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ANISOTROPIC SCALING OF THE RANDOM GRAIN MODEL WITH APPLICATION TO NETWORK TRAFFIC

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Abstract

We obtain a complete description of anisotropic scaling limits of the random grain model on the plane with heavy-tailed grain area distribution. The scaling limits have either independent or completely dependent increments along one or both coordinate axes and include stable, Gaussian, and 'intermediate' infinitely divisible random fields. The asymptotic form of the covariance function of the random grain model is obtained. Application to superimposed network traffic is included.

Keywords: Random grain model; anisotropic scaling; long-range dependence; Lévy sheet; fractional Brownian sheet; workload process

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Secondary 60G22; 60G51; 60K25

1. Introduction

It is well known that many random fields (RFs) exhibit different scaling behaviors in different directions. Important examples of RFs with such a behavior are the fractional Brownian sheet (FBS) and various classes of stochastic partial differential equations driven by FBS; see, e.g. [2] and the references therein. For a stationary RF $Y = \{Y(t, s),\ (t, s) \in \mathbb{R}^2\}$, the simplest form of anisotropic scaling is obtained by taking partial integrals $S_{\lambda,\gamma}(x, y) = \int_{(0,\lambda x]\times(0,\lambda^\gamma y]} Y(t, s)\, \mathrm{d}t\, \mathrm{d}s$ over rectangles $(0, \lambda x] \times (0, \lambda^\gamma y] \subset \mathbb{R}^2_+$ whose sides grow with $\lambda \to \infty$ at different rate $O(\lambda)$ and $O(\lambda^\gamma)$ (provided $\gamma \neq 1$). Throughout this paper, we have $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}^2_+ := (0, \infty)^2$. The (large-scale) behavior of Y is reflected in the scaling limit

$$a^{-1}_{\lambda,\gamma} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} V_\gamma(x, y) \quad \text{as } \lambda \to \infty, \tag{1.1}$$

where $a_{\lambda,\gamma} \to \infty$ is a normalization and '$\xrightarrow{\mathrm{D}}$' denotes weak convergence of finite-dimensional distributions. Moreover, if $a_{\lambda,\gamma}$ is regularly varying at ∞ with exponent $H(\gamma) > 0$, the limit RF V_γ in (1.1) has stationary rectangular increments and satisfies the self-similarity property

$$V_\gamma(\lambda x, \lambda^\gamma y) \overset{\mathrm{D}}{=} \lambda^{H(\gamma)} V_\gamma(x, y) \quad \text{for each } \lambda > 0,$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in finite-dimensional distributions; see [21], which is a particular case of the operator scaling RF property introduced in Biermé *et al.* [4].

Puplinskaitė and Surgailis [21] observed that for many RFs Y on $\mathbb{Z}^2$ or $\mathbb{R}^2$, (nontrivial) scaling limits in (1.1) exist for any $\gamma > 0$, resulting in a one-dimensional family $\{V_\gamma,\ \gamma > 0\}$ of scaling limits termed the *scaling diagram of Y* below. Since scaling limits characterize the

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dependence structure and large-scale properties of the underlying random process, the scaling diagram provides a more complete 'large-scale summary of Y' compared to the (isotropic or anisotropic) scaling with fixed $\gamma > 0$ discussed in [1], [2], [5], [6], [12], [14], [22], [23], and elsewhere. Scaling diagrams of some classes of long-range dependent (LRD) Gaussian and aggregated nearest-neighbor autoregressive RFs on $\mathbb{Z}^2$ were identified in [20] and [21]. It turned out that for these RFs, there exists a unique point $\gamma_0 > 0$ such that the scaling limits $V_\gamma \overset{\mathrm{D}}{=} V_\pm$ do not depend on γ for $\gamma < \gamma_0, \gamma > \gamma_0$, and $V_+ \overset{\mathrm{D}}{\neq} V_-$. In [21] this phenomenon was termed the *scaling transition* (at $\gamma = \gamma_0$). The scaling transition also arises under joint temporal and contemporaneous aggregation of independent LRD processes in telecommunication and economics, see [9], [10], [16], [17], [19]; see also ([21, Remark 2.3]. In this paper we obtain a different kind of scaling diagram (see Figure 1) with *two* change-points $\gamma_- < \gamma_+$ of scaling limits which shows that this concept might be more complex requiring further study.

In this paper we study the scaling limits (scaling diagram) of the *random grain model*

$$X(t,s) := \sum_i \mathbf{1}\{((t-x_i)/R_i^p, (s-y_i)/R_i^{1-p}) \in B\}, \qquad (t,s) \in \mathbb{R}^2, \tag{1.2}$$

where $B \subset \mathbb{R}^2$ ('generic grain') is a measurable bounded set of finite Lebesgue measure $\mathrm{Leb}(B) < \infty$, $0 < p < 1$ is a shape parameter, $\{(x_i, y_i), R_i\}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity $\mathrm{d}x\,\mathrm{d}y F(\mathrm{d}r)$, and $\mathbf{1}$ is the indicator function. We assume that F is a probability distribution on $\mathbb{R}_+$ having a density function f such that

$$f(r) \sim c_f r^{-1-\alpha} \quad \text{as } r \to \infty, \tag{1.3}$$

for some $1 < \alpha < 2, c_f > 0$. The sum in (1.2) counts the number of uniformly scattered and randomly dilated grains $(x_i, y_i) + R_i^P B$ containing (t, s), where $R^P B := \{(R^p x, R^{1-p} y) \colon (x, y) \in B\} \subset \mathbb{R}^2$ is the dilation of B by factors R^p and R^{1-p} in the horizontal and vertical directions, respectively. The $p = 1/2$ case corresponds to uniform or isotropic dilation. Note that the area $\mathrm{Leb}(R^P B) = \mathrm{Leb}(B)R$ of a generic randomly dilated grain is proportional to R and does not depend on p and has a heavy-tailed distribution with finite mean $\mathbb{E}\,\mathrm{Leb}(R^P B) < \infty$ and infinite second moment $\mathbb{E}\,\mathrm{Leb}(R^P B)^2 = \infty$ according to (1.3). Condition (1.3) also guarantees that the covariance of the random grain model is not integrable, i.e. $\int_{\mathbb{R}^2} |\mathrm{cov}(X(0,0), X(t,s))|\,\mathrm{d}t\,\mathrm{d}s = \infty$, see Section 3; hence, (1.2) is an LRD RF. Examples of the grain set B are the unit ball and the unit square, leading respectively to the *random ellipses model*

$$X(t,s) = \sum_i \mathbf{1}\{(t-x_i)^2/R_i^{2p} + (s-y_i)^2/R_i^{2(1-p)} \le 1\}$$

and the *random rectangles model*

$$X(t,s) = \sum_i \mathbf{1}\{x_i < t \le x_i + R_i^p, y_i < s \le y_i + R_i^{1-p}\}.$$

Note that for $p \neq \frac{1}{2}$ the ratio $R^p/R^{1-p} = R^{2p-1}$ of the sides of a generic rectangle tends to 0 or ∞ as $R \to \infty$ implying that large rectangles are 'elongated' or 'flat' and resulting in a strong anisotropy of the random rectangles model. A similar observation applies to the general random grain model in (1.2).

Our main results are summarized in Figure 1 in which we show a panorama of scaling limits V_γ in (1.1) of the centered random grain model $Y(t,s) = X(t,s) - \mathbb{E}X(t,s)$ as γ changes between 0 and ∞. Precise formulations pertaining to Figure 1 and the terminology therein are given in Section 2. Below we explain the most important facts about this diagram.

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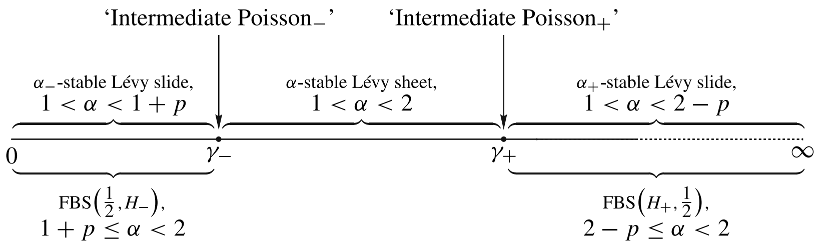

FIGURE 1: Scaling diagram of a random grain model.

First, note that, due to the symmetry of the random grain model in (1.2), the scaling limits in (1.1) are symmetric under simultaneous exchange $x \leftrightarrow y, \gamma \leftrightarrow 1/\gamma, p \leftrightarrow 1 - p$, and a reflection transformation of B. This symmetry is reflected in Figure 1, where the left region $0 < \gamma \le \gamma_-$ and the right region $\gamma_+ \le \gamma < \infty$ including the change points of the scaling limits

$$\gamma_- := \frac{1-p}{\alpha - (1-p)}, \qquad \gamma_+ := \frac{\alpha}{p} - 1, \tag{1.4}$$

are symmetric with respect to the above transformations. The middle region $\gamma_- < \gamma < \gamma_+$ in Figure 1 corresponds to an *α-stable Lévy sheet* defined as a stochastic integral over $(0, x] \times (0, y]$ with respect to an α-stable random measure on $\mathbb{R}^2_+$. According to Figure 1, for $\gamma > \gamma_+$ the scaling limits in (1.1) exhibit a dichotomy depending on parameters α, p, featuring a Gaussian (FBS) limit for $2 - p \le \alpha < 2$, and an α_+-stable limit for $1 < \alpha < 2 - p$ with stability parameter

$$\alpha_+ := \frac{\alpha - p}{1 - p} > \alpha \tag{1.5}$$

larger than the parameter α. The terminology *$\alpha_\pm$-stable Lévy slide* refers to a RF of the form $xL_+(y)$ or $yL_-(x)$ 'sliding' linearly to 0 along one of the coordinate axes, where $L_\pm$ are $\alpha_\pm$-stable Lévy processes (see Section 2 for the definition). Finally, the 'intermediate Poisson' limits in Figure 1 at $\gamma = \gamma_\pm$ are not stable although infinitely divisible RFs given by stochastic integrals with respect to a Poisson random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $c_f \,\mathrm{d}u\, \mathrm{d}v r^{-1-\alpha}\, \mathrm{d}r$.

The results of this paper are related to those in, for example, [3], [7], [9]–[11], [16], [17], and [19]–[21] in which different scaling regimes occur for various classes of LRD models, in particular, heavy-tailed duration models. Isotropic scaling limits ($\gamma = 1$ case) of the random grain and random balls models in arbitrary dimension were discussed in Kaj *et al.* [11] and Biermé *et al.* [3]. Lifshits [15] provided a nice discussion of limit behavior of heavy-tailed duration models whose spatial version is the random grain model in (1.2). From an application viewpoint, probably the most interesting is the study of different scaling regimes of superposed network traffic models [7], [9], [10], [16]. In these studies, it is assumed that traffic is generated by independent sources and the problem concerns the limit distribution of the aggregated traffic as the time scale T and the number of sources M both tend to ∞, possibly at different rates. In this paper we extend the above-mentioned work, by considering the limit behavior of the aggregated workload process

$$A_{M,K}(Tx) := \int_0^{Tx} W_{M,K}(t)\,\mathrm{d}t, \tag{1.6}$$

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TABLE 1: Limit distribution of the workload process in (1.6) with $M = T^\gamma$, $K = T^\beta$ at slow connection rate $0 < \gamma < \gamma_+$.

