J. Plasma Phys. (2017), *vol.* 83, 905830503
doi:10.1017/S0022377817000733

Dispersion relation for electrostatic waves in plasmas with isotropic and anisotropic Kappa distributions for electrons and ions

L. F. Ziebell[1,†], R. Gaelzer[1] and F. J. R. Simões Jr[2]

[1]Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970, Porto Alegre, RS, Brazil

[2]Instituto de Física e Matemática, Universidade Federal de Pelotas, 96010-900, Pelotas, RS, Brazil

(Received 22 July 2017; revised 11 September 2017; accepted 12 September 2017)

Velocity distribution functions which feature extended tails with power-law dependence have been consistently observed in the solar wind environment and are frequently modelled by the so-called Kappa distributions. Different forms of Kappa distributions are commonly employed in analytical studies, and despite their similarities, they can produce different effects on the dispersion properties that occur in a plasma. We consider two different and widely used forms of Kappa distributions, in both isotropic and anisotropic cases, and systematically discuss their effects on the dispersion relations of Langmuir and ion-sound waves. It is shown that in the case of Langmuir waves, one of the forms leads to the expression for the Bohm–Gross dispersion relation, valid for plasmas with Maxwellian velocity distributions, while the other form of Kappa functions leads to a dispersion relation with significant difference regarding the Maxwellian case, particularly in the case of small values of the kappa index. For ion-sound waves, the dispersion relations obtained with the different forms of Kappa distributions are different among themselves, and also different from the Maxwellian case, with difference which increases for small values of the kappa index. Some results obtained from numerical solution of the dispersion relations are presented, which illustrate the magnitude of the perceived differences. Some results obtained with relativistic particle-in-cell simulations are also presented, which allow the comparison between the dispersion relations obtained from analytical calculations and the frequency–wavelength distribution of wave fluctuations which are observed in numerical experiments.

Key words: plasma waves, space plasma physics

1. Introduction

Observations made in the space environment consistently show plasma particles with velocity distributions that have non-thermal tails, and frequently with anisotropies which are not well described by Maxwellian distributions or bi-Maxwellian distributions. These observed non-thermal features are usually described by distributions featuring power-law tails, which are generically known as Kappa distributions

† Email address for correspondence: luiz.ziebell@ufrgs.br

(Olbert 1968; Vasyliunas 1968; Summers & Thorne 1991; Mace & Hellberg 1995; Leubner & Schupfer 2000, 2001; Leubner 2002, 2004).

It is common knowledge that Kappa distributions have been introduced to describe non-thermal features of velocity distributions in the pioneering works by Olbert (1968) and Vasyliunas (1968), but are nowadays written in terms of different mathematical distributions. Isotropic Kappa distributions are usually written in terms of two different forms, one which can be found in Summers & Thorne (1991), Mace & Hellberg (1995), and the other which can be found in Leubner (2002, 2004). These two different forms of Kappa distributions have been used by the plasma physics community, and have been the subject of a number of theoretical discussion in recent years (Hellberg *et al.* 2009; Hapgood *et al.* 2011; Livadiotis & McComas 2013; Livadiotis 2015). Anisotropic particle distributions can also have extended non-thermal features, which are also usually described in terms of two different forms. In one of these forms, which is known as a bi-Kappa distribution (BK), the anisotropy is associated with parameters related to the temperature, with a single kappa index (Leubner & Schupfer 2000, 2001; Lazar & Poedts 2009*a*,*b*; Lazar, Poedts & Schlickeiser 2011; Lazar 2012; Lazar & Poedts 2014). In the second form, which is known as product-bi-Kappa distribution (PBK), anisotropic kappa indexes are introduced, in addition to anisotropic temperature parameters (Lazar *et al.* 2012; Lazar & Poedts 2014; dos Santos, Ziebell & Gaelzer 2014, 2015, 2016).

In any of the mentioned forms, Kappa distributions are frequently employed in an empirical way, fitting observed distributions, as in Maksimovic, Pierrard & Riley (1997). It is also possible to find theoretical studies dedicated to basic fundamental properties of Kappa distributions (Hellberg & Mace 2002; Mace & Hellberg 2003; Hau & Fu 2007; Hau, Fu & Chuang 2009; Hellberg *et al.* 2009; Mace & Hellberg 2009), or to the relationship between Kappa distributions and wave phenomena in plasmas. Among studies with the latter characteristics, mention can be made of studies on low-frequency electromagnetic instabilities, dealing with the ion-cyclotron instability, as in Lazar (2012), Lazar & Poedts (2014), or with the ion firehose instability, as in Lazar & Poedts (2009*a*,*b*), Lazar *et al.* (2011), or with Langmuir waves, as in Thorne & Summers (1991), or obliquely propagating generalized Langmuir waves, as in Mace & Hellberg (2003), or studies based on a quasi-linear formulation aiming to investigate bursty solar wave emission phenomena, as in Li & Cairns (2014), or analyses on the effects of superthermal particles on waves in magnetized plasmas, in space environment (Hellberg, Mace & Cattaert 2006).

As mentioned, the Kappa distributions used in the literature are frequently represented in different forms, which have in common the power-law dependence for high values of velocity. In the present paper, we use a generic form of Kappa distribution, which in particular cases can reproduce the two widely used forms already mentioned, and consider the isotropic and the anisotropic cases, discussing in each case the dispersion relations for Langmuir (L) and ion-sound (S) waves in a systematic way. The description of these waves is obtained from the dispersion relation for electrostatic (ES) waves, given by $\varepsilon_{zz}=0$, where

$$\varepsilon_{zz}=1-\frac{1}{z^2}\sum_{\beta}\frac{\omega_{p\beta}^2}{\Omega_*^2}\frac{1}{n_{\beta 0}}\left(I-J(0,0,2;f_\beta)\right). \tag{1.1}$$

The $J(0,0,2;f_\beta)$ in (1.1) is a particular case of a generic integral form, which is valid both for magnetized and unmagnetized environments,

$$J(n,m,h;f_\beta)\equiv z\int \mathrm{d}^3u\,\frac{u_\parallel^h u_\perp^{2(m-1)}u_\perp L(f_\beta)}{z-nr_\beta-q_\parallel u_\parallel}, \tag{1.2}$$

and I is defined as follows

$$I = \int \mathrm{d}^3 u \, \frac{u_\parallel}{u_\perp} \mathcal{L}(f_\beta). \tag{1.3}$$

In these expressions, $\mathcal{L}$ and L are differential operators,

$$\mathcal{L} = \left(u_\parallel \frac{\partial}{\partial u_\perp} - u_\perp \frac{\partial}{\partial u_\parallel} \right) \tag{1.4}$$

$$L = \left[\left(1 - \frac{q_\parallel}{z} u_\parallel \right) \frac{\partial}{\partial u_\perp} + \frac{q_\parallel}{z} u_\perp \frac{\partial}{\partial u_\parallel} \right] \tag{1.5}$$

and the following dimensionless variables have been introduced

$$z = \frac{\omega}{\Omega_*}, \quad u_{\parallel,\perp} = \frac{v_{\parallel,\perp}}{v_*}, \quad q_\parallel = \frac{k_\parallel v_*}{\Omega_*}, \quad r_\beta = \frac{\Omega_\beta}{\Omega_*}, \tag{1.6a–d}$$

where ω and k are the wave angular frequency and wavenumber, respectively, Ω_β is the angular cyclotron frequency for particles of species β, and Ω_* and v_* are some characteristic angular frequency and velocity, respectively. Moreover, $\omega_{p\beta}$ is the angular plasma frequency and f_β the velocity distribution function, for particles of species β. In the present case, for the study of ES waves, it is convenient to assume $\Omega_* = \omega_{pe}$ and $v_* = v_e$, the angular electron plasma frequency and the electrons thermal velocity, respectively. As usual, the thermal velocities for particles of species β are defined as $v_\beta^2 = 2T_\beta/m_\beta$, where T_β is a parameter which characterizes the temperature. Notice that in the present notation the dependence of the velocity distribution functions appear in the dispersion relation through the integral quantities (1.2) and (1.3).

The paper is organized as follows. In § 2 we introduce a generic form of Kappa distribution, considering the isotropic case and also the anisotropic BK and PBK cases of this generic distribution. Section 2 also presents analytical results for the dispersion relation of L and S waves, both exact and approximated, obtained with the use of these generic distributions. In § 3 we consider two particular isotropic cases of the generic distribution introduced in § 2, and present the corresponding analytical expressions for the dispersion relation of L and S waves, both exact and approximated. In § 4 we introduce and discuss anisotropic forms of Kappa distribution which are particular cases of the generic distribution presented in § 2, and also present the corresponding exact and approximated forms of the dispersion relations for L and S waves. In § 5 we present some results obtained from numerical solution of the dispersion relation obtained for different forms of the velocity distribution, and also present some results obtained from particle-in-cell (PIC) simulations, regarding electrostatic fluctuations. Section 6 summarizes the results obtained.

2. Generic forms of isotropic and anisotropic Kappa distributions and the corresponding dispersion relations for Langmuir and ion-sound waves

Let us assume the following form of isotropic Kappa distribution for ions and electrons (Gaelzer & Ziebell 2014, 2016),

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2} w_{\beta,\kappa}^3} \frac{\Gamma(\kappa_\beta + \alpha_\beta)}{\Gamma(\kappa_\beta + \alpha_\beta - 3/2)} \left(1 + \frac{v^2}{\kappa_\beta w_{\beta,\kappa}^2} \right)^{-(\kappa_\beta + \alpha_\beta)}, \tag{2.1}$$

where α_β is a constant and $w_{\beta,\kappa}$ is a parameter with the same physical dimension as the particle thermal velocity, and which reduces to the Maxwellian thermal velocity in the limit $\kappa_\beta \to \infty$. The distribution function given by (2.1) is normalized such that $\int \mathrm{d}^3v f_\beta = n_{\beta 0}$.

