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Least energy solution for a scalar field equation with a singular nonlinearity

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We are concerned with a nonnegative solution to the scalar field equation

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0.$$

A classical existence result by Berestycki and Lions [3] considers only the case when f is continuous. In this paper, we are interested in the existence of a solution when f is singular. For a singular nonlinearity f, Gazzola, Serrin and Tang [8] proved an existence result when $f \in L^1_{loc}(\mathbb{R}) \cap \mathrm{Lip}_{loc}(0,\infty)$ with $\int_0^u f(s)\,\mathrm{d}s < 0$ for small $u > 0$. Since they use a shooting argument for their proof, they require the property that $f \in \mathrm{Lip}_{loc}(0,\infty)$. In this paper, using a purely variational method, we extend the previous existence results for $f \in L^1_{loc}(\mathbb{R}) \cap C(0,\infty)$. We show that a solution obtained through minimization has the least energy among all radially symmetric weak solutions. Moreover, we describe a general condition under which a least energy solution has compact support.

Keywords: Least energy solution; scalar field equation; singular nonlinearity; compact supported solution

1. Introduction and statement of the main result

In this paper, we are interested in the following scalar field equation:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N, \\ u \geqslant 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1}$$

Due to its relation with many other problems, there have been extensive studies on the above scalar field equation. When $f \in C(\mathbb{R})$, an almost optimal existence result for (1.1) was obtained by Berestycki and Lions [3]. In fact, under the following assumptions:

(F1-1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(t) = 0$ for $t \leqslant 0$ and $-\infty < \liminf_{t\to 0+} f(t)/t \leqslant \limsup_{t\to 0+} f(t)/t < 0$;

(F2-1) $\limsup_{t\to\infty} f(t)/t^l \leqslant 0$ for $l = (N+2)/(N-2)$;

(F3) there exists $T > 0$ such that $F(T) > 0$, where $F(t) = \int_0^t f(\sigma)\, d\sigma$;

Berestycki and Lions constructed a least energy positive solution in C^2 to (1.1) that is radially symmetric and decays exponentially to 0 at infinity. It is well known that the condition (F3) is necessary for existence of a solution to (1.1). It is easy to see that for $\lim_{t\to 0+} f(t)/t > 0$, there exists no finite energy solution. For a zero mass case, that is, $\lim_{t\to 0} f(t)/t = 0$, Berestycki and Lions also obtained a least energy solution to (1.1) when (F3) and the following (F1-2), (F2-2) hold.

(F1-2) $f : [0,\infty) \to \mathbb{R}$ is continuous, $f(t) = 0$ for $t \leqslant 0$ and $\limsup_{t\to 0+} f(t)/t^{(N+2)/(N-2)} \leqslant 0$;

(F2-2) $\limsup_{t\to\infty} f(t)/t^{(N+2)/(N-2)} = 0$.

When $f(t) = t - t^\alpha$ with $\alpha \in (0,1)$, the solution obtained by Berestycki and Lions in [3] has compact support. As a related problem, Gui [10] and Cortazar–Elgueta–Felmer [6] proved the radial symmetry of a solution for the overdetermined boundary value problem:

$$\begin{cases} \Delta u + u - u^\alpha = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0, \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the outward normal derivative on $\partial\Omega$ and $0 < \alpha < 1$. Kaper–Kwong–Li [11] also studied the symmetry properties when f is the sum of a continuous nondecreasing function and a Lipschitz continuous function on $[0,\infty)$. The following problem with a more singular nonlinearity was studied by Serrin–Tang [16] and Davila–Montenegro [7]:

$$\begin{cases} \Delta u + u^p - u^{-q} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0, \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1,\ 1 < p < (N+2)/(N-2)$. A general existence result for a non-negative solution to (1.1) with a singular nonlinearity was obtained by Gazzola–Serrin–Tang [8]. Their existence result was obtained when f is locally Lipschitz continuous on $(0,\infty)$ and in $L^1_{loc}(\mathbb{R})$ with $\int_0^u f(s)\, ds < 0$ for small $u > 0$. They use a purely ODE argument for the proof in [8]. Chung–Kim–Kwon–Pan [5] recently

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study a one-dimensional case of (1.1) with a discontinuous nonlinearity $f(x)$ at $x = 0$. Their result is motivated by the Allee effect in mathematical ecology and shows that the discontinuity of f represents the very strong Allee effect.

In this paper, by using a variational argument we extend the previous existence results by assuming only that $f \in L^1_{loc}(\mathbb{R}) \cap C(0, \infty)$. Moreover, we find a general condition under which a least positive energy solution has compact support. In fact, for $f \in L^1_{loc}(\mathbb{R})$, we define $F(t) = \int_0^t f(s)ds$ for $t \geqslant 0$, $F(t) = 0$ for $t \leqslant 0$. We assume that for $N > 2$,

(f1) $f \in L^1_{loc}(\mathbb{R}) \cap C(0, \infty)$, $f(t) = 0$ for $t \leqslant 0$;

(f2) there exists $T > 0$ such that $F(T) > 0$;

(f3) $\limsup_{t\to 0} F(t)/t^{2N/(N-2)} \leqslant 0$ and $\limsup_{t\to 0} f(t) < \infty$;

(f4) if there exists no $S > T$ with $F(S) = 0$, $\lim_{t\to\infty} |f(t)|/t^{(N+2)/(N-2)} = 0$;

and for $N = 2$,

(f3-2) $\limsup_{t\to 0} F(t)/t^2 = -m < 0$ and $\limsup_{t\to 0} f(t) < \infty$;

(f4-2) if there exists no $S > T$ with $F(S) = 0$, $\limsup_{t\to\infty} |f(t)|e^{-ct^2} < \infty$ for any $c > 0$.

If there exists $S > T$ with $F(S) = 0$, we do not require (f4) and (f4-2); then we assume that $F(t) = 0$ for $t \geqslant S$. We define a space $\mathbf{H} = \mathcal{D}^{1,2}(\mathbb{R}^N)$ for $N \geqslant 3$ and $\mathbf{H} = H^1(\mathbb{R}^2)$ for $N = 2$. We say that $u \in \mathbf{H}$ is a *weak solution* of (1.1) if $f(u) \in L^1_{loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi - f(u)\phi \,\mathrm{d}x = 0 \text{ for all } \phi \in C_0^\infty(\mathbb{R}^N).$$

For $u \in \mathbf{H}$, the corresponding energy $E(u)$ is defined by

$$E(u) \equiv \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

Our main result in this paper is the following.