Parameter region		Limit process
$(1+\gamma)(1-p) < \alpha\beta \le \infty$	$1 < \alpha < 2$	α-stable Lévy process
$0 < \alpha\beta < (1+\gamma)(1-p)$	$1 < \alpha < 2p$	(α/p)-stable Lévy process
	$1 \vee 2p < \alpha < 2$	Brownian motion

TABLE 2: Limit distribution of the workload process in (1.6) with $M = T^\gamma$, $K = T^\beta$ at fast connection rate $\gamma_+ < \gamma < \infty$.

Parameter region		Limit process
$0 < \alpha_+\beta < \gamma_+$	$1 < \alpha < 2p$	Fractional Brownian motion, $H = (3 - (\alpha/p))/2$
	$1 \vee 2p < \alpha < 2$	Brownian motion
$\gamma_+ < \alpha_+\beta < \gamma$	$1 < \alpha < 2 - p$	Gaussian line
$\gamma < \alpha_+\beta \le \infty$		α_+-stable line
$\gamma_+ < \alpha_+\beta \le \infty$	$2 - p < \alpha < 2$	Fractional Brownian motion, $H = (2 - \alpha + p)/2p$

where $W_{M,K}(t) := \sum_i (R_i^{1-p} \wedge K)\,\mathbf{1}\{x_i < t \le x_i + R_i^p,\ 0 < y_i < M\}$, $t \ge 0$, and $\{(x_i, y_i), R_i\}$ is the same Poisson point process as in (1.2). The quantity $W_{M,K}(t)$ in (1.6) can be interpreted as the active workload at time t from sources arriving at x_i with $0 < y_i < M$ and transmitting at rate $R_i^{1-p} \wedge K$ during time interval $(x_i, x_i + R^p]$. Thus, the transmission rate in (1.6) is a (deterministic) function $(R^p)^{(1-p)/p} \wedge K$ of the transmission duration R^p depending on parameter $0 < p \le 1$, with $0 < K \le \infty$ playing the role of the maximal rate bound. The limiting case $p = 1$ in (1.6) corresponds to a constant rate workload from the stationary M/G/∞ queue. In Theorems 4.1 and 4.2 we obtain the limit distributions of the centered and properly normalized process $\{A_{M,K}(Tx),\ x \ge 0\}$ with heavy-tailed distribution of R in (1.3) when the time scale T, the source intensity M, and the maximal source rate K tend jointly to ∞ so as $M = T^\gamma$, $K = T^\beta$ for some $0 < \gamma < \infty, 0 < \beta \le \infty$. The results of Theorems 4.1 and 4.2 are summarized in Tables 1 and 2, respectively. The workload process in (1.6) featuring a power-law dependence between transmission rate and duration is closely related to the random rectangles model with $B = (0, 1]^2$, the last fact being reflected in Tables 1 and 2, where most (but not all) of the limit processes can be linked to the scaling limits in Figure 1 and where γ_+, α_+ are the same as in (1.4) and (1.5).

The rest of the paper is organized as follows. Section 2 contains rigorous formulations (Theorems 2.1–2.5) of the asymptotic results pertaining to Figure 1. In Section 3 we discuss the LRD properties and asymptotics of the covariance function of the random grain model. In Section 4 we obtain limit distributions of the aggregated workload process in (1.6). All proofs are relegated to Section 5.

Throughout, C stands for a generic positive constant which may assume different values at various locations and whose precise value has no importance.

2. Scaling limits of the random grain model

We can write (1.2) as the stochastic integral

$$X(t,s) = \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathbf{1}\{((t-u)/r^p, (s-v)/r^{1-p}) \in B\} N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r), \qquad (t,s) \in \mathbb{R}^2 \quad (2.1)$$

with respect to a Poisson random measure $N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r)$ on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\mathbb{E}N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) = \mathrm{d}u\,\mathrm{d}v F(\mathrm{d}r)$. The integral in (2.1) is well defined and follows a Poisson distribution with mean $\mathbb{E}X(t,s) = \mathrm{Leb}(B)\int_0^\infty rF(\mathrm{d}r)$. The RF X in (2.1) is stationary with finite variance and covariance function

$$\begin{aligned}&\mathrm{cov}(X(0,0), X(t,s))\\ &\quad= \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathbf{1}\{(u/r^p, v/r^{1-p}) \in B, ((u-t)/r^p, (v-s)/r^{1-p}) \in B\}\,\mathrm{d}u\,\mathrm{d}v F(\mathrm{d}r). \qquad (2.2)\end{aligned}$$

Let

$$\begin{aligned}S_{\lambda,\gamma}(x,y) &:= \int_0^{\lambda x}\int_0^{\lambda^\gamma y} (X(t,s) - \mathbb{E}X(t,s))\,\mathrm{d}t\,\mathrm{d}s \qquad (2.3)\\ &= \int_{\mathbb{R}^2\times\mathbb{R}_+}\bigg\{\int_0^{\lambda x}\int_0^{\lambda^\gamma y} \mathbf{1}\{((t-u)/r^p, (s-v)/r^{1-p}) \in B\}\,\mathrm{d}t\,\mathrm{d}s\bigg\}\widetilde{N}(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r)\end{aligned}$$

for $(x,y) \in \mathbb{R}_+^2$, where $\widetilde{N}(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) = N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) - \mathbb{E}N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r)$ is the centered Poisson random measure in (2.1). Recall the definition of $\gamma_\pm$, i.e.

$$\gamma_- := \frac{1-p}{\alpha-(1-p)}, \qquad \gamma_+ := \frac{\alpha}{p} - 1.$$

In Theorems 2.1–2.5 we specify the limit RFs V_γ and normalizations $a_{\lambda,\gamma}$ in (1.1) for all $\gamma > 0$ and $\alpha \in (1,2)$, $p \in (0,1)$ in Figure 1. Throughout this paper we assume that B is a bounded Borel set whose boundary ∂B has zero Lebesgue measure, i.e. $\mathrm{Leb}(\partial B) = 0$.

2.1. The $\gamma_- < \gamma < \gamma_+$ case

For $1 < \alpha < 2$, we introduce an α-stable Lévy sheet

$$L_\alpha(x,y) := Z_\alpha((0,x]\times(0,y]), \qquad (x,y) \in \mathbb{R}_+^2 \qquad (2.4)$$

as a stochastic integral with respect to an α-stable random measure $Z_\alpha(\mathrm{d}u\,\mathrm{d}v)$ on $\mathbb{R}^2$ with control measure $\sigma^\alpha\,\mathrm{d}u\,\mathrm{d}v$ and skewness parameter 1, where the constant σ^α is given in (5.5) below. Thus, $\mathbb{E}\exp\{\mathrm{i}\theta Z_\alpha(A)\} = \exp\{-\mathrm{Leb}(A)\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2))\}$, $\theta \in \mathbb{R}$, for any Borel set $A \subset \mathbb{R}^2$ of finite Lebesgue measure $\mathrm{Leb}(A) < \infty$. Note $\mathbb{E}Z_\alpha(A) = 0$.

Theorem 2.1. *Let $\gamma_- < \gamma < \gamma_+$, $1 < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \xrightarrow{\mathrm{D}} L_\alpha(x,y) \quad \text{as } \lambda \to \infty, \qquad (2.5)$$

where $H(\gamma) := (1+\gamma)/\alpha$ and L_α is an α-stable Lévy sheet defined in (2.4).

2.2. The $\gamma > \gamma_+$, $1 < \alpha < 2 - p$, and $\gamma < \gamma_-$, $1 < \alpha < 1 + p$, cases

For $1 < \alpha < 2 - p$ and $1 < \alpha < 1 + p$ we introduce the totally skewed stable Lévy processes $\{L_+(y),\ y \ge 0\}$ and $\{L_-(x),\ x \ge 0\}$ with respective stability indices $\alpha_\pm \in (1,2)$ defined as

$$\alpha_+ := \frac{\alpha - p}{1-p}, \qquad \alpha_- := \frac{\alpha - 1 + p}{p}$$

and characteristic functions

$$\mathbb{E}\exp\{\mathrm{i}\theta L_\pm(1)\} := \exp\bigg\{-\sigma^{\alpha_\pm}|\theta|^{\alpha_\pm}\bigg(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan\bigg(\frac{\pi\alpha_\pm}{2}\bigg)\bigg)\bigg\}, \qquad \theta \in \mathbb{R}, \qquad (2.6)$$

where σ^{α_+} is given in (5.10) and σ^{α_-} can be found by symmetry; see (5.1) below.

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Theorem 2.2. (i) *Let* $\gamma > \gamma_+$, $1 < \alpha < 2 - p$. *Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} x L_+(y) \quad \text{as } \lambda \to \infty, \tag{2.7}$$

where $H(\gamma) := 1 + \gamma/\alpha_+$ *and* L_+ *is the* α_+*-stable Lévy process defined by (2.6).*

(ii) *Let* $0 < \gamma < \gamma_-$, $1 < \alpha < 1 + p$. *Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} y L_-(x) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma + 1/\alpha_-$ *and* L_- *is the* α_-*-stable Lévy process defined by (2.6).*

2.3. The $\gamma > \gamma_+$, $2 - p \le \alpha < 2$, and $\gamma < \gamma_-$, $1 + p \le \alpha < 2$, cases

A (standard) FBS B_{H_1,H_2} with Hurst indices $0 < H_1, H_2 \le 1$ is defined as a Gaussian process with zero mean and covariance

$$\mathbb{E} B_{H_1,H_2}(x_1, y_1) B_{H_1,H_2}(x_2, y_2)$$
$$= \tfrac{1}{4}(x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1})(y_1^{2H_2} + y_2^{2H_2} - |y_1 - y_2|^{2H_2}),$$

$(x_i, y_i) \in \mathbb{R}^2_+$, $i = 1, 2$. The constants σ_+ and $\widetilde{\sigma}_+$ appearing in Theorems 2.3(i) and 2.4(i) are defined in (5.14) and (5.16), respectively. The corresponding constants σ_- and $\widetilde{\sigma}_-$ in Theorems 2.3(ii) and 2.4(ii) can be found by symmetry (see (5.1)).

Theorem 2.3. (i) *Let* $\gamma > \gamma_+, 2 - p < \alpha < 2$. *Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} \sigma_+ B_{H_+,1/2}(x, y) \quad \text{as } \lambda \to \infty, \tag{2.8}$$

where $H(\gamma) := H_+ + \gamma/2$, $H_+ := 1/p - \gamma_+/2 = (2 - \alpha + p)/2p \in (\frac{1}{2}, 1)$, *and* $B_{H_+,1/2}$ *is an FBS with parameters* $(H_+, \frac{1}{2})$.

(ii) *Let* $\gamma < \gamma_-, 1 + p < \alpha < 2$. *Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} \sigma_- B_{1/2,H_-}(x, y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma H_- + \frac{1}{2}$, $H_- := 1/(1 - p) + (1 - p - \alpha)/2(1 - p) \in (\frac{1}{2}, 1)$, *and* $B_{1/2,H_-}$ *is an FBS with parameters* $(\frac{1}{2}, H_-)$.