The quantity I in (1.1) vanishes in the case of an isotropic distribution, as is the case of (2.1). After evaluation of the $J(0, 0, 2; f_\beta)$ integral for this distribution function (details can be seen in appendix A), the dispersion relation for ES waves propagating along a given direction denominated as z axis, with wavenumber $\boldsymbol{k} = k_\parallel \boldsymbol{e}_z$, can be written as follows,

$$1 + 2\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2}(\zeta_\beta^0)^2 \left(\frac{\kappa_\beta + \alpha_\beta - 3/2}{\kappa_\beta} + \frac{\Gamma(\kappa_\beta + \alpha_\beta)}{\kappa_\beta \Gamma(\kappa_\beta)} \right. \\ \left. \times \frac{\Gamma(\kappa_\beta - 1/2)}{\Gamma(\kappa_\beta + \alpha_\beta - 3/2)} \zeta_\beta^0 Z_{\kappa_\beta}^{(\alpha_\beta)}(\zeta_\beta^0) \right) = 0, \tag{2.2}$$

where $\zeta_\beta^0 = \omega/(k_\parallel w_{\beta,\kappa})$, and where $Z_{\kappa_\beta}^{(\alpha_\beta)}$ is obtained from the general definition

$$Z_\kappa^{(m)}(\xi) = \frac{\mathrm{i}\Gamma(\kappa)\Gamma(\kappa + m + 1/2)}{\kappa^{1/2}\Gamma(\kappa - 1/2)\Gamma(\kappa + m + 1)} \\ \times {}_2F_1 \left[1, 2\kappa + 2m; \kappa + m + 1; \frac{1}{2}\left(1 + \frac{\mathrm{i}\xi}{\kappa^{1/2}}\right)\right], \tag{2.3}$$

valid for $\kappa > -m - 1/2$, for $m = \alpha$.

An approximated form of the dispersion relation of L waves can be obtained if we take into account that the L waves are waves with large phase velocity, $|\zeta_\beta^0| \gg 1$. Expanding the integrand of the $J(0, 0, 2; f_\beta)$ integral (details can be seen in appendix A), the dispersion relation for L waves turns out to be given as follows, in terms of dimensional quantities,

$$1 - \sum_\beta \frac{\omega_{p\beta}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta + \alpha_\beta - 5/2}\frac{1}{(\zeta_\beta^0)^2}\right) = 0. \tag{2.4}$$

In order to obtain an approximated expression to be used for S waves, we consider the distribution function given by (2.2) and waves satisfying the conditions $|\zeta_e^0| \ll 1$ and $|\zeta_i^0| \gg 1$. By expansion of the integrand of the $J(n, m, h, f_\beta)$ integrals considering the appropriate limits (details can be seen in appendix A), the dispersion relation for S waves can be given as follows

$$1 - \frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{\kappa_i}{\kappa_i + \alpha_i - 5/2}\frac{1}{(\zeta_i^0)^2}\right) + 2\frac{\omega_{pe}^2}{\omega^2}\frac{\kappa_e + \alpha_e - 3/2}{\kappa_e}(\zeta_e^0)^2 = 0. \tag{2.5}$$

Particular cases of the distribution (2.1), which correspond to forms of Kappa distributions which are widely used in plasma physics investigations, can be obtained by suitable choice of the parameters α_β and $w_{\beta,\kappa}$. These particular cases will be considered in the sections which follow.

A generic form of BK distribution can also be defined, similar to the distribution given by (2.1) (Gaelzer, Ziebell & Meneses 2016),

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2} w_{\beta,\kappa,\perp}^2 w_{\beta,\kappa,\parallel}} \frac{\Gamma(\kappa_\beta+\alpha_\beta)}{\Gamma(\kappa_\beta+\alpha_\beta-3/2)} \times \left(1+\frac{v_\parallel^2}{\kappa_\beta w_{\beta,\kappa,\parallel}^2}+\frac{v_\perp^2}{\kappa_\beta w_{\beta,\kappa,\perp}^2}\right)^{-(\kappa_\beta+\alpha_\beta)}, \tag{2.6}$$

where $w_{\beta,\kappa,\perp}$ and $w_{\beta,\kappa,\parallel}$ are parameters with the same physical dimension as the particle thermal velocity. The BK distribution is azimuthally symmetric around a given direction, assumed here to be the z axis, and is in general anisotropic, becoming isotropic if $w_{\beta,\kappa,\parallel} = w_{\beta,\kappa,\perp}$.

The integrals I and J which are used in the dispersion relation can be evaluated considering distribution function (2.6) by a procedure similar to that used in appendix A. As a result, the dispersion relation for ES waves with wavenumber $\boldsymbol{k} = k_\parallel\ \boldsymbol{e}_z$ can be written as follows,

$$1+2\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2}(\zeta_{\beta,\kappa,\parallel}^0)^2 \left(\frac{\kappa_\beta+\alpha_\beta-3/2}{\kappa_\beta}+\frac{\Gamma(\kappa_\beta+\alpha_\beta)}{\kappa_\beta\Gamma(\kappa_\beta)} \times \frac{\Gamma(\kappa_\beta-1/2)}{\Gamma(\kappa_\beta+\alpha_\beta-3/2)}\zeta_{\beta,\kappa,\parallel}^0 Z_{\kappa_\beta}^{(\alpha_\beta)}(\zeta_{\beta,\kappa,\parallel}^0)\right) = 0, \tag{2.7}$$

where $\zeta_{\beta,\kappa,\parallel}^0 = \omega/(k_\parallel w_{\beta,\kappa,\parallel})$.

For a dispersion relation valid for L waves, we can consider the expanded form of the dispersion relation, which is obtained taking into account the fact that for L waves the phase velocity of the waves is much greater than the thermal velocity. The procedure is similar to that shown in appendix A, and the dispersion relation obtained is as follows

$$1-\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2}\left(1+\frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta+\alpha_\beta-5/2}\frac{1}{(\zeta_{\beta,\kappa,\parallel}^0)^2}\right) = 0. \tag{2.8}$$

In order to obtain an approximated expression to be used for S waves in the case of BK distributions as given by (2.6), we proceed by expansion of the integrand of the $J(n, m, h, f_\beta)$ integrals considering the appropriate limits $|\zeta_{e,\kappa,\parallel}^0| \ll 1$ and $|\zeta_{i\kappa,\parallel}^0| \gg 1$ (details can be seen in appendix A). The dispersion relation for S waves can therefore be given as follows

$$1-\frac{\omega_{pi}^2}{\omega^2}\left(1+\frac{3}{2}\frac{\kappa_i}{\kappa_i+\alpha_i-5/2}\frac{1}{(\zeta_{i,\kappa,\parallel}^0)^2}\right)+2\frac{\omega_{pe}^2}{\omega^2}\frac{\kappa_e+\alpha_e-3/2}{\kappa_e}(\zeta_{e,\kappa,\parallel}^0)^2 = 0. \tag{2.9}$$

Product-bi-Kappa distributions can also be defined using the parameter α_β, similarly to what has been done for BK distributions in (2.6),

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_{\beta\perp}\kappa_{\beta\parallel}^{1/2} w_{\beta,\kappa_\perp}^2 w_{\beta,\kappa_\parallel}} \frac{\Gamma(\kappa_{\beta\perp}+\alpha_\beta)\Gamma(\kappa_{\beta\parallel}+\alpha_\beta)}{\Gamma(\kappa_{\beta\perp}+\alpha_\beta-1)\Gamma(\kappa_{\beta\parallel}+\alpha_\beta-1/2)} \times \left(1+\frac{v_\parallel^2}{\kappa_{\beta\parallel} w_{\beta,\kappa_\parallel}^2}\right)^{-(\kappa_{\beta\parallel}+\alpha_\beta)} \left(1+\frac{v_\perp^2}{\kappa_{\beta\perp} w_{\beta,\kappa_\perp}^2}\right)^{-(\kappa_{\beta\perp}+\alpha_\beta)}. \tag{2.10}$$

PBK distributions are inherently anisotropic, even in the case of $w_{\beta\kappa_{\beta\parallel}} = w_{\beta,\kappa_\perp}$.

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

Upon evaluation of the I and J integrals, considering the distributions given by (2.10), the dispersion relation for electrostatic waves with wavenumber $\boldsymbol{k} = k_\parallel \, \boldsymbol{e}_z$ can be given as follows

$$1 + 2\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2}(\zeta_{\beta,\kappa_\parallel}^0)^2 \left(\frac{\kappa_{\beta\parallel} + \alpha_\beta - 1/2}{\kappa_{\beta\parallel}} + \frac{\Gamma(\kappa_{\beta\parallel} + \alpha_\beta)}{\Gamma(\kappa_{\beta\parallel})} \right.$$
$$\left. \times \frac{\Gamma(\kappa_{\beta\parallel} - 1/2)}{\Gamma(\kappa_{\beta\parallel} + \alpha_\beta - 1/2)} \frac{\kappa_{\beta\parallel} + \alpha_\beta}{\kappa_{\beta\parallel}} \zeta_{\beta,\kappa_\parallel}^0 Z_{\kappa_{\beta\parallel}}^{(\alpha_\beta+1)}(\zeta_{\beta,\kappa_\parallel}^0) \right) = 0, \tag{2.11}$$

where $\zeta_{\beta,\kappa_\parallel}^0 = \omega/(k_\parallel w_{\beta,\kappa_\parallel})$.

