y\|. \qquad (36)\end{aligned}$$

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By (35) and (36) we can set

$$C_m(x, y) = \frac{2D^3 e^{b+\varepsilon+3\varepsilon|m|}}{(1-e^{-a})^2}\|y\|\left(1 + \frac{De^{b+\varepsilon+2\varepsilon|m|}}{1-e^{-a}}\max\{D, 1\}\|x\|\right)$$
$$\times \frac{1}{|s(y)-r(y)|\cdot|s(x)-r(x)|} + \frac{2}{\|x\|}.$$

This completes the proof of Lemma 1. □

5.3. *Construction of the maps $\hat{u}_m$.* Each map $\hat{u}_m$ is constructed in a similar manner to that of the map u_m. Namely, for each $m \in \mathbb{Z}$ we define

$$\hat{u}_m : [0, 1] \times \mathcal{S} \to \overline{\hat{\mathcal{R}}_m} \quad \text{by} \quad \hat{u}_m(t, x) = \hat{\tau}_m(t, x)x, \tag{37}$$

where

$$\hat{\tau}_m(t, x) = \frac{t}{\hat{r}(x)} + \frac{1-t}{\hat{s}(x)},$$

having set for simplicity

$$\hat{r}(x) = \hat{q}_m(x) \quad \text{and} \quad \hat{s}(x) = \hat{q}_{m-1}(\hat{A}_{m-1}^{-1}x).$$

LEMMA 2. *There exist continuous functions $D_m, E_m : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for every $t, s \in [0, 1]$ and $x, y \in \mathcal{S}$ we have*

$$\|\hat{u}_m(t, x) - \hat{u}_m(s, y)\| \le D_m(x, y)|t-s| + E_m(x, y)\|x-y\|.$$

Proof. We have

$$\begin{aligned}\|\hat{u}_m(t, x) - \hat{u}_m(s, y)\| &\le \|\hat{u}_m(t, x) - \hat{u}_m(t, y)\| + \|\hat{u}_m(t, y) - \hat{u}_m(s, y)\| \\ &\le \hat{\tau}_m(t, x)\|x-y\| + |\hat{\tau}_m(t, x) - \hat{\tau}_m(t, y)|\cdot\|y\| \\ &\quad + |\hat{\tau}_m(t, y) - \hat{\tau}_m(s, y)|\cdot\|y\|.\end{aligned}$$

Proceeding as in (25) and using (4) we obtain

$$\begin{aligned}\hat{s}(x) &\ge \alpha\|\hat{A}_{m-1}^{-1}x\|e^{-\varepsilon|m-1|} \\ &\ge \alpha\|\hat{A}_{m-1}\|^{-1}\|x\|e^{-\varepsilon|m-1|} \\ &= \alpha\|\hat{A}(m, m-1)\|^{-1}e^{-\varepsilon|m-1|} \ge \frac{\alpha}{D}e^{a-2\varepsilon|m-1|}.\end{aligned}$$

Using again (25) and since $\|x\| = 1$, this implies that

$$\begin{aligned}\hat{\tau}_m(t, x) &\le \frac{t}{\hat{r}(x)} + \frac{1-t}{\hat{s}(x)} \\ &\le t\alpha^{-1}e^{\varepsilon|m|} + (1-t)\alpha^{-1}De^{-a+2\varepsilon|m-1|} \\ &\le \alpha^{-1}e^{2\varepsilon|m|}\max\{1, De^{-a+2\varepsilon}\}.\end{aligned} \tag{38}$$

Furthermore,

$$|\hat{\tau}_m(t, x) - \hat{\tau}_m(t, y)| \le \frac{t}{\hat{r}(x)\hat{r}(y)}|\hat{r}(y) - \hat{r}(x)| + \frac{1-t}{\hat{s}(x)\hat{s}(y)}|\hat{s}(y) - \hat{s}(x)|.$$