Theorem 2.4. (i) *Let* $\gamma > \gamma_+, \alpha = 2 - p$. *Then*

$$\lambda^{-H(\gamma)} (\log \lambda)^{-1/2} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} \widetilde{\sigma}_+ B_{1,1/2}(x, y) \quad \text{as } \lambda \to \infty, \tag{2.9}$$

where $H(\gamma) := 1 + \gamma/2$, $B_{1,1/2}$ *is an FBS with parameters* $(1, \frac{1}{2})$.

(ii) *Let* $\gamma < \gamma_-, \alpha = 1 + p$. *Then*

$$\lambda^{-H(\gamma)} (\log \lambda)^{-1/2} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} \widetilde{\sigma}_- B_{1/2,1}(x, y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma + \frac{1}{2}$ *and* $B_{1/2,1}$ *is an FBS with parameters* $(\frac{1}{2}, 1)$.

2.4. The $\gamma = \gamma_\pm$ cases

Define 'intermediate Poisson' RFs $I_\pm = \{I_\pm(x, y),\ (x, y) \in \mathbb{R}^2_+\}$ as stochastic integrals

$$I_+(x, y) := \int_{\mathbb{R}\times(0,y]\times\mathbb{R}_+} \widetilde{M}(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) \int_{(0,x]\times\mathbb{R}} \mathbf{1}\{((t-u)/r^p, s/r^{1-p}) \in B\}\,\mathrm{d}t\,\mathrm{d}s, \quad (2.10)$$

$$I_-(x, y) := \int_{(0,x]\times\mathbb{R}\times\mathbb{R}_+} \widetilde{M}(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) \int_{\mathbb{R}\times(0,y]} \mathbf{1}\{(t/r^p, (s-v)/r^{1-p}) \in B\}\,\mathrm{d}t\,\mathrm{d}s$$

with respect to the centered Poisson random measure $\widetilde{M}(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) = M(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) - \mathbb{E}M(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r)$ on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\mathbb{E}M(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) = c_f\,\mathrm{d}u\,\mathrm{d}v r^{-(1+\alpha)}\,\mathrm{d}r$.

Proposition 2.1. (i) *The RF I_+ in (2.10) is well defined for $1 < \alpha < 2, 0 < p < 1$, and $\mathbb{E}|I_+(x, y)|^q < \infty$ for any $0 < q < \alpha_+ \wedge 2$. Moreover, if $2-p < \alpha < 2$ then $\mathbb{E}|I_+(x, y)|^2 < \infty$ and*

$$\mathbb{E}I_+(x_1, y_1)I_+(x_2, y_2) = \sigma_+^2 \mathbb{E}B_{H_+,1/2}(x_1, y_1)B_{H_+,1/2}(x_2, y_2), \qquad (x_i, y_i) \in \mathbb{R}^2_+,\ i = 1, 2, \quad (2.11)$$

where σ_+, H_+ are the same as in Theorem 2.3(i).

(ii) *The RF I_- in (2.10) is well defined for $1 < \alpha < 2, 0 < p < 1$, and $\mathbb{E}|I_-(x, y)|^q < \infty$ for any $0 < q < \alpha_- \wedge 2$. Moreover, if $1 + p < \alpha < 2$ then $\mathbb{E}|I_-(x, y)|^2 < \infty$ and*

$$\mathbb{E}I_-(x_1, y_1)I_-(x_2, y_2) = \sigma_-^2 \mathbb{E}B_{1/2,H_-}(x_1, y_1)B_{1/2,H_-}(x_2, y_2), \qquad (x_i, y_i) \in \mathbb{R}^2_+, i = 1, 2,$$

where σ_-, H_- are the same as in Theorem 2.3(ii).

Theorem 2.5. (i) *Let $\gamma = \gamma_+$, $1 < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} I_+(x, y) \quad \text{as } \lambda \to \infty, \quad (2.12)$$

where $H(\gamma) := 1/p$ and RF I_+ is defined in (2.10).

(ii) *Let $\gamma = \gamma_-$, $1 < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} I_-(x, y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma_-/(1 - p)$ and RF I_- is defined in (2.10).

Remark 2.1. The normalizing exponent $H(\gamma) \equiv H(\gamma, \alpha, p)$ in Theorems 2.1–2.5 is a jointly continuous (albeit nonanalytic) function of $(\gamma, \alpha, p) \in (0, \infty) \times (1, 2) \times (0, 1)$.

Remark 2.2. Restriction $\alpha < 2$ is crucial for our results. Indeed, if $\alpha > 2$ then for *any* $\gamma > 0$, $p \in (0, 1)$ the normalized integrals $\lambda^{-(1+\gamma)/2} S_{\lambda,\gamma}(x, y) \xrightarrow{\mathrm{D}} \sigma B_{1/2,1/2}(x, y)$ tend to a classical Brownian sheet with variance $\sigma^2 = \mathrm{Leb}(B)^2 \int_0^\infty r^2 F(\mathrm{d}r)$. We omit the proof of the last result which follows a general scheme of the proofs in Section 5.

3. LRD properties of the random grain model

One of the most common definitions of the LRD property pertains to stationary random processes with a nonsummable (nonintegrable) autocovariance function. In the case of anisotropic RFs, the autocovariance function may decay at different rates in different directions, motivating a more detailed classification of LRD as in Definition 3.1 below. In this section we also verify these LRD properties for the random grain model in (1.2) and (1.3) and relate them to the change of the scaling limits or the dichotomies in Figure 1; see Remark 3.1 below.

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Definition 3.1. Let $Y = \{Y(t,s),\ (t,s) \in \mathbb{R}^2\}$ be a stationary RF with finite variance and nonnegative covariance function $\rho_Y(t,s) := \operatorname{cov}(Y(0,0),\ Y(t,s)) \geq 0$. We say that:

(i) Y has a *short-range dependence (SRD)* property if $\int_{\mathbb{R}^2} \rho_Y(t,s)\,\mathrm{d}t\,\mathrm{d}s < \infty$; otherwise we say that Y has a *long-range dependence (LRD)* property;

(ii) Y has a *vertical SRD* property if $\int_{[-Q,Q]\times\mathbb{R}} \rho_Y(t,s)\,\mathrm{d}t\,\mathrm{d}s < \infty$ for any $0 < Q < \infty$; otherwise we say that Y has a *vertical LRD* property;

(iii) Y has a *horizontal SRD* property if $\int_{\mathbb{R}\times[-Q,Q]} \rho_Y(t,s)\,\mathrm{d}t\,\mathrm{d}s < \infty$ for any $0 < Q < \infty$; otherwise we say that Y has a *horizontal LRD* property.

The main result of this section is Theorem 3.1 providing the asymptotics of the covariance function of the random grain model in (1.2) and (1.3) as $|t| + |s| \to \infty$ and enabling the verification of its integrability properties in Definition 3.1. Let

$$w := (|t|^{1/p} + |s|^{1/(1-p)})^p \quad \text{for } (t,s) \in \mathbb{R}^2.$$

For $p = \frac{1}{2}$, w is the Euclidean norm and $(w, \arccos(t/w))$ are the polar coordinates of $(t,s) \in \mathbb{R}^2$, $s \geq 0$. Introduce a function $b(z)$, $z \in [-1,1]$ by

$$b(z) := c_f \int_0^\infty \operatorname{Leb}\biggl(B \cap \biggl(B + \biggl(\frac{z}{r^p}, \frac{(1-|z|^{1/p})^{1-p}}{r^{1-p}}\biggr)\biggr)\biggr) r^{-\alpha}\,\mathrm{d}r, \tag{3.1}$$

playing the role of the 'angular function' in the asymptotics (3.2). For the random balls model with $p = \frac{1}{2}$ and $B = \{x^2 + y^2 \leq 1\}$, $b(z)$ is a constant function independent of z.

Theorem 3.1. *Let* $1 < \alpha < 2$, $0 < p < 1$.

(i) *The function* $b(z)$ *in (3.1) is bounded, continuous, and strictly positive on* $[-1,1]$.

(ii) *The covariance function* $\rho(t,s) := \operatorname{cov}(X(0,0), X(t,s))$ *in (2.2) has the following asymptotics:*

$$\rho(t,s) \sim b\biggl(\frac{\operatorname{sgn}(s)t}{w}\biggr) w^{-(\alpha-1)/p} \quad \text{as } |t| + |s| \to \infty. \tag{3.2}$$

Theorem 3.1 implies the following bound for covariance function $\rho(t,s) = \operatorname{cov}(X(0,0), X(t,s))$ of the random grain model: there exist $Q > 0$ and strictly positive constants $0 < C_- < C_+ < \infty$ such that, for any $|t| + |s| > Q$,

$$C_-(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha} \leq \rho(t,s) \leq C_+(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha}. \tag{3.3}$$

The bounds in (3.3) together with the easy integrability properties of the function $(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha}$ on $\{|t| + |s| > Q\}$ imply the following corollary.

Corollary 3.1. *The random grain model in (1.2) and (1.3) has*

(i) *the LRD property for any* $1 < \alpha < 2$, $0 < p < 1$*;*

(ii) *the vertical LRD property for* $1 < \alpha \leq 2 - p$ *and vertical SRD property for* $2 - p < \alpha < 2$ *and any* $0 < p < 1$*;*

(iii) *the horizontal LRD property for* $1 < \alpha \leq 1 + p$ *and horizontal SRD property for* $1 + p < \alpha < 2$ *and any* $0 < p < 1$.

Remark 3.1. The above corollary indicates that the dichotomy at $\alpha = 2 - p$ in Figure 1, region $\gamma > \gamma_+$ is related to the change from the vertical LRD to the vertical SRD property in the random grain model. Similarly, the dichotomy at $\alpha = 1 + p$ in Figure 1, region $\gamma < \gamma_+$ is related to the change from the horizontal LRD to the horizontal SRD property.

Puplinskaitė and Surgailis [21] introduced a type-I distributional LRD property for a RF Y with two-dimensional 'time' in terms of dependence properties of rectangular increments of V_γ, $\gamma > 0$. The increment of a RF $V = \{V(x, y),\ (x, y) \in \mathbb{R}^2_+\}$ on rectangle $K = (u, x] \times (v, y] \subset \mathbb{R}^2_+$ is defined as the double difference $V(K) = V(x, y) - V(u, y) - V(x, v) + V(u, v)$. Let $\ell \subset \mathbb{R}^2$ be a line, $(0, 0) \in \ell$. According to [21, Definition 2.2], a RF

$$V = \{V(x, y),\ (x, y) \in \mathbb{R}^2_+\}$$

is said to have

(i) *independent rectangular increments in direction* ℓ if $V(K)$ and $V(K')$ are independent for any two rectangles $K, K' \subset \mathbb{R}^2_+$ which are separated by an orthogonal line $\ell' \perp \ell$;

(ii) *invariant rectangular increments in direction* ℓ if $V(K) = V(K')$ for any two rectangles K, K' such that $K' = (x, y) + K$ for some $(x, y) \in \ell$;

(iii) *properly dependent rectangular increments* if V has neither independent nor invariant increments in arbitrary direction ℓ.

Further on, a stationary RF Y on $\mathbb{Z}^2$ is said to have *type-I distributional LRD* [21, Definition 2.4] if there exists a unique point $\gamma_0 > 0$ such that its scaling limit V_{γ_0} has properly dependent rectangular increments while all other scaling limits $V_\gamma, \gamma \neq \gamma_0$ have either independent or invariant rectangular increments in some direction $\ell = \ell(\gamma)$. The above definition trivially extends to a RF Y on $\mathbb{R}^2$.