Dispersion relations valid for L and S waves for the case of the generic PBK distribution (2.10) can be obtained by considering the appropriate limits $|\zeta_{\beta,\kappa_\parallel}^0| \gg 1$ for L waves,

$$1 - \sum_\beta \frac{\omega_{p\beta}^2}{\omega^2} \left(1 + \frac{3}{2} \frac{\kappa_{\beta\parallel}}{\kappa_{\beta\parallel} + \alpha_\beta - 3/2} \frac{1}{(\zeta_{\beta,\kappa_\parallel}^0)^2} \right) = 0, \tag{2.12}$$

and $|\zeta_{i,\kappa_\parallel}^0| \gg 1$, $|\zeta_{e,\kappa_\parallel}^0| \ll 1$, for S waves,

$$1 - \frac{\omega_{pi}^2}{\omega^2} \left(1 + \frac{3}{2} \frac{\kappa_{i\parallel}}{\kappa_{i\parallel} + \alpha_i - 3/2} \frac{1}{(\zeta_{i,\kappa_\parallel}^0)^2} \right) + 2\frac{\omega_{pe}^2}{\omega^2} \frac{\kappa_{e\parallel} + \alpha_e - 1/2}{\kappa_{e\parallel}} (\zeta_{e,\kappa_\parallel}^0)^2 = 0. \tag{2.13}$$

The generic forms of isotropic and anisotropic Kappa distributions, given by (2.1), (2.6) and (2.10) have also been used in a recent study instabilities in the ion-cyclotron range (Ziebell & Gaelzer 2017). As far as we are aware, with the exception of (3.3), the other generic dispersion relations for electrostatic waves which appear in the present section are original, both in the exact forms and in the approximated forms valid for L and S waves. In some particular cases, discussed in the following section, the generic form leads to a form which is already known. These cases will be acknowledged when they appear.

3. Two particular forms of isotropic Kappa distributions

3.1. *The isotropic Kappa distribution as defined by Olbert (1968), Vasyliunas (1968)*

With the choice of $\alpha_\beta = 1$, equation (2.1) becomes as follows

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2} w_{\beta,\kappa}^3} \frac{\Gamma(\kappa_\beta + 1)}{\Gamma(\kappa_\beta - 1/2)} \left(1 + \frac{v^2}{\kappa_\beta w_{\beta,\kappa}^2} \right)^{-(\kappa_\beta+1)}, \tag{3.1}$$

which corresponds to an isotropic distribution which is widely used in the literature (Olbert 1968; Vasyliunas 1968; Summers & Thorne 1991; Mace & Hellberg 1995; Podesta 2015), as long as the quantity $w_{\beta,\kappa}$ appearing in this expression is written as a κ_β-dependent effective thermal velocity, given by

$$w_{\beta,\kappa}^2 = \frac{\kappa_\beta - 3/2}{\kappa_\beta} v_\beta^2, \tag{3.2}$$

where $v_\beta = (2T_\beta/m_\beta)^{1/2}$, with T_β being the temperature of species β (Livadiotis & McComas 2011). The average value of the squared particle speed is related to the physical temperature, given by $\langle v^2\rangle_\beta = 3v_\beta^2/2$. This quantity is independent of the index κ_β, and it is seen that distribution (3.1) corresponds to the isotropic limit of distributions characterized as 'case A' in the work by Lazar, Fichtner & Yoon (2016). It is interesting to notice that the effective thermal velocity $w_{\beta,\kappa}^2$ actually decreases for small values of κ_β and tends to zero for $\kappa_\beta \to 3/2$, a feature which is in contrast with the existence of a significant non-thermal tail in the velocity distribution function.

For reference, we will call the distribution given by (3.1) the isotropic Kappa distribution of type I. In the case of this distribution, the dispersion relation for ES waves is a particular case of (2.2), and is given by

$$1 + 2\sum_\beta \frac{\omega_\beta^2}{\omega^2}(\zeta_\beta^0)^2\left(\frac{\kappa_\beta - 1/2}{\kappa_\beta} + \zeta_\beta^0 Z_{\kappa_\beta}^{(1)}(\zeta_\beta^0)\right) = 0, \tag{3.3}$$

where $Z_{\kappa_\beta}^{(1)}$ is obtained from the general definition (2.3) for $m=1$ and for $\kappa_\beta > -3/2$.

In the limit $\kappa_\beta \to \infty$, $w_{\beta,\kappa} \to v_\beta$, $Z_\kappa^{(1)}(\zeta_\beta^0)$ tends to the well-known plasma dispersion function, $Z(\zeta_\beta^0)$, and the dispersion relation becomes the conventional dispersion relation for ES waves, usually found in textbooks as obtained considering the case of isotropic Maxwellian distributions. Equation (3.3) corresponds to the dispersion relation appearing as equation (27) in Mace (2003).

An approximated form of the dispersion relation for L waves can be obtained as a particular case of (2.4), and becomes as follows

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left(1 + \frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta - 3/2}\frac{k_\parallel^2 w_{\beta,\kappa}^2}{\omega^2}\right) = 0. \tag{3.4}$$

Taking into account the definition of the effective thermal velocities $w_{\beta,\kappa}$, it is seen that the dispersion relation obtained is formally the same as the well-known expanded form of the dispersion relation for L waves, obtained in the case of isotropic Maxwellian distributions, depending on the thermodynamic temperature T_β,

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k_\parallel^2 v_\beta^2}{\omega^2}\right) = 0. \tag{3.5}$$

This coincidence of the form of the dispersion relation obtained in the case of thermal plasmas and in the case of plasmas described by the Kappa distribution given by (3.1) is well known, and reported in a recently published paper (Li & Cairns 2014).

The dispersion relation can be further simplified by considering that the L waves are high-frequency waves, so that it is possible to neglect the effect of the ions in the dispersion relation. The dispersion relation therefore is approximated as follows

$$1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k_\parallel^2 v_e^2}{\omega^2}\right) = 0, \tag{3.6}$$

which may become even more familiar if we take into account that ω is not far from ω_{pe}, and therefore

$$\omega^2 \simeq \omega_{pe}^2\left(1 + \frac{3}{2}\frac{k_\parallel^2 v_e^2}{\omega_{pe}^2}\right). \tag{3.7}$$

In the limit $\kappa_\beta \to \infty$, of course, the result is the same, since the expression obtained does not depend on κ. The dispersion relation as given by (3.7) is frequently denominated as the Bohm–Gross dispersion relation.

In order to obtain an approximated expression to be used for S waves, we utilize (2.5) with $\alpha_\beta = 1$ and $w^2_{\beta,\kappa}$ given by (3.2), obtaining the following

$$1 - \frac{\omega_{pi}^2}{\omega^2} + 2\frac{\kappa_e - 1/2}{\kappa_e}\frac{\omega_{pe}^2}{k_\parallel^2 w_{e,\kappa}^2} = 0, \tag{3.8}$$

where we have neglected the term with $(\zeta_i^0)^{-2}$. The dispersion relation obtained can be written in terms of the thermal velocities v_β by taking into account the definition of the $w_{\beta,\kappa}$,

$$1 - \frac{\omega_{pi}^2}{\omega^2} + 2\frac{\kappa_e - 1/2}{\kappa_e - 3/2}\frac{\omega_{pe}^2}{k_\parallel^2 v_e^2} = 0. \tag{3.9}$$

This expression can be cast in a more familiar form by definition of the electron Debye length λ_{De} and the ion-sound velocity c_s

$$\lambda_{De}^2 = \frac{v_e^2}{2\omega_{pe}^2}, \quad c_s = \sqrt{\frac{T_e}{m_i}} \tag{3.10a,b}$$

and by assuming $n_{i0} = n_{e0}$ and ion charge number $Z = 1$,

$$\omega^2 = \frac{k_\parallel^2 c_{s\kappa}^2}{\dfrac{\kappa_e - 1/2}{\kappa_e - 3/2} + k_\parallel^2 \lambda_{De}^2}. \tag{3.11}$$

We notice that, differently to what occurs for the case of L waves, the dispersion relation obtained does not correspond to the usual dispersion relation obtained for S waves in the case of Maxwellian distributions. The usual and well-known result is obtained in the case of $\kappa_e \to \infty$.

3.2. *The isotropic Kappa distribution as defined by Leubner (2002)*

With the choices $\alpha_\beta = 0$, $w_{\beta,\kappa} = v_\beta$, equation (2.1) becomes the isotropic Kappa distribution which is used, for instance, in equation Leubner (2002, 2004),

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2} v_\beta^3}\frac{\Gamma(\kappa_\beta)}{\Gamma(\kappa_\beta - 3/2)}\left(1 + \frac{v^2}{\kappa_\beta v_\beta^2}\right)^{-\kappa_\beta}. \tag{3.12}$$

This distribution will be denominated the isotropic Kappa distribution of type II. As is well known and easy to show, the relationship between the parameter v_β and the thermal velocity spread of the distribution of particles of species β is as follows,

$$\langle v^2 \rangle_\beta = \frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta - 5/2} v_\beta^2. \tag{3.13}$$

This thermal velocity spread increases for small κ_β, a feature which is consistent with the occurrence of an extended non-thermal tail in the distribution function. It is interesting to notice that for large velocities the distribution function given by (3.12) features a power-law dependence given by $v^{-2\kappa}$, which is reminiscent of distributions generated by diffusion in a superthermal radiation field (Hasegawa, Mima & Duongvan 1985), while the distribution function given by (3.1) leads to an symptotic dependence given by $v^{-2(\kappa+1)}$. The dependency of the kinetic temperature on the κ_β index characterizes the distribution given by (3.12) as the isotropic limit of 'case B' distributions, according to the denomination used in Lazar *et al.* (2016). Moreover, it can also be noticed that the distribution given by (3.12) can be obtained as a result of the use of a non-extensive statistical mechanics as formulated in Tsallis (1988), Silva, Plastino & Lima (1998), Leubner (2002), while the distribution function given by (3.1) results from a modified approach to non-extensive statistical mechanics which utilizes the so-called escort probability functions (Tsallis, Mendes & Plastino 1998; Livadiotis & McComas 2009).