THEOREM 1.1. *Assume that (f1)–(f4) hold if $N > 2$ and, (f1), (f2), (f3-2) and (f4-2) hold if $N = 2$. Then there exists a radially symmetric nonzero weak solution $u \in C^1(\mathbb{R}^N) \cap \mathbf{H}$ of problem (1.1), which has the least positive energy among all radially symmetric nonzero weak solutions of (1.1). Moreover, $u \in C^2(\{r \geqslant 0 \mid u(r) > 0\})$, and u has compact support if $\int_0^\delta \mathrm{d}t/\sqrt{F_-(t)} < \infty$ for some $\delta > 0$.*

REMARK 1.2. The similar sufficient conditions for a solution to have compact support as in Theorem 1.1 are described in [**1**, **13**–**15**, **17**] for quasilinear equations.

This paper is organized as follows. In Section 2, we prove the main theorem. In Section 3, we consider a one-dimensional case and we give some remarks about properties of such solutions.

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2. Proof of the main result

In this section, we assume that (f1)–(f4) hold for $N > 2$, and (f1), (f2), (f3-2) and (f4-2) hold for $N = 2$. For the existence of a solution for $N \geqslant 2$, we follow the minimization arguments in [2] and [3]. We consider the following minimization problems:

$$I_N \equiv \inf\left\{ \int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x \,\middle|\, \int_{\mathbb{R}^N} F(w) \,\mathrm{d}x = 1, w \in \mathcal{D}^{1,2}(\mathbb{R}^N) \right\} \text{ for } N > 2;$$

$$I_2 \equiv \inf\left\{ \int_{\mathbb{R}^2} |\nabla w|^2 \,\mathrm{d}x \,\middle|\, \int_{\mathbb{R}^2} F(w) \,\mathrm{d}x = 0, w \not\equiv 0, w \in H^1(\mathbb{R}^2) \right\} \text{ for } N = 2.$$

Proposition 2.1. *I_N is attained by a radially symmetric minimizer for $N \geqslant 2$.*

Proof. Let $\{v_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of I_N. By Schwartz symmetrization, we may assume that v_n is non-negative, radially symmetric and non-increasing with respect to $r = |x|$. When $N = 2$, we define $v_n^t(x) \equiv v_n(tx)$ for $t > 0$. Then, since for each $t > 0$,

$$\int_{\mathbb{R}^2} F(v_n^t) \,\mathrm{d}x = 0, \quad \int_{\mathbb{R}^2} |\nabla v_n^t|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} |\nabla v_n|^2 \,\mathrm{d}x,$$

we may assume that $\int_{\mathbb{R}^2} (v_n)^2 \,\mathrm{d}x = 1$. Thus, we may assume that the minimizing sequence $\{v_n\}$ is bounded. Now, taking a subsequence if necessary, we may assume that v_n converges weakly to v in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ for $N \geqslant 3$, in $H^1(\mathbb{R}^2)$ for $N = 2$ and v_n converges pointwise to v a.e. as $n \to \infty$.

For any radially symmetric $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N > 2$,

$$|w(r)| = \left| \int_\infty^r w'(s) ds \right| \leqslant \left| \int_\infty^r s^{-(N-1)} ds \right|^{1/2} \left| \int_\infty^r s^{N-1} (w'(s))^2 ds \right|^{1/2}.$$

Thus, there exists $C = C(N) > 0$ such that for any radially symmetric $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N > 2$,

$$|w(r)| \leqslant \frac{C}{r^{(N-2)/2}} \left(\int_{\mathbb{R}^N \setminus B(0,r)} |\nabla w|^2 \,\mathrm{d}x \right)^{1/2}. \tag{2.1}$$

When $N = 2$, for any radially symmetric $w \in H^1(\mathbb{R}^2)$,

$$-\left(rw^2(r)\right)_r = -(w(r))^2 - 2rw(r)w_r(r) \leqslant r(w(r))^2 + r(w_r(r))^2. \tag{2.2}$$

Then, integrating (2.2) on (r, ∞), it follows that

$$|w(r)| \leqslant \frac{C}{\sqrt{r}} \|w\|_{H^1(\mathbb{R}^2)}, \quad r \geqslant 1 \tag{2.3}$$

for some constant $C > 0$, independent of $w \in H^1(\mathbb{R}^2)$. The above inequalities (2.1) and (2.3) imply that $v_n(r)$ converges to 0 uniformly as $r \to \infty$.

First, we consider the case $N > 2$. We define $F_+(t) = \max\{F(t), 0\}$, $F_-(t) = \max\{-F(t), 0\}$. Note that $|F(t)| \leqslant A(1 + t^{2N/(N-2)})$ for some constant $A > 0$. Thus, for each $R > 0$, there exists $C(R) > 0$ such that $\int_{B(0,R)} F_+(v_n) + F_-(v_n)\,\mathrm{d}x \leqslant C(R)$ by the Sobolev inequality. Since $\{v_n\}$ is bounded in $H^1(B(0,R))$, we see that

$$\lim_{n\to\infty} \int_{B(0,R)} F_+(v_n)\,\mathrm{d}x = \int_{B(0,R)} F_+(v)\,\mathrm{d}x.$$

Note from (f3) that for some $\delta(R) > 0$ with $\lim_{R\to\infty} \delta(R) = 0$,

$$\int_{\mathbb{R}^N\setminus B(0,R)} F_+(v_n)\,\mathrm{d}x \leqslant \delta(R) \int_{\mathbb{R}^N\setminus B(0,R)} v_n^{2N/(N-2)}\,\mathrm{d}x.$$

Thus, $\lim_{R\to\infty} \int_{\mathbb{R}^N\setminus B(0,R)} F_+(v_n)\,\mathrm{d}x = 0$ uniformly for $n \geqslant 1$. This implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F_+(v_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} F_+(v)\,\mathrm{d}x.$$

Since $\int_{\mathbb{R}^N} F_+(v_n)\,\mathrm{d}x = 1 + \int_{\mathbb{R}^N} F_-(v_n)\,\mathrm{d}x$, it follows from Fatou's lemma that

$$\int_{\mathbb{R}^N} F_+(v)\,\mathrm{d}x \geqslant 1 + \int_{\mathbb{R}^N} F_-(v)\,\mathrm{d}x.$$

Now it holds that $\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x \leqslant I_N$ and $\int_{\mathbb{R}^N} F(v)\,\mathrm{d}x \geqslant 1$. If $\int_{\mathbb{R}^N} F(v)\,\mathrm{d}x > 1$, there exists $\sigma > 1$ such that for $v_\sigma(x) \equiv v(\sigma x)$, $\int_{\mathbb{R}^N} F(v_\sigma)\,\mathrm{d}x = 1$. In this case, we have

$$\int_{\mathbb{R}^N} |\nabla v_\sigma|^2\,\mathrm{d}x = \sigma^{2-N} \int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x < I_N,$$

which is a contradiction. This implies that $\int_{\mathbb{R}^N} F(v)\,\mathrm{d}x = 1$ and v is a radially symmetric minimizer of I_N.