We end this section with the observation that *all scaling limits of the random grain model in (1.2) and (1.3) in Theorems 2.1–2.5 have either independent or invariant rectangular increments in the direction of one or both coordinate axes.* This last fact is immediate from stochastic integral representations in (2.4), (2.10), the covariance function of FBS with Hurst indices H_1, H_2 equal to 1 or $\frac{1}{2}$ (see also [21, Example 2.3]) and the limit RFs in (2.7). We conclude that the random grain model in (1.2) and (1.3) *does not have type-I distributional LRD* in contrast to Gaussian and other classes of LRD RFs discussed in [21] and [20]. This last conclusion is not surprising since similar facts about scaling limits of heavy-tailed duration models with one-dimensional time are well known; see, e.g. [13].

4. Limit distributions of an aggregated workload process

We write the accumulated workload in (1.6) as the integral

$$A_{M,K}(Tx) = \int_{\mathbb{R}\times(0,M]\times\mathbb{R}_+} \left\{ (r^{1-p} \wedge K) \int_0^{Tx} \mathbf{1}\{u < t \le u + r^p\}\,\mathrm{d}t \right\} N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r), \qquad (4.1)$$

where $N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r)$ is the same Poisson random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity $\mathbb{E}N(\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r) = \mathrm{d}u\,\mathrm{d}v F(\mathrm{d}r)$ as in (1.2). We assume that $F(\mathrm{d}r)$ has a density $f(r)$ satisfying (1.3) with $1 < \alpha < 2$ as in Section 2. We let $p \in (0, 1]$ in (4.1) and, thus, the parameter may take value $p = 1$ as well. We assume that K and M grow with T in such a way that

$$M = T^\gamma, \qquad K = T^\beta \quad \text{for some } 0 < \gamma < \infty,\ 0 < \beta \le \infty.$$

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We are interested in the limit distribution

$$b_T^{-1}(A_{M,K}(Tx) - \mathbb{E}A_{M,K}(Tx)) \xrightarrow{\mathrm{D}} \mathcal{A}(x) \quad \text{as } T \to \infty, \tag{4.2}$$

where $b_T \equiv b_{T,\gamma,\beta} \to \infty$ is a normalization.

Recall from (1.4) and (1.5) the definitions

$$\gamma_+ = \frac{\alpha}{p} - 1, \qquad \alpha_+ = \frac{\alpha - p}{1 - p}.$$

For $p = 1$, let $\alpha_+ := \infty$. By assumption (1.3), transmission durations R_i^p, $i \in \mathbb{Z}$ have a heavy-tailed distribution with tail parameter $\alpha/p > 1$. Following the terminology in [7], [9], [10], and [16], the regions $\gamma < \gamma_+$ and $\gamma > \gamma_+$ will be respectively referred to as the *slow connection rate* and the *fast connection rate*. For each of these 'regimes', Theorems 4.1 and 4.2 detail the limit processes and normalizations in (4.2) depending on parameters β, α, p.

The RFs defined in Section 2 reappear in Theorems 4.1 and 4.2 for the certain grain set; namely, the unit square $B = (0, 1]^2$. Recall that a homogeneous Lévy process $\{L(x),\ x \geq 0\}$ is completely specified by its characteristic function $\mathbb{E}\exp\{\mathrm{i}\theta L(1)\}$, $\theta \in \mathbb{R}$. A (standard) fractional Brownian motion with Hurst parameter $H \in (0, 1]$ is a Gaussian process $\{B_H(x),\ x \geq 0\}$ with zero mean and covariance function $(\frac{1}{2})(x^{2H} + y^{2H} - |x - y|^{2H})$, $x, y \geq 0$.

Theorem 4.1. (Slow connection rate.) *Let $0 < \gamma < \gamma_+$. The convergence in (4.2) holds with the limit $\mathcal{A}$ and normalization $b_T = T^{\mathcal{H}}$ specified in* (i)–(iii) *below.*

(i) *Let $(1 + \gamma)(1 - p) < \alpha\beta \leq \infty$. Then $\mathcal{H} := (1 + \gamma)/\alpha$ and $\mathcal{A} := \{L_\alpha(x, 1),\ x \geq 0\}$ is an α-stable Lévy process defined by (2.4).*

(ii) *Let $0 < \alpha\beta < (1 + \gamma)(1 - p)$ and $1 < \alpha < 2p$. Then $\mathcal{H} := \beta + (1 + \gamma)p/\alpha$ and $\mathcal{A} := \{L_{\alpha/p}(x),\ x \geq 0\}$ is an (α/p)-stable Lévy process with characteristic function given by (5.20).*

(iii) *Let $0 < \alpha\beta < (1 + \gamma)(1 - p)$ and $1 \vee 2p < \alpha < 2$. Then $\mathcal{H} := (\frac{1}{2})(1 + \gamma + \beta(2 - \alpha)/(1 - p))$ and $\mathcal{A} := \{\sigma_1 B(x),\ x \geq 0\}$ is a Brownian motion with variance σ_1^2 given by (5.21).*

Theorem 4.2. (Fast connection rate.) *Let $\gamma_+ < \gamma < \infty$. The convergence in (4.2) holds with the limit $\mathcal{A}$ and normalization $b_T := T^{\mathcal{H}}$ specified in* (i)–(v) *below.*

(i) *Let $0 < \alpha_+\beta < \gamma_+$ and $1 < \alpha < 2p$. Then $\mathcal{H} := H + \beta + \gamma/2$ and $\mathcal{A} := \{\sigma_2 B_H(x),\ x \geq 0\}$ is a fractional Brownian motion with $H = (3 - \alpha/p)/2$ and variance σ_2^2 given by (5.22).*

(ii) *Let $0 < \alpha_+\beta < \gamma_+$ and $1 \vee 2p < \alpha < 2$. Then $\mathcal{H}$ and $\mathcal{A}$ are the same as in Theorem 4.1(iii).*

(iii) *Let $\gamma_+ < \alpha_+\beta < \gamma$ and $1 < \alpha < 2 - p$. Then $\mathcal{H} := 1 + (\frac{1}{2})(\gamma + \beta(2 - \alpha - p)/(1 - p))$ and $\mathcal{A} := \{xZ,\ x \geq 0\}$ is a Gaussian line with random slope $Z \sim N(0, \sigma_3^2)$ and σ_3^2 given in (5.23).*

(iv) *Let $\gamma < \alpha_+\beta \leq \infty$ and $1 < \alpha < 2 - p$. Then $\mathcal{H} := 1 + \gamma/\alpha_+$ and $\mathcal{A} := \{xL_+(1),\ x \geq 0\}$ is an α_+-stable line with random slope $L_+(1)$ having α_+-stable distribution defined by (2.6).*

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(v) *Let* $\gamma_+ < \alpha_+\beta \le \infty$ *and* $2 - p < \alpha < 2$. *Then* $\mathcal{H} := H_+ + \gamma/2$ *and* $\mathcal{A} := \{\sigma_+ B_{H_+,1/2}(x, 1),\ x \ge 0\}$ *is a fractional Brownian motion with* $H = H_+ = (2 - \alpha + p)/2p$ *and variance* σ_+^2 *given by (5.14).*

Remark 4.1. For $\gamma = \gamma_+$ we have $(1 + \gamma)(1 - p)/\alpha = \gamma_+/\alpha_+ = (1 - p)/p$. A complete description of limit workload processes in (4.2) for *all* values $1 < \alpha < 2$, $0 < p \le 1$ and $0 < \gamma < \infty$, $0 \le \beta \le \infty$ including the case of *intermediate connection rate* $\gamma = \gamma_+$ can be found in the extended version of this paper [18]. Apart from the classical Gaussian and stable limit processes given in Theorems 4.1 and 4.2, this description includes some 'intermediate' infinitely divisible processes given by stochastic integrals with respect to Poisson or Gaussian random measures, in particular, the intermediate process discussed in [8] and [9].

Remark 4.2. Note that $p = 1$ implies that $\gamma_+ = \alpha - 1$. In this case, Theorem 4.1 reduces to the α-stable limit in (i), whereas Theorem 4.2 reduces to the fractional Brownian motion limit in (v) discussed in [16] and other papers. A similar dichotomy appears for β close to 0 and $1 < \alpha < 2p$ with the difference that α is now replaced by α/p. Intuitively, it can be explained as follows. For small $\beta > 0$, the workload process $W_{M,K}(t)$ in (1.6) behaves like a constant rate process $K \sum_i \mathbf{1}\{x_i < t \le x_i + R_i^p,\ 0 < y_i < M\}$ with transmission lengths R_i^p that are independent and identically distributed and follow the same distribution $\mathbb{P}(R_i^p > r) = \mathbb{P}(R_i > r^{1/p}) \sim (c_f/\alpha) r^{-(\alpha/p)}$, $r \to \infty$ with tail parameter $1 < \alpha/p < 2$. Therefore, for small β our results agree with [16], including the Gaussian limit in Theorems 4.1(iii) and 4.2(ii) arising when the R_i^p have finite variance.

Remark 4.3. As it follows from the proof, the random line limits in Theorem 4.2(iii) and 4.2(iv) are caused by extremely long sessions starting in the past at times $x_i < 0$ and lasting $R_i^p = O(T^{\alpha_+\beta/\gamma_+})$, $\gamma_+ < \alpha_+\beta < \gamma$ or $R_i^p = O(T^{\gamma/\gamma_+})$, $\gamma_+ < \gamma < \alpha_+\beta$, respectively, so that typically these sessions end at times $x_i + R_i^p \gg T$.

5. Proofs

5.1. Proofs of Sections 2 and 3

Let

$$X^*(t, s) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathbf{1}\{((t - u)/r^{1-p}, (s - v)/r^p) \in B^*\} N(\mathrm{d}u\, \mathrm{d}v\, \mathrm{d}r), \qquad (t, s) \in \mathbb{R}^2,$$

be a 'reflected' version of (2.1), with B replaced by $B^* := \{(u, v) \in \mathbb{R}^2 \colon (v, u) \in B\}$, p replaced by $1 - p$ and the same Poisson random measure $N(\mathrm{d}u\, \mathrm{d}v\, \mathrm{d}r)$ as in (2.1). Let

$$S^*_{\lambda_*,\gamma_*}(x, y) := \int_0^{\lambda_* x} \int_0^{\lambda_*^{\gamma_*} y} (X^*(t, s) - \mathbb{E}X^*(t, s))\, \mathrm{d}t\, \mathrm{d}s, \qquad (x, y) \in \mathbb{R}_+^2$$

be the corresponding partial integral in (2.3). If λ_*, γ_* are related to λ, γ as $\lambda_* = \lambda^\gamma$, $\gamma_* = 1/\gamma$ then

$$S^*_{\lambda_*,\gamma_*}(y, x) \overset{\mathrm{D}}{=} S_{\lambda,\gamma}(x, y) \tag{5.1}$$

holds by the symmetry property of the Poisson random measure. As noted in the introduction, (5.1) allows us to reduce the limits of $S_{\lambda,\gamma}(x, y)$ as $\lambda \to \infty$ and $\gamma \le \gamma_-$ to the limits of $S^*_{\lambda_*,\gamma_*}(y, x)$ as $\lambda_* \to \infty$ and $\gamma_* \ge \gamma_{*+} := \alpha/(1 - p) - 1$. As a consequence, the proofs of parts (ii) of Theorems 2.2–2.5 can be omitted since they can be deduced from parts (i) of the corresponding statements.

