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On the decay properties of solutions to the non-stationary Navier–Stokes equations in $\mathbb{R}^3$

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In this paper, we study the asymptotic decay properties in both spatial and temporal variables for a class of weak and strong solutions, by constructing the weak and strong solutions in corresponding weighted spaces. It is shown that, for the strong solution, the rate of temporal decay depends on the rate of spatial decay of the initial data. Such rates of decay are optimal.

1. Introduction

We consider the decay properties of solutions to the non-stationary Navier–Stokes equations in $\mathbb{R}^3 \times [0, +\infty)$:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u = -\nabla p, &\quad \text{in } \mathbb{R}^3 \times (0,\infty),\\ \operatorname{div} u = 0, &\quad \text{in } \mathbb{R}^3 \times (0,\infty),\\ u \to 0, &\quad \text{as } |x| \to +\infty,\\ u(x,0) = a(x), &\quad \text{in } \mathbb{R}^3. \end{aligned}\right\} \tag{1.1}$$

Here $u = u(x,t) = (u_1,u_2,u_3)$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$ respectively, while $\nu > 0$ is the viscosity and $a(x)$ is a given initial velocity vector field. For simplicity, let $\nu = 1$.

Since Leray [25] constructed the weak solutions for (1.1) in $\mathbb{R}^3$, and posed the question about the decay properties of his weak solution, there is large literature discussing the decay properties of weak solutions and strong solutions to the non-stationary Navier–Stokes equations [1–4, 7, 8, 10, 11, 14–20, 22, 23, 26–34, 40, 42]. The L^2 decay properties have been extensively studied and the decay rates, similar to

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that of heat equation, are also obtained. Their results show: for each $a \in \mathring{J}^2(\mathbb{R}^3)$, subspace of $L^2(\mathbb{R}^3)$ consisting of all solenoidal vector fields, there exists a weak solution u such that

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{1.2}$$

If $a \in L^r(\mathbb{R}^3)(1 \leqslant r < 2)$, then

$$\|u(t)\|_2 \leqslant Ct^{-(3/r-3/2)/2}. \tag{1.3}$$

Here and after, $\|\cdot\|_r$ denotes the norm in $L^r(\mathbb{R}^3)$. Furthermore, if $a \in \mathring{J}^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, then

$$\|u(t)\|_r \leqslant C(1+t)^{-(3-3/r)/2} \tag{1.4}$$

for $1 < r \leqslant 2$ and

$$\lim_{t\to\infty} \|u(t)\|_1 = 0. \tag{1.5}$$

If $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\beta}$ for some $\beta > 0$ with A being the Stokes operator, then

$$\|u(t)\|_r \leqslant C(1+t)^{-(3-3/r+\gamma)/2}, \qquad \gamma = \min\{1, 2\beta\}, \tag{1.6}$$

for $1 \leqslant r \leqslant 2$. See [2–4,19,26,27,29,31–33,41]. Schonbek gave the estimates of upper and the lower bound of decay rates for weak solutions in a series of works [32–34]. Moreover, the decay rates of high-order derivatives have been studied in [22, 23] for weak solutions after sufficient large time. Recently, Miyakawa studied the decay rates in Hardy space and established the decay results for weak solutions in some 'L^r-like space' with $r < 1$. For details, see [29, 30] and references therein. The time-decay properties are therefore well understood. However, there are few results on the spatial decay properties. Farwing and Sohr [11] showed a class of weighted ($|x|^\alpha$) weak solutions with second derivatives about spatial various and one order derivatives about time variable in $L^s(0, +\infty; L^q)$ for $1 < q < 3/2$, $1 < s < 2$ and $0 \leqslant 3/q + 2/s - 4 \leqslant \alpha < \min\{1/2, 3 - 3/q\}$, in the case of exterior domain. In [10], they also showed that there exists a class of weak solutions satisfying

$$\||x|^{\alpha/2}u\|_2^2 + \int_0^t \||x|^{\alpha/2}\nabla u\|_2^2 \,\mathrm{d}\tau \leqslant \begin{cases} C(a, f, \alpha) & \text{if } 0 \leqslant \alpha < \frac{1}{2}, \\ C(a, f, \alpha', \alpha)t^{\alpha'/2-1/4} & \text{if } \frac{1}{2} \leqslant \alpha < \alpha' < 1, \\ C(a, f)(t^{1/4} + t^{1/2}) & \text{if } \alpha = 1. \end{cases} \tag{1.7}$$

While in [15], a class of weak solutions

$$(1 + |x|^2)^{1/4}u \in L^\infty(0, +\infty; L^p(\mathbb{R}^3)) \tag{1.8}$$

was constructed for $6/5 \leqslant p < 3/2$, which satisfies $(1.7)_3$ for $f = 0$.

The first purpose of this paper is to construct weak solutions in weighted spaces. We show that, if $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$ and $|x|a \in L^2(\mathbb{R}^3)$, there exists a class of weak solution u satisfying

$$\|(1 + |x|^2)^{1/2}u(t)\|_2^2 + \int_0^t \|(1 + |x|^2)^{1/2}\nabla u(\tau)\|_2^2 \,\mathrm{d}\tau \leqslant C,$$

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which improves the corresponding local weighted estimate as $(1.7)_3$ in [15]. The interpolation inequality implies that u satisfies

$$(1+|x|)^\alpha u(t) \in L^\infty(0,+\infty; L^p(\mathbb{R}^3))$$

for $1 < p \leqslant 2$ and $\alpha = 2(p-1)/p$, which also improves estimate (1.8). Furthermore, if $|x|^{3/2}a \in L^2(\mathbb{R}^3)$, then we show that there exists a class of weak solutions satisfying

$$\|(1+|x|^2)^{\alpha/2}u(t)\|_2^2 + \int_0^t \|(1+|x|^2)^{\alpha/2}\nabla u(\tau)\|_2^2 \,\mathrm{d}\tau \leqslant C(1+\log(1+t)), \quad \forall t \geqslant 0, \tag{1.9}$$

for $0 \leqslant \alpha \leqslant 3$. If $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\gamma}$ for some $\gamma > 0$, then the right-hand side of (1.9) can be replaced by a constant independent of t. It should be noted that Schonbek and Schonbek [35] studied the decay properties of the moment estimate $\||x|^{\alpha/2}u\|_2$ for smooth solutions.

The second purpose of this paper is to establish the estimates of decay rate of weighted norm for strong solutions in $L^p(\mathbb{R}^3)$. For the existence and decay properties of strong solutions, there are many results, see [1, 5–9, 12–18, 20, 21, 28, 40, 42]. In particular, Heywood showed the existence and decay properties of strong solutions as $\|a\|_{H^1}$ is small in [17]. Miyakawa [28] improved Heywood's results by a different method. On the other hand, many results about the existence and decay properties of strong solution have been obtained in space $L^p(0,T;L^q(R^n))$ for $q \geqslant n \geqslant 3$ and $1/p+n/2q \leqslant 1/2$ in [1,6–9,12,14,16,18,20,21,42]. In particular, let $BC(0,T)(T>0)$ denote the set of bounded and continuous functions defined in $(0,T)$, Kato [20] first obtains the following decay rates for strong solutions in $L^p(R^n)$

$$t^{(1-n/q)/2}u \in BC([0,+\infty; L^q(R^n)) \qquad \forall n \leqslant q \leqslant +\infty, \tag{1.10}$$

$$t^{(1-n/q)/2+1/2}\nabla u \in BC([0,+\infty; L^q(R^n)) \qquad \forall n \leqslant q < +\infty \tag{1.11}$$

provided that $a \in L^n(R^n)$ and $\|a\|_n$ is small. In the case that $a \in L^n(R^n) \cap L^p(R^n)(1 < p < n)$, then

$$t^{(3/p-n/q)/2}u \in BC([1,+\infty; L^q(R^n)) \qquad \forall p \leqslant q \leqslant +\infty, \tag{1.12}$$

$$t^{(n/p-n/q)/2+1/2}\nabla u \in BC([1,+\infty; L^q(R^n)) \qquad \forall p \leqslant q < +\infty \tag{1.13}$$

provided the exponent of t in (1.12) and (1.13) is smaller than 1; otherwise, (1.12) and (1.13) are valid for any positive number less than 1 as the exponent of t is bigger than 1. By adding a correction term, Carpio [7] showed that, for initial data a satisfying (i) $\|a\|_n (n \geqslant 3)$ is small; (ii) $a \in L^p(R^n) \cap L^n(R^n)(1 \leqslant p < n)$, the solution to the Navier–Stokes equations behaves like the solutions of the heat equations taking the same initial data. Thus Carpio [7] removes Kato's restriction on exponent of t and extends the decay estimate to reach the case $p = 1$. So this problem is also well understood. While for the spatial decay properties at large distance, a class of weighted strong solutions was constructed, which satisfies that $(1+|x|^2)u \in L^\infty(0,+\infty; L^2(\mathbb{R}^3))$ and $t^{1/2}\nabla u \in L^\infty(0,+\infty; L^2(\mathbb{R}^3))$, if $\|a\|_2 + \|a\|_p$ small for some $1 \leqslant p < 2$ in [15], and similar results for exterior domains in [16]. Recently, Takahashi [39] showed that if u is a smooth solution, then

$$t^\beta |x|^\alpha |u(x,t)| \leqslant C \quad \text{for } |x|+t \text{ large} \tag{1.14}$$