For $N = 2$, let $F_1(t) \equiv F(t) + m't^2$ for $m' \in (0, m)$. Then, we define

$$F_{1+}(t) = \max\{F_1(t), 0\}, \quad F_{1-}(t) = \max\{-F_1(t), 0\}.$$

Condition (f4-2) implies that for any $c > 0$, there exists a constant $A > 0$ such that $|F_1(t)| \leqslant A(1 + e^{ct^2}), t \in \mathbb{R}$. Thus, for each $R > 0$, there exists $C(R) > 0$ such that $\int_{B(0,R)} |F_1(v_n)|\,\mathrm{d}x \leqslant C(R)$ by the Trudinger–Moser inequality. Since $\{v_n\}$ is bounded in $H^1(B(0,R))$, we see that

$$\lim_{n\to\infty} \int_{B(0,R)} F_{1+}(v_n)\,\mathrm{d}x = \int_{B(0,R)} F_{1+}(v)\,\mathrm{d}x.$$

We deduce from (f3-2) that for some $\delta(R) > 0$ with $\lim_{R\to\infty} \delta(R) = 0$,

$$\int_{\mathbb{R}^2\setminus B(0,R)} F_{1+}(v_n)\,\mathrm{d}x \leqslant \delta(R) \int_{\mathbb{R}^2\setminus B(0,R)} v_n^2\,\mathrm{d}x \leqslant \delta(R).$$

Thus, $\lim_{R\to\infty} \int_{\mathbb{R}^N\setminus B(0,R)} F_{1+}(v_n) dx = 0$ uniformly for $n \geqslant 1$. This implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F_{1+}(v_n)\,\mathrm{d}x = \int_{\mathbb{R}^2} F_{1+}(v)\,\mathrm{d}x.$$

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Since $\int_{\mathbb{R}^2} F_{1+}(v_n)\,\mathrm{d}x = m'\int_{\mathbb{R}^2}(v_n)^2\,\mathrm{d}x + \int_{\mathbb{R}^2} F_{1-}(v_n)\,\mathrm{d}x$, it follows from Fatou's lemma that

$$\int_{\mathbb{R}^2} F_{1+}(v)\,\mathrm{d}x \geqslant m'\int_{\mathbb{R}^2} v^2\,\mathrm{d}x + \int_{\mathbb{R}^2} F_{1-}(v)\,\mathrm{d}x.$$

Hence, we have

$$\int_{\mathbb{R}^2} F_1(v)\,\mathrm{d}x \geqslant m'\int_{\mathbb{R}^2} v^2\,\mathrm{d}x > 0.$$

In particular, $v \not\equiv 0$. Also it follows that

$$\int_{\mathbb{R}^2} v^2\,\mathrm{d}x \leqslant 1, \quad \int_{\mathbb{R}^2} F(v)\,\mathrm{d}x \geqslant 0 \text{ and } \int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x \leqslant I_2.$$

If $\int_{\mathbb{R}^2} F(v)\,\mathrm{d}x > 0$, then there exists $\lambda \in (0,1)$ such that $\int_{\mathbb{R}^2} F(\lambda v)\,\mathrm{d}x = 0$ since $F(t) < 0$ near zero. In this case, we have

$$\int_{\mathbb{R}^2} |\nabla \lambda v|^2\,\mathrm{d}x = \lambda^2\int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x < I_2,$$

which is a contradiction. This implies that $\int_{\mathbb{R}^N} F(v)\,\mathrm{d}x = 0$ and v is a radially symmetric minimizer of I_2. □

For a minimizer v of I_N, let $R_0 > 0$ be a number such that $v(r) > 0$ for $r < R_0$ and $v(r) = 0$ for $r \geqslant R_0$. If there is no such a finite R_0, we define $R_0 = \infty$.

Proposition 2.2. *There exists a constant* $\theta > 0$ *such that* $\int_{B(0,R_0)} \nabla v\cdot\nabla\phi - \theta f(v)\phi\,\mathrm{d}x = 0$ *for any* $\phi \in C_0^\infty(B(0,R_0))$.

Proof. We take any $R \in (0,R_0)$. Then, $v(r) \geqslant v(R) > 0$ for $r \leqslant R$. From (f4) if $N > 2$ and from (f4-2) if $N = 2$, we see that for some constant $C > 0$,

$$\left|\int_{\mathbb{R}^N} f(v)\phi\,\mathrm{d}x\right| \leqslant C\left(\int_{B(0,R)} |\nabla\phi|^2\,\mathrm{d}x\right)^{1/2}, \quad \phi \in C_0^\infty(B(0,R)).$$

Thus, there exists $w \in H_0^1(B(0,R))$ such that for any $\phi \in H_0^1(B(0,R))$,

$$\int_{B(0,R)} \nabla w\cdot\nabla\phi - f(v)\phi\,\mathrm{d}x = 0. \tag{2.4}$$

For any $\phi \in C_0^\infty(B(0,R))$, we define

$$\phi_1 \equiv \phi - w\frac{\int_{B(0,R)} \nabla w\cdot\nabla\phi\,\mathrm{d}x}{\int_{B(0,R)} |\nabla w|^2\,\mathrm{d}x}, \quad \phi_2 \equiv \phi - \phi_1.$$

Then, we see that

$$\int_{B(0,R)} f(v)\phi_1\,\mathrm{d}x = \int_{B(0,R)} \nabla w\cdot\nabla\phi_1\,\mathrm{d}x = 0. \tag{2.5}$$

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Choose a test function $\psi_1 \in C_0^\infty(B(0,R))$ satisfying $\int_{\mathbb{R}^N} f(v)\psi_1 \,\mathrm{d}x \neq 0$ and define

$$g(t,\sigma) \equiv \int_{\mathbb{R}^N} F(v + t\phi_1 + \sigma\psi_1)\,\mathrm{d}x.$$