The dispersion relation for ES waves in the case of an isotropic distribution of type II is obtained from (2.2) for $\alpha_\beta = 0$ and $w_{\beta,\kappa} = v_\beta$, and is written as follows,

$$1 + 2\sum_\beta \frac{\omega_\beta^2}{\omega^2}\frac{\kappa_\beta - 3/2}{\kappa_\beta}(\zeta_\beta^0)^2\left(1 + \zeta_\beta^0 Z_{\kappa_\beta}^{(0)}(\zeta_\beta^0)\right), \tag{3.14}$$

where $Z_{\kappa_\beta}^{(0)}$ is obtained from the general definition (2.3) for $m = 0$ and for $\kappa_\beta > -1/2$. As in the case discussed in the previous section, in the case $\kappa \to \infty$ this dispersion relation corresponds to the dispersion relation for ES waves obtained in the case of isotropic Maxwellian distributions.

The expanded form valid in the case of large phase velocity represents a dispersion relation for L waves, is similarly obtained as a particular case of (2.4),

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left(1 + \frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta - 5/2}\frac{k_\parallel^2 v_\beta^2}{\omega^2}\right) = 0. \tag{3.15}$$

Differently from the result obtained in the previous section, in the case of finite value of κ_β this dispersion relation does not correspond to the conventional dispersion for L waves. The difference can become very meaningful, for κ_β approaching the value 5/2.

For completeness we write the approximate form which is obtained by neglecting the effect of the ions in the dispersion relation, and by considering that ω is not far from ω_{pe},

$$\omega^2 \simeq \omega_{pe}^2\left(1 + \frac{3}{2}\frac{\kappa_e}{\kappa_e - 5/2}\frac{k_\parallel^2 v_e^2}{\omega_{pe}^2}\right) \tag{3.16}$$

which is different from the result obtained in the Maxwellian case, particularly for small values of κ_e. In analogy to the case of (3.7), we call (3.16) the Kappa–Bohm–Gross dispersion relation.

For the dispersion relation appropriated for S waves in the case of distribution (3.12), we write down the corresponding particular case of (2.5), neglecting the term with $(\zeta_i^0)^{-2}$.

$$1 - \frac{\omega_{pi}^2}{\omega^2} + 2\frac{\kappa_e - 3/2}{\kappa_e}\frac{\omega_{pe}^2}{k_\parallel^2 v_e^2} = 0. \tag{3.17}$$

By assuming $n_{i0} = n_{e0}$ $Z = 1$, and using the definitions of Debye length and ion-sound velocity, equations (3.10*a,b*), the dispersion relation for S waves in the case of the Kappa distribution of type II becomes as follows

$$\omega^2 = \frac{k_\parallel^2 c_s^2}{\dfrac{\kappa_e - 3/2}{\kappa_e} + k_\parallel^2 \lambda_{De}^2}. \tag{3.18}$$

This expression is different from the dispersion relation obtained in the Maxwellian case, and also different from (3.11). One notices that the difference between the result obtained for type II distributions and the result obtained for type I distributions can become very impressive, for κ approaching the value 3/2.

4. Four particular forms of anisotropic Kappa distributions

4.1. *The anisotropic bi-Kappa distribution, similar to the isotropic distribution defined by Olbert (1968), Vasyliunas (1968)*

In the present section we will assume bi-Kappa distributions for ions and electrons which can be obtained from (2.6) with the choice $\alpha_\beta = 1$

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2} w_{\beta,\kappa,\perp}^2 w_{\beta,\kappa,\parallel}} \frac{\Gamma(\kappa_\beta + 1)}{\Gamma(\kappa_\beta - 1/2)} \left(1 + \frac{v_\parallel^2}{\kappa_\beta w_{\beta,\kappa,\parallel}^2} + \frac{v_\perp^2}{\kappa_\beta w_{\beta,\kappa,\perp}^2}\right)^{-(\kappa_\beta+1)}, \tag{4.1}$$

and considering

$$w_{\beta,\kappa,\parallel}^2 = \frac{\kappa_\beta - 3/2}{\kappa_\beta} v_{\beta\parallel}^2, \quad w_{\beta,\kappa,\perp}^2 = \frac{\kappa_\beta - 3/2}{\kappa_\beta} v_{\beta\perp}^2, \tag{4.2a,b}$$

with

$$v_{\beta\parallel}^2 = \frac{2T_{\beta\parallel}}{m_\beta}, \quad v_{\beta\perp}^2 = \frac{2T_{\beta\perp}}{m_\beta}, \tag{4.3a,b}$$

where $T_{\beta\parallel}$ and $T_{\beta\perp}$ are the parallel and perpendicular physical temperatures. This distribution will be denominated the bi-Kappa distribution of type I. The quantities $v_{\beta\parallel}^2$ and $v_{\beta\perp}^2$ are related to the average values of $v_\parallel^2$ and $v_\perp^2$,

$$\langle v_\parallel^2 \rangle_\beta = \frac{1}{2} v_{\beta\parallel}^2, \quad \langle v_\perp^2 \rangle_\beta = v_{\beta\perp}^2. \tag{4.4a,b}$$

The dispersion relation for ES waves in the case of BK distributions of type I is obtained as a particular case of (2.7), and can be written as follows,

$$1 + 2\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2} (\zeta_{\beta,\kappa,\parallel}^0)^2 \left(\frac{\kappa_\beta - 1/2}{\kappa_\beta} + \zeta_{\beta,\kappa,\parallel}^0 Z_{\kappa_\beta}^{(2)}(\zeta_{\beta,\kappa,\parallel}^0)\right) = 0, \tag{4.5}$$

where $Z_{\kappa_\beta}^{(2)}$ is obtained from the general definition (2.3) for $m = 2$ and for $\kappa_\beta > -5/2$.

For a dispersion relation valid for L waves, we take the particular case of (2.8)

$$1-\sum_\beta \frac{\omega_\beta^2}{\omega^2}\left[1+\frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta-3/2}\frac{k_\parallel^2 w_{\beta,\kappa,\parallel}^2}{\omega^2}\right]=0, \tag{4.6}$$

which may also be written in terms of the thermal velocities,

$$1-\sum_\beta \frac{\omega_\beta^2}{\omega^2}\left[1+\frac{3}{2}\frac{k_\parallel^2 v_{\beta\parallel}^2}{\omega^2}\right]=0. \tag{4.7}$$

It is interesting to notice that the approximated form given by (4.7) is independent of κ_β, and corresponds to the dispersion relation obtained in the case of a bi-Maxwellian distribution. Moreover, the more precise form given by (4.5) also corresponds to the dispersion relation obtained for a bi-Maxwellian distribution, in the limit $\kappa_\beta \to \infty$.

If the effect of the ions is neglected in the dispersion relation and if we take into account that ω must be close to ω_{pe}, a simpler form is obtained from (4.7),

$$\omega^2 \simeq \omega_{pe}^2\left(1+\frac{3}{2}\frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{4.8}$$

The dispersion relation for S waves is obtained as a particular case of (2.9), and can be given as follows

$$\omega^2\left(1+2\frac{\kappa_e-1/2}{\kappa_e}\frac{\omega_{pe}^2}{k_\parallel^2 w_{e,\kappa,\parallel}^2}\right)=\omega_{pi}^2 \tag{4.9}$$

which can be rewritten in terms of the thermal velocities by use of the definition of $w_{\beta,\kappa,\parallel}$,

$$\omega^2\left(1+2\frac{\kappa_e-1/2}{\kappa_e-3/2}\frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right)=\omega_{pi}^2. \tag{4.10}$$

Introducing a definition for the parallel Debye length, similar to that of (3.10*a*,*b*) but using $T_{\beta\parallel}$ instead of T_β, the corresponding definition of a parallel ion-sound velocity, we obtain

$$\omega^2=\frac{k_\parallel^2 c_{s\kappa\parallel}^2}{\dfrac{\kappa_e-1/2}{\kappa_e-3/2}+k_\parallel^2\lambda_{De\parallel}^2}. \tag{4.11}$$

4.2. *The anisotropic bi-Kappa distribution, similar to the isotropic distribution defined by Leubner (2002)*

Let us assume bi-Kappa distributions for ions and electrons which are obtained from (2.6) with the choice $\alpha_\beta=0$, $w_{\beta,\kappa,\parallel}=v_{\beta\parallel}$ and $w_{\beta,\kappa,\perp}=v_{\beta\perp}$,

$$f_\beta(\boldsymbol{v})=\frac{n_{\beta 0}}{\pi^{3/2}\kappa_\beta^{3/2}v_{\beta\perp}^2 v_{\beta\parallel}}\frac{\Gamma(\kappa_\beta)}{\Gamma(\kappa_\beta-3/2)}\left(1+\frac{v_\parallel^2}{\kappa_\beta v_{\beta\parallel}^2}+\frac{v_\perp^2}{\kappa_\beta v_{\beta\perp}^2}\right)^{-\kappa_\beta}. \tag{4.12}$$

This distribution will be denominated the bi-Kappa distribution of type II. The velocity spreads along parallel and perpendicular directions can be easily obtained by averaging over the distribution function,

$$\langle v_\parallel^2\rangle_\beta = \frac{1}{2}\frac{\kappa_\beta}{\kappa_\beta - 5/2} v_{\beta\parallel}^2, \quad \langle v_\perp^2\rangle_\beta = \frac{\kappa_\beta}{\kappa_\beta - 5/2} v_{\beta\perp}^2. \tag{4.13a,b}$$

The generic form of the dispersion relation for parallel propagating ES waves in the case of type II BK distributions is obtained as follows, from (2.7),

$$1 + 2\sum_\beta \frac{\omega_\beta^2}{\omega^2} (\zeta_{\beta,\kappa,\parallel}^0)^2 \frac{\kappa_\beta - 3/2}{\kappa_\beta} \left(1 + \zeta_{\beta,\kappa,\parallel}^0 Z_{\kappa_\beta}^{(1)}(\zeta_{\beta,\kappa,\parallel}^0)\right) = 0, \tag{4.14}$$

where $Z_{\kappa_\beta}^{(1)}$ is obtained from the general definition (2.3) for $m = 1$ and for $\kappa_\beta > -3/2$.