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Observe that by (34) and (25) the second summand can be estimated by

$$\frac{e^{3\varepsilon|m-1|}}{\alpha^2\|\hat{A}_{m-1}^{-1}x\|\cdot\|\hat{A}_{m-1}^{-1}y\|}\beta\|\hat{A}_{m-1}^{-1}\|\cdot\|x-y\| \leq \frac{\beta e^{3\varepsilon|m-1|}}{\alpha^2}\|\hat{A}_{m-1}^{-1}\|\cdot\|\hat{A}_{m-1}\|^2\|x-y\|.$$

Therefore,

$$|\hat{\tau}_m(t,x)-\hat{\tau}_m(t,y)| \leq \frac{\beta e^{3\varepsilon|m|}}{\alpha^2}\|x-y\| + \frac{\beta D^3 e^{-2a+b+6\varepsilon|m-1|}}{\alpha^2}\|x-y\|. \tag{39}$$

On the other hand, proceeding as in (38) we obtain

$$\begin{aligned}|\hat{\tau}_m(t,y)-\hat{\tau}_m(s,y)| &\leq \left(\frac{1}{\hat{r}(y)}+\frac{1}{\hat{s}(y)}\right)|t-s| \\ &\leq \alpha^{-1}(e^{\varepsilon|m|}+De^{-a+2\varepsilon|m-1|})|t-s|. \end{aligned}\tag{40}$$

The desired statement follows readily from (38), (39) and (40). □

5.4. *Construction of the maps H_m.* It follows from Property P for the pair $(A_m, \hat{A}_m)$ that there is a Lipschitz curve $\gamma_m\colon [0,1] \to \mathcal{B}(X)$ in the set of invertible operators with bounded inverse such that $\gamma_m(0) = \hat{A}_m$ and $\gamma_m(1) = A_m$. We define

$$H_{m+1}\colon [0,1]\times\mathcal{S}\to[0,1]\times\mathcal{S} \quad \text{by} \quad H_{m+1}(t,x) = \left(t, \frac{\gamma_m(t)A_m^{-1}x}{\|\gamma_m(t)A_m^{-1}x\|}\right). \tag{41}$$

We also define $G_{m+1}\colon [0,1]\times\mathcal{S}\to[0,1]\times\mathcal{S}$ by

$$G_{m+1}(t,y) = \left(t, \frac{A_m\gamma_m(t)^{-1}y}{\|A_m\gamma_m(t)^{-1}y\|}\right).$$

We observe that G_m is the inverse of H_m. Indeed, given $(t,y)\in[0,1]\times\mathcal{S}$ we have

$$(H_m\circ G_m)(t,y) = \left(t, \frac{y}{\|y\|}\right) = (t,y).$$

LEMMA 3. *There exist continuous functions $D'_m, E'_m\colon \mathcal{S}\times\mathcal{S}\to\mathbb{R}$ such that for every $t,s\in[0,1]$ and $x,y\in\mathcal{S}$ we have*

$$\|H_m(t,x)-H_m(s,y)\| \leq D'_m(x,y)\|x-y\| + E'_m(x,y)|t-s|.$$

Proof. Set

$$u=\gamma_m(t)A_m^{-1}x, \quad v=\gamma_m(t)A_m^{-1}y \quad \text{and} \quad w=\gamma_m(s)A_m^{-1}y.$$

We have

$$\begin{aligned}&\|H_{m+1}(t,x)-H_{m+1}(s,y)\| \\ &\quad\leq \|H_{m+1}(t,x)-H_{m+1}(t,y)\| + \|H_{m+1}(t,y)-H_{m+1}(s,y)\| \\ &\quad\leq \frac{F}{\|u\|\cdot\|v\|} + |t-s| + \|A_m^{-1}y\|\frac{G}{\|v\|\cdot\|w\|},\end{aligned}$$

where

$$
\begin{aligned}
F &= \big\|(\|v\| - \|u\|)u + \|u\|(u - v)\big\| \\
&\le \|\gamma_m(t)A_m^{-1}(y - x)\| \cdot \|u\| + \|u\| \cdot \|\gamma_m(t)A_m^{-1}(x - y)\|, \\
&\le \|\gamma_m(t)A_m^{-1}\|^2\|x - y\|
\end{aligned}
$$

and

$$
\begin{aligned}
G &= \big\|(\|w\| - \|v\|)\gamma_m(t) + \|v\|(\gamma_m(t) - \gamma_m(s))\big\| \\
&\le 2\|\gamma_m(t)\| \cdot \|A_m^{-1}y\| \cdot \|\gamma_m(t) - \gamma_m(s)\|.
\end{aligned}
$$