Then, we see that

$$g(0,0) = \begin{cases} 1, & \text{if } N > 2, \\ 0, & \text{if } N = 2, \end{cases}$$

and $\partial g(0,0)/\partial\sigma = \int_{\mathbb{R}^N} f(v)\psi_1 \,\mathrm{d}x \neq 0$. By the implicit function theorem, there exist $\delta > 0$ and a C^1-function $\sigma : (-\delta,\delta) \to \mathbb{R}$ such that $\sigma(0) = 0$ and

$$g(t,\sigma(t)) = \begin{cases} 1, & \text{if } N > 2, \\ 0, & \text{if } N = 2, \end{cases}$$

for $t \in (-\delta,\delta)$. Note that $\mathrm{d}/\mathrm{d}t|_{t=0} \int_{\mathbb{R}^N} |\nabla(v + t\phi_1 + \sigma(t)\psi_1)|^2 \,\mathrm{d}x = 0$. Since $\sigma'(0) = 0$ by (2.5), it follows that $\int_{\mathbb{R}^N} \nabla v \cdot \nabla\phi_1 \,\mathrm{d}x = 0$. Thus, for any $\phi \in C_0^\infty(B(0,R))$,

$$\int_{\mathbb{R}^N} \nabla v \cdot \nabla\phi_1 \,\mathrm{d}x = \int_{\mathbb{R}^N} f(v)\phi_1 \,\mathrm{d}x = 0. \tag{2.6}$$

Defining

$$\theta \equiv \frac{\int_{B(0,R)} \nabla v \cdot \nabla w \,\mathrm{d}x}{\int_{B(0,R)} |\nabla w|^2 \,\mathrm{d}x},$$

we see from (2.4) and (2.6) that

$$\begin{aligned}
\int_{B(0,R)} \nabla v \cdot \nabla\phi - \theta f(v)\phi \,\mathrm{d}x &= \int_{B(0,R)} \nabla v \cdot \nabla\phi_2 - \theta f(v)\phi_2 \,\mathrm{d}x \\
&= \frac{\int_{B(0,R)} \nabla w \cdot \nabla\phi \,\mathrm{d}x}{\int_{B(0,R)} |\nabla w|^2 \,\mathrm{d}x} \int_{B(0,R)} \nabla v \cdot \nabla w - \theta f(v) w \,\mathrm{d}x \\
&= \frac{\int_{B(0,R)} \nabla w \cdot \nabla\phi \,\mathrm{d}x}{\int_{B(0,R)} |\nabla w|^2 \,\mathrm{d}x} \int_{B(0,R)} \nabla v \cdot \nabla w - \theta |\nabla w|^2 \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

This proves that for any $\phi \in C_0^\infty(B(0,R_0))$,

$$\int_{B(0,R_0)} \nabla v \cdot \nabla\phi - \theta f(v)\phi \,\mathrm{d}x = 0.$$

It remains to show that $\theta > 0$. Obviously $\theta \neq 0$. Suppose that $\theta < 0$. Choose a test function $\phi \in C_0^\infty(B(0,R_0))$ such that $\int_{B(0,R_0)} f(v)\phi \,\mathrm{d}x > 0$. Then, for small $\epsilon > 0$,

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it holds that

$$\int_{B(0,R_0)} F(v+\epsilon\phi)\,\mathrm{d}x > \int_{B(0,R_0)} F(v) = \begin{cases} 1, & \text{if } N>2, \\ 0, & \text{if } N=2. \end{cases}$$

Since $\int_{B(0,R_0)} \nabla v \cdot \nabla\phi\,\mathrm{d}x = \theta\int_{B(0,R_0)} f(v)\phi\,\mathrm{d}x < 0$, we also get

$$\int_{B(0,R_0)} |\nabla(v+\epsilon\phi)|^2\,\mathrm{d}x < \int_{B(0,R_0)} |\nabla v|^2\,\mathrm{d}x = I_N,$$

for small $\epsilon > 0$.

First, consider s case $N > 2$. Note that there exists $\sigma = \sigma(\epsilon) > 1$ such that

$$\int_{B(0,R_0)} F((v+\epsilon\phi)(\sigma x))\,\mathrm{d}x = 1.$$

Then it follows that

$$\int_{B(0,R_0)} |\nabla(v+\epsilon\phi)(\sigma x)|^2\,\mathrm{d}x = \sigma^{2-N}\int_{B(0,R_0)} |\nabla(v+\epsilon\phi)(x)|^2\,\mathrm{d}x < I_N,$$

which is a contradiction.

If $N = 2$, we choose $\lambda \in (0, 1)$ such that

$$\int_{B(0,R_0)} F(\lambda(v+\epsilon\phi)(x))\,\mathrm{d}x = 0.$$

Then we get

$$\int_{B(0,R_0)} |\nabla(\lambda(v+\epsilon\phi))(x)|^2\,\mathrm{d}x = \lambda^2\int_{B(0,R_0)} |\nabla(v+\epsilon\phi)(x)|^2\,\mathrm{d}x < I_2,$$

which is a contradiction. Hence, we conclude that $\theta > 0$. This completes the proof. □

PROPOSITION 2.3. *$v \in C^2(B(0,R_0))$ and $\lim_{r\to R_0} v_r(r) = 0$.*

Proof. First, we show that $v \in C^2(B(0,R_0))$. Since v satisfies the equation in Proposition 2.2, standard elliptic regularity theory [9] shows that $v \in C^{1,\alpha}(B(0,R_0))$. Note that for $r \in (0,\infty)$, v satisfies the equation

$$v_{rr} + \frac{N-1}{r}v_r = -\theta f(v). \tag{2.7}$$

Thus, it is enough to show that v_{rr} is continuous at 0. Since $\mathrm{d}/\mathrm{d}r(r^{N-1}v_r) = r^{N-1}(v_{rr} + (N-1)/rv_r)$, integrating (2.7) on $(0,r)$ we get

$$r^{N-1}v_r = -\int_0^r s^{N-1}\theta f(v(s))\,\mathrm{d}s.$$

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Letting $s = rt$, we have

$$r^{-1} v_r = -\int_0^1 t^{N-1} \theta f(v(rt))\,\mathrm{d}t.$$

This implies that

$$v_{rr}(0) = \lim_{r\to 0} \frac{v_r}{r} = -\frac{\theta f(v(0))}{N}. \tag{2.8}$$

From (2.7) and (2.8), we also obtain

$$\lim_{r\to 0} v_{rr} = -(N-1)\lim_{r\to 0}\frac{v_r}{r} - \theta f(v(0)) = -\frac{\theta f(v(0))}{N}.$$

Hence, $v \in C^2(B(0,R_0))$.

