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A NOTE ON ASYMPTOTIC NORMAL STRUCTURE AND CLOSE-TO-NORMAL STRUCTURE

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ABSTRACT. A closed convex subset X of a Banach space E is said to have (i) asymptotic normal structure if for each bounded closed convex subset C of X containing more than one point and for each sequence $(x_n)_{n=1}^{\infty}$ in C satisfying $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, there is a point $x \in C$ such that $\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}_{\| \|}(C)$; (ii) close-to-normal structure if for each bounded closed convex subset C of X containing more than one point, there is a point $x \in C$ such that $\|x - y\| < \operatorname{diam}_{\| \|}(C)$ for all $y \in C$. While asymptotic normal structure and close-to-normal structure are both implied by normal structure, they are not related. The example that a reflexive Banach space which has asymptotic normal structure but not close-to normal structure provides us a non-empty weakly compact convex set which does not have close-to-normal structure. This answers an open question posed by Wong in [9] and hence also provides us a Kannan map defined on a weakly compact convex set which does not have a fixed point.

1. Introduction. The notion of normal structure was first introduced by Brodskii and Milman in [3]: A closed convex subset X of a Banach space $(E, \| \|)$ is said to have normal structure if for each bounded closed convex subset C of X with more than one point, there is a point $x \in C$ such that $\sup\{\|x - y\| : y \in C\} < \operatorname{diam}_{\| \|}(C)$. In [4], Kirk proved that every non-expansive mapping defined on a weakly compact convex subset X of a Banach space with normal structure has a fixed point. (A mapping $f: X \to X$ is non-expansive if $\|f(x) - f(y)\| \le \|x - y\|$, for all $x, y \in X$.) In [2], Baillon and Schöneberg introduced the notion of asymptotic normal structure as follows: A closed convex subset X of a Banach space $(E, \| \|)$ is said to have asymptotic normal structure if for each bounded closed convex subset C of X with more than one point and for each sequence $(x_n)_{n=1}^{\infty}$ in C satisfying $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, there is a point $x \in C$ such that $\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}_{\| \|}(C)$. They proved that Kirk's result remains true if "normal structure"; is replaced by "asymptotic

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normal structure" and at the same time showed that the class of spaces which have normal structure is properly contained in the class of spaces which have asymptotic normal structure.

In [8] and [9], Wong introduced the notion of close-to-normal structure: A closed convex subset X of a Banach space $(E, \| \ \|)$ is said to have close-to-normal structure if for each bounded closed convex subset C of X with more than one point, there is a point $x \in C$ such that $\|x-y\| < \operatorname{diam}_{\| \ \|}(C)$ for all $y \in C$. He showed that every Banach space $(E, \| \ \|)$ which is either strictly convex or separable or which has the property A (i.e. for every sequence $(x_n)_{n=1}^{\infty}$ in E, $x \in E$, if $x_n \to x$ (weakly) and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$) has close-to-normal structure. He also proved that in a normed space $(E, \| \ \|)$, every weakly compact convex subset of E has close-to-normal structure if and only if every Kannan map on a non-empty weakly compact convex subset X of E has a unique fixed point. (A mapping $f: X \to X$ is a Kannan map if $\|f(x)-f(y)\| \le \frac{1}{2}(\|x-f(x)\|+\|y-f(y)\|)$, for all $x, y \in X$).

In this note, it is our purpose to show that while both asymptotic normal structure and close-to-normal structure are implied by normal structure, they are not related. Incidentally, the example that a reflexive Banach space which has asymptotic normal structure but not close-to-normal structure provides us a non-empty weakly compact convex set which does not have close-to-normal structure. This answers an open question posed in [9] and hence also provides us a Kannan map defined on a weakly compact convex set which does not have a fixed point.

2. Main results. A fundamental open question in fixed point theory has been whether or not every weakly compact convex subset of a Banach space has a fixed point property for non-expansive mappings. Recently, Alspach [1] (see also [5]) has answered this in the negative by providing the following example:

Example 1. Let $E = L^1[0, 1]$ and $X = \{f \in E : \int_0^1 f = 1 \text{ and } 0 \le f \le 2 \text{ a.e.}\}$. Then K is weakly compact convex. Define $T: X \to X$ by

$$Tf(t) = \begin{cases} \min\{2f(2t), 2\}, & 0 \le t \le \frac{1}{2} \\ \max\{2f(2t-1)-2, 0\}, & \frac{1}{2} < t \le 1. \end{cases}$$

Then T is non-expansive (in fact, isometric) on K which has no fixed point.

Remark 2. By Baillon and Schöneberg's result, the set K in Example 1 can not possess asymptotic normal structure. Since $L^1[0, 1]$ is separable, Wong's result shows that K does possess close-to-normal structure. Incidentally, Alspach's example shows that a weakly compact convex subset of a Banach space with close-to-normal structure need not have fixed point property for non-expansive mappings. This Example 1 also settles an open question posed by Wong in [9].