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The convergence of normalized partial integrals in (1.1) is equivalent to the convergence of characteristic functions, i.e.

$$\mathbb{E}\exp\bigg\{\mathrm{i}a_{\lambda,\gamma}^{-1}\sum_{i=1}^{m}\theta_i S_{\lambda,\gamma}(x_i,y_i)\bigg\}\to\mathbb{E}\exp\bigg\{\mathrm{i}\sum_{i=1}^{m}\theta_i V_\gamma(x_i,y_i)\bigg\}\quad\text{as }\lambda\to\infty \tag{5.2}$$

for all $m=1,2,\ldots,(x_i,y_i)\in\mathbb{R}_+^2,\theta_i\in\mathbb{R},i=1,\ldots,m$. We restrict the proof of (5.2) to a one-dimensional convergence for $m=1,(x,y)\in\mathbb{R}_+^2$ only. The general case of (5.2) follows analogously. We have

$$\begin{aligned}W_{\lambda,\gamma}(\theta)&:=\log\mathbb{E}\exp\{\mathrm{i}\theta a_{\lambda,\gamma}^{-1}S_{\lambda,\gamma}(x,y)\}\\&=\int_{\mathbb{R}^2\times\mathbb{R}_+}\Psi\bigg(\frac{\theta}{a_{\lambda,\gamma}}\int_0^{\lambda x}\int_0^{\lambda^\gamma y}\mathbf{1}\{((t-u)/r^p,(s-v)/r^{1-p})\\&\qquad\in B\}\,\mathrm{d}t\,\mathrm{d}s\bigg)\,\mathrm{d}u\,\mathrm{d}v f(r)\,\mathrm{d}r,\end{aligned}\tag{5.3}$$

where $\Psi(z):=\exp(\mathrm{i}z)-1-\mathrm{i}z$, $z\in\mathbb{R}$. We shall use the following inequality:

$$|\Psi(z)|\le\min\bigg(2|z|,\frac{z^2}{2}\bigg),\qquad z\in\mathbb{R}.\tag{5.4}$$

Proof of Theorem 2.1. In the integrals on the right-hand side of (5.3) we change the variables, i.e.

$$\frac{t-u}{r^p}\to t,\qquad\frac{s-v}{r^{1-p}}\to s,\qquad u\to\lambda u,\qquad v\to\lambda^\gamma v,\qquad r\to\lambda^{H(\gamma)}r.$$

This yields $W_{\lambda,\gamma}(\theta)=\int_0^\infty g_\lambda(r)f_\lambda(r)\,\mathrm{d}r$, where

$$f_\lambda(r):=\lambda^{(1+\alpha)H(\gamma)}f(\lambda^{H(\gamma)}r)\to c_f r^{-(1+\alpha)},\qquad\lambda\to\infty$$

according to (1.3), and

$$g_\lambda(r):=\int_{\mathbb{R}^2}\Psi(\theta h_\lambda(u,v,r))\,\mathrm{d}u\,\mathrm{d}v,$$

$$h_\lambda(u,v,r):=r\int_B\mathbf{1}\{0<u+\lambda^{-\delta_1}r^pt\le x,\ 0<v+\lambda^{-\delta_2}r^{1-p}s\le y\}\,\mathrm{d}t\,\mathrm{d}s,$$

where the exponents $\delta_1:=1-H(\gamma)p=(\gamma_+-\gamma)/(1+\gamma_+)>0$, $\delta_2:=\gamma-H(\gamma)(1-p)=(\gamma-\gamma_-)/(1+\gamma_-)>0$. Clearly,

$$h_\lambda(u,v,r)\to\mathrm{Leb}(B)r\,\mathbf{1}\{0<u\le x,\ 0<v\le y\},\qquad\lambda\to\infty$$

for any fixed $(u,v,r)\in\mathbb{R}^2\times\mathbb{R}_+$, $u\notin\{0,x\}$, $v\notin\{0,y\}$, implying

$$g_\lambda(r)\to xy\Psi(\theta\,\mathrm{Leb}(B)r)\quad\text{for any }r>0.$$

Since we have $\int_{\mathbb{R}^2}h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v=xyr\,\mathrm{Leb}(B)$ and $h_\lambda(u,v,r)\le Cr$, the dominating bound $|g_\lambda(r)|\le C\min(r,r^2)$ follows by (5.4). Whence and from Lemma 5.1, we conclude that

$$W_{\lambda,\gamma}(\theta)\to W_\gamma(\theta):=xyc_f\int_0^\infty(\exp(\mathrm{i}\theta\,\mathrm{Leb}(B)r)-1-\mathrm{i}\theta\,\mathrm{Leb}(B)r)r^{-(1+\alpha)}\,\mathrm{d}r.$$

It remains to verify that

$$W_\gamma(\theta) = -xy\sigma^\alpha|\theta|^\alpha\left(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha}{2}\right)\right) = \log\mathbb{E}\exp\{\mathrm{i}\theta L_\alpha(x,y)\},$$

where

$$\sigma^\alpha := \frac{c_f\,\mathrm{Leb}(B)^\alpha\cos(\pi\alpha/2)\Gamma(2-\alpha)}{\alpha(1-\alpha)}. \tag{5.5}$$

This proves the one-dimensional convergence in (2.5) and Theorem 2.1, too. □

Proof of Theorem 2.2. In (5.3), change the variables as follows:

$$\begin{aligned} t \to \lambda t, \qquad s - v \to \lambda^{(1-p)\gamma/(\alpha-p)}s,\\ u \to \lambda^{p\gamma/(\alpha-p)}u, \qquad v \to \lambda^\gamma v, \qquad r \to \lambda^{\gamma/(\alpha-p)}r. \end{aligned} \tag{5.6}$$

This yields $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r)f_\lambda(r)\,\mathrm{d}r$, where

$$f_\lambda(r) := \lambda^{(1+\alpha)\gamma/(\alpha-p)}f(\lambda^{\gamma/(\alpha-p)}r) \to c_f\,r^{-(1+\alpha)}, \qquad \lambda\to\infty \tag{5.7}$$

and $g_\lambda(r) := \int_{\mathbb{R}^2}\Psi(\theta h_\lambda(u,v,r))\,\mathrm{d}u\,\mathrm{d}v$ with

$$h_\lambda(u,v,r) := \int_0^x \mathrm{d}t\int_{\mathbb{R}}\mathbf{1}\{((\lambda^{-\delta_1}t-u)/r^p, s/r^{1-p})\in B\}\,\mathbf{1}\{0<v+\lambda^{-\delta_2}s<y\}\,\mathrm{d}s, \tag{5.8}$$

where $\delta_1 := p\gamma/(\alpha-p) - 1 = (\gamma-\gamma_+)/\gamma_+ > 0$ and $\delta_2 := \gamma(\alpha-1)/(\alpha-p) > 0$. Let $B(u) := \{v\in\mathbb{R}\colon (u,v)\in B\}$ and write $\mathrm{Leb}_1(A)$ for the Lebesgue measure of a set $A\subset\mathbb{R}$. By the dominated convergence theorem,

$$\begin{aligned} h_\lambda(u,v,r) &\to h(u,v,r)\\ &:= x\,\mathbf{1}\{0<v<y\}\int_{\mathbb{R}}\mathbf{1}\{(-u/r^p, s/r^{1-p})\in B\}\,\mathrm{d}s\\ &= x\,\mathbf{1}\{0<v<y\}r^{1-p}\,\mathrm{Leb}_1\left(B\left(\frac{-u}{r^p}\right)\right) \end{aligned} \tag{5.9}$$

for any $(u,v,r)\in\mathbb{R}^2\times\mathbb{R}_+$, $v\notin\{0,y\}$, implying

$$g_\lambda(r)\to g(r) := \int_{\mathbb{R}^2}\Psi(\theta h(u,v,r))\,\mathrm{d}u\,\mathrm{d}v = yr^p\int_{\mathbb{R}}\Psi(\theta x r^{1-p}\,\mathrm{Leb}_1(B(u)))\,\mathrm{d}u$$

for any $r>0$. Indeed, since B is bounded, for fixed $r>0$ the function $(u,v)\mapsto h_\lambda(u,v,r)$ has a bounded support uniformly in $\lambda\ge 1$. Therefore, it is easy to verify the domination criterion for the above convergence. Combining $h_\lambda(u,v,r)\le Cr^{1-p}$ with $\int_{\mathbb{R}^2}h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v = xyr\,\mathrm{Leb}(B)$ gives $|g_\lambda(r)|\le C\min(r, r^{2-p})$ by (5.4). Hence, and by Lemma 5.1, $W_{\lambda,\gamma}(\theta)\to W_\gamma(\theta) := c_f\int_0^\infty g(r)r^{-(1+\alpha)}\,\mathrm{d}r$. By a change of variable, the last integral can be written as

$$\begin{aligned} W_\gamma(\theta) &= c_f yx^{\alpha_+}(1-p)^{-1}\int_{\mathbb{R}}\mathrm{Leb}_1(B(u))^{\alpha_+}\,\mathrm{d}u\int_0^\infty(\exp(\mathrm{i}\theta w)-1-\mathrm{i}\theta w)w^{-(1+\alpha_+)}\,\mathrm{d}w\\ &= -(yx^{\alpha_+})\sigma^{\alpha_+}|\theta|^{\alpha_+}\left(1-\mathrm{i}\,\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_+}{2}\right)\right)\\ &= \log\mathbb{E}\exp\{\mathrm{i}\theta x L_+(y)\}, \end{aligned}$$

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where

$$\sigma^{\alpha_+} := \frac{c_f \Gamma(2-\alpha_+)\cos(\pi\alpha_+/2)}{(1-p)\alpha_+(1-\alpha_+)} \int_{\mathbb{R}} \mathrm{Leb}_1(B(u))^{\alpha_+}\,\mathrm{d}u, \tag{5.10}$$

thus completing the proof of the one-dimensional convergence in (2.7). □

Proof of Theorem 2.3. In (5.3), change the variables as follows:

$$t \to \lambda t, \qquad s - v \to \lambda^{(1/p)-1} s, \qquad u \to \lambda u, \qquad v \to \lambda^{\gamma} v, \qquad r \to \lambda^{1/p} r. \tag{5.11}$$

We have $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r)\,\mathrm{d}r$, where $f_\lambda(r) := \lambda^{(1+\alpha)/p} f(\lambda^{1/p} r)$,

$$g_\lambda(r) := \int_{\mathbb{R}^2} \lambda^{2(H(\gamma)-1/p)} \Psi(\theta\lambda^{(1/p)-H(\gamma)} h_\lambda(u,v,r))\,\mathrm{d}u\,\mathrm{d}v, \tag{5.12}$$

with

$$\begin{aligned} h_\lambda(u,v,r) &:= \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\{0 < v + \lambda^{-\delta} s < y\}\mathbf{1}\{((t-u)/r^p, s/r^{1-p}) \in B\}\,\mathrm{d}s \\ &\to \mathbf{1}\{0 < v < y\} \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\{((t-u)/r^p, s/r^{1-p}) \in B\}\,\mathrm{d}s \\ &= \mathbf{1}\{0 < v < y\} r^{1-p} \int_0^x \mathrm{Leb}_1\Big(B\Big(\frac{t-u}{r^p}\Big)\Big)\,\mathrm{d}t \\ &=: h(u,v,r) \quad \text{as } \lambda \to \infty \end{aligned} \tag{5.13}$$

for all $(u,v,r) \in \mathbb{R}^2 \times \mathbb{R}_+$, $v \notin \{0,y\}$, since $\delta := 1+\gamma-(1/p) > 0$. Note that $2(H(\gamma)-1/p) = \gamma - \gamma_+ > 0$ and, hence,

$$\lambda^{2(H(\gamma)-1/p)} \Psi(\theta\lambda^{(1/p)-H(\gamma)} h_\lambda(u,v,r)) \to -\Big(\frac{\theta^2}{2}\Big) h^2(u,v,r), \qquad \lambda \to \infty.$$