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for all $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that $\alpha + 2\beta < 3$. Meanwhile, Miyakawa [30] pointed out that if $a = \partial b/\partial x_1$, then

$$\sup_{t>0} |e^{-tA}a| \sim C|x|^{-1-n} \quad \text{and} \quad \sup_{x} |e^{-tA}a| \sim Ct^{-(n+1)/2},$$

as $|x| \to \infty$ and $t \to \infty$. Thus he conjectured that if u is a smooth solution, then u should satisfy that

$$|u(x,t)| \sim C|x|^{-\alpha}t^{-\beta} \qquad \text{as } |x| + t \to \infty, \tag{1.15}$$

for all $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that $\alpha + 2\beta = n + 1$.

In this paper, we show that, with the hypothesis that $\|a\|_1 + \|a\|_2$ is small, there exists a unique strong solution u such that

$$t^{\beta}\|(1+|x|^2)^{\alpha/2}u(t)\|_p \leqslant C \quad \text{for } t \geqslant 0 \tag{1.16}$$

and

$$t^{\beta+1/2}\|(1+|x|^2)^{\alpha/2}\nabla u(t)\|_p \leqslant C \quad \text{for } t \geqslant 0 \tag{1.17}$$

with $\alpha = 3 - 3/p_0$, $\beta = (3/p_0 - 3/p)/2$ for $1 \leqslant p_0 \leqslant p \leqslant +\infty$ and $3 < p$. So $\alpha + 2\beta = 3 - 3/p$. Therefore, if $p = \infty$, our results improve (1.14). Moreover, the time decay rates of weighted norm of the solutions also improve the results of Kato's in the sense that there is no restriction on exponent of t and p_0 can be taken to be 1. If $a = \partial b/\partial x_i$ for some $i = 1, 2, 3$, we show that, if $\|a\|_1 + \|a\|_2$ small, there exists a unique strong solution u, which satisfies

$$t^{\beta+1/2}\|(1+|x|^2)^{\alpha/2}u(t)\|_p \leqslant C \quad \text{for } t \geqslant 0. \tag{1.18}$$

In the special case $p = \infty$, our result (1.18) shows that Miyakawa's conjecture (1.15) is right. Finally, it should be noted that in order to obtain the global strong solutions, we assume the smallness of the initial data, i.e. $\|a\|_1 + \|a\|_2 \leqslant \delta$, which is different from any previous known small assumptions.

The paper is organized as follows. In § 2, we state our main results. A new class of approximate solutions is constructed and then the integral representations are delivered in § 3. In §§ 4 and 5, we establish the main weighted estimates for weak and strong solutions.

We conclude this introduction by listing some notation used in the rest of the paper.

Let $L^p(\mathbb{R}^3)$, $1 \leqslant p \leqslant +\infty$, represent the usual Lebesgue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^{\infty}_{0,\sigma}(\mathbb{R}^3)$ denote the set of all C^{∞} real vector-valued functions $\phi = (\phi_1, \phi_2, \phi_3)$ with compact support in $\mathbb{R}^3$, such that $\operatorname{div}\phi = 0$. $\mathring{J}^p(\mathbb{R}^3)$, $1 \leqslant p < \infty$, is the closure of $C^{\infty}_{0,\sigma}(\mathbb{R}^3)$ with respect to $\|\cdot\|_p$. $H^m(\mathbb{R}^3)$ denotes the usual Sobolev Space. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1 \leqslant p \leqslant +\infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|_X^p \, dt < +\infty$. Let P be the Helmholtz projection from $L^p(\mathbb{R}^3)$ to $\mathring{J}^p(\mathbb{R}^3)$. Then the Stokes operator A is defined by $A = -P\Delta$ with $D(A) = H^2(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$. Let

$$D^2 = \sum_{i,j}\left|\frac{\partial^2}{\partial x_i \partial x_j}\right| \quad \text{and} \quad D^3 = \sum_{jik}\left|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\right|.$$

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At last, we denote a generic constant by C, the value of which is inessential to our aims, and which may change from line to line.

2. The main results

We first give the definitions of weak and strong solutions.

DEFINITION 2.1. u is called a weak solution to the Cauchy problem (1.1) if

(i) $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ for any $T > 0$;

(ii) u satisfies the equations (1.1) in the sense of distribution, i.e.

$$\int_0^\infty \int_{\mathbb{R}^3} \left(-\frac{\partial \phi}{\partial \tau} u + \nabla u \cdot \nabla \phi + (u \cdot \nabla) u \cdot \phi \right) \mathrm{d}x \, \mathrm{d}\tau = \int_{\mathbb{R}^3} \phi(x,0) a(x)$$

for every $\phi \in C^\infty_{0,\sigma}(R \times \mathbb{R}^3)$.

(iii) $\operatorname{div} u = 0$ in the sense of distribution, i.e.

$$\int_{\mathbb{R}^3} u(x,t) \nabla \psi(x) = 0$$

for every $\psi \in C_0^\infty(\mathbb{R}^3)$.

DEFINITION 2.2. u is called a strong solution to the Cauchy problem (1.1) if $u \in L^\infty(0,T;L^p(\mathbb{R}^3))$ for $3 < p \leqslant +\infty$ and any $T > 0$, and (ii) and (iii) in definition 2.1 hold for u.

The main results in this paper are described in the following theorems. Our first result concerns the existence and global estimates of weak solutions in a weighted L^2-space.

THEOREM 2.3. *Let $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$ and $(1+|x|^2)^{1/2} a \in L^2(\mathbb{R}^3)$. Then there exists a weak solution u in $L^\infty(0,+\infty;L^2(\mathbb{R}^3))$ to (1.1) such that*

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, \mathrm{d}\tau \leqslant 4\|a\|_2^2 \tag{2.1}$$

and

$$\|(1+|x|^2)^{1/2} u\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2} \nabla u\|_2^2 \, \mathrm{d}\tau \leqslant C A_1 \tag{2.2}$$

for any $t \geqslant 0$. Moreover, for $t \geqslant 0$,

$$\|u(t)\|_2 \leqslant C N (1+t)^{-3/4} \tag{2.3}$$

and

$$\|u(t)\|_1 \leqslant C B. \tag{2.4}$$

Here

$$\begin{aligned} A_1 &= \mathrm{e}^{C\|a\|_2^2} (\|(1+|x|^2)^{1/2} a\|_2^2 + N^2), \\ N &= \|a\|_1 + \|a\|_1^2 + \|a\|_2 + \|a\|_2^2, \\ B &= \|a\|_1 + \|a\|_2 N. \end{aligned}$$

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In the case that the initial velocity field possesses higher-order moments, we have the following estimates.

THEOREM 2.4. *Let* $a \in L^2(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$ *and* $|x|^{3/2}a \in L^2(\mathbb{R}^3)$, *then there exists a weak solution* u *in* $L^\infty(0,+\infty; L^2(\mathbb{R}^3))$ *to (1.1), which satisfies that*

$$\int_{\mathbb{R}^3} (1+|x|^2)^{3/2} u^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{3/2} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}\tau \leqslant C(A_2 + B^{2/3}N^{4/3}\log(1+t)) \tag{2.5}$$

for any $t \geqslant 0$, *(2.1)-(2.4) are valid for* u. *Moreover, if* $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\gamma}$ *for some* $\gamma > 0$, *then*

$$\int_{\mathbb{R}^3} (1+|x|^2)^{3/2} u^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{3/2} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}\tau \leqslant C(A_2 + B^{2/3}N^{4/3}) \tag{2.6}$$

for any $t \geqslant 0$ *with* $A_2 = A_1^{3/2}(\|a\|_2^{5/4}N^{3/4} + N\|a\|_2^{5/6})$.