Now we show that $\lim_{r\to R_0} v_r(r) = 0$. If $R_0 = \infty$, it directly follows from the facts that $v > 0$, $v_r \leqslant 0$ and $\lim_{r\to\infty} v(r) = 0$. Therefore, we may assume $R_0 < \infty$. First, we note from (2.7) that for any $0 < r_1 < r_2 < R_0$,

$$\frac{1}{2}((v_r(r_2))^2 - (v_r(r_1))^2) + \int_{r_1}^{r_2} \frac{N-1}{r}(v_r(r))^2 \mathrm{d}r = -\theta \int_{v(r_1)}^{v(r_2)} f(t)\mathrm{d}t.$$

Then, since $v \in H^1(B(0,R_0))$ and $f \in L^1_{loc}(\mathbb{R})$, we see that $v_r \in L^\infty$ and $\lim_{r\uparrow R_0} v_r(r)$ exists. To the contrary, suppose that $\lim_{r\uparrow R_0} v_r(r) < 0$. Then, for small $\epsilon > 0$, there exists a constant $C_0 > 0$ such that $v_r(r) < -C_0 < 0$ for $r \in (R_0-\epsilon, R_0)$. We define a function $\tilde{v}$ by

$$\tilde{v}(r) = \begin{cases} v\left(\frac{r+(R_0-\epsilon)}{2}\right), & \text{if } |R_0 - r| < \epsilon, \\ v(r), & \text{otherwise.} \end{cases}$$

We then see that

$$\begin{aligned} \int_{\mathbb{R}^N} F(\tilde{v})\,\mathrm{d}x &= \int_{B(0,R_0-\epsilon)} F(v)\,\mathrm{d}x + |S^{N-1}| \int_{R_0-\epsilon}^{R_0+\epsilon} F(\tilde{v}(r)) r^{N-1}\,\mathrm{d}r \\ &= \int_{B(0,R_0-\epsilon)} F(v)\,\mathrm{d}x + |S^{N-1}| \int_{R_0-\epsilon}^{R_0} F(v(r))\,(2r-(R_0-\epsilon))^{N-1}\,2\,\mathrm{d}r \\ &= \int_{B(0,R_0)} F(v)\,\mathrm{d}x + |S^{N-1}| \\ &\quad \times \int_{R_0-\epsilon}^{R_0} F(v(r)) \left\{ 2\,(2r-(R_0-\epsilon))^{N-1} - r^{N-1} \right\} \mathrm{d}r \\ &= \begin{cases} 1+a, & \text{if } N > 2, \\ a, & \text{if } N = 2, \end{cases} \end{aligned}$$

where $|S^{N-1}|$ is the volume of the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$ and

$$a = |S^{N-1}| \int_{R_0-\epsilon}^{R_0} F(v(r)) \left\{ 2\,(2r-(R_0-\epsilon))^{N-1} - r^{N-1} \right\} \mathrm{d}r.$$

Note that $a = o(\epsilon)$ since $F(0) = 0$ and $F(v(r))$ is continuous on $[R_0-\epsilon, R_0]$.

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When $N > 2$, there exists $\sigma = 1 + o(\epsilon)$ such that for $\tilde{v}_\sigma \equiv \tilde{v}(\sigma r)$, $\int_{\mathbb{R}^N} F(\tilde{v}_\sigma)\,\mathrm{d}x = 1$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \tilde{v}_\sigma|^2\,\mathrm{d}x = \sigma^{2-N}\int_{\mathbb{R}^N} |\nabla \tilde{v}|^2\,\mathrm{d}x \\
&= \sigma^{2-N}\left\{\int_{B(0,R_0-\epsilon)} |\nabla v|^2\,\mathrm{d}x + |S^{N-1}|\int_{R_0-\epsilon}^{R_0+\epsilon} |\tilde{v}_r(r)|^2 r^{N-1}\,\mathrm{d}r\right\} \\
&= \sigma^{2-N}\left\{\int_{B(0,R_0-\epsilon)} |\nabla v|^2\,\mathrm{d}x + |S^{N-1}|\int_{R_0-\epsilon}^{R_0} |v_r(r)|^2 \frac{(2r-(R_0-\epsilon))^{N-1}}{2}\,\mathrm{d}r\right\} \\
&= \sigma^{2-N}\left\{\int_{B(0,R_0)} |\nabla v|^2\,\mathrm{d}x + |S^{N-1}|\right. \\
&\quad\left.\times\int_{R_0-\epsilon}^{R_0} |v_r(r)|^2\left(\frac{(2r-(R_0-\epsilon))^{N-1}}{2} - r^{N-1}\right)\mathrm{d}r\right\} \\
&= \sigma^{2-N}\left\{I_N + |S^{N-1}|\int_{R_0-\epsilon}^{R_0} |v_r(r)|^2\left(\left(\frac{2r-(R_0-\epsilon)}{2^{1/(N-1)}}\right)^{N-1} - r^{N-1}\right)\mathrm{d}r\right\}.
\end{aligned}$$

Since there exists a constant $C_1 > 0$, independent of small $\epsilon > 0$, such that

$$\left(\frac{2r-(R_0-\epsilon)}{2^{1/(N-1)}}\right)^{N-1} - r^{N-1} < -C_1 < 0 \text{ on } (R_0-\epsilon, R_0),$$

it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \tilde{v}_\sigma|^2\,\mathrm{d}x &\leqslant \sigma^{2-N} I_N - \sigma^{2-N}|S^{N-1}|\int_{R_0-\epsilon}^{R_0} C_1 |v_r(r)|^2\,\mathrm{d}r \\
&\leqslant (1+o(\epsilon))I_N - \sigma^{2-N}|S^{N-1}|C_0^2 C_1\epsilon \\
&\leqslant I_N + o(\epsilon) - C\epsilon,
\end{aligned}$$

where $C = (|S^{N-1}|C_0^2 C_1)/2$. Thus, for such small $\epsilon > 0$, we have

$$\int_{\mathbb{R}^N} F(\tilde{v}_\sigma)\,\mathrm{d}x = 1 \text{ and } \int_{\mathbb{R}^N} |\nabla \tilde{v}_\sigma|^2\,\mathrm{d}x < I_N.$$

This is a contradiction.

Now we consider the remaining case of $N = 2$. We can choose $\phi \in C_0^\infty(B(0,R_0))$ such that $\int_{B(0,R_0)} f(v)\phi\,\mathrm{d}x > 0$. Then, there exists $t = o(\epsilon)$ such that