Therefore, since

$$
\|v\| \ge \frac{\|y\|}{\|(\gamma_m(t)A_m^{-1})^{-1}\|} \ge \frac{1}{\|A_m\| \cdot \|\gamma_m(t)^{-1}\|},
$$

we obtain

$$
\begin{aligned}
&\|H_{m+1}(t, x) - H_{m+1}(s, y)\| \\
&\quad \le \|A_m\|^2\|\gamma_m(t)^{-1}\|^2\|\gamma_m(t)A_m^{-1}\|^2\|x - y\| + |t - s| \\
&\quad\quad + 2\|A_m\|^2\|\gamma_m(t)^{-1}\| \cdot \|\gamma_m(s)^{-1}\| \cdot \|A_m^{-1}y\| \cdot \|\gamma_m(t) - \gamma_m(s)\|.
\end{aligned}
$$

Since γ_m is Lipschitz we obtain the desired statement. □

5.5. *Construction of the maps h_m.* Using the transformations u_m, H_m and $\hat{u}_m$ constructed above we can define maps $z_m : \mathcal{R}_m \to \hat{\mathcal{R}}_m$ by (32). In fact it also follows from the above constructions that each map z_m has a continuous extension to the closure of $\mathcal{R}_m$. Without loss of generality we continue to denote it by z_m. We can easily verify that the extension takes the closure of $\mathcal{R}_m$ onto the closure of $\hat{\mathcal{R}}_m$.

LEMMA 4. *The map z_m is locally Lipschitz on $\overline{\mathcal{R}_m}$.*

Proof. This is immediate from Lemmas 1, 2 and 3. □

Set now $\mathcal{S}_m = \{x \in X : q_m(x) = 1\}$.

LEMMA 5. *We have $z_{m+1}(A_m x) = \hat{A}_m z_m(x)$ whenever $x \in \mathcal{S}_m$.*

Proof. Take $x \in \mathcal{S}_m$. By (33) we have $v_m(x) = (1, x/\|x\|)$. Therefore,

$$
\hat{A}_m z_m(x) = \hat{A}_m(\hat{u}_m \circ H_m \circ v_m)(x) = (\hat{A}_m\hat{u}_m \circ H_m)\left(1, \frac{x}{\|x\|}\right).
$$

By (41) we obtain $H_m(1, x/\|x\|) = (1, x/\|x\|)$, and thus

$$
\hat{A}_m z_m(x) = \hat{A}_m\hat{u}_m\left(1, \frac{x}{\|x\|}\right) = \frac{\hat{A}_m x}{\hat{q}_m(x)}. \tag{42}
$$

On the other hand, since $q_m(x) = 1$ we have

$$
v_{m+1}(A_m x) = (0, A_m x/\|A_m x\|).
$$

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Therefore,

$$\begin{aligned} z_{m+1}(A_m x) &= (\hat{u}_{m+1} \circ H_{m+1} \circ v_{m+1})(A_m x) \\ &= (\hat{u}_{m+1} \circ H_{m+1})\left(0, \frac{A_m x}{\|A_m x\|}\right), \end{aligned}$$

and since

$$H_m\left(0, \frac{A_m x}{\|A_m x\|}\right) = \left(0, \frac{\hat{A}_m x}{\|\hat{A}_m x\|}\right),$$

we obtain

$$z_{m+1}(A_m x) = \hat{u}_{m+1}\left(0, \frac{\hat{A}_m x}{\|\hat{A}_m x\|}\right) = \frac{\hat{A}_m x}{\hat{q}_m(x)}. \tag{43}$$