Next, the weighted norm (both in time and in space) estimates of strong solutions are established in the following theorem.

THEOREM 2.5. *Let* $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$, $(1+|x|^2)^{1/2}a \in L^2(\mathbb{R}^3)$ *and* $(1+|x|^2)^{\alpha/2}a \in L^{p_0}(\mathbb{R}^3)$ *for* $1 \leqslant p_0 \leqslant +\infty$ *and* $\alpha = 3 - 3/p_0$. *Then there exists a constant* $\lambda > 0$, *such that if* $\|a\|_1 + \|a\|_2 \leqslant \lambda$, *then there exists a unique strong solution* $u \in L^\infty(0,+\infty; L^p(\mathbb{R}^3))$ *for* $3 < p \leqslant +\infty$, *which satisfies the estimate*

$$\begin{aligned} &\|t^\beta(1+|x|^2)^{\alpha/2}u\|_p + \|t^{1/2+\beta}(1+|x|^2)^{\alpha/2}\nabla u\|_p \\ &\qquad \leqslant C(\|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N) \end{aligned} \tag{2.7}$$

for any $t \geqslant 0$ *and* $\beta = (3/p_0 - 3/p)/2$.

Finally, for a class of special initial data, the results in theorem 2.5 can be improved by the following theorem.

THEOREM 2.6. *Let* $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$ *and* $(1+|x|^2)^{1/2}a \in L^2(\mathbb{R}^3)$. *Let* $a = \partial b/\partial x_i$ *for some* $i = 1,2,3$ *with* $b \in L^1(\mathbb{R}^3)$ *and* $(1+|x|^2)^{\alpha/2}b \in L^{p_0}(\mathbb{R}^3)$ *for* $1 \leqslant p_0 \leqslant +\infty$ *and* $\alpha = 3 - 3/p_0$. *Then there exists a constant* $\lambda_0 > 0$, *such that if* $\|a\|_1 + \|a\|_2 \leqslant \lambda_0$, *then there exists a unique strong solution* $u \in L^\infty(0,+\infty; L^p(\mathbb{R}^3))$ *for* $3 < p \leqslant +\infty$, *which satisfies the estimate*

$$\|t^{1/2+\beta}(1+|x|^2)^{\alpha/2}u\|_p \leqslant C(\|(1+|x|^2)^{\alpha/2}b\|_{p_0} + \|b\|_1 + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N) \tag{2.8}$$

for any $t \geqslant 0$ *and* $\beta = (3/p_0 - 3/p)/2$.

REMARK 2.7.

(i) By the interpolation inequality, (2.3) and (2.4) imply that the solution u, obtained in theorems 2.3 and 2.4, satisfies the decay property (1.4), which for weak solutions has already been obtained in, for example, [3, 19]. However, (1.5) is not a consequence.

(ii) The weak solution u, obtained in theorem 2.3, also satisfies that

$$(1+|x|)^{\alpha}u \in L^{\infty}(0,+\infty; L^p(\mathbb{R}^3))$$

for $1 < p \leqslant 2$ and $\alpha = 2(p-1)/p$, which also improves the corresponding results in [10, 11, 15].

(iii) Schonbek and Schonbek [35] studied the decay properties of the moment estimate $\||x|^{\alpha}u\|_2$ for $0 \leqslant \alpha \leqslant 3/2$, when u is a smooth solution. While we get estimates (2.5) and (2.6) for the weak solution. Moreover, the weak solution u, obtained in theorem 2.4, satisfies that, for any $0 \leqslant \alpha \leqslant 3$,

$$\int_{\mathbb{R}^3} |x|^{\alpha}|u|^2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} |x|^{\alpha}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

can be dominated by the terms on the right-hand sides of (2.5) or (2.6), respectively.

(iv) In theorem 2.4, the assumption $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\gamma}$ holds for some $\gamma > 0$, if $a = A^{\gamma}b$ for some $b \in L^1(\mathbb{R}^3)$.

(v) In theorem 2.5, $\alpha + 2\beta = 3 - 3/p$. When $p = +\infty$, our result improves Takahashi's result. Moreover, we obtain the decay rate similar to that of Kato, for weighted norm. At same time, we remove the restriction on the exponent of t and extend the decay rate to reach to the case $p_0 = 1$. By adding a correction term, Carpio [7] have showed the solution to the Navier–Stokes equations behaves like the solution of the heat equation, with the same initial data, in $L^q(R^n)(n \geqslant 3)$ for $q \geqslant p$, as initial data a satisfying (i) $\|a\|_n$ is small, (ii) $a \in L^p(R^n) \cap L^n(R^n)$ for $1 \leqslant p < n$. But our results are different from that in [7].

(vi) Taking $p = \infty$ in theorem 2.6, then estimate (2.8) yields (1.15), which gives an assertive answer to Miyakawa's conjecture.

(vii) Let $p_0 = 1$ and $p = \infty$ in theorem 2.6, then

$$\|u(t)\|_{\infty} = O(t^{-2}),$$

which has been proved by Miyakawa under different assumptions on initial data a. See theorem 1.9(i) in [30].

Applying the weighted estimates obtained in §§ 4 and 5, the proof of theorems 2.3–2.6 are standard. So we will only deduce the necessary weighted estimates, and omit the details of the procedure of the proof.

3. The approximation solutions and their integral representations

In this section, we construct a sequence of approximate solutions by using the linearized Navier–Stokes equations in $\mathbb{R}^3$, and derive the integral representations of the approximate solutions. First, let $a \in \mathring{J}^p(\mathbb{R}^3) \cap \mathring{J}^q(\mathbb{R}^3)$ $(1 \leqslant p, q \leqslant +\infty)$. We select $a^k \in C_{0,\sigma}^{\infty}(\mathbb{R}^3)$, such that

$$a^k \to a \qquad \text{in } \mathring{J}^p(\mathbb{R}^3) \cap \mathring{J}^q(\mathbb{R}^3) \text{ strongly}$$

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and

$$\|a^k\|_p \leqslant 2\|a\|_p, \qquad \|a^k\|_q \leqslant 2\|a\|_q. \tag{3.1}$$

The approximate solutions are defined as follows: let (u^0, p^0) solve the Cauchy problem of the Stokes equations,

$$\left.\begin{aligned} \frac{\partial u^0}{\partial t} - \Delta u^0 &= -\nabla p^0, && \text{in } \mathbb{R}^3 \times (0,\infty),\\ \operatorname{div} u^0 &= 0, && \text{in } \mathbb{R}^3 \times (0,\infty),\\ u^0 &\to 0, && \text{as } |x| \to +\infty,\\ u^0(x,0) &= a^0(x), && \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{3.2}$$

and (u^k, p^k) $(k \geqslant 1)$ solve the Cauchy problem for the linearized Navier–Stokes equations

$$\left.\begin{aligned} \frac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1}\cdot\nabla)u^k &= -\nabla p^k, && \text{in } \mathbb{R}^3 \times (0,\infty),\\ \operatorname{div} u^k &= 0, && \text{in } \mathbb{R}^3 \times (0,\infty),\\ u^k &\to 0, && \text{as } |x| \to +\infty,\\ u^k(x,0) &= a^k(x), && \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{3.3}$$

for $k \geqslant 1$. It is well known (cf. [24]) that there exists a unique solution u^k $(k \geqslant 0)$ to (3.2) and (3.3) satisfying

$$\frac{\partial u^k}{\partial t}, \qquad \frac{\partial u^k}{\partial x_i}, \qquad \frac{\partial^2 u^k}{\partial x_i \partial x_j}, \qquad \frac{\partial p^k}{\partial x_i} \in L^2(0,T;L^2(\mathbb{R}^3)) \tag{3.4}$$

for $i,j = 1,2,3$, $k \geqslant 0$ and any $T > 0$.