Next, by the dominated convergence theorem

$$g_\lambda(r) \to g(r) := -\frac{\theta^2}{2} \int_{\mathbb{R}^2} h^2(u,v,r)\,\mathrm{d}u\,\mathrm{d}v \quad \text{for any } r > 0.$$

Using $\int_{\mathbb{R}^2} h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v = xy\,\mathrm{Leb}(B) r$ and $h_\lambda(u,v,r) \le C\min(r^{1-p}, r)$ similarly as in the proof of Theorem 2.2, we obtain $|g_\lambda(r)| \le C\int_{\mathbb{R}^2} h_\lambda^2(u,v,r)\,\mathrm{d}u\,\mathrm{d}v \le C\min(r^{2-p}, r^2)$. Then, by Lemma 5.1,

$$W_{\lambda,\gamma}(\theta) \to W_\gamma(\theta) := c_f \int_0^\infty g(r) r^{-(1+\alpha)}\,\mathrm{d}r = -\Big(\frac{\theta^2}{2}\Big)\sigma_+^2 x^{2H_+} y,$$

where

$$\sigma_+^2 := c_f \int_{\mathbb{R}} \mathrm{d}u \int_0^\infty \Big(\int_0^1 \mathrm{Leb}_1\Big(B\Big(\frac{t-u}{r^p}\Big)\Big)\,\mathrm{d}t\Big)^2 r^{1-\alpha-2p}\,\mathrm{d}r, \tag{5.14}$$

where the last integral converges. (Indeed, since $u \mapsto \mathrm{Leb}_1(B(u)) = \int \mathbf{1}\{(u,v) \in B\}\,\mathrm{d}v$ is a bounded function with compact support, the inner integral in (5.14) does not exceed $C(1 \wedge r^p)\mathbf{1}\{|u| < K(1+r^p)\}$ for some $C, K > 0$ implying $\sigma_+^2 \le C\int_0^\infty (1 \wedge r^p)^2 (1 + r^p) r^{1-\alpha-2p}\,\mathrm{d}r < \infty$ since $2 - p < \alpha < 2$.) This ends the proof of the one-dimensional convergence in (2.8). Theorem 2.3 is proved. □

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Proof of Theorem 2.4. After the same change of variables as in (5.6), i.e.

$$t \to \lambda t, \qquad s - v \to \lambda^{\gamma/2} s, \qquad u \to \lambda^{p\gamma/2(1-p)} u, \qquad v \to \lambda^{\gamma} v, \qquad r \to \lambda^{\gamma/2(1-p)} r,$$

we obtain $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r)\,\mathrm{d}r$ with $f_\lambda(r)$ as in (5.7), and

$$g_\lambda(r) := \int_{\mathbb{R}^2} \Psi(\theta(\log\lambda)^{-1/2} h_\lambda(u, v, r))\,\mathrm{d}u\,\mathrm{d}v,$$

where

$$h_\lambda(u, v, r) := \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\{((\lambda^{-\delta_1} t - u)/r^p, s/r^{1-p}) \in B\}\,\mathbf{1}\{0 < v + \lambda^{-\delta_2} s < y\}\,\mathrm{d}s,$$

$\delta_1 := p\gamma/2(1-p) - 1 = (\gamma - \gamma_+)/\gamma_+ > 0$, and $\delta_2 := \gamma/2 > 0$ are the same as in (5.8). Then

$$h_\lambda(u, v, r) \to h(u, v, r) := x\,\mathbf{1}\{0 < v < y\} \int_{\mathbb{R}} \mathbf{1}\{(-u/r^p, s/r^{1-p}) \in B\}\,\mathrm{d}s;$$

see (5.9). Below we prove that the main contribution to the limit of $W_{\lambda,\gamma}(\theta)$ comes from the interval $\lambda^{-\delta_1/p} < r < 1$; namely, that $W_{\lambda,\gamma}(\theta) - W^0_{\lambda,\gamma}(\theta) \to 0$, where

$$\begin{aligned} W^0_{\lambda,\gamma}(\theta) &:= \int_{\lambda^{-\delta_1/p}}^1 g_\lambda(r) f_\lambda(r)\,\mathrm{d}r \\ &\sim -\frac{\theta^2}{2}\frac{c_f}{\log\lambda}\int_{\lambda^{-\delta_1/p}}^1 \frac{\mathrm{d}r}{r^{3-p}} \int_{\mathbb{R}^2} h^2(u, v, r)\,\mathrm{d}u\,\mathrm{d}v \\ &= -\frac{\theta^2}{2} x^2 y c_f \int_{\mathbb{R}} (\mathrm{Leb}_1(B(u)))^2\,\mathrm{d}u \frac{1}{\log\lambda}\int_{\lambda^{-\delta_1/p}}^1 r^{-1}\,\mathrm{d}r \\ &= -\frac{\theta^2}{2}\widetilde{\sigma}_+^2 x^2 y \\ &=: W_\gamma(\theta), \end{aligned} \tag{5.15}$$

where

$$\widetilde{\sigma}_+^2 := \frac{c_f(\gamma - \gamma_+)}{2(1-p)} \int_{\mathbb{R}} \mathrm{Leb}(B \cap (B + (0, u)))\,\mathrm{d}u, \tag{5.16}$$

and where we used the fact that

$$\int_{\mathbb{R}^2} h^2(u, v, r)\,\mathrm{d}u\,\mathrm{d}v = x^2 y r^{2-p} \int_{\mathbb{R}} \mathrm{Leb}_1(B(u))^2\,\mathrm{d}u = x^2 y r^{2-p} \int_{\mathbb{R}} \mathrm{Leb}(B \cap (B + (0, u)))\,\mathrm{d}u.$$

Accordingly, write $W_{\lambda,\gamma}(\theta) = W^0_{\lambda,\gamma}(\theta) + W^-_{\lambda,\gamma}(\theta) + W^+_{\lambda,\gamma}(\theta)$, where

$$W^-_{\lambda,\gamma}(\theta) := \int_0^{\lambda^{-\delta_1/p}} g_\lambda(r) f_\lambda(r)\,\mathrm{d}r \quad \text{and} \quad W^+_{\lambda,\gamma}(\theta) := \int_1^\infty g_\lambda(r) f_\lambda(r)\,\mathrm{d}r$$

are the remainder terms. Indeed, using (5.4),

$$\int_{\mathbb{R}^2} h_\lambda(u, v, r)\,\mathrm{d}u\,\mathrm{d}v = xyr\,\mathrm{Leb}(B), \quad \text{and} \quad h_\lambda(u, v, r) \le C(\lambda^{\delta_1} r) \wedge r^{1-p}, \tag{5.17}$$

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it follows that

$$|W^{+}_{\lambda,\gamma}(\theta)| \le \frac{C}{(\log\lambda)^{1/2}} \int_1^\infty \frac{\mathrm{d}r}{r^{3-p}} \int_{\mathbb{R}^2} h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v = O((\log\lambda)^{-1/2}) = o(1).$$

Similarly,

$$\begin{aligned}|W^{-}_{\lambda,\gamma}(\theta)| &\le \frac{C\lambda^{\delta_1}}{\log\lambda} \int_0^{\lambda^{-\delta_1/p}} r f_\lambda(r)\,\mathrm{d}r \int_{\mathbb{R}^2} h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{C\lambda^{\delta_1}}{\log\lambda} \int_0^{\lambda^{-\delta_1/p}} r^2 f_\lambda(r)\,\mathrm{d}r \\ &= \frac{C}{\lambda\log\lambda} \int_0^{\lambda^{1/p}} r^2 f(r)\,\mathrm{d}r \\ &= O((\log\lambda)^{-1}) \\ &= o(1),\end{aligned}$$

since $\delta_1 = p\gamma/2(1-p) - 1$.

Consider the main term $W^0_{\lambda,\gamma}(\theta)$ in (5.15). Let

$$\widetilde{W}_{\lambda,\gamma}(\theta) := -\frac{\theta^2}{2\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r)\,\mathrm{d}r \int_{\mathbb{R}^2} h^2_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v.$$

Then using (5.17) and $|\Psi(z) + z^2/2| \le |z|^3/6$, we obtain

$$\begin{aligned}|W^0_{\lambda,\gamma}(\theta) - \widetilde{W}_{\lambda,\gamma}(\theta)| &\le \frac{C}{(\log\lambda)^{3/2}} \int_{\lambda^{-\delta_1/p}}^1 r^{2-2p} f_\lambda(r)\,\mathrm{d}r \int_{\mathbb{R}^2} h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{C}{(\log\lambda)^{3/2}} \int_{\lambda^{-\delta_1/p}}^1 r^{3-2p} f_\lambda(r)\,\mathrm{d}r \\ &\le \frac{C}{(\log\lambda)^{3/2}} \int_0^1 r^{-p}\,\mathrm{d}r \\ &= O((\log\lambda)^{-3/2}) \\ &= o(1).\end{aligned}$$

Finally, it remains to estimate the difference $|\widetilde{W}_{\lambda,\gamma}(\theta) - W_\gamma(\theta)| \le C(J'_\lambda + J''_\lambda)$, where

$$J'_\lambda := \frac{1}{\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r)\,\mathrm{d}r \int_{\mathbb{R}^2} |h^2_\lambda(u,v,r) - h^2(u,v,r)|\,\mathrm{d}u\,\mathrm{d}v,$$

$$J''_\lambda := \frac{1}{\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 r^{2-p} |f_\lambda(r) - c_f r^{p-3}|\,\mathrm{d}r.$$

Let

$$\widetilde{h}_\lambda(u,v,r) := x \int_{\mathbb{R}} \mathbf{1}\{(-u/r^p, s/r^{1-p}) \in B\} \mathbf{1}\{0 < v + \lambda^{-\delta_2} s < y\}\,\mathrm{d}s.$$

Then $J'_\lambda \le J'_{\lambda 1} + J'_{\lambda 2}$, where

$$J'_{\lambda 1} := (\log\lambda)^{-1} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r)\,\mathrm{d}r \int_{\mathbb{R}^2} |h^2_\lambda(u,v,r) - \widetilde{h}^2_\lambda(u,v,r)|\,\mathrm{d}u\,\mathrm{d}v$$

and

$$J'_{\lambda 2} := (\log\lambda)^{-1}\int_{\lambda^{-\delta_1/p}}^{1} f_\lambda(r)\,\mathrm{d}r\int_{\mathbb{R}^2}|\widetilde{h}^2_\lambda(u,v,r)-h^2(u,v,r)|\,\mathrm{d}u\,\mathrm{d}v.$$