An approximate form of the dispersion relation, obtained taking into account waves whose phase velocity is much greater than the thermal velocities, is obtained as a particular case from (2.8),

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left[1 + \frac{3}{2}\frac{\kappa_\beta}{\kappa_\beta - 5/2}\frac{k_\parallel^2 v_{\beta\parallel}^2}{\omega^2}\right] = 0\ . \tag{4.15}$$

It is interesting to notice that in the limit $\kappa_\beta \to \infty$, both forms of the dispersion relation, the more precise form given by (4.14) and the approximated form given by (4.15), are independent of κ_β and correspond to the dispersion relation obtained for a bi-Maxwellian distribution. However, for small values of κ_β, the dispersion relation obtained for L waves in a plasma with anisotropic bi-Kappa distribution as given by (4.12) can be significantly different from the dispersion relation of the Maxwellian case, regarding the dependence of wavenumber.

If the effect of the ions is neglected in the dispersion relation and if we take into account that ω must be close to ω_{pe}, a simpler form is obtained from (4.15),

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2}\frac{\kappa_e}{\kappa_e - 5/2}\frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{4.16}$$

The dispersion relation valid for S waves is obtained by considering the pertinent expansions in the integrand of the J integrals for electrons and ions, and the contributions of the I integrals, with the distribution function given by (4.12). It is obtained from (2.9) as follows

$$\omega^2 \left(1 + 2\frac{\kappa_e - 3/2}{\kappa_e}\frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right) = \omega_{pi}^2. \tag{4.17}$$

By taking into account the definitions of $c_{s\parallel}$ and $\lambda_{De\parallel}$, the dispersion relation for S waves, in the case of distribution function (4.12), is written as follows

$$\omega^2 = \frac{k_\parallel^2 c_{s\parallel}^2}{\dfrac{\kappa_e - 3/2}{\kappa_e} + k_\parallel^2 \lambda_{De\parallel}^2}. \tag{4.18}$$

This expression is different from the dispersion relation obtained in the case of a bi-Maxwellian plasma, and also different from the form obtained in the case of distribution functions given by (4.1). The difference is particularly significant for small value of the kappa index in the electron distribution, κ_e.

4.3. *The anisotropic product-bi-Kappa distribution, similar to the isotropic distribution defined by Olbert (1968), Vasyliunas (1968)*

In the present section we will assume product bi-Kappa distributions for ions and electrons which can be obtained from (2.10) with the choice $\alpha_\beta = 1$

$$f_\beta(\boldsymbol{v}) = \frac{n_{\beta 0}}{\pi^{3/2}\kappa_{\beta\perp}\kappa_{\beta\parallel}^{1/2} w_{\beta,\kappa_\perp}^2 w_{\beta,\kappa_\parallel}} \frac{\Gamma(\kappa_{\beta\perp}+1)\Gamma(\kappa_{\beta\parallel}+1)}{\Gamma(\kappa_{\beta\perp})\Gamma(\kappa_{\beta\parallel}+1/2)} \times \left(1+\frac{v_\parallel^2}{\kappa_{\beta\parallel} w_{\beta,\kappa_\parallel}^2}\right)^{-(\kappa_{\beta\parallel}+1)} \left(1+\frac{v_\perp^2}{\kappa_{\beta\perp} w_{\beta,\kappa_\perp}^2}\right)^{-(\kappa_{\beta\perp}+1)}, \tag{4.19}$$

and considering

$$w_{\beta,\kappa,\parallel}^2 = \frac{\kappa_{\beta\parallel}-1/2}{\kappa_{\beta\parallel}} v_{\beta\parallel}^2, \quad w_{\beta,\kappa,\perp}^2 = \frac{\kappa_{\beta\perp}-1}{\kappa_{\beta\perp}} v_{\beta\perp}^2, \tag{4.20a,b}$$

with

$$v_{\beta\parallel}^2 = \frac{2T_{\beta\parallel}}{m_\beta}, \quad v_{\beta\perp}^2 = \frac{2T_{\beta\perp}}{m_\beta}. \tag{4.21a,b}$$

This distribution will be denominated the product-bi-Kappa distribution of type I. The quantities $v_{\beta\parallel}^2$ and $v_{\beta\perp}^2$ are related to the average values of $v_\parallel^2$ and $v_\perp^2$,

$$\langle v_\parallel^2\rangle_\beta = \tfrac{1}{2} v_{\beta\parallel}^2, \quad \langle v_\perp^2\rangle_\beta = v_{\beta\perp}^2. \tag{4.22a,b}$$

The dispersion relation for ES waves in the case of PBK distributions of type I is obtained as a particular case of (2.11), and can be written as follows,

$$1 + 2\sum_\beta \frac{\omega_{p\beta}^2}{\omega^2} (\zeta_{\beta,\kappa_\parallel}^0)^2 \left(\frac{\kappa_{\beta\parallel}+1/2}{\kappa_{\beta\parallel}} + \frac{\kappa_{\beta\parallel}+1}{\kappa_{\beta\parallel}-1/2} \zeta_{\beta,\kappa_\parallel}^0 Z_{\kappa_{\beta\parallel}}^{(2)}(\zeta_{\beta,\kappa_\parallel}^0)\right) = 0, \tag{4.23}$$

where $Z_{\kappa_{\beta\parallel}}^{(2)}$ is obtained from the general definition (2.3) for $m = 2$ and for $\kappa_{\beta\parallel} > -5/2$.

For a dispersion relation valid for L waves, we take the particular case of (2.12)

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left[1 + \frac{3}{2}\frac{\kappa_{\beta\parallel}}{\kappa_{\beta\parallel}-1/2}\frac{k_\parallel^2 w_{\beta,\kappa,\parallel}^2}{\omega^2}\right] = 0, \tag{4.24}$$

which may also be written in terms of the thermal velocities,

$$1 - \sum_\beta \frac{\omega_\beta^2}{\omega^2}\left[1 + \frac{3}{2}\frac{k_\parallel^2 v_{\beta\parallel}^2}{\omega^2}\right] = 0. \tag{4.25}$$

It is interesting to notice that the approximated form given by (4.25) corresponds to the dispersion relation obtained with a bi-Maxwellian distribution. Moreover, it is seen that the more precise form given by (4.23) is independent of $\kappa_{\beta\perp}$, and in the limit $\kappa_{\beta\parallel} \to \infty$ also correspond to the dispersion relation obtained for a bi-Maxwellian distribution.

If the effect of the ions is neglected in the dispersion relation and if we take into account that ω must be close to ω_{pe}, a simpler form is obtained from (4.25),

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2} \frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2} \right). \tag{4.26}$$

The dispersion relation for S waves is obtained as a particular case of (2.13), and is given as follows

$$\omega^2 \left(1 + 2 \frac{\kappa_{e\parallel} + 1/2}{\kappa_{e\parallel}} \frac{\omega_{pe}^2}{k_\parallel^2 w_{e,\kappa\parallel}^2} \right) = \omega_{pi}^2, \tag{4.27}$$

which can be rewritten in terms of the thermal velocities by use of the definition of $w_{\beta,\kappa\parallel}$,

$$\omega^2 \left(1 + 2 \frac{\kappa_{e\parallel} + 1/2}{\kappa_{e\parallel} - 1/2} \frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2} \right) = \omega_{pi}^2. \tag{4.28}$$

Using the definition of the parallel Debye length, we obtain

$$\omega^2 = \frac{k_\parallel^2 C_{s\kappa\parallel}^2}{\dfrac{\kappa_{e\parallel} + 1/2}{\kappa_{e\parallel} - 1/2} + k_\parallel^2 \lambda_{De\parallel}^2}. \tag{4.29}$$

4.4. *The anisotropic product-bi-Kappa distribution, similar to the isotropic distribution defined by Leubner (2002)*

Let us assume product bi-Kappa distributions for ions and electrons which are obtained from (2.10) with the choice $\alpha_\beta = 0$, $w_{\beta,\kappa_\parallel} = v_{\beta\parallel}$ and $w_{\beta,\kappa_\perp} = v_{\beta\perp}$,

$$\begin{aligned} f_\beta(\boldsymbol{v}) = {} & \frac{n_{\beta 0}}{\pi^{3/2} \kappa_{\beta\perp} \kappa_{\beta\parallel}^{1/2} v_{\beta\perp}^2 v_{\beta\parallel}} \frac{\Gamma(\kappa_{\beta\perp}) \Gamma(\kappa_{\beta\parallel})}{\Gamma(\kappa_{\beta\perp} - 1) \Gamma(\kappa_{\beta\parallel} - 1/2)} \\ & \times \left(1 + \frac{v_\parallel^2}{\kappa_{\beta\parallel} v_{\beta\parallel}^2} \right)^{-\kappa_{\beta\parallel}} \left(1 + \frac{v_\perp^2}{\kappa_{\beta\perp} v_{\beta\perp}^2} \right)^{-\kappa_{\beta\perp}}. \end{aligned} \tag{4.30}$$

This distribution will be denominated the product-bi-Kappa distribution of type II. The velocity spreads along parallel and perpendicular directions can be easily obtained by averaging over the distribution function,

$$\langle v_\parallel^2 \rangle_\beta = \frac{1}{2} \frac{\kappa_{\beta\parallel}}{\kappa_{\beta\parallel} - 3/2} v_{\beta\parallel}^2, \quad \langle v_\perp^2 \rangle_\beta = \frac{\kappa_{\beta\perp}}{\kappa_{\beta\perp} - 2} v_{\beta\perp}^2. \tag{4.31a,b}$$

The generic form of the dispersion relation for parallel propagating ES waves in the case of type II PBK distributions is obtained as follows, from (2.11),

$$1 + 2 \sum_\beta \frac{\omega_\beta^2}{\omega^2} (\zeta_{\beta,\kappa_\parallel}^0)^2 \left(\frac{\kappa_{\beta\parallel} - 1/2}{\kappa_{\beta\parallel}} + \zeta_{\beta,\kappa_\parallel}^0 Z_{\kappa_{\beta\parallel}}^{(1)}(\zeta_{\beta,\kappa_\parallel}^0) \right) = 0, \tag{4.32}$$

where $Z^{(1)}_{\kappa_{\beta\|}}$ is obtained from the general definition (2.3) for $m = 1$ and for $\kappa_{\beta\|} > -3/2$.