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In [6], Routledge proved the following

THEOREM 3. *Let A be a non-empty set, of finite diameter δ, in l_2 (real or complex). Then there is a unique sphere of minimum radius containing A, and this radius does not exceed $\delta/\sqrt{2}$. Furthermore, this constant is the best possible.*

Even though Routledge proved the above theorem for the separable Hilbert space l_2, his proof can be modified to hold for any Hilbert space. Thus we have:

THEOREM 3'. *Let H be any Hilbert space (real or complex), and A be a non-empty closed convex subset of H with finite diameter δ. Then there is a unique point $a \in A$ and a minimum radius $\gamma \leq \delta/\sqrt{2}$ such that the sphere with center a and radius γ contains A.*

For a simple, direct and unified proof of the above general Theorem 3', see [7].

EXAMPLE 4. Let Γ be uncountable and $\lambda \geq 1$. Let X_λ be the space

$$l_2(\Gamma) = \left\{ x = (x_\alpha)_{\alpha \in \Gamma} : \sum_{\alpha \in \Gamma} |x_\alpha|^2 < \infty \right\}$$

with norm $\| \ \|$ defined by

$$\|x\| = \max\{\|x\|_2, \lambda \|x\|_\infty\},$$

where $\|x\|_\infty = \sup\{|x_\alpha| : \alpha \in \Gamma\}$ and $\|x\|_2 = (\sum_{\alpha \in \Gamma} |x_\alpha|^2)^{1/2}$. Let C be a non-empty bounded closed convex subset of X_λ with

$$0 < d = \text{diam}_{\| \ \|}(C) < \infty.$$

Since $\|x\|_2 \leq \|x\| \leq \lambda \|x\|_2$ for all $x \in X_\lambda$, $\| \ \|_2$ and $\| \ \|$ are equivalent norms on X_λ, so that C is also bounded under $\| \ \|_2$; thus by Theorem 3', $\exists x \in C$, such that

$$(*) \qquad \sup\{\|x - y\|_2 : y \in C\} \leq \text{diam}_{\| \ \|_2}(C)/\sqrt{2},$$

$$\begin{aligned} \therefore \sup\{\|x - y\| : y \in C\} &\leq \lambda \sup\{\|x - y\|_2 : y \in C\} \\ &\leq \frac{\lambda}{\sqrt{2}} \text{diam}_{\| \ \|_2}(C) \\ &\leq \frac{\lambda}{\sqrt{2}} \text{diam}_{\| \ \|}(C) \\ &< \text{diam}_{\| \ \|}(C), \text{ if } 1 \leq \lambda < \sqrt{2}, \end{aligned}$$

so that X_λ has normal structure and hence also close-to-normal structure for $1 \leq \lambda < \sqrt{2}$.

Suppose $1 \leq \lambda < 2$. Let $(x_n)_{n=1}^\infty$ be any sequence in C such that $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. By (*), we have for all $n = 1, 2, \ldots,$

$$\|x - x_n\|_2 \leq \text{diam}_{\| \ \|_2}(C)/\sqrt{2} \leq \text{diam}_{\| \ \|}(C)/\sqrt{2} = d/\sqrt{2}.$$

If $\liminf_{n\to\infty} \|x-x_n\|<d$, we are done. Suppose $\liminf_{n\to\infty} \|x-x_n\|=d$, then $\lim_{n\to\infty} \|x-x_n\|=d$ so that

$$\lim_{n\to\infty} \|x-x_n\|_\infty = d/\lambda.$$

Since $l_2(\Gamma)$, being a Hilbert space, is reflexive and $\| \ \|$ and $\| \ \|_2$ are equivalent, C is weakly compact in X_λ. Let $(x_{n_i})_{i=1}^\infty$ be any subsequence of $(x_n)_{n=1}^\infty$ and $\bar{x}\in C$ such that $x_{n_i}\to\bar{x}$ weakly as $i\to\infty$. Then the proof in [2] (with j_n replaced by $\alpha_n\in\Gamma$ for each $n=1,2,\ldots$) can be used to show $\|\bar{x}-x\|_2\geq d/\lambda>d/2$. It follows that

$$\limsup_{i\to\infty} \|x_{n_i}-\bar{x}\|_2^2 \leq (d/\sqrt{2})^2-\|\bar{x}-x\|_2^2<d^2/4,$$

so that

$$\limsup_{i\to\infty} \|x_{n_i}-\bar{x}\| \leq \lambda \limsup_{i\to\infty} \|x_{n_i}-\bar{x}\|_2 \leq \lambda\ d/2<d \text{ since } \lambda<2,$$

so that

$$\liminf_{n\to\infty} \|x_n-\bar{x}\| \leq \limsup_{i\to\infty} \|x_{n_i}-\bar{x}\|<d.$$