Using the fact that B is a bounded set with $\mathrm{Leb}(\partial B)=0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}&|h_\lambda(u,v,r)-\widetilde{h}_\lambda(u,v,r)|\,\mathrm{d}u\,\mathrm{d}v\\ &\le yr\int_0^x \mathrm{d}t\int_{\mathbb{R}^2}|\mathbf{1}\{(\lambda^{-\delta_1}t/r^p-u,s)\in B\}-\mathbf{1}\{(-u,s)\in B\}|\,\mathrm{d}u\,\mathrm{d}s\\ &\le r\varepsilon\bigg(\frac{1}{\lambda^{\delta_1}r^p}\bigg),\end{aligned}$$

where $\varepsilon(z), z\ge 0$ is a bounded function with $\lim_{z\to 0}\varepsilon(z)=0$. We also have $h_\lambda(u,v,r)+\widetilde{h}_\lambda(u,v,r)\le Cr^{1-p}$ as in (5.17). Using these bounds together with $f_\lambda(r)\le Cr^{p-3}, r>\lambda^{-\delta_1/p}$, we obtain

$$J'_{\lambda 1}\log\lambda\le C\int_{\lambda^{-\delta_1/p}}^{1}\varepsilon\bigg(\frac{1}{\lambda^{\delta_1}r^p}\bigg)r^{-1}\,\mathrm{d}r = C\int_{\lambda^{-\delta_1}}^{1}\varepsilon(z)z^{-1}\,\mathrm{d}z = o(\log\lambda),$$

proving $J'_{\lambda 1}\to 0$ as $\lambda\to\infty$. In a similar way, using

$$\begin{aligned}\int_{\mathbb{R}^2}&|\widetilde{h}_\lambda(u,v,r)-h(u,v,r)|\,\mathrm{d}u\,\mathrm{d}v\\ &\le xr\int_{\mathbb{R}^3}\mathbf{1}\{(-u,s)\in B\}|\mathbf{1}\{0<v+\lambda^{-\delta_2}r^{1-p}s<y\}-\mathbf{1}\{0<v<y\}|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\\ &\le Cr^{2-p}\lambda^{-\delta_2},\end{aligned}$$

we obtain $J'_{\lambda 2}\log\lambda\le C\lambda^{-\delta_2}\int_0^1 r^{-p}\,\mathrm{d}r = O(\lambda^{-\delta_2})$, proving $J'_{\lambda 2}\to 0$ and, hence, $J'_\lambda\to 0$. Finally,

$$J''_\lambda = (\log\lambda)^{-1}\int_{\lambda^{1/p}}^{\infty} r^{2-p}|f(r)-c_f r^{p-3}|\,\mathrm{d}r\to 0$$

follows from (1.3). This proves the limit $\lim_{\lambda\to\infty}W_{\lambda,\gamma}(\theta)=W_\gamma(\theta)=-(\theta^2/2)\widetilde{\sigma}^2_+x^2y$ for any $\theta\in\mathbb{R}$, or one-dimensional convergence in (2.9). Theorem 2.4 is proved. □

Proof of Proposition 2.1. (i) We use well-known properties of Poisson stochastic integrals and [17, Equation (3.3)]. Accordingly, $I_+(x,y)$ is well defined and satisfies $\mathbb{E}|I_+(x,y)|^q\le 2J_q(x,y)$ $(1\le q\le 2)$ provided that

$$\begin{aligned}J_q(x,y) &:= c_f\int_0^\infty r^{-(1+\alpha)}\,\mathrm{d}r\int_{\mathbb{R}\times(0,y]}\mathrm{d}u\,\mathrm{d}v\bigg|\int_{(0,x]\times\mathbb{R}}\mathbf{1}\{((t-u)/r^p, s/r^{1-p})\in B\}\,\mathrm{d}t\,\mathrm{d}s\bigg|^q\\ &= c_f y\int_0^\infty r^{q(1-p)-(1+\alpha)}\,\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}u\bigg|\int_0^x \mathrm{Leb}_1\bigg(B\bigg(\frac{t-u}{r^p}\bigg)\bigg)\,\mathrm{d}t\bigg|^q\\ &<\infty.\end{aligned}$$

Split $J_q(x,y)=c_f y[\int_0^1\mathrm{d}r+\int_1^\infty]\cdots\mathrm{d}r =: c_f y[J'+J'']$. Then

$$J''\le C\int_1^\infty r^{q(1-p)-(1+\alpha)}\,\mathrm{d}r\int\mathbf{1}\{|u|\le Cr^p\}\,\mathrm{d}u\le C\int_1^\infty r^{q(1-p)-(1+\alpha)+p}\,\mathrm{d}r<\infty$$

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provided that $q < (\alpha - p)/(1-p)$. Similarly,

$$J' \le C \int_0^1 r^{q(1-p)-(1+\alpha)}\,\mathrm{d}r \left| \int \mathbf{1}\{|t| \le Cr^p\}\,\mathrm{d}t \right|^q \le C \int_0^1 r^{q(1-p)-(1+\alpha)+qp}\,\mathrm{d}r < \infty$$

provided that $\alpha < q$. Note that $\alpha < (\alpha-p)/(1-p) \le 2$ for $1 < \alpha \le 2-p$ and $(\alpha-p)/(1-p) > 2$ for $2-p < \alpha < 2$. Equation (2.11) follows from (2.8) and $J_2(x,y) = \sigma_+^2 y x^{2H_+}$ by a change of variables.

(ii) The proof is analogous. □

Proof of Theorem 2.5. Using the change of variables as in (5.11), we obtain $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r)\,\mathrm{d}r$ with the same $f_\lambda(r), g_\lambda(r)$ as in (5.12) and $h_\lambda(u,v,r)$ satisfying (5.13). (Note that $H(\gamma) = H(\gamma_+) = 1/p$; hence, $\lambda^{H(\gamma_+)-(1/p)} = 1$ in the definition of $g_\lambda(r)$ in (5.12).) In particular, $\Psi(\theta h_\lambda(u,v,r)) \to \Psi(\theta h(u,v,r))$ for any $(u,v,r) \in \mathbb{R}^2 \times \mathbb{R}_+, v \notin \{0,y\}$. Then $g_\lambda(r) \to g(r) := \int_{\mathbb{R}^2} \Psi(\theta h(u,v,r))\,\mathrm{d}u\,\mathrm{d}v$ follows by the dominated convergence theorem. Using $\int_{\mathbb{R}^2} h_\lambda(u,v,r)\,\mathrm{d}u\,\mathrm{d}v = xyr\,\mathrm{Leb}(B)$ and $h_\lambda(u,v,r) \le Cr$ we obtain $|g_\lambda(r)| \le C\min(r, r^2)$ and, hence,

$$W_{\lambda,\gamma}(\theta) \to \int_0^\infty g(r) r^{-(1+\alpha)}\,\mathrm{d}r = \log \mathbb{E} \exp\{\mathrm{i}\theta I_+(x,y)\},$$

proving the one-dimensional convergence in (2.12). □

Proof of Theorem 3.1. (i) Write $D_r(x,y) := \{(u,v) \in \mathbb{R}^2 \colon (u-x)^2 + (v-y)^2 \le r^2\}$ for a ball in $\mathbb{R}^2$ centered at (x,y) and having radius r. Recall that B is bounded. Note that $\inf_{z\in[-1,1]}(|z|/r^p + (1-|z|^{1/(p-1)})^{1-p}/r^{1-p}) \ge c_0 \min(r^{-p}, r^{-(1-p)})$ for some constant $c_0 > 0$. Therefore, there exists $r_0 > 0$ such that for all $0 < r < r_0$ the intersection $B_{z,r} := B \cap (B + (z/r^p, (1-|z|^{1/p})^{1-p}/r^{1-p})) = \varnothing$ in (3.1). Hence, $b(z) \le C < \infty$ uniformly in $z \in [-1,1]$.

Let $(x,y) \in B \setminus \partial B$. Then $D_{2r}(x,y) \subset B$ for all $r < r_0$ and some $r_0 > 0$. If we translate B by distance r_0 at most, the translated set still contains the ball $D_{r_0}(x,y)$. Since $\sup_{z\in[-1,1]}(|z|/r^p + (1-|z|^{1/p})^{1-p}/r^{1-p}) \le 2\max(r^{-p}, r^{-(1-p)})$, there exists $r_1 > 0$ for which $\inf_{r>r_1} \mathrm{Leb}(B_{z,r}) \ge \pi r_0^2$, proving $\inf_{z\in[-1,1]} b(z) > 0$. The continuity of $b(z)$ follows from the above argument and the continuity of the mapping $z \mapsto \mathrm{Leb}(B_{z,r})\colon [-1,1] \to \mathbb{R}_+$ for each $r > 0$.

(ii) Let $s \ge 0$. In the integral (2.2) we change the variables, i.e. $u \to r^p u$, $v \to r^{1-p} v$, and $r \to w^{1/p} r$. Then

$$\rho(t,s) = w^{-(\alpha-1)/p} \int_0^\infty \mathrm{Leb}(B_{t/w,r}) f_w(r) r\,\mathrm{d}r,$$

where $f_w(r) := w^{(1+\alpha)/p} f(w^{1/p} r) \to c_f r^{-(1+\alpha)}$, $w \to \infty$. Then (3.2) follows by Lemma 5.1 and the aforementioned properties of $\mathrm{Leb}(B_{t/w,r})$. □

In this paper we often use the following lemma which is a version of [11, Lemma 2] or [3, Lemma 2.4]. The proof of Lemma 5.1 can be found in the extended version of this paper [18].

Lemma 5.1. *Let F be a probability distribution that has a density function f satisfying (1.3). Set $f_\lambda(r) := \lambda^{1+\alpha} f(\lambda r)$ for $\lambda \ge 1$. Assume that g and g_λ are measurable functions on $\mathbb{R}_+$*

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such that $g_\lambda(r) \to g(r)$ *as* $\lambda \to \infty$ *for all* $r > 0$ *and such that* $|g_\lambda(r)| \le C(r^{\beta_1} \wedge r^{\beta_2})$ *holds for all* $r > 0$ *and some* $0 < \beta_1 < \alpha < \beta_2$*, where* C *does not depend on* r, λ*. Then*

$$\int_0^\infty g_\lambda(r) f_\lambda(r)\,\mathrm{d}r \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\,\mathrm{d}r \quad \textit{as } \lambda \to \infty.$$

5.2. Proofs of Section 4

Proof of Theorem 4.1. We have

$$\begin{aligned} W_{T,\gamma,\beta}(\theta) &:= \log \mathbb{E} \exp\{\mathrm{i}\theta T^{-\mathcal{H}}(A_{M,K}(Tx) - \mathbb{E}A_{M,K}(Tx))\} \\ &= T^\gamma \int_{\mathbb{R}\times\mathbb{R}_+} \Psi\bigg(\theta T^{-\mathcal{H}}(r^{1-p} \wedge T^\beta) \int_0^{Tx} \mathbf{1}\{u < t < u + r^p\}\,\mathrm{d}t\bigg)\,\mathrm{d}u f(r)\,\mathrm{d}r, \end{aligned} \tag{5.18}$$

where $\Psi(z) = \exp(\mathrm{i}z) - 1 - \mathrm{i}z$, $z \in \mathbb{R}$ as in Section 5.1.