An approximate form of the dispersion relation, obtained taking into account waves whose phase velocity is much greater than the thermal velocities, is obtained as a particular case from (2.12),

$$1 - \sum_{\beta} \frac{\omega_\beta^2}{\omega^2}\left[1 + \frac{3}{2}\frac{\kappa_{\beta\|}}{\kappa_{\beta\|} - 3/2}\frac{k_\|^2 v_{\beta\|}^2}{\omega^2}\right] = 0\ . \tag{4.33}$$

It is interesting to notice that both forms of the dispersion relation, the more precise form given by (4.32) and the approximated form given by (4.33), are independent of $\kappa_{\beta\perp}$, and in the limit $\kappa_{\beta\|} \to \infty$ correspond to the dispersion relation obtained for a bi-Maxwellian distribution. However, for small values of $\kappa_{\beta\|}$, the dispersion relation obtained for L waves in a plasma with anisotropic product-bi-Kappa distribution as given by (4.30) can be significantly different from the dispersion relation of the Maxwellian case, regarding the dependence of wavenumber.

If the effect of the ions is neglected in the dispersion relation and if we take into account that ω must be close to ω_{pe}, a simpler form is obtained from (4.33),

$$\omega^2 \simeq \omega_{pe}^2\left(1 + \frac{3}{2}\frac{\kappa_{\beta\|}}{\kappa_{\beta\|} - 3/2}\frac{k_\|^2 v_{e\|}^2}{\omega_{pe}^2}\right). \tag{4.34}$$

The dispersion relation valid for S waves is obtained by considering the pertinent expansions in the integrand of the J integrals for electrons and ions, and the contributions of the I integrals, with the distribution function given by (4.30). It is obtained from (2.13) as follows

$$\omega^2\left(1 + 2\frac{\kappa_{e\|} - 1/2}{\kappa_{e\|}}\frac{\omega_{pe}^2}{k_\|^2 v_{e\|}^2}\right) = \omega_{pi}^2. \tag{4.35}$$

By taking into account the definitions of $c_{s\|}$ and $\lambda_{De\|}$, the dispersion relation for S waves, in the case of distribution function (4.30), is written as follows

$$\omega^2 = \frac{k_\|^2 c_{s\|}^2}{\dfrac{\kappa_{e\|} - 1/2}{\kappa_{e\|}} + k_\|^2\lambda_{De\|}^2}. \tag{4.36}$$

This expression is different from the dispersion relation obtained in the case of a bi-Maxwellian plasma, and also different from the form obtained in the case of distribution functions given by (4.19). The difference is particularly significant for small value of the parallel kappa index in the electron distribution, $\kappa_{e\|}$.

5. Numerical results

In order to illustrate the effects of the different superthermal features associated with different forms of the Kappa distribution functions on electrostatic waves which can occur in a plasma, we start by considering the case of isotropic Kappa distributions.

As a first step, we show in figure 1 the plots of Kappa distributions of type I and type II, for $\kappa_\beta = 3$ and $\kappa_\beta = 10$. The black lines represent type I distributions, given

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

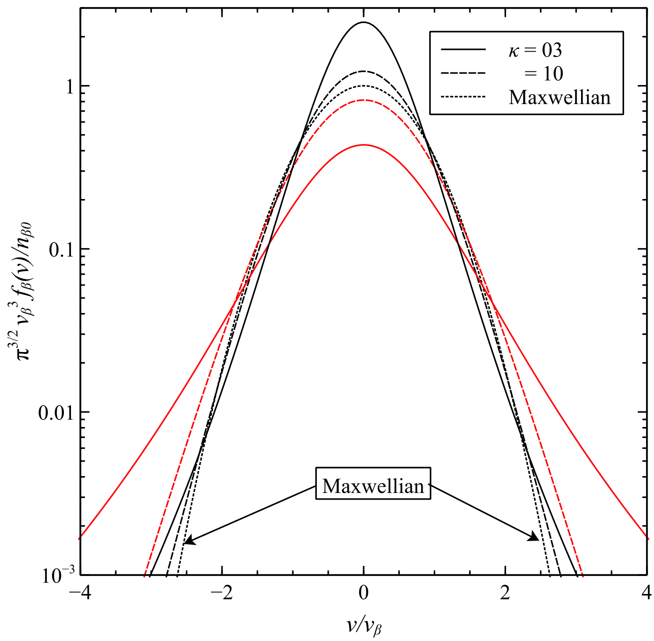

FIGURE 1. Normalized values of isotropic Kappa distributions, versus normalized velocity, for $\kappa = 3$ and $\kappa = 10$. Black lines are obtained using (3.1) (type I), and red lines are obtained using (3.12) (type II). The dashed line represents a Maxwellian distribution.

by (3.1), and red lines represent type II distributions, given by (3.12). A Maxwellian distribution is also shown for comparison, represented by a dashed line. It is seen that the population of particles in the small velocity region is above that of the thermal case, for distributions of type I, and below that of the thermal case, for type II distributions. Both types of distributions feature superthermal tails, with particle population above that of the Maxwellian case, but it is seen that for sufficiently large velocities type II distributions feature more pronounced superthermal populations than type I distributions. A discussion on possible mechanisms associated with the generation of these different distributions of particle velocities can be found in Lazar *et al.* (2016), where these distributions are denoted as Kappa A and Kappa B, respectively.

In the sequence, we present the analysis of the dispersion properties, starting with the dispersion properties of Langmuir (L) waves. Figure 2 shows real and imaginary parts of the normalized wave frequency as function of the normalized parallel wavenumber, for the case of plasma particles with isotropic Kappa distributions of type I, given by (3.1). The results shown in figure 2 are obtained from numerical solution of (3.3). The wavenumbers are normalized as $q_\parallel = k_\parallel v_e/\sqrt{2}\omega_{pe}$. The figure shows the results obtained in the case of distributions with $\kappa_e = \kappa_i = \kappa$, with $\kappa \to \infty$, corresponding to results of the usual dispersion relation obtained for the Maxwellian case, and also shows the values obtained for $\kappa = 20$, 15, 10, 05 and 03. It is seen that the real part of the dispersion relation for L waves is relatively insensitive to the value of κ, in the region of small wavenumbers (large wavelengths), but becomes quite different from the conventional relation obtained for Maxwellian plasmas for values of the normalized wavenumber which are above 0.25, particularly for small

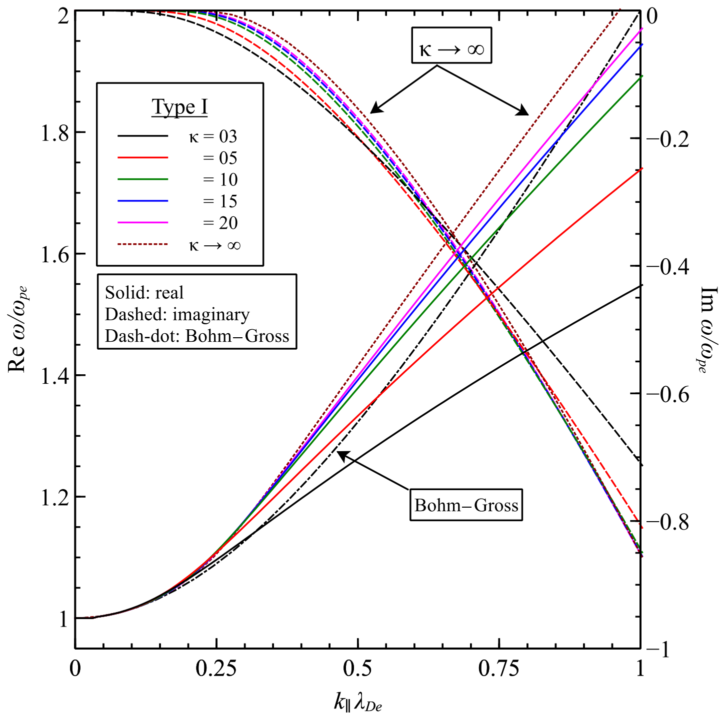

FIGURE 2. Real and imaginary parts of the wave frequency versus normalized parallel wavenumber, for Langmuir waves, for different values of the index κ. The results have been obtained by numerical solution of (3.3), considering electrons and ions described by isotropic Kappa distribution functions of type I, given by (3.1). The figure also shows results obtained with the Bohm–Gross form of the dispersion relation, which is (3.7).

values of κ. The effect on the imaginary part of the frequency is not so large, but it is also significant. For instance, one sees from figure 2 that for $\kappa = 3$ there is significant damping for waves with normalized wavenumber $\simeq 0.3$, while the damping is negligible at these wavenumbers, for $\kappa \to \infty$.