In order to derive an integral expression for u^k, one can use the singular integral expression of the projection operator $P : L^2(\mathbb{R}^3) \to \mathring{J}^2(\mathbb{R}^3)$, that is,

$$P\phi = \phi + \frac{1}{4\pi}\nabla \operatorname{div} \int_{\mathbb{R}^3} \frac{\phi(y)}{|x-y|}\,\mathrm{d}y \tag{3.5}$$

for any $\phi \in L^2(\mathbb{R}^3)$ (cf. [24]). Applying the fundamental solution of the heat equation, we can rewrite the solution to the Cauchy problem for the Stokes equations,

$$\left.\begin{aligned} \frac{\partial v}{\partial t} - \Delta v &= -\nabla p + f,\\ \operatorname{div} v &= 0,\\ v(x,0) &= 0, \end{aligned}\right\}$$

as

$$v_i = \int_0^t \int_{\mathbb{R}^3} V^i(x-y,t-\tau)\cdot f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau, \qquad i = 1,2,3, \tag{3.6}$$

where

$$\left.\begin{aligned} V^i(x,t) &= \Gamma(x,t)e^i + \frac{1}{4\pi}\nabla\frac{\partial}{\partial x_i}\int_{\mathbb{R}^3}\frac{\Gamma(x-z,t)}{|z|}\,\mathrm{d}z = P(\Gamma e^i),\\ \Gamma(x,t) &= (4\pi t)^{-3/2}\mathrm{e}^{-|x|^2/4t}, \end{aligned}\right\} \tag{3.7}$$

and e^i is the unit vector along x_i-axis. It is easy to see that

$$V^i(x,t) = \operatorname{curl}(\operatorname{curl}\omega^i) = -\Delta\omega^i + \nabla \operatorname{div}\omega^i, \qquad i = 1,2,3$$

with

$$\left.\begin{aligned} \omega^i(x,t) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Gamma(x-z,t)}{|z|}\,\mathrm{d}z\, e^i, \\ \bar{\theta}(x,t) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Gamma(x-z,t)}{|z|}\,\mathrm{d}z. \end{aligned}\right\} \tag{3.8}$$

For the detailed derivation of (3.7) and (3.8), see Ladyzhenskaya [24].

For simplicity, we drop the right upper label k of the solution u^k of (3.3) and use b to denote u^{k-1}. Let y and τ denote the variables in equations (3.3). We multiply both sides of (3.3) by $V^i(x-y,t-\tau)$, then integrate for $y \in \mathbb{R}^3$ and $\tau \in [0, t-\varepsilon]$ for arbitrary $0 < \varepsilon < t$, to get

$$\begin{aligned} &\int_0^{t-\varepsilon}\int_{\mathbb{R}^3}\left(\frac{\partial u}{\partial \tau} - \Delta u\right)(y,\tau)V^i(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\qquad = \int_0^{t-\varepsilon}\int_{\mathbb{R}^3}(-\nabla_y p - (b\cdot\nabla)u)(y,\tau)V^i(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

Since $(-\partial/\partial\tau - \Delta_y)V^i = 0$ and $V^i = P(\Gamma e^i)$, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^3} u(y,t-\varepsilon)V^i(x-y,\varepsilon)\,\mathrm{d}y - \int_{\mathbb{R}^3} a(y)V^i(x-y,t)\,\mathrm{d}y \\ &\qquad = -\int_0^{t-\varepsilon}\int_{\mathbb{R}^3}(b\cdot\nabla)u(y,\tau)V^i(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

Since u is divergence free, so it follows, by the structure of $V^i(x-y,t-\tau)$, that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} u(y,t-\varepsilon)V^i(x-y,\varepsilon)\,\mathrm{d}y = u_i(x,t),$$

where u_i denotes the ith component of the vector u. Thus,

$$u_i = -\int_0^t\int_{\mathbb{R}^3}(b\cdot\nabla)u(y,\tau)V^i(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau + \int_{\mathbb{R}^3} a(y)V^i(x-y,t)\,\mathrm{d}y.$$

Substituting (3.7) into above equation, we get that

$$\begin{aligned} u_i = &-\int_0^t\int_{\mathbb{R}^3}(b\cdot\nabla)u(y,\tau)\Gamma(x-y,t-\tau)e^i\,\mathrm{d}y\,\mathrm{d}\tau \\ &- \int_0^t\int_{\mathbb{R}^3}(b\cdot\nabla)u(y,\tau)\nabla\frac{\partial}{\partial y_i}\bar{\theta}(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_{\mathbb{R}^3} a(y)\Gamma(x-y,t)e^i\,\mathrm{d}y. \end{aligned} \tag{3.9}$$

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By integration by parts, we arrive at the desired integral representation form:

$$u_i = \int_0^t \int_{\mathbb{R}^3} \sum_j b_j u_i(y,\tau) \frac{\partial}{\partial y_j} \Gamma(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \tag{3.10}$$

$$+ \int_0^t \int_{\mathbb{R}^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau) \frac{\partial^3}{\partial y_i \partial y_l \partial y_k} \bar{\theta}(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \tag{3.11}$$

$$+ \int_{\mathbb{R}^3} a(y)\Gamma(x-y,t)e^i\,\mathrm{d}y. \tag{3.12}$$

Let

$$\left.\begin{aligned} I_1^k &= \int_{\mathbb{R}^3} |a^k|(y)\Gamma(x-y,t)\,\mathrm{d}y, \\ I_2^k &= \int_0^{t/2} \int_{\mathbb{R}^3} |u^{k-1}||u^k|(y,\tau)(|\nabla\Gamma| + |D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau, \\ I_3^k &= \int_{t/2}^t \int_{\mathbb{R}^3} |u^{k-1}||u^k|(y,\tau)(|\nabla\Gamma| + |D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}\right\} \tag{3.13}$$

Thus

$$|u^k(x,t)| \leqslant C(I_1^k + I_2^k + I_3^k). \tag{3.14}$$

For Γ and $\bar{\theta}$, direct calculations show that

$$\left.\begin{aligned} |D^m\Gamma(x,t)| &\leqslant C_m(|x|^2+t)^{-(m+3)/2}, \\ |D^m\bar{\theta}(x,t)| &\leqslant C_m(|x|^2+t)^{-(m+1)/2}. \end{aligned}\right\} \tag{3.15}$$

for $m \in N$.

4. Weighted estimates for the approximate solutions I

In this section, we establish some *a priori* estimates for the approximate solutions constructed in § 3, which result in theorems 2.3 and 2.4 by standard compactness argument. First, standard energy estimates yield the following.

LEMMA 4.1. *Let* $a \in \mathring{J}^2(\mathbb{R}^3)$*. Then the estimates,*

$$\left.\begin{aligned} \|u^k(t)\|_2 \leqslant 2\|a\|_2 \qquad \forall t > 0, \\ \int_0^\infty \|\nabla u^k(s)\|_2^2\,\mathrm{d}s \leqslant 4\|a\|_2^2, \end{aligned}\right\} \tag{4.1}$$

hold uniformly for $k \geqslant 0$.

The next lemma follows from (3.12)–(3.15), (4.1) and standard convolution estimates.

LEMMA 4.2. *Let* $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$*. Then we have*

$$\|u^k(t)\|_2 \leqslant CNt^{-3/4} \tag{4.2}$$

and

$$\|u^k(t)\|_1 \leqslant CB \tag{4.3}$$

holds uniformly for $k \geqslant 0$ *and* $t > 0$. *Furthermore, if* $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\gamma}$ *for some* $\gamma > 0$, *then*

$$\|u^k(t)\|_2 \leqslant Ct^{-3/4-\gamma_1/2}, \tag{4.4}$$

for $\gamma_1 = \min\{1, 2\gamma\}$, *where* $N = \|a\|_1 + \|a\|_1^2 + \|a\|_2 + \|a\|_2^2$ *and* $B = \|a\|_1 + \|a\|_2 N$.

We now turn to the main weighted norm estimates in theorems 2.3 and 2.4.

LEMMA 4.3. *Let* $a \in L^1(\mathbb{R}^3)$ *and* $(1+|x|^2)^{1/2}a \in L^2(\mathbb{R}^3)$. *Then*

$$\|(1+|x|^2)^{1/2}u^k\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2}|\nabla u^k|\|_2^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqslant CA_1 \tag{4.5}$$

for any $k \geqslant 0$ *and* $t \geqslant 0$ *with* $A_1 = \mathrm{e}^{C\|a\|_2^2}(\|(1+|x|^2)^{1/2}a\|_2^2 + N^2)$.

Proof. Taking the divergence of the first equations of (3.3) yields

$$-\Delta p^k = \sum_{ij=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i^{k-1}u_j^k). \tag{4.6}$$

Then the standard Calderón–Zygmund estimate gives

$$\|p^k\|_r \leqslant C\|u^{k-1}\|_{2r}\|u^k\|_{2r}$$

for $1 < r < +\infty$.