$\int_{\mathbb{R}^N} F(\tilde{v}+t\phi)\,\mathrm{d}x = 0$ since $\int_{\mathbb{R}^N} F(\tilde{v})\,\mathrm{d}x = a = o(\epsilon)$. Then, it follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla(\tilde{v}+t\phi)|^2\,\mathrm{d}x &= \int_{\mathbb{R}^2} |\nabla\tilde{v}|^2\,\mathrm{d}x + 2t\int_{\mathbb{R}^2} \nabla\tilde{v}\cdot\nabla\phi\,\mathrm{d}x + \int_{\mathbb{R}^2} t^2|\nabla\phi|^2\,\mathrm{d}x \\ &\leqslant |S^1|\int_0^{R_0+\epsilon} |\tilde{v}_r(r)|^2 r\,\mathrm{d}r + o(\epsilon) + o(\epsilon^2) \\ &\leqslant \int_{B(0,R_0)} |\nabla v|^2\,\mathrm{d}x + |S^1| \\ &\quad\times \int_{R_0-\epsilon}^{R_0} |v_r(r)|^2 \left(\frac{(2r-(R_0-\epsilon))}{2} - r\right)\mathrm{d}r + o(\epsilon) \\ &\leqslant I_2 - \frac{R_0|S^1|}{4}\int_{R_0-\epsilon}^{R_0} |v_r(r)|^2\,\mathrm{d}r + o(\epsilon) \\ &\leqslant I_2 - C\epsilon + o(\epsilon) \\ &< I_2\end{aligned}$$

for sufficiently small $\epsilon > 0$, where $C = R_0|S^1|C_0^2/4$. Thus, for such small $\epsilon > 0$, we have

$$\int_{\mathbb{R}^2} F(\tilde{v}+t\phi)\,\mathrm{d}x = 0 \text{ and } \int_{\mathbb{R}^2} |\nabla(\tilde{v}+t\phi)|^2\,\mathrm{d}x < I_2.$$

This is a contradiction for $N = 2$. Thus, $\lim_{r\to R_0} v(r) = 0$; this completes the proof. □

Now we prove the Pohozaev identity for a weak solution of (1.1).

Proposition 2.4. *Let $w \in \mathbf{H}$ be a radially symmetric weak solution of (1.1) with $F_-(w) \in L^1(\mathbb{R}^N)$. Then w satisfies the following identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2\,\mathrm{d}x = N\int_{\mathbb{R}^N} F(w)\,\mathrm{d}x.$$

Proof. We see from (f3) and (f3-2) that $F_+(w) \in L^1$ since $u \in \mathbf{H}$. Thus we have that $|F(w)| \in L^1(\mathbb{R}^N)$. Let $\{\Omega_i\}$ be the connected components of $\{x \in \mathbb{R}^N \mid w > 0\}$ and choose an arbitrary component $\Omega \in \{\Omega_i\}$. Then Ω should be one of three possible cases, $B(0,R_1)$, $B(0,R_2)\setminus B(0,R_3)$, $\mathbb{R}^N\setminus B(0,R_4)$. We first prove that for any $i = 1,\cdots,4$, $\lim_{r\to R_i} w_r(r) = 0$.

Note from the equation

$$w_{rr} + \frac{N-1}{r}w_r = -f(w)$$

that for any $r_1, r_2 \in \Omega$,

$$\frac{1}{2}((w_r(r_2))^2 - (w_r(r_1))^2) + \int_{r_1}^{r_2} \frac{N-1}{r}(w_r(r))^2\,\mathrm{d}r = -\int_{w(r_1)}^{w(r_2)} f(t)\,\mathrm{d}t.$$

Since $w \in \mathbf{H}$ and $f \in L^1_{loc}(\mathbb{R})$, we see that for any $i = 1,\cdots,4$, $\lim_{r\to R_i} w_r(r)$ exists. As in the proof of Proposition 2.3, we can show that $u \in C^2(\{x \in \mathbb{R}^N \mid w > 0\})$.

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Choose a radially symmetric nonnegative function $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi > 0$ on $\partial\Omega$. We define $\Omega^\delta \equiv \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}$. Then, it follows from Green's identity that

$$\begin{aligned}\int_\Omega f(w)\phi \,\mathrm{d}x &= \lim_{\delta\to 0}\int_{\Omega^\delta} f(w)\phi\,\mathrm{d}x = \lim_{\delta\to 0}\int_{\Omega^\delta}(-\Delta w)\phi\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\left(\int_{\Omega^\delta}\nabla w\cdot\nabla\phi\,\mathrm{d}x - \int_{\partial\Omega^\delta}\frac{\partial w}{\partial\nu}\phi\,\mathrm{d}x\right) \\ &= \int_\Omega \nabla w\cdot\nabla\phi\,\mathrm{d}x - \lim_{\delta\to 0}\int_{\partial\Omega^\delta}\frac{\partial w}{\partial\nu}\phi\,\mathrm{d}x, \end{aligned} \tag{2.9}$$

where ν is the outward normal to $\partial\Omega^\delta$. Since $w(r) = 0$ for $r \in \partial\Omega$, we see that

$$\lim_{\delta\to 0}\int_{\partial\Omega^\delta}\frac{\partial w}{\partial\nu}\phi\,\mathrm{d}x \leqslant 0.$$

Then, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} f(w)\phi\,\mathrm{d}x = \sum_i\int_{\Omega_i} f(w)\phi\,\mathrm{d}x &= \sum_i\int_{\Omega_i}\nabla w\cdot\nabla\phi\,\mathrm{d}x - \sum_i\lim_{\delta\to 0}\int_{\partial\Omega_i^\delta}\frac{\partial w}{\partial\nu}\phi\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\nabla w\cdot\nabla\phi\,\mathrm{d}x - \sum_i\lim_{\delta\to 0}\int_{\partial\Omega_i^\delta}\frac{\partial w}{\partial\nu}\phi\,\mathrm{d}x.\end{aligned}$$

Since w is a weak solution, we get $\sum_i \lim_{\delta\to 0}\int_{\partial\Omega_i^\delta}\partial w/\partial\nu\phi\,\mathrm{d}x = 0$. This implies that as $r \to \partial\Omega_i$ for each i, $w_r(r)$ converges to 0.

Now we prove the Pohozaev identity. First, we consider the cases such that $\Omega = B(0, R_1)$ or $\Omega = B(0, R_2) \setminus B(0, R_3)$. For each $\delta > 0$, we again use the notation $\Omega^\delta \equiv \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}$. It follows from integration by parts that

$$\int_{\Omega^\delta} f(w)(x\cdot\nabla w)\,\mathrm{d}x = -N\int_{\Omega^\delta}F(w)\,\mathrm{d}x + \int_{\partial\Omega^\delta}F(w)(x\cdot\nu)\,\mathrm{d}S$$

and

$$2\int_{\Omega^\delta}\Delta w(x\cdot\nabla w)\,\mathrm{d}x = (N-2)\int_{\Omega^\delta}|\nabla w|^2\,\mathrm{d}x + \int_{\partial\Omega^\delta}\left(\frac{\partial w}{\partial\nu}\right)^2(x\cdot\nu)\,\mathrm{d}S.$$