Figure 3 is similar to figure 2, but was obtained considering plasma particles with velocity distributions given by (3.12), that is, isotropic Kappa distributions of type II, and by numerical solution of the dispersion relation (3.14). The values of κ (for $\kappa_e = \kappa_i$) which have been considered are the same as those used in the case of figure 2. A difference when comparing to the case depicted in figure 2 is that figure 3 shows that for distributions of type II, the real part of the wave frequency is different from that obtained in the Maxwellian case, in the region of small wavenumbers, with the difference increasing for small values of κ. It is interesting to notice that the real part of the frequency is above the Maxwellian result, for normalized wavenumber $\leqslant 0.5$, and below the Maxwellian result, for normalized wavenumber above $\simeq 0.5$. Regarding the imaginary part of the frequency, figure 3 shows that the damping of the L waves is considerably increased, in comparison with the damping in an Maxwellian plasma, and increases with the decrease of κ.

In figure 4 we investigate the validity of the approximate dispersion relations for L waves, obtained by expansion of the integrand of velocity integrals which appear in the components of the dielectric tensor, for different values of κ, for $\kappa_e = \kappa_i$. In

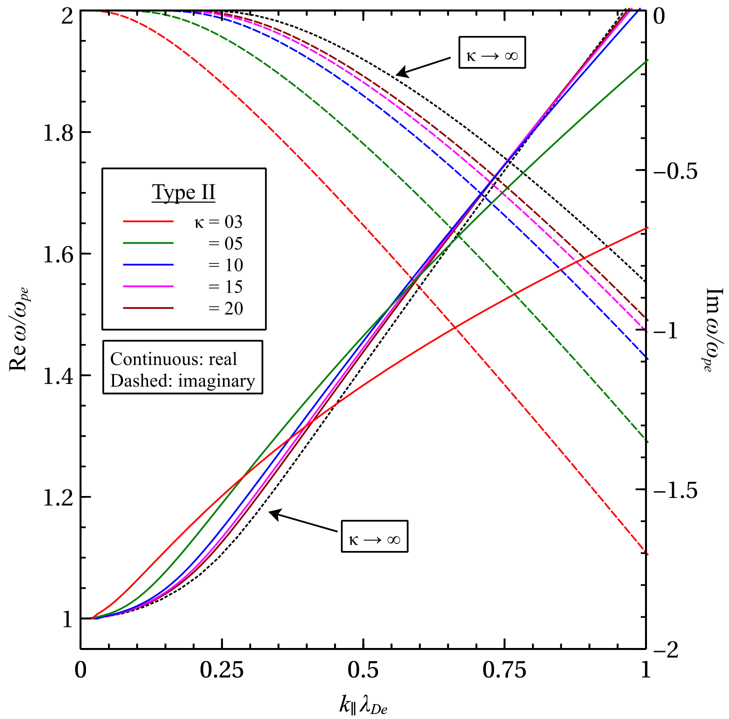

FIGURE 3. Real and imaginary parts of the wave frequency versus normalized parallel wavenumber, for Langmuir waves, for different values of the index κ. The results have been obtained by numerical solution of (3.14), considering electrons and ions described by isotropic Kappa distribution functions of type II, given by (3.12).

order to obtain the results presented in figure 4, which represent the real part of the normalized wave frequency versus wavenumbers normalized as in figure 2, we have considered plasma particles with type II distributions, given by (3.12), and have used the dispersion relation given by (3.14) and the approximated form given by (3.15). The continuous lines show the values obtained by numerical solution of (3.14), and the dashed lines show the values obtained using the approximated dispersion relation given by (3.15). It is seen that for large κ, represented by the line indicated by $\kappa \rightarrow \infty$, the line obtained from the approximated dispersion relation lies below the continuous line which indicates the numerical solution, for the whole range of normalized wavenumbers shown in the figure. The exact and the approximated solutions are very close for small wavenumber, and separate appreciably for larger wavenumbers. For $\kappa = 15$, the behaviour of the solutions is similar to that obtained for $\kappa \rightarrow \infty$, but it is seen that the approximated result become larger than the exact numerical result, for normalized wavenumber $\leqslant$0.75. For $\kappa = 5$, figure 4 shows that the approximated result is below the exact result, but relatively close, for small wavenumbers, and is above the exact result for normalized wavenumber above $\simeq$0.25, with the difference increasing with the increase in wavenumber. For very small values of κ, illustrated by the case of $\kappa = 3$ in figure 4, the real frequency predicted by the approximated dispersion relation is already above the exact numerical result even for moderate wavenumbers (normalized values above $q \simeq 0.05$), with enormous difference for larger wavenumbers.

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

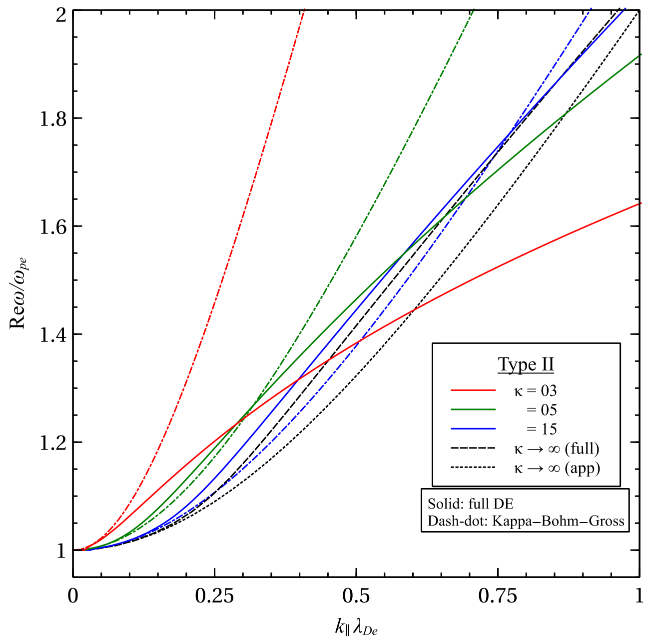

FIGURE 4. Real part of the wave frequency versus normalized parallel wavenumber, for Langmuir waves, for different values of the index κ, in the case of electrons and ions described by isotropic Kappa distribution functions of type II, given by (3.12). For each value of κ, the continuous line shows the values obtained by numerical solution of (3.14), and the dashed line shows the values obtained using the approximated dispersion relation given by (3.15). The figure also shows results obtained with the Kappa–Bohm–Gross form of the dispersion relation for distributions of type II, which is (3.16).

A figure corresponding to figure 4, but for the case of the isotropic distribution function of type I, equation (3.1), is not necessary, because a previous figure already displays the information about the difference between the exact and the approximated dispersion relation. In figure 2 the solid lines display the results obtained with the use of the exact dispersion relation given by (3.3) for $\kappa = 3$, 5, 10, 15 and 20, and there is a dashed line displaying the exact result for the Maxwellian limit, while a dot-dashed curve displays the approximated result, which is given by (3.7) and is independent of κ.

In the following, we continue with the analysis with the dispersion properties of ion-sound (S) waves.

Figure 5 shows the real and imaginary parts of the normalized wave frequency for S waves, obtained from numerical solution of (3.3), as function of the normalized parallel wave number, for plasma particles with isotropic Kappa distributions of type I, given by (3.1). The figure shows the case of distributions with $\kappa_e = \kappa_i = \kappa$, with $\kappa \to \infty$, and depicts results obtained with $\kappa = \infty$, 20, 15, 10, 5 and 3, as in figure 2. It is seen that the real part of the dispersion relation for S waves is relatively insensitive to the value of κ for very small wavenumbers. The difference increases for increasing value of $k_\parallel$, and is more significant for small values of κ ($\kappa \leqslant 5.0$). The imaginary part of the frequency, associated with the wave damping, has greater magnitude in the case of finite κ than in the case of $\kappa \to \infty$, for normalized wavenumber $q \leqslant 2.0$, and smaller magnitude for larger wavenumbers.

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

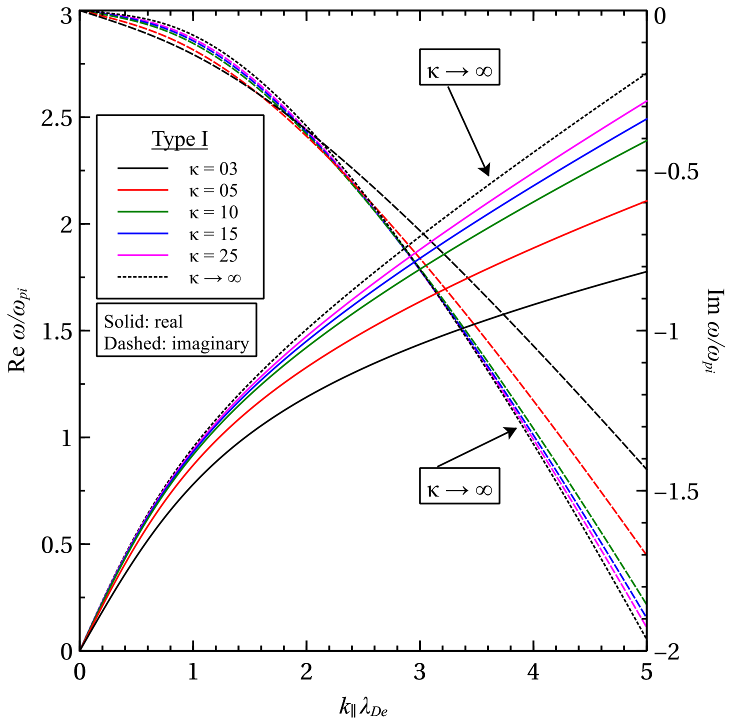

FIGURE 5. Real and imaginary parts of the wave frequency versus normalized parallel wavenumber, for ion-sound waves, for different values of the index κ. The results have been obtained by numerical solution of equation (3.3), considering electrons and ions described by isotropic Kappa distribution functions of type I, given by (3.1).