The desired identity follows from (42) and (43). □

For each $m \in \mathbb{Z}$ we now define the map $h_m : X \to X$ by

$$h_m(x) = \begin{cases} \hat{\mathcal{A}}(m, \tau_{m,x}) z_{\tau_{m,x}}(\mathcal{A}(\tau_{m,x}, m)x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{44}$$

where $\tau = \tau_{m,x}$ is the unique integer for which $\mathcal{A}(\tau, m)x \in \mathcal{R}_\tau$.

LEMMA 6. *The maps h_m satisfy (10).*

Proof. It follows from the identity $\mathcal{A}(\tau, m+1)A_m x = \mathcal{A}(\tau, m)x$ that $\tau_{m+1, A_m x} = \tau_{m,x}$. Therefore,

$$\begin{aligned} h_{m+1}(A_m x) &= \hat{\mathcal{A}}(m+1, \tau_{m,x}) z_{\tau_{m,x}}(\mathcal{A}(\tau_{m,x}, m+1)A_m x) \\ &= \hat{\mathcal{A}}(m+1, \tau_{m,x}) z_{\tau_{m,x}}(\mathcal{A}(\tau_{m,x}, m)x) \\ &= \hat{A}_m \hat{\mathcal{A}}(m, \tau_{m,x}) z_{\tau_{m,x}}(\mathcal{A}(\tau_{m,x}, m)x) = \hat{A}_m h_m(x), \end{aligned}$$

as we wanted to show. □

LEMMA 7. *Each map h_m is invertible.*

Proof. We will indicate explicitly the inverse of h_m. For each $m \in \mathbb{Z}$ we define the map $g_m : X \to X$ by

$$g_m(x) = \begin{cases} \mathcal{A}(m, \hat{\tau}_{m,x}) w_{\hat{\tau}_{m,x}}(\hat{\mathcal{A}}(\hat{\tau}_{m,x}, m)x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $w_m = u_m \circ G_m \circ \hat{u}_m^{-1}$ and where $\hat{\tau} = \hat{\tau}_{m,x} \in \mathbb{Z}$ is the unique integer such that

$$\hat{\mathcal{A}}(\hat{\tau}, m)x \in \hat{\mathcal{R}}_{\hat{\tau}}. \tag{45}$$

We will show that g_m is the inverse of h_m. Set $\tau = \tau_{m, g_m(x)}$ and $\hat{\tau} = \hat{\tau}_{m,x}$. By the definitions of $g_m(x)$ and τ we have

$$\mathcal{A}(\tau, \hat{\tau}) w_{\hat{\tau}}(\hat{\mathcal{A}}(\hat{\tau}, m)x) \in \mathcal{R}_\tau. \tag{46}$$

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On the other hand, by the construction of the map z_m (see (32)) we have that $w_m = z_m^{-1} : \hat{\mathcal{R}}_m \to \mathcal{R}_m$. By (45) we obtain

$$w_{\hat{\tau}}(\hat{\mathcal{A}}(\hat{\tau}, m)x) \in \mathcal{R}_{\hat{\tau}}. \tag{47}$$

Furthermore, in view of the construction of $\tau_{m,x}$ (see §5.1), for each $y = w_{\hat{\tau}}(\hat{\mathcal{A}}(\hat{\tau}, m)x)$ there is a unique integer $k \in \mathbb{Z}$ such that $\mathcal{A}(k, \hat{\tau})y \in \mathcal{R}_k$. But since $y \in \mathcal{R}_{\hat{\tau}}$ (see (47)) we must have $k = \hat{\tau}$. It follows from (46) that $k = \tau = \hat{\tau}$. We now show that g_m is the inverse of h_m. We have

$$h_m(g_m(x)) = \hat{\mathcal{A}}(m, \tau)z_\tau(\mathcal{A}(\tau, m)g_m(x)), \tag{48}$$

and, since $\tau = \hat{\tau}$,

$$\mathcal{A}(\tau, m)g_m(x) = w_\tau(\hat{\mathcal{A}}(\tau, m)x).$$

The map w_τ is the inverse of z_τ, and thus, by (48),

$$h_m(g_m(x)) = \hat{\mathcal{A}}(m, \tau)(z_\tau \circ w_\tau)\hat{\mathcal{A}}(\tau, m)x = x.$$