Therefore X_λ has asymptotic normal structure for all $1\leq\lambda<2$. Finally we shall now show that X_λ does not have close-to-normal structure for $\sqrt{2}\leq\lambda<2$. Let $D=\{x\in X_\lambda : \|x\|_2\leq 1 \text{ and } x_\alpha\geq 0 \text{ for all } \alpha\in\Gamma\}$, then clearly D is non-empty bounded closed and convex, and hence weakly compact convex (as X_λ is reflexive). If $x, y\in D$,

$$\begin{aligned}\|x-y\|_2 &= \Big(\sum_{\alpha\in\Gamma} |x_\alpha-y_\alpha|^2\Big)^{1/2}\\ &\leq \Big(\sum_{\alpha\in\Gamma} (|x_\alpha|^2+|y_\alpha|^2)\Big)^{1/2}, \qquad \because x_\alpha\geq 0 \quad\text{and}\quad y_\alpha\geq 0,\\ &\leq\sqrt{2},\end{aligned}$$

$$\lambda\ \|x-y\|_\infty = \lambda \sup_{\alpha\in\Gamma} |x_\alpha-y_\alpha| \leq \lambda, \qquad \because 0\leq x_\alpha,\ y_\alpha\leq 1,$$

It follows that $\operatorname{diam}_{\| \|}(D)\leq\max\{\sqrt{2},\lambda\}=\lambda$, as $\lambda\geq\sqrt{2}$. On the other hand, for $\alpha\in\Gamma$, define

$$e_\gamma^{(\alpha)}=\begin{cases}0, & \text{if } \gamma\neq\alpha,\\ 1, & \text{if } \gamma=\alpha,\end{cases}$$

then $e^{(\alpha)}\in D$, $\forall\alpha\in\Gamma$. Take any $\alpha, \beta\in\Gamma$ with $\alpha\neq\beta$, we have

$$\|e^{(\alpha)}-e^{(\beta)}\|_2=\sqrt{2}, \quad \text{and} \quad \lambda\ \|e^{(\alpha)}-e^{(\beta)}\|_\infty=\lambda$$

so that $\operatorname{diam}_{\| \|}(D)\geq\max\{\sqrt{2},\lambda\}=\lambda$. Hence $\operatorname{diam}_{\| \|}(D)=\lambda$. Now let $x\in D$, then $\Gamma_x=\{\alpha\in\Gamma \mid x_\alpha\neq 0\}$ is countable. Choose any $\alpha(x)\in\Gamma$ such that $\alpha(x)\notin\Gamma_x$, then

$e^{(\alpha(x))} \in D$ and

$$\begin{aligned}\|x - e^{(\alpha(x))}\| &= \max\{\|x - e^{(\alpha(x))}\|_2,\ \lambda\ \|x - e^{(\alpha(x))}\|_\infty\} \\ &= \max\left\{\left(1 + \sum x_\alpha^2\right)^{1/2},\ \lambda\ \max\{1, \|x\|_\infty\}\right\} \\ &= \lambda = \operatorname{diam}_{\|\ \|}(D).\end{aligned}$$

Therefore X_λ (and D) does not possess close-to-normal structure.

REMARK 5. In the above example, the set D is non-empty weakly compact convex which does not have close-to-normal structure. This answers an open question posed by Wong in [9] in the negative. If we define $f: D \to D$ by $f(x) = e^{(\alpha(x))}$, for all $x \in D$, then

$$\begin{aligned}\tfrac{1}{2}(\|x - f(x)\| + \|y - f(y)\|) &= \tfrac{1}{2}(\|x - e^{(\alpha(x))}\| \\ &\quad + \|y - e^{(\alpha(y))}\|) = \tfrac{1}{2}(\lambda + \lambda) = \lambda \geq \|e^{(\alpha(x))} - e^{(\alpha(y))}\| = \|f(x) - f(y)\|,\end{aligned}$$

for all x, $y \in D$, so that f is a Kannan map on D which fails to have a fixed point.

Added in proof. Since every non-separable Hilbert space H is isometrically isomorphic to some $l_2(\Gamma)$ (where Γ is the Hilbert dimension of H), together with Wong's results in [8], we have

THEOREM 6. A Hilbert space is isomorphic to a Banach space which does not have close-to-normal structure if and only if it is not separable.

Wong announced the above result in [10] without proof; however, in that paper, "not separable" was misprinted as "separable".

REFERENCES

1. D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981) 423–424.
2. J.-B. Baillon and R. Schöneberg, *Asymptotic normal structure and fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **81** (1981) 257–264.
3. M. S. Brodskii and D. P. Milman, On the centre of a convex set, Dokl. Akad. Nauk. SSSR 59 (1948), pp. 837–840, MR9 #448.
4. W. A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly **72** (1968), pp. 1004–1006.
5. W. A. Kirk, *Fixed point theory for nonexpansive mappings*, preprint.
6. N. A. Routledge, *A result in Hilbert space*, Quart. J. Math. **3** (1952), pp. 12–18.
7. K. K. Tan, *A short proof of a theorem of Routledge*, submitted.
8. C. S. Wong, *Close-to-normal structure and its applications*, J. Functional Analysis **16** (1974), pp. 353–358.
9. C. S. Wong, *On Kannan maps*, Proc. Amer. Math. Soc. **47** (1975), pp. 105–111.
10. C. S. Wong, *Fixed points and characterizations of certain maps*, Pacific J. Math. **54** (1974), pp. 305–312.

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