(i) Let $0 < p < 1$, $\delta_1 := \beta - (1+\gamma)(1-p)/\alpha > 0$, $\delta_2 := 1 - (1+\gamma)p/\alpha > 0$. Using the change of variables $(t-u)/r^p \to t$, $u \to Tu$, and $r \to T^{(1+\gamma)/\alpha} r$ in (5.18), we obtain

$$W_{T,\gamma,\beta}(\theta) = \int_0^\infty g_T(r) f_T(r)\,\mathrm{d}r, \tag{5.19}$$

where $f_T(r) := T^{(1+\alpha)(1+\gamma)/\alpha} f(T^{(1+\gamma)/\alpha} r)$, and

$$g_T(r) := \int_{\mathbb{R}} \Psi(\theta(r^{1-p} \wedge T^{\delta_1}) r^p h_T(u,r))\,\mathrm{d}u,$$

where

$$h_T(u,r) := \int_0^1 \mathbf{1}\{0 < u + T^{-\delta_2} r^p t < x\}\,\mathrm{d}t \to \mathbf{1}\{0 < u < x\}$$

for fixed $(u,r) \in \mathbb{R} \times \mathbb{R}_+$, $u \notin \{0, x\}$. Hence, $g_T(r) \to g(r) := x\Psi(\theta r)$ follows by the dominated convergence theorem. The bound $|g_T(r)| \le C\min(r, r^2)$ follows from (5.4) and $\int_{\mathbb{R}} h_T(u,r)\,\mathrm{d}u = x$ with $h_T(u,r) \le 1$. Finally, by Lemma 5.1, $W_{T,\gamma,\beta}(\theta) \to x c_f \int_0^\infty \Psi(\theta r) r^{-(1+\alpha)}\,\mathrm{d}r = \log \mathbb{E} \exp\{\mathrm{i}\theta L_\alpha(x,1)\}$, proving (i) for $0 < p < 1$. The $p = 1$ case follows similarly.

(ii) By the same change of variables as in (i), we obtain $W_{T,\gamma,\beta}(\theta)$ as in (5.19), where

$$g_T(r) := \int_{\mathbb{R}} \Psi(\theta((T^{-\delta_1} r^{1-p}) \wedge 1) r^p h_T(u,r))\,\mathrm{d}u,$$

where δ_1, $f_T(r)$, and $h_T(u,r)$ are the same as in (5.19) except that now $\delta_1 < 0$. Next, $g_T(r) \to x\Psi(\theta r^p)$ by the dominated convergence theorem, while $|g_T(r)| \le C\min(r^p, r^{2p})$ follows by (5.4) and $\int_{\mathbb{R}} \min(h_T(u,r), h_T^2(u,r))\,\mathrm{d}u \le C$. Then

$$W_{T,\gamma,\beta}(\theta) \to W_{\gamma,\beta}(\theta) := x c_f \int_0^\infty \Psi(\theta r^p) r^{-(1+\alpha)}\,\mathrm{d}r$$

follows by Lemma 5.1. To finish the proof of (ii) it suffices to check that

$$\begin{aligned} W_{\gamma,\beta}(\theta) &= -x \frac{c_f \Gamma(2-\alpha/p)}{\alpha(1-\alpha/p)} \cos\left(\frac{\pi\alpha}{2p}\right) |\theta|^{\alpha/p} \left(1 - \mathrm{i}\,\mathrm{sgn}(\theta) \tan\left(\frac{\pi\alpha}{2p}\right)\right) \\ &=: \log \mathbb{E} \exp\{\mathrm{i}\theta L_{\alpha/p}(x)\}. \end{aligned} \tag{5.20}$$

(iii) Denote $\delta_1 := 1+\gamma-\alpha\beta/(1-p) > 0$ and $\delta_2 := 1-p\beta/(1-p) > 0$. Then by a change of variables, i.e. $(t-u)/r^p \to t$, $u \to Tu$, $r \to T^{\beta/(1-p)}r$, we write $W_{T,\gamma,\beta}(\theta)$ as in (5.19), where

$$f_T(r) := T^{(1+\alpha)\beta/(1-p)} f(T^{\beta/(1-p)}r)$$
$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}(r^{1-p}\wedge 1)r^p h_T(u,r))\,\mathrm{d}u$$

with $h_T(u,r) := \int_0^1 \mathbf{1}\{0 < u + T^{-\delta_2}r^p t < x\}\,\mathrm{d}t \to \mathbf{1}\{0 < u < x\}$. Then

$$g_T(r) \to -\frac{\theta^2}{2}(r^{1-p}\wedge 1)^2 r^{2p}x$$

by the dominated convergence theorem using the bounds $|\Psi(z)| \le z^2/2$, $z \in \mathbb{R}$, and $h_T(u,r) \le \mathbf{1}\{-r^p < u < x\}$. Moreover, $|g_T(r)| \le C\min(r^{2p}, r^2)$ holds in view of

$$\int_{\mathbb{R}} h_T^2(u,r)\,\mathrm{d}u \le C.$$

Using Lemma 5.1, we obtain

$$W_{T,\gamma,\beta}(\theta) \to -\left(\frac{\theta^2}{2}\right)xc_f\int_0^\infty (r^{1-p}\wedge 1)^2 r^{2p-(1+\alpha)}\,\mathrm{d}r = -\left(\frac{\theta^2}{2}\right)\sigma_1^2 x,$$

where

$$\sigma_1^2 := \frac{2c_f(1-p)}{(2-\alpha)(\alpha-2p)} < \infty \tag{5.21}$$

since $\max(1,2p) < \alpha < 2$.

This proves (iii) and Theorem 4.1, too. □

Proof of Theorem 4.2. (i) Denote $\delta_1 := 1+\gamma-\alpha/p = \gamma-\gamma_+ > 0$ and $\delta_2 := (1-p)/p - \beta > 0$. By changing the variables in (5.3), i.e. $t \to Tt$, $u \to Tu$, $r \to T^{1/p}r$, we write $W_{T,\gamma,\beta}(\theta)$ as in (5.19), where $f_T(r) := T^{(1+\alpha)/p} f(T^{1/p}r)$ and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}((T^{\delta_2}r^{1-p})\wedge 1)h(u,r))\,\mathrm{d}u$$

with $h(u,r) := \int_0^x \mathbf{1}\{u < t < u+r^p\}\,\mathrm{d}t$. The dominated convergence $g_T(r) \to g(r) := -(\theta^2/2)\int_{\mathbb{R}} h^2(u,r)\,\mathrm{d}u$ follows by (5.4). The latter combined with

$$\int_{\mathbb{R}} h^2(u,r)\,\mathrm{d}u \le C\min(1,r^p)\int_{\mathbb{R}} h(u,r)\,\mathrm{d}u \le C\min(r^p, r^{2p})$$

gives the bound $|g_T(r)| \le C\min(r^p, r^{2p})$. Finally, by Lemma 5.1, $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\sigma_2^2 x^{2H}$, where

$$\begin{aligned}\sigma_2^2 &:= c_f\int_{\mathbb{R}\times\mathbb{R}}\left(\int_0^1 \mathbf{1}\{u < t < u+r^p\}\,\mathrm{d}t\right)^2\frac{\mathrm{d}u\,\mathrm{d}r}{r^{1+\alpha}}\\ &= \frac{2c_f}{\alpha(2-\alpha/p)(3-\alpha/p)(\alpha/p-1)},\end{aligned} \tag{5.22}$$

proving (i).

(ii) The proof is the same as that of Theorem 4.1(iii).

(iii) Let $\delta_1 := \gamma - \alpha_+\beta > 0$ and $\delta_2 := \alpha_+\beta/\gamma_+ - 1 > 0$. By a change of variables, i.e. $t \to Tt, u \to T^{\beta p/(1-p)}u, r \to T^{\beta/(1-p)}r$, we obtain (5.19) with

$$f_T(r) := T^{(1+\alpha)\beta/(1-p)} f(T^{\beta/(1-p)}r)$$

and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}(r^{1-p} \wedge 1)h_T(u, r))\, \mathrm{d}u,$$

with $h_T(u, r) := \int_0^x \mathbf{1}\{0 < (T^{-\delta_2}t - u)/r^p < 1\}\, \mathrm{d}t \to h(u, r) := x\,\mathbf{1}\{-r^p < u < 0\}$. Then (5.4) and $h_T^2(u, r) \le x\,\mathbf{1}\{-r^p < u < 1\}$ justify the dominated convergence

$$g_T(r) \to -\frac{\theta^2}{2}(r^{1-p} \wedge 1)^2 r^p x^2.$$

By (5.4) and

$$\int_{\mathbb{R}} h_T^2(u, r)\, \mathrm{d}u \le C \int_{\mathbb{R}} h_T(u, r)\, \mathrm{d}u \le Cr^p,$$

we have $|g_T(r)| \le C \min(r^p, r^{2-p})$. Finally, by Lemma 5.1

$$W_{T,\gamma,\beta}(\theta) \to -\bigg(\frac{\theta^2}{2}\bigg)x^2 c_f \int_0^\infty (r^{1-p} \wedge 1)^2 r^{p-(1+\alpha)}\, \mathrm{d}r = -\bigg(\frac{\theta^2}{2}\bigg)x^2\sigma_3^2$$

with

$$\sigma_3^2 := \frac{2c_f(1-p)}{(2-p-\alpha)(\alpha-p)}, \tag{5.23}$$

proving (iii).

(iv) Denote $\delta_1 := \beta - \gamma/\alpha_+ > 0$ and $\delta_2 := \gamma/\gamma_+ - 1 > 0$. By a change of variables, i.e. $t \to Tt, u \to T^{\gamma/\gamma_+}u, r \to T^{\gamma/\gamma_+ p}r$, we obtain (5.19) with

$$f_T(r) := T^{(1+\alpha)\gamma/\gamma_+ p} f(T^{\gamma/\gamma_+ p}r), \qquad g_T(r) := \int_{\mathbb{R}} \Psi(\theta(r^{1-p} \wedge T^{\delta_1})h_T(u, r))\, \mathrm{d}u,$$

where

$$h_T(u, r) := \int_0^x \mathbf{1}\{u < T^{-\delta_2}t < u + r^p\}\, \mathrm{d}t \to h(u, r) := x\,\mathbf{1}\{-r^p < u < 0\}.$$

Then

$$g_T(r) \to g(r) := \int_{\mathbb{R}} \Psi(\theta x r^{1-p}\,\mathbf{1}\{-r^p < u < 0\})\, \mathrm{d}u,$$

$$W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\, \mathrm{d}r = \log \mathbb{E}\exp\{\mathrm{i}\theta x L_+(1)\}$$

similarly to the proof of Theorem 2.2(ii).

(v) We follow the proof of (i). By the same change of variables, we obtain (5.19) with

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}(r^{1-p} \wedge T^{-\delta_2})h(u, r))\, \mathrm{d}u,$$

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and the same $h(u, r)$, $\delta_1 > 0$, $\delta_2 < 0$ and $f_T(r)$ as in (i). Then

$$g_T(r) \to g(r) := -\left(\frac{\theta^2}{2}\right) \int_{\mathbb{R}} r^{2(1-p)} h^2(u, r)\,\mathrm{d}u$$

$$W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\,\mathrm{d}r = -\left(\frac{\theta^2}{2}\right)\sigma_+^2 x^{2H_+}$$

similarly to the proof of Theorem 2.3(i). The proof of Theorem 4.2 is complete. □

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