If $(1+|x|^2)^{1/2}u^{k-1} \in L^\infty_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^3))$, we can show that $(1+|x|^2)^{1/2}u^k \in L^\infty_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^3))$, by (3.12)–(3.15). By induction,

$$\int_{\mathbb{R}^3}(1+|x|^2)|u^k|^2\,\mathrm{d}x \quad \text{and} \quad \int_0^t\int_{\mathbb{R}^3}(1+|x|^2)|\nabla u^k|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

are well defined. Multiplying the first equation in (3.3) by $(1+|x|^2)u^k$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\|(1+|x|^2)^{1/2}u^k\|_2^2 + \int_{\mathbb{R}^3}(1+|x|^2)|\nabla u^k|^2\,\mathrm{d}x \\
&\leqslant C\|u^k\|_2^2 + C\|(1+|x|^2)^{1/2}u^k\|_2\|u^{k-1}\|_4\|u^k\|_4 \\
&\leqslant 2\|u^k\|_2^2 + C\|(1+|x|^2)^{1/2}u^k\|_2\|u^{k-1}\|_2^{1/4}\|u^k\|_2^{1/4}\|\nabla u^{k-1}\|_2^{3/4}\|\nabla u^k\|_2^{3/4} \\
&\leqslant \|(1+|x|^2)^{1/2}u^k\|_2^2\|\nabla u^{k-1}\|_2\|\nabla u^k\|_2 + C\|u^k\|_2^2 \\
&\qquad + C\|u^{k-1}\|_2\|u^k\|_2 + \|\nabla u^{k-1}\|_2\|\nabla u^k\|_2.
\end{aligned} \tag{4.7}$$

Now (4.5) follows from (4.1), (4.2) and (4.7), by Gronwall's inequality. □

The higher-order moments are estimated as follows.

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LEMMA 4.4. *Let* $a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3)$ *and* $|x|^{3/2}a \in L^2(\mathbb{R}^3)$. *Then*

$$\int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|u^k|^2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|\nabla u^k|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant CA_2 + B^{3/2}N^{4/3}\log(1+t) \quad (4.8)$$

hold uniformly for $k \geqslant 0$ *and* $t \geqslant 0$ *with* $A_2 = A_1^{3/2}(\|a\|_2^{5/4}N^{3/4} + \|a\|_2^{5/6}N)$.

Moreover, if $\|\mathrm{e}^{-tA}a\|_1 \leqslant C(1+t)^{-\gamma}$, *then*

$$\int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|u^k|^2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|\nabla u^k|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant CA_2 + N^2(\tfrac{3}{2}) \quad (4.9)$$

hold uniformly for $k \geqslant 0$ *and* $t \geqslant 0$.

Proof. Applying estimate (3.15) and the inequality

$$(1+|x|^2)^{\alpha/2} \leqslant 2^{\alpha/2}((1+|y|^2)^{\alpha/2} + |x-y|^{\alpha}) \quad \text{for } \alpha \geqslant 0, \quad (4.10)$$

we can show by using (3.12)–(3.14) and lengthy calculation that

$$(1+|x|^2)^{3/4}u^k \in L^\infty_{\mathrm{loc}}(0,+\infty; L^2(\mathbb{R}^3))$$

and

$$(1+|x|^2)^{3/4}|\nabla u^k| \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\mathbb{R}^3))$$

as long as $(1+|x|^2)^{3/4}u^{k-1} \in L^\infty_{\mathrm{loc}}(0,+\infty; L^2(\mathbb{R}^3))$. By induction,

$$\int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|u^k|^2\,\mathrm{d}x \quad \text{and} \quad \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|\nabla u^k|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

are well defined.

We now multiply both sides of (3.3) by $(1+|x|^2)^{3/2}u^k$ and integrate over $\mathbb{R}^3$ to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|u^k|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} (1+|x|^2)^{3/2}|\nabla u^k|^2\,\mathrm{d}x \\ \leqslant 3\int_{\mathbb{R}^3} (1+|x|^2)|u^k||\nabla u^k|\,\mathrm{d}x + 3\int_{\mathbb{R}^3} (1+|x|^2)|u^{k-1}||u^k|^2\,\mathrm{d}x \\ + 3\int_{\mathbb{R}^3} (1+|x|^2)|u^k||p^k|\,\mathrm{d}x. \quad (4.11)\end{aligned}$$

Employing the weighted estimates on singular integral operators (cf. [36]), we deduce from (4.6) that

$$\|(1+|x|^2)^{1/2}p^k\|_2 \leqslant C\|(1+|x|^2)^{1/2}|u^{k-1}||u^k|\|_2.$$

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Thus

$$\begin{aligned}\int_{\mathbb{R}^3}(1+|x|^2)|p^k||u^k|\,\mathrm{d}x &\leqslant \|(1+|x|^2)^{1/2}u^k\|_2\|(1+|x|^2)^{1/2}p^k\|_2\\ &\leqslant C\|(1+|x|^2)^{1/2}u^k\|_2\|(1+|x|^2)^{1/2}|u^{k-1}||u^k|\|_2\\ &\leqslant C\|(1+|x|^2)^{1/2}u^k\|_2\|(1+|x|^2)^{3/4}u^k\|_6^{2/3}\|u^k\|_{24/7}^{1/3}\|u^{k-1}\|_{24/7}\\ &\leqslant CA_1\|\nabla\{(1+|x|^2)^{3/4}u^k\}\|_2^{2/3}\|u^k\|_{24/7}^{1/3}\|u^{k-1}\|_{24/7}\\ &\leqslant \tfrac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CA_1^{3/2}\|u^k\|_{24/7}^{1/2}\|u^{k-1}\|_{24/7}^{3/2}\\ &\quad + CA_1\|(1+|x|^2)^{1/4}u^k\|_2\|u^k\|_{24/7}^{1/3}\|u^{k-1}\|_{24/7}\\ &\leqslant \tfrac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CA_1^{3/2}\|u^k\|_2^{3/16}\|u^{k-1}\|_2^{9/16}\|\nabla u^k\|_2^{5/16}\|\nabla u^{k-1}\|_2^{15/16}\\ &\quad + CA_1^{3/2}\|u^k\|_2^{5/8}\|u^{k-1}\|_2^{3/8}\|\nabla u^k\|_2^{5/24}\|\nabla u^{k-1}\|_2^{5/8},\end{aligned}$$

where we have used lemma 4.3 and the inequality,

$$\|u\|_{24/7} \leqslant \|u\|_2^{3/8}\|\nabla u\|_2^{5/8}.$$

By (4.1) and (4.2), we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3}&(1+|x|^2)|p^k||u^k|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant \frac{1}{4}\int_0^t \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2\,\mathrm{d}\tau + CA_1^{3/2}(N^{3/4}\|a\|_2^{5/4} + N\|a\|_2^{5/6}).\end{aligned} \tag{4.12}$$

Similarly,

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3}&(1+|x|^2)|u^{k-1}||u^k|^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant \frac{1}{4}\int_0^t \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2\,\mathrm{d}\tau + CA_1^{3/2}(N^{3/4}\|a\|_2^{5/4} + N\|a\|_2^{5/6}).\end{aligned} \tag{4.13}$$

Finally, we estimate the first term on the right-hand side of (4.11). By Hölder's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^3}(1+|x|^2)|u^k||\nabla u^k|\,\mathrm{d}x &= \int_{\mathbb{R}^3}(1+|x|^2)^{3/4}|\nabla u^k|(1+|x|^2)^{1/4}|u^k|\,\mathrm{d}x\\ &\leqslant \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2\|(1+|x|^2)^{1/4}u^k\|_2\\ &\leqslant \tfrac{1}{8}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + C\|(1+|x|^2)^{1/4}u^k\|_2^2.\end{aligned}$$

Since

$$\begin{aligned}\|(1+|x|^2)^{1/4}u^k\|_2^2 &= \int_{\mathbb{R}^3}(1+|x|^2)^{1/2}|u^k|^{2/3}|u^k|^{4/3}\,\mathrm{d}x\\ &\leqslant \|(1+|x|^2)^{3/4}u^k\|_6^{2/3}\|u^k\|_{3/2}^{4/3}\\ &\leqslant C\|\nabla\{(1+|x|^2)^{3/4}u^k\}\|_2^{2/3}\|u^k\|_{3/2}^{4/3}\\ &\leqslant \varepsilon\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + \tfrac{1}{2}\|(1+|x|^2)^{1/4}u^k\|_2^2 + C\|u^k\|_{3/2}^2,\end{aligned}$$