Figure 6 is similar to figure 5, but was obtained considering plasma particles with velocity distributions given by (3.12), that is, isotropic Kappa distributions of type II, and therefore numerical solution of (3.14). The values of κ (for $\kappa_e = \kappa_i$) which have been considered are the same as those used in the case of figure 5. In the case shown in figure 6, the real part of the wave frequency is relatively insensitive to the value of κ, for small wavenumbers. Significant difference only occurs for normalized wavenumbers above $q \simeq 2$, for small values of κ, smaller than $\kappa \simeq 5.0$. The effect on the imaginary part of the frequency is much more impressive, starting from small values of wavenumber, and increases with the decrease of κ.

In figure 7 we compare results obtained with the approximate dispersion relation for S waves given by (3.8) with results obtained using the exact dispersion relation (3.3), for isotropic type I distributions, given by (3.1). Figure 7 shows that the exact dispersion relation and the approximate dispersion relation predict similar results for the real part of the dispersion relation, for normalized wavenumber $q \leqslant 1.0$. For larger wavenumbers, figure 7 shows that the approximate dispersion relation converges to a value of $\mathrm{Re}\,(\omega)/\omega_{pe} \simeq 2.0$, independently of the value of κ, while the exact dispersion relation shows that the value of the real part of the frequency continues to increase for increasing wavenumber. The results predicted in the case of distributions with finite κ values lie below the results predicted in the case of Maxwellian distributions ($\kappa \rightarrow \infty$), with the difference increasing for smaller values of κ.

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

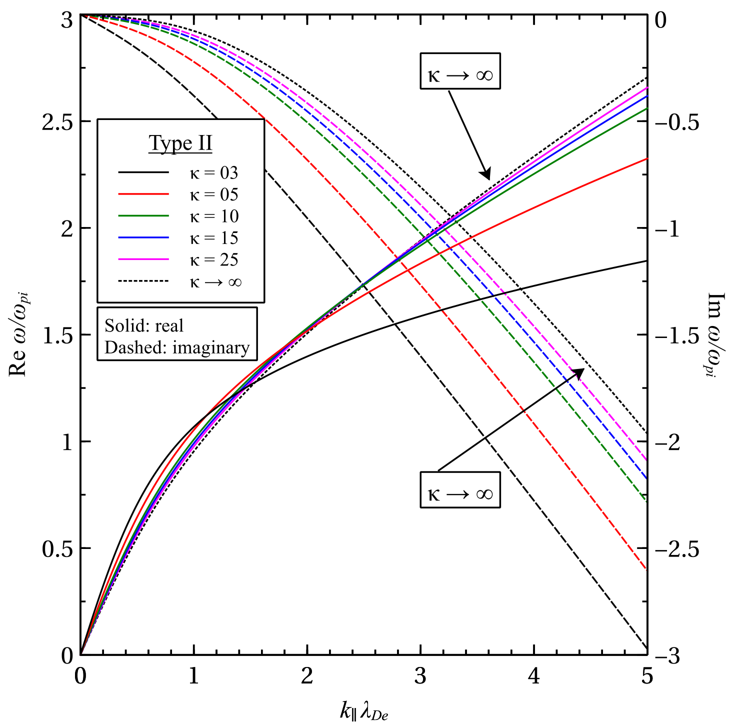

FIGURE 6. Real and imaginary parts of the wave frequency versus normalized parallel wavenumber, for ion-sound waves, for different values of the index κ. The results have been obtained by numerical solution of equation (3.14), considering electrons and ions described by isotropic Kappa distribution functions of type II, given by (3.12).

Figure 8 is dedicated to an analysis similar to that made in figure 7, but considering the case of isotropic type II distributions, given by (3.12). That is, figure 8 compare results obtained with the approximate dispersion relation given by (3.17) with results obtained using the exact dispersion relation (3.14). The results obtained and appearing in figure 8 are qualitatively similar to those shown in figure 7, except that in the case of figure 8 the results predicted by the approximate dispersion relation are close to those predicted for the Maxwellian case for κ between ∞ and $\kappa = 5.0$, and depart appreciably for very small value of κ, exemplified in figure 8 by $\kappa = 3.0$.

As a verification of the theoretical dispersion relations obtained for L waves in the case of different forms of Kappa distributions, we have carried out some tests with a particle-in-cell numerical code. The numerical scheme which is utilized is based on the one-dimensional electromagnetic PIC code KEMPO (Kyoto ElectroMagnetic Particle Code) (Omura & Matsumoto 1993), but in a modified version to include at the initial time the Kappa distribution and relativistic effects for particles and fields.

To avoid deleterious effects due to numerical noise in the PIC code we included relativistic effects. The importance of relativistic effects is particularly important in the case of small kappa index, $\kappa < 5$, for which the particles of suprathermal tails are accelerated to relativistic velocities in few time steps. It is also necessary to solve the Poisson's equation at every time step, in order to eliminate a non-physical force along the simulation run.

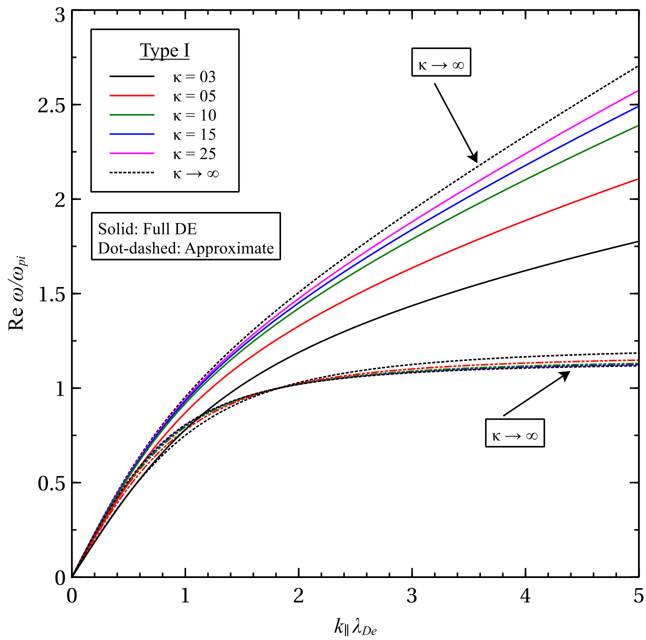

FIGURE 7. Real part of the wave frequency versus normalized parallel wavenumber, for ion-sound waves, for different values of the index κ, in the case of electrons and ions described by isotropic Kappa distribution functions of type I, given by (3.1). For each value of κ, the continuous line shows the values obtained by numerical solution of equation (3.3), and the dot-dashed line shows the values obtained using the approximated dispersion relation given by (3.8).

We have used 8192 spatial grid points with distances normalized by λ_{De}, the grid spacing being $\Delta x = 2.0\lambda_{De}$, and we have run the simulation for a total of 16 384 time steps, with $\Delta t = 0.02\omega_{pe}^{-1}$, which means that the system evolves until $\omega_{pe} t = 327.68$. The ambient magnetic field is assumed to be zero, the frequencies are normalized by plasma frequency and the velocities are normalized by c. We have taken into account only the electron dynamics, assuming thermal velocity $v_e = 0.0125\ c$ and considering 512 super-particles per grid cell. The ions were included just for charge neutrality.

To start the PIC simulation, we have generated the Kappa distribution using the OCTAVE free software, that has a statistical package with the function T-student. For our application, we have exchanged the T-student for a Kappa function, following the method proposed by Abdul & Mace (2014).

Figure 9 shows the (ω, k) diagram obtained from the x (longitudinal) component of the electric field by Fourier transforming in space along the x axis and in time, for different forms of the isotropic Kappa distribution for electrons. Panels (*a,c,e*) present the results obtained assuming that the electron distribution is a Kappa distribution of type I, as given by (3.1), panels (*b,d,f*) present the results obtained using a Kappa distribution of type II, as given by (3.12). Panels (*a,b*) show the results obtained considering $\kappa = 3$, panels (*c,d*) show results obtained using $\kappa = 5$ and panels (*e,f*) show the results from the simulation made with $\kappa = 20$. For reference, the panels display dotted lines which represent the analytical dispersion relation for L waves, obtained in the case of Maxwellian distribution, and continuous lines which represent the L wave dispersion obtained in the case of a Kappa distribution. In the case of

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

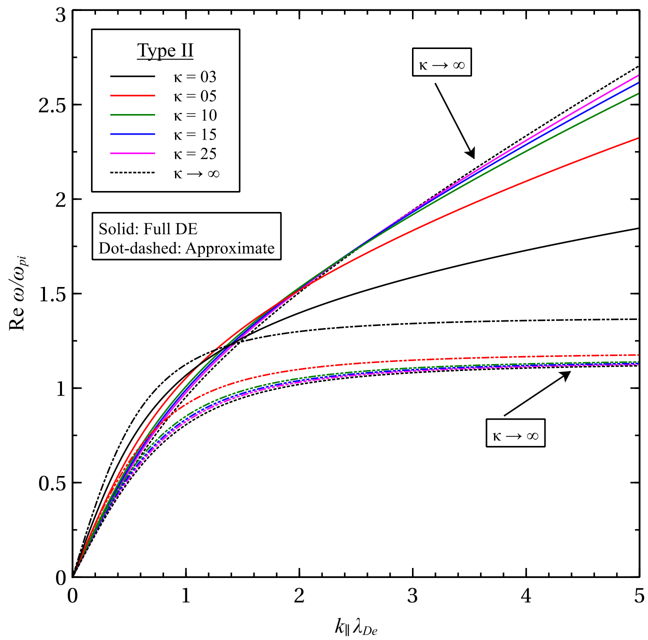

FIGURE 8. Real part of the wave frequency versus normalized parallel wavenumber, for ion-sound waves, for different values of the index κ, in the case of electrons and ions described by isotropic Kappa distribution functions of type II, given by (3.12). For each value of κ, the continuous line shows the values obtained by numerical solution of equation (3.14), and the dot-dashed line shows the values obtained using the approximated dispersion relation given by (3.17).

type I distributions, appearing at the left-hand side, the dispersion relations for Kappa and Maxwellian distributions coincide