This completes the proof of the lemma. □

5.6. *Hölder regularity.* We now show that the maps h_m are locally Hölder, and locally Lipschitz outside the origin.

LEMMA 8. *For each $m \in \mathbb{Z}$ there exists a constant $C > 0$ such that $\|h_m(x)\| \le C\|x\|^\alpha$ for all sufficiently small x, with α as in (11).*

Proof. By (27), for a fixed m and x sufficiently close to zero we have $\tau_{m,x} < m$, and $\tau_{m,x} \to -\infty$ when $x \to 0$. Let $\tau = \tau_{m,x}$ and $y = z_\tau(\mathcal{A}(\tau, m)x)$. We have $\hat{q}_\tau(y) \le 1$, and it follows from (26) that

$$\begin{aligned} \|h_m(x)\| &\le \|\hat{\mathcal{A}}(m, \tau)\| \cdot \|y\| \le \|\hat{\mathcal{A}}(m, \tau)\| \frac{\hat{q}_\tau(y)}{\alpha} e^{\varepsilon|\tau|} \\ &\le D\frac{e^{\varepsilon|\tau|}}{\alpha} e^{-a(m-\tau)+\varepsilon|\tau|} \\ &\le D\frac{e^{2\varepsilon|m|}}{\alpha} e^{-(a-2\varepsilon)(m-\tau)}. \end{aligned} \tag{49}$$

By (28), when $x \ne 0$ we have

$$1 < q_{\tau-1}(\mathcal{A}(\tau - 1, m)x) \le \beta e^{\varepsilon|\tau-1|}\|\mathcal{A}(\tau - 1, m)x\|.$$

Using (4) we obtain

$$\begin{aligned} 1 &< \beta D e^{2\varepsilon|\tau-1|} e^{b(m-\tau+1)+\varepsilon|m|}\|x\| \\ &\le \beta D e^{2\varepsilon+b+3\varepsilon|m|} e^{(b+2\varepsilon)(m-\tau)}\|x\|, \end{aligned}$$

which yields

$$e^{m-\tau} \ge \left(\frac{1}{\beta D e^{2\varepsilon+b+3\varepsilon|m|}\|x\|}\right)^{1/(b+2\varepsilon)}.$$

By (49) we find that

$$\|h_m(x)\| \le \frac{D}{\alpha} e^{2\varepsilon|m|}(\beta D e^{2\varepsilon+b+3\varepsilon|m|})^{(a-2\varepsilon)/(b+2\varepsilon)}\|x\|^{(a-2\varepsilon)/(b+2\varepsilon)}.$$

This completes the proof of the lemma. □

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LEMMA 9. *For each $m \in \mathbb{Z}$ the function h_m is locally Hölder with Hölder exponent as in (11), and is locally Lipschitz outside zero.*

Proof. We first observe that the sets $J_n = \mathcal{A}(m, n)\mathcal{R}_n$, $n \in \mathbb{Z}$, are pairwise disjoint. Furthermore, their union is $X \setminus \{0\}$. These are easy consequences of the discussion in §5.1. Clearly, for each $x \in J_n$ we have $\tau_{m,x} = n$. Thus, it follows readily from the definition of h_m that

$$h_m | J_n = \hat{\mathcal{A}}(m, n) z_n(\mathcal{A}(n, m)x).$$

In view of Lemma 4, the map h_m is locally Lipschitz on each J_n. Furthermore, also by Lemma 4, h_m has a unique continuous extension $\bar{h}_m$ to $\overline{J_n}$, and this remains Lipschitz. Thus, to show that h_m is locally Lipschitz outside the origin it remains to verify that

$$\bar{h}_m = h_m \quad \text{on} \quad \overline{J_n} \setminus J_n = \mathcal{A}(m, n)\mathcal{S}_{n-1} \subset J_{n-1}.$$