Thus, since $u \in C^2(\Omega)$ and $\Delta w + f(w) = 0$ in Ω, taking $\delta \to 0$, we get from the continuity of $F(w), w_r$ on $\overline{\Omega}$ that

$$\begin{aligned}&\frac{(N-2)}{2}\int_\Omega|\nabla w|^2\,\mathrm{d}x - N\int_\Omega F(w)\,\mathrm{d}x \\ &\quad = -\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial w}{\partial\nu}\right)^2(x\cdot\nu)\,\mathrm{d}S - \int_{\partial\Omega}F(w)(x\cdot\nu)\,\mathrm{d}S = 0.\end{aligned}$$

Now, consider the remaining case $\Omega = \mathbb{R}^N \setminus B(0, R_4)$. Then, applying the above argument to $B(0, R_5) \setminus B(0, R_4)$ with $R_5 > R_4$, we see that

$$\int_{B(0,R_5)\setminus B(0,R_4)} \left[\frac{N-2}{2}|\nabla w|^2 - NF(w)\right] \mathrm{d}x$$
$$= -R_5 \left\{\int_{\partial B(0,R_5)} \left[\frac{1}{2}(w_r)^2 + F(w)\right] \mathrm{d}S\right\}.$$

Since

$$\int_{\mathbb{R}^N} |\nabla w|^2 + |F(w)| \,\mathrm{d}x = \int_0^\infty \left\{\int_{\partial B(0,R)} |\nabla w|^2 + |F(w)| \,\mathrm{d}S\right\} \mathrm{d}R < \infty,$$

there exists a sequence $\{R_{5,n}\}_n$ such that as $n \to \infty$,

$$R_{5,n} \to \infty \text{ and } R_{5,n} \int_{\partial B(0,R_{5,n})} |\nabla w|^2 + F(w) \,\mathrm{d}s \to 0.$$

Then, taking the limit $n \to \infty$, we get the identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N\setminus B(0,R_4)} |\nabla w|^2 \,\mathrm{d}x - N\int_{\mathbb{R}^N\setminus B(0,R_4)} F(w) \,\mathrm{d}x = 0.$$

Thus, adding the identity over Ω_i with respect to i, we obtain that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x - N\int_{\mathbb{R}^N} F(w) \,\mathrm{d}x = 0. \qquad \square$$

Then we are ready to prove the main theorem.

Completion of the proof for the main theorem. Let $u(x) = v(\sqrt{\theta}x)$. Then by Proposition 2.2, we see that for any $\phi \in C_0^\infty(B(0, R_0/\sqrt{\theta}))$,

$$\int_{B(0,R_0/\sqrt{\theta})} \nabla u \cdot \nabla \phi - f(u)\phi \,\mathrm{d}x = 0. \tag{2.10}$$

By Proposition 2.3, we know that $u \in C^2(B(0, R_0/\sqrt{\theta}))$ is a classical solution. If $R_0 = \infty$, there is nothing to prove, so we assume $R_0 < \infty$ from now on.

First, we show that $f(u) \in L^1(\mathbb{R}^N)$. Since $f \in C(0,\infty)$, it is enough to consider the integrability near $R_0/\sqrt{\theta}$. By (f3) or (f3-2), there exists a constant $M > 0$ such that $f_+(t) \leqslant M$ near $t = 0$. Then, for $r \in (R_0/(2\sqrt{\theta}), R_0/\sqrt{\theta})$, we get

$$u_{rr} + \frac{N-1}{r}u_r + M \geqslant u_{rr} + \frac{N-1}{r}u_r + f_+(u) = f_-(u).$$

Integrating both sides, we can see that $f_-(u)$ is L^1 near $R_0/\sqrt{\theta}$ since u_r is integrable near $R_0/\sqrt{\theta}$ and $\lim_{r\to R_0} u_r = 0$ exists. Therefore, $f(u) \in L^1(\mathbb{R}^N)$.

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To show that u is a weak solution of (1.1), we have to show that u satisfies (2.10) on $\mathbb{R}^N$. We deduce from Proposition 2.3 that for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi - f(u)\phi \,\mathrm{d}x &= \int_{B(0,R_0/\sqrt{\theta})} \nabla u \cdot \nabla \phi - f(u)\phi \,\mathrm{d}x \\ &= \int_{\partial B(0,R_0/\sqrt{\theta})} u_r \phi \,\mathrm{d}S - \int_{B(0,R_0/\sqrt{\theta})} (\Delta u + f(u))\phi \,\mathrm{d}x \\ &= 0.\end{aligned}$$

Now we want to show that u has the least energy among radially symmetric weak solutions of (1.1). We see from (f3), (f4) and (f3-2), (f4-2) that for any $w \in \mathbf{H}$, $\int_{\mathbb{R}^n} F_+(w)\,\mathrm{d}x < \infty$. Thus, if $F(w) \notin L^1(\mathbb{R}^N)$, we get $F_-(w) \notin L^1(\mathbb{R}^N)$. In this case, we get that

$$E(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F_+(w)\,\mathrm{d}x + \int_{\mathbb{R}^N} F_-(w)\,\mathrm{d}x = \infty.$$

Hence, we may assume that $F(w) \in L^1(\mathbb{R}^N)$ for a weak solution w of (1.1).

First, we consider a case $N > 2$. Recall that v is the minimizer of the following problem:

$$I_N = \inf\left\{ \int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x \,\middle|\, \int_{\mathbb{R}^N} F(w)\,\mathrm{d}x = 1, w \in \mathcal{D}^{1,2}(\mathbb{R}^N) \right\}.$$

By a change of variables, we see that

$$E(u) = \frac{1}{2} I_N \theta^{-(N-2)/2} - \theta^{-(N/2)}.$$

Define a function $h : (0, \infty) \to \mathbb{R}$ by $h(t) = 1/2 I_N t^{-(N-2)/2} - t^{-(N/2)}$, which has a maximum at $t = 2N/((N-2)I_N)$, where the maximum value is

$$h\left(\frac{2N}{(N-2)I_N}\right) = \frac{2}{N-2}\left(\frac{2N}{(N-2)I_N}\right)^{-(N/2)}.$$

Let w be an arbitrary radial weak solution of (1.1) such that $F(w) \in L^1(\mathbb{R}^N)$. Then it holds that

$$\int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x = NE(w), \quad \int_{\mathbb{R}^N} F(w)\,\mathrm{d}x = \frac{N-2}{2} E(w)$$

by Proposition 2.4. For $\sigma = ((N-2)/2E(w))^{1/N}$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} F(w(\sigma x))\,\mathrm{d}x = \sigma^{-N} \int_{\mathbb{R}^N} F(w)\,\mathrm{d}x = 1, \\ &\int_{\mathbb{R}^N} |\nabla w(\sigma x)|^2 \,\mathrm{d}x = \sigma^{2-N} \int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x = NE(w)^{2/N} \left(\frac{N-2}{2}\right)^{(2-N)/N}.\end{aligned}$$

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Since $\int_{\mathbb{R}^N} |\nabla w(\sigma x)|^2 \,\mathrm{d}x \geqslant I_N$, it follows that

$$E(w) \geqslant \left(\frac{I_N}{N}\right)^{N/2} \left(\frac{N-2}{2}\right)^{(N-2)/2} = h\left(\frac{2N}{(N-2)I_N}\right).$$

Therefore, we have

$$E(w) \geqslant h\left(\frac{2N}{(N-2)I_N}\right) \geqslant h(\theta) = E(u)$$

since h has a maximum at $2N/((N-2)I_N)$.