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for some $\varepsilon > 0$. Thus,

$$\|(1+|x|^2)^{1/4}u^k\|_2^2 \leqslant 2\varepsilon\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + C\|u^k\|_1^{2/3}\|u^k\|_2^{4/3}.$$

Taking $2\varepsilon = 1/8$ yields

$$\int_{\mathbb{R}^3}(1+|x|^2)|u^k||\nabla u^k|\,\mathrm{d}x \leqslant \tfrac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CB^{2/3}\|u^k\|_2^{4/3}. \tag{4.14}$$

Substituting (4.1)–(4.14) into (4.11), we obtain estimates (4.8) and (4.9), by lemma 4.2. □

5. Weighted estimates for approximate solutions II

In this section, we establish the decay rates estimates of weighted norms for approximate solutions in $L^p(\mathbb{R}^3)$ ($p > 3$), which are used to show the existence of corresponding strong solutions. To this end, we first recall some basic estimates on Γ and $\bar{\theta}$. Applying the inequality $\tau^\alpha \mathrm{e}^{-C\tau} \leqslant C^\alpha \mathrm{e}^{-1}$ for $\alpha > 0$, we can verify directly that

$$\left.\begin{aligned} \||x|^\alpha \Gamma\|_p &\leqslant Ct^{\alpha/2-(3-3/p)/2}, \\ \||x|^\alpha \nabla\Gamma\|_p &\leqslant Ct^{(\alpha-1)/2-(3-3/p)/2}, \end{aligned}\right\} \tag{5.1}$$

for $1 \leqslant p \leqslant +\infty$ and $\alpha \geqslant 0$. By the weighted estimates about the singular integrals (cf. [36–38]) and (3.8), we have

$$\left.\begin{aligned} \||x|^\alpha D^2\bar{\theta}\|_p &\leqslant C\||x|^\alpha \Gamma\|_p \leqslant Ct^{\alpha/2-(3/2)(1-1/p)}, \\ \||x|^\alpha D^3\bar{\theta}\|_p &\leqslant C\||x|^\alpha |\nabla\Gamma|\|_p \leqslant Ct^{(\alpha-1)/2-(3/2)(1-1/p)}, \end{aligned}\right\} \tag{5.2}$$

for $1 < p < +\infty$ and $-1/p < \alpha < 3 - 3/p$.

Now we can deduce the main estimates needed for theorems 2.5 and 2.6.

Lemma 5.1. *Let*

$$a \in L^1(\mathbb{R}^3) \cap \mathring{J}^2(\mathbb{R}^3), \quad (1+|x|^2)^{1/2}a \in L^2(\mathbb{R}^3) \quad \textit{and} \quad (1+|x|^2)^{\alpha/2}a \in L^{p_0}(\mathbb{R}^3)$$

for $1 \leqslant p_0 \leqslant +\infty$ and $\alpha = 3 - 3/p_0$. Then there exists a constant $\lambda > 0$, such that if $N \leqslant \lambda$, then, for $\beta = (3/p_0 - 3/p)/2$, the estimate

$$\|t^\beta(1+|x|^2)^{\alpha/2}u^k\|_p \leqslant C(\|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2} + B^{1/2}A_1N) \tag{5.3}$$

holds uniformly for $k \geqslant 0$ and for $3 < p \leqslant +\infty$.

Proof. By (3.14) and Minkowski inequality, we obtain

$$\|(1+|x|^2)^{\alpha/2}u^k\|_p \leqslant C\sum_{i=1}^{3}\|(1+|x|^2)^{\alpha/2}I_i^k\|_p. \tag{5.4}$$

By the Minkowski inequality, (4.10) and the basic L^p-estimates for convolutions, we have, for $3 < p \leqslant +\infty$,

$$\begin{aligned}\|(1+|x|^2)^{\alpha/2} I_1^k\|_p &\leqslant C\bigg\|\int_{\mathbb{R}^3}(1+|y|^2)^{\alpha/2}|a(y)|\varGamma(x-y,t)\,\mathrm{d}y\bigg\|_p \\ &\quad + C\bigg\|\int_{\mathbb{R}^3}|a(y)||x-y|^{\alpha}\varGamma(x-y,t)\,\mathrm{d}y\bigg\|_p \\ &\leqslant C\|(1+|x|^2)^{\alpha/2}a\|_{p_0}t^{-(3/2)(1/p_0-1/p)} + C\|a\|_1 t^{-3/2+3/(2p)+\alpha/2} \\ &\leqslant C(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0})t^{-(3/2)(1/p_0-1/p)}, \end{aligned} \tag{5.5}$$

where we have used the estimates (5.1) (if $p=\infty$, we use (3.15) to estimate the second term of the first step in (5.5)).

By (4.10) and the Minkowski inequality, we can get, for $3 < p \leqslant +\infty$,

$$\begin{aligned}&\|(1+|x|^2)^{\alpha/2} I_2^k\|_p \\ &\quad\leqslant C\bigg\|\int_0^{t/2}\int_{\mathbb{R}^3}(1+|y|^2)^{\alpha/2}|u^{k-1}||u^k|(|\nabla\varGamma|+|D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg\|_p \\ &\qquad + C\bigg\|\int_0^{t/2}\int_{\mathbb{R}^3}|u^{k-1}||u^k||x-y|^{\alpha}(|\nabla\varGamma|+|D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg\|_p \\ &\quad\triangleq C(\|I_{21}^k\|_p + \|I_{22}^k\|_p). \end{aligned} \tag{5.6}$$

Similarly,

$$\begin{aligned}&\|(1+|x|^2)^{\alpha/2} I_3^k\|_p \\ &\quad\leqslant C\bigg\|\int_{t/2}^{t}\int_{\mathbb{R}^3}(1+|y|^2)^{\alpha/2}|u^{k-1}||u^k|(|\nabla\varGamma|+|D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg\|_p \\ &\qquad + C\bigg\|\int_{t/2}^{t}\int_{\mathbb{R}^3}|u^{k-1}||u^k||x-y|^{\alpha}(|\nabla\varGamma|+|D^3\bar{\theta}|)(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg\|_p \\ &\quad\triangleq C(\|I_{31}^k\|_p + \|I_{32}^k\|_p). \end{aligned} \tag{5.7}$$

Let $J_p^k \triangleq \|(1+|x|^2)^{\alpha/2}u^k\|_p$. In order to estimate I_2^k and I_3^k, we discuss three separated cases: (i) $p=+\infty$ and $1 \leqslant p_0 \leqslant +\infty$; (ii) $1 \leqslant p_0 < p < \infty$ and $p > 3$; (iii) $3 < p_0 = p < \infty$.

Case I. $p = +\infty$ and $1 \leqslant p_0 \leqslant +\infty$.

In order to establish the uniform estimates on $t^{\beta}\|(1+|x|^2)^{\alpha/2}u^k\|_\infty$ with $\beta = 3/(2p_0)$, the singular factor $t^{-\beta}$ will later appear in the integral. So it seems necessary to treat two cases: $p_0 > 3/2$ and $1 \leqslant p_0 \leqslant 3/2$. Similarly, to establish the uniform estimates on $t^{1/2+\beta}\|(1+|x|^2)^{\alpha/2}\nabla u^k\|_\infty$, the singular factor $t^{-1/2-\beta}$ will appear in this procedure, we need to distinguish three cases: $p_0 > 3$, $3/2 < p_0 \leqslant 3$ and $1 \leqslant p_0 \leqslant 3$.

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First, it follows from (5.1), (5.2) and lemma 4.2, that

$$\begin{aligned}\|I_{21}^k\|_\infty &\leqslant C\int_0^{t/2}\|u^{k-1}\|_2 J_\infty^k\||\nabla\Gamma|+|D^3\bar{\theta}|\|_2\,\mathrm{d}\tau\\ &\leqslant C\int_0^{t/2}\|u^k\|_2 J_\infty^{k-1}(t-\tau)^{-5/4}\,\mathrm{d}\tau\\ &\leqslant CN\int_0^{t/2}J_\infty^{k-1}(1+\tau)^{-3/4}(t-\tau)^{-5/4}\,\mathrm{d}\tau \end{aligned}\tag{5.8}$$

which yields the desired estimate for $p_0 > 3$. If $1 \leqslant p_0 \leqslant 3/2$, then $\alpha \leqslant 1$ and $1 \leqslant \beta \leqslant 3/2$. By (3.15), we get, with the help of lemmas 4.2 and 4.3, that

$$\begin{aligned}\|I_{21}^k\|_\infty &\leqslant C\int_0^{t/2}\|u^k\|_{3/2}\|(1+|y|^2)^{\alpha/2}u^{k-1}\|_2^{2/3}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}\,\mathrm{d}\tau\\ &\leqslant CA_1^{2/3}\int_0^{t/2}\|u^k\|_1^{1/3}\|u^k\|_2^{2/3}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}\,\mathrm{d}\tau\\ &\leqslant CA_1^{2/3}B^{1/3}N^{2/3}\int_0^{t/2}(J_\infty^{k-1})^{1/3}(1+\tau)^{-1/2}(t-\tau)^{-2}\,\mathrm{d}\tau. \end{aligned}\tag{5.9}$$

If $3/2 < p_0 \leqslant 3$, then $\alpha - 1 \in (0, 1]$ and $1/2 \leqslant \beta \leqslant 1$. By (3.15), we obtain, with the aid of lemmas 4.2 and 4.3, that

$$\|I_{21}^k\|_\infty \leqslant CA_1^{5/3}B\int_0^{t/2}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}\,\mathrm{d}\tau,\tag{5.10}$$

where we have used the inequality

$$\|(1+|x|^2)^{\gamma_0/2}u\|_p \leqslant \|u\|_q\|(1+|x|^2)^{1/2}u\|_2\tag{5.11}$$

for $1 \leqslant q < p < 2$ and $\gamma_0 = 2(p-q)/(p(2-q))$, which follows from the Hölder inequality.

The $\|I_{22}^k\|_\infty$ can be estimated as

$$\begin{aligned}\|I_{22}^k\|_\infty &\leqslant C\left\|\int_0^{t/2}|u^{k-1}||u^k||x-y|^\alpha(|x-y|^2+(t-\tau))^{-2}\,\mathrm{d}y\,\mathrm{d}\tau\right\|_\infty\\ &\leqslant C\int_0^{t/2}\|u^{k-1}\|_2\|u^k\|_2(t-s)^{-2+\alpha/2}\,\mathrm{d}\tau\\ &\leqslant CN^2\int_0^{t/2}(1+\tau)^{-3/2}(t-\tau)^{-1/2-3/(2p_0)}\,\mathrm{d}\tau\\ &\leqslant CN^2t^{-1-3/(2p_0)}, \end{aligned}\tag{5.12}$$

where one has used (3.15).

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Next, we estimate I_3^k. By (5.1) and (5.2), we have

$$\begin{aligned}\|I_{31}^k\|_\infty &\leqslant C\int_{t/2}^t \|(1+|x|^2)^{\alpha/2}|u^{k-1}||u^k|\|_4(t-\tau)^{-7/8}\,\mathrm{d}\tau\\ &\leqslant C\int_{t/2}^t J_\infty^{k-1}(J_\infty^k)^{1/2}\|u^k\|_2^{1/2}(t-\tau)^{-7/8}\,\mathrm{d}\tau\\ &\leqslant CN^{1/2}\int_{t/2}^t J_\infty^{k-1}(J_\infty^k)^{1/2}(1+\tau)^{-3/8}(t-\tau)^{-7/8}\,\mathrm{d}\tau. \end{aligned}\tag{5.13}$$

Note that in the estimate of I_{32}^k, the singular factor $(t-\tau)^{-2+\alpha/2}$ will appear in the integral with $\alpha = 3-3/p_0$. In order to control this singularity, we will separate two cases: $p_0 > 3$ and $1 \leqslant p_0 \leqslant 3$.

If $p_0 > 3$, it follows from the estimate (3.15) that

$$\|I_{32}^k\|_\infty \leqslant C\int_{t/2}^t \|u^{k-1}\|_2\|u^k\|_2(t-\tau)^{-1/2-3/(2p_0)}\,\mathrm{d}\tau \leqslant CN^2t^{-1-\beta}. \tag{5.14}$$

If $1 \leqslant p_0 \leqslant 3$, using (5.1) and (5.2), we can get

$$\|I_{32}^k\|_\infty \leqslant C\int_{t/2}^t \||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau. \tag{5.15}$$

If $5/3 \leqslant p_0 \leqslant 3$, then $(3p_0+1)/(3p_0-2) \leqslant 2$. By lemma 4.2, we get

$$\begin{aligned}\||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)} &\leqslant J_\infty^{k-1}\|u^k\|_1^{(3p_0-5)/(3p_0+1)}\|u^k\|_2^{6/(3p_0+1)}\\ &\leqslant J_\infty^{k-1}B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}(1+\tau)^{-9/(6p_0+2)}. \end{aligned}\tag{5.16}$$

If $1 \leqslant p_0 < 5/3$, then $(3p_0+1)/(3p_0-2) > 2$. Then,

$$\begin{aligned}&\||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)}\\ &\quad\leqslant J_\infty^{k-1}(J_\infty^k)^{(5-3p_0)/(3p_0+1)}\|u^k\|_2^{2(3p_0-2)/(3p_0+1)}\\ &\quad\leqslant J_\infty^{k-1}(J_\infty^k)^{(5-3p_0)/(3p_0+1)}N^{2(3p_0-2)/(3p_0+1)}(1+\tau)^{-3(3p_0-2)/(2(3p_0+1))}. \end{aligned}\tag{5.17}$$

CASE II. $1 \leqslant p_0 < p < +\infty$ and $3 < p$.

This can be achieved in a similar way as case I with slight modification. Indeed, we deal with three cases for p_0. For $p_0 > 3$, (5.1), (5.2) and lemma 4.2 then imply that

$$\begin{aligned}\|I_{21}^k\|_p &\leqslant C\int_0^{t/2} \|u^k\|_2 J_p^{k-1}\||\nabla\Gamma| + |D^3\bar\theta|\|_2\,\mathrm{d}\tau\\ &\leqslant C\int_0^{t/2} \|u^k\|_2 J_p^{k-1}(t-\tau)^{-5/4}\,\mathrm{d}\tau\\ &\leqslant CN\int_0^{t/2} J_p^{k-1}(1+\tau)^{-3/4}(t-\tau)^{-5/4}\,\mathrm{d}\tau. \end{aligned}\tag{5.18}$$

If $1 \leqslant p_0 \leqslant 3/2$, then $\alpha \leqslant 1$. It now follows from (5.1), (5.2) and lemmas 4.2 and 4.3 that

$$\begin{aligned}\|I_{21}^k\|_p &\leqslant C\int_0^{t/2} \|u^k\|_{4p/(3p-2)}\|(1+|y|^2)^{\alpha/2}u^{k-1}\|_2^{1/2}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau\\ &\leqslant CA_1^{1/2}\int_0^{t/2} \|u^k\|_1^{(p-2)/(2p)}\|u^k\|_2^{(p+2)/(2p)}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau\\ &\leqslant CA_1^{1/2}B^{(p-2)/(2p)}N^{(p+2)/(2p)}\\ &\qquad\times\int_0^{t/2}(J_p^{k-1})^{1/2}(1+\tau)^{-3/8-3/(4p)}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau \qquad (5.19)\end{aligned}$$

If $3/2 < p_0 \leqslant 3$, then $\alpha - 1 \in (0, 1]$. By (5.1) and (5.2), we obtain, with the help of lemmas 4.2 and 4.3 and inequality (5.11), that

$$\begin{aligned}\|I_{21}^k\|_p &\leqslant C\int_0^{t/2} \|(1+|y|^2)^{1/2}u^{k-1}\|_2^{1/2}\|(1+|y|^2)^{(\alpha-1)/4}u^k\|_{4p/(3p-2)}\\ &\qquad\qquad\times(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau\\ &\leqslant CA_1^{1/2}\int_0^{t/2}\|u^k\|_1\|(1+|x|^2)^{1/2}u^k\|_2(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau\\ &\leqslant CA_1^{3/2}B\int_0^{t/2}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}\,\mathrm{d}\tau. \qquad (5.20)\end{aligned}$$

Applying (5.1) and (5.1), we can estimate the $\|I_{22}^k\|_p$ as

$$\|I_{22}^k\|_p \leqslant C\int_0^{t/2}\|u^{k-1}\|_2\|u^k\|_2(t-s)^{-2+\alpha/2+3/(2p)}\,\mathrm{d}\tau \leqslant CN^2t^{-1-\beta}. \qquad (5.21)$$

Now we estimate I_3^k. Let $r = 6(p-1)/(p+1)$. By (5.1) and (5.2), we have

$$\begin{aligned}\|I_{31}^k\|_p &\leqslant C\int_{t/2}^{t}\|(1+|x|^2)^{\alpha/2}|u^{k-1}||u^k|\|_{pr/(p+r)}(t-\tau)^{-1/2-3/(2r)}\,\mathrm{d}\tau\\ &\leqslant C\int_{t/2}^{t}J_p^{k-1}(J_p^k)^{p(r-2)/(r(p-2))}\|u^k\|_2^{2(p-r)/(r(p-2))}(t-\tau)^{-1/2-3/(2r)}\,\mathrm{d}\tau\\ &\leqslant CN^{(p-3)/(3(p-1))}\int_{t/2}^{t}J_p^{k-1}(J_p^k)^{2p/(3(p-1))}\\ &\qquad\qquad\times(1+\tau)^{-(p-3)/(4(p-1))}(t-\tau)^{-1/2-3/(2r)}\,\mathrm{d}\tau. \qquad (5.22)\end{aligned}$$

For the estimate on I_{32}^k, We also consider two cases. If $p_0 \geqslant 3$, we have, by (5.1) and (5.2),

$$\|I_{32}^k\|_p \leqslant C\int_{t/2}^{t}\|u^{k-1}\|_2\|u^k\|_2(t-\tau)^{-1/2-\beta}\,\mathrm{d}\tau \leqslant CN^2t^{-1-\beta}. \qquad (5.23)$$

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By Young's inequality and estimates (5.1) and (5.2), we get

$$\|I_{32}^k\|_p \leqslant C\int_{t/2}^t \||u^{k-1}||u^k|\|_l(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau$$

for $1/l = 1/p + (3p_0-2)/(3p_0+1)$. If $5/3 \leqslant p_0 \leqslant 3$, by lemma 4.2,

$$\begin{aligned}\|I_{32}^k\|_p &\leqslant C\int_{t/2}^t J_p^{k-1}\|u^k\|_{(3p_0+1)/(3p_0-2)}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau\\ &\leqslant C\int_{t/2}^t J_p^{k-1}\|u^k\|_1^{(3p_0-5)/(3p_0+1)}\|u^k\|_2^{6/(3p_0+1)}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau\\ &\leqslant CB^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}\int_{t/2}^t J_p^{k-1}(1+\tau)^{-9/(2(3p_0+1))}\\ &\qquad\qquad\times(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau.\end{aligned} \tag{5.24}$$

If $1 \leqslant p_0 < 5/3$, then

$$\begin{aligned}\|I_{32}^k\|_p &\leqslant C\int_{t/2}^t J_p^{k-1}\|u^k\|_{(3p_0+1)/(3p_0-2)}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau\\ &\leqslant C\int_{t/2}^t J_p^{k-1}(J_p^k)^{(5-3p_0)/(3p_0+1)}\|u^k\|_2^{2(3p_0-2)/(3p_0+1)}\\ &\qquad\times(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau\\ &\leqslant CN^{2(3p_0-2)/(3p_0+1)}\int_{t/2}^t J_p^{k-1}(J_p^k)^{(5-3p_0)/(3p_0+1)}\\ &\qquad\times(1+\tau)^{-3(3p_0-2)/(2(3p_0+1))}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}\,\mathrm{d}\tau.\end{aligned} \tag{5.25}$$

CASE III. $3 < p_0 = p < +\infty$.

The estimates of $\|I_{21}^k\|_p$ and $\|I_{31}^k\|_p$ in this case are same as (5.18) and (5.22). So we only give the estimate of $\|I_{22}^k\|_p$ and $\|I_{32}^k$. Applying (3.15) and the theory of singular integral operator (cf. [37]), we get

$$\begin{aligned}\|I_{22}^k\|_p + \|I_{32}^k\|_p &\leqslant C\left\|\int_0^t |u^{k-1}||u^k||x-y|^\alpha(|x-y|^2+(t-\tau))^{-2}\,\mathrm{d}y\,\mathrm{d}\tau\right\|_p\\ &\leqslant C\left\|\int_0^t |u^{k-1}||u^k||x-y|^{-1-3/p}\,\mathrm{d}y\,\mathrm{d}\tau\right\|_p\\ &\leqslant C\int_0^t \||u^{k-1}||u^k|\|_{3/2}\,\mathrm{d}\tau\\ &\leqslant C\|a\|_2Nt^{-1/4}.\end{aligned} \tag{5.26}$$

Summarizing the above estimates, it is obvious that, if $t^\beta J_p^{k-1} \leqslant C$, then $t^\beta J_p^k \leqslant C$, by the Gronwall inequality. Therefore, it holds that $t^\beta J_p^k \in L^\infty_{\mathrm{loc}}(0,+\infty)$, by

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induction. Thus

$$
\begin{aligned}
t^\beta J_\infty^k \leqslant C\Big(&\|a\|_1+\|(1+|x|^2)^{\alpha/2}a\|_{p_0}+N^2(1+t)^{-1}\\
&+N^{1/2}(1+t)^{-\beta/2-1/4}\max_{0\leqslant t\leqslant\infty}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^k)^{1/2}\Big)\\
&+\begin{cases}
C\Big(N(1+t)^{-1}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})+N^2(1+t)^{-1}\Big), & \text{as } p_0>3;\\
C\Big(A_1^{5/3}B(1+t)^{-1+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})\Big), & \text{as } \tfrac{5}{3}\leqslant p_0\leqslant 3;\\
C\Big(A_1^{5/3}B(1+t)^{-1+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +N^{2(3p_0-2)/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^k)^{(5-3p_0)/(3p_0+1)}\Big), & \text{as } \tfrac{3}{2}<p_0<\tfrac{5}{3};\\
C\Big(B^{1/3}A_1^{2/3}N^{2/3}(1+t)^{-3/2+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +N^{2(3p_0-2)/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^k)^{(5-3p_0)/(3p_0+1)}\Big), & \text{as } 1\leqslant p_0\leqslant \tfrac{3}{2}.
\end{cases}
\end{aligned}
$$

By Young's inequality, we deduce that

$$
\begin{aligned}
\max_{t\in[0,\infty]}(t^\beta J_\infty^k) \leqslant C\Big(&\|a\|_1+\|(1+|x|^2)^{\alpha/2}a\|_{p_0}+N^2+N\max_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^2\Big)\\
&+\begin{cases}
C\Big(N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})+N^2\Big), & \text{as } p_0>3;\\
C\Big(A_1^{5/3}B\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})\Big), & \text{as } \tfrac{5}{3}\leqslant p_0\leqslant 3;\\
C\Big(A_1^{5/3}B\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{(3p_0+1)/(2(3p_0-2))}\Big), & \text{as } \tfrac{3}{2}<p_0<\tfrac{5}{3};\\
C\Big(B^{1/3}A_1^{2/3}N^{2/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} \\
\quad +N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{(3p_0+1)/(2(3p_0-2))}\Big), & \text{as } 1\leqslant p_0\leqslant \tfrac{3}{2}.
\end{cases}
\end{aligned}
$$

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Therefore, if N is suitably small, then

$$\max(t^\beta J_\infty^k) \leqslant C(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N),$$

which yields the desired estimate for the case $p=\infty$. The argument for $3<p<\infty$ is similar. □

By a lengthy but similar calculation, we can show the following lemma.

Lemma 5.2. *Assume the conditions in lemma 5.1 are satisfied. Then if $N \leqslant \lambda$, the estimates, for $t \geqslant 0$,*

$$t^{1/2+\beta}\|\nabla u^k\|_p \leqslant C(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N)$$

hold uniformly for $k \geqslant 0$ and $3 < p \leqslant +\infty$.

If $a = (\partial b)/(\partial x_i)$ for some $i=1,2,3$, we can show, by similar discussion, the following.

Lemma 5.3. *Assume the conditions in lemma 5.1 hold. If $a = (\partial b)/(\partial x_i)$ for some $i=1,2,3$ with $b \in L^1(\mathbb{R}^3)$ and $(1+|x|^2)^{\alpha/2} \in L^{p_0}(\mathbb{R}^3)$, then there exists a constant $\lambda_0 > 0$ such that if $N \leqslant \lambda_0$, the estimate*

$$t^{1/2+\beta}\|u^k\|_p \leqslant C(\|b\|_1 + \|(1+|x|^2)^{\alpha/2}b\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N), \quad t \in [0,\infty)$$

holds uniformly for $k \geqslant 0$ and $3 < p \leqslant +\infty$.

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