For $x \in \mathcal{A}(m, n)\mathcal{S}_{n-1}$, it follows from Lemma 5 that

$$\begin{aligned}
&\hat{\mathcal{A}}(m, n) z_n(\mathcal{A}(n, m)x) \\
&\quad = \hat{\mathcal{A}}(m, n) z_n(A_{n-1}\mathcal{A}(n-1, m)x) \\
&\quad = \hat{\mathcal{A}}(m, n)\hat{A}_{n-1} z_{n-1}(\mathcal{A}(n-1, m)x) \\
&\quad = \hat{\mathcal{A}}(m, n-1) z_{n-1}(\mathcal{A}(n-1, m)x) = h_m(x),
\end{aligned}$$

since $x \in J_{n-1}$. This completes the proof of the Lipschitz property.

It remains to verify that h_m is locally Hölder. For $x, y \in X$ with $x \neq 0$ and $y = 0$ this is the content of Lemma 8. For $x, y \in X \setminus \{0\}$ the Hölder property is an immediate consequence of the Lipschitz property. □

5.7. *Proofs of the theorems.*

Proof of Theorem 2. It follows from Lemmas 6, 7 and 9 that there exist homeomorphisms h_m satisfying (10). The regularity properties of h_m follow also from Lemma 9. Interchanging in the proofs the roles of the matrices A_m and $\hat{A}_m$, we find that $h_m^{-1} = g_m$ has the properties in Lemma 9.

To establish Property A assume that $A_m = A$ and $\hat{A}_m = \hat{A}$ for every $m \in \mathbb{Z}$ and some linear operators A and $\hat{A}$. Choosing the curves $\gamma_m : [0, 1] \to \mathcal{B}(X)$ independent of m, it follows from (44) that, for $x \neq 0$,

$$h_m(x) = (\hat{A}^{m-\tau} \circ \hat{u}_\tau \circ H_\tau \circ v_\tau \circ A^{\tau-m})(x), \tag{50}$$

where $\tau = \tau_{m,x}$ is determined by $A^{\tau-m}x \in \mathcal{R}_\tau$. Furthermore, by (37), (41) and (33), the maps $\hat{u}_\tau$, H_τ and v_τ are independent of τ. By (31), $\mathcal{R}_\tau$ is also independent of τ, and thus $\tau - m = \tau_{m,x} - \tau$ is independent of m. It follows readily from (50) that $h_m(x)$ is independent of m. □

Proof of Theorem 6. We first obtain conjugacies separately in the stable and unstable components. By (21), (22) and Theorem 2, there exist homeomorphisms $h_m^- : E \to E$, $m \in \mathbb{Z}$, such that

$$h_{m+1}^- \circ B_m = \hat{B}_m \circ h_m^-, \quad m \in \mathbb{Z}, \tag{51}$$

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and homeomorphisms $h_m^+ \colon F \to F$, $m \in \mathbb{Z}$, such that

$$h_{m+1}^+ \circ C_{-m} = \hat{C}_{-m} \circ h_m^+, \quad m \in \mathbb{Z}. \tag{52}$$

One can easily verify that for each $m \in \mathbb{Z}$ the map $h_m \colon X \to X$ defined by

$$h_m(x, y) = h_m^-(x) + h_{-m}^+(y), \quad (x, y) \in E \times F, \tag{53}$$

is a homeomorphism. The identities in (10) follow readily from (51) and (52).

Furthermore, again by Theorem 2, for each $m \in \mathbb{Z}$ the homeomorphisms h_m^- and h_{-m}^+ are locally Hölder, and are locally Lipschitz outside zero. The same happens with their inverses. Using (53) we readily obtain the same properties for h_m. This completes the proof of the theorem. □

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