Second, we consider the case $N = 2$. Recall that v is the minimizer of the problem

$$I_2 = \inf\left\{\int_{\mathbb{R}^2} |\nabla w|^2 \,\mathrm{d}x \,\middle|\, \int_{\mathbb{R}^2} F(w)\,\mathrm{d}x = 0, w \not\equiv 0, w \in H^1(\mathbb{R}^2)\right\}.$$

Note that if w is a radial weak solution of (1.1) with $F(w) \in L^1(\mathbb{R}^N)$, then $\int_{\mathbb{R}^2} F(w)\,\mathrm{d}x = 0$ by Proposition 2.4. Hence, we immediately obtain $E(w) \geqslant E(v) = E(u)$.

Lastly, we prove that if $\int_0^\delta \mathrm{d}t/\sqrt{F_-(t)} < \infty$ for small $\delta > 0$, then $R_0 < \infty$. This result was proved by many authors for more general type of quasilinear equations (see for example [**1**, **13**–**15**, **17**]). For completeness, we write a short proof. To the contrary, suppose that $R_0 = \infty$. Then, since $v_r \leqslant 0$ on $\mathbb{R}$, we see that $v_{rr} \geqslant -\theta f(v)$. Since $\lim_{r\to\infty} v_r(r) = 0$ and $(|v_r|^2)_r \leqslant -2\theta(F(v))_r$, we have $((v_r)^2) \geqslant -2\theta(F(v))$. Thus,

$$v_r(r) \leqslant -\sqrt{2\theta F_-(v(r))}. \tag{2.11}$$

Integrating (2.11) on (r, ∞) for large $r > 0$, we see that

$$\int_0^{v(r)} \frac{\mathrm{d}t}{\sqrt{F_-(t)}} = -\int_r^\infty \frac{v_r(s)}{\sqrt{F_-(v(s))}}\,\mathrm{d}s \geqslant \sqrt{2\theta}\int_r^\infty \mathrm{d}s = \infty.$$

This is a contradiction, proving that $R_0 < \infty$ if $\int_0^\delta \mathrm{d}t/(\sqrt{F_-(t)}) < \infty$ for small $\delta > 0$. □

3. A one-dimensional case and concluding remarks

For the one-dimensional case, we can obtain a similar result. For $N = 1$, we assume the following conditions:

(f1-1) $f \in L^1_{loc}(\mathbb{R}) \cap C(0, \infty)$

(f2-1) there exists $T_0 \equiv \inf\{t > 0 : F(t) = 0\} < \infty$ such that $T_0 > 0$ and $f(T_0) > 0$.

Note that, by (f2-1), $F(t) < 0$ on $(0, T_0)$. Then, we define

$$R_1 \equiv \int_0^{T_0} \frac{\mathrm{d}s}{\sqrt{-2F(s)}} \in (0, \infty].$$

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THEOREM 3.1. *If (f1-1), (f2-1) hold, there exists an even solution u of the problem (1.1) for $N = 1$. Furthermore, this solution satisfies*

(i) $u(0) = T_0$, $u'(x) < 0$ *on* $(0, R_1)$ *for some* $R_1 \in (0, \infty]$ *and* $u(x) = 0$ *for* $x \geqslant R_1$,

(ii) $u \in C^2((-R_1, R_1)) \cap C^1(\mathbb{R})$.

Proof. Since F is C^1 near T_0 and $f(T_0) > 0$, $1/\sqrt{-2F(s)}$ is integrable on (x, T_0) for any $x \in (0, T_0)$. Then, a function

$$g(t) \equiv \int_t^{T_0} \mathrm{d}s/(\sqrt{-2F(s)})$$

is well-defined and C^1 on $(0, T_0)$. Let u be the inverse function of g on $[0, R_1)$, where $R_1 \equiv \int_0^{T_0} \mathrm{d}s/(\sqrt{-2F(s)}) \in (0, \infty]$. Then, we extend the function $u(x)$ on $(-R_1, 0]$ so that the extended function is even on (R_1, R_1). If $R_1 < \infty$, we extend u to the whole real line by setting $u(x) = 0$ for $x \in \mathbb{R} \setminus (-R_1, R_1)$. Now, by construction, u satisfies *(i)*.

By direct differentiation, we have

$$\frac{u'(x)}{\sqrt{-2F(u(x))}} = -1 \text{ for } x \in (-R_1, R_1). \tag{3.1}$$

Then, since $F \circ u$ is in $C^1((-R_1, R_1))$, we see that $u \in C^2((-R_1, R_1))$. Hence if we differentiate it again, we obtain

$$u'(x)(u''(x) + f(u(x))) = 0.$$

Since $u'(x) \neq 0$ on $(0, R_1)$, u satisfies (1.1). It remains to show that $\lim_{x\uparrow R_1} u'(x) = 0$ when $R_1 < \infty$. This immediately follows from (3.1) since $\lim_{x\uparrow R_1} F(u(x)) = 0$. □

REMARK 3.2. The additional condition $\limsup_{t\to 0} f(t) < \infty$ in (f3) and (f3-2) was used only to prove $f(u) \in L^1_{loc}(\mathbb{R}^N)$ for a solution u of (1.1). Without the additional condition, our proof shows that for $R_0, \theta > 0$ given in the previous section, there exists a classical solution of the following overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B(0, R_0/\sqrt{\theta}), \\ u > 0 & \text{in } B(0, R_0), \\ u = 0, \ \partial_\nu u = 0 & \text{on } \partial B(0, R_0/\sqrt{\theta}). \end{cases}$$

REMARK 3.3. In [**4**], the authors proved the radial symmetry of a least energy solution for (1.1) when $f \in C(\mathbb{R})$. To apply the symmetry result in [**4**] to our problem with a singular nonlinearity f, for any least energy solution u of (1.1), it is necessary to prove the Pohozaev identity [**3**, **12**]

$$(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x = 2N \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

For the details, see [**4**]. On the other hand, when f is not continuous at 0, we do not know whether or not the Pohozaev identity holds for any least energy solution of (1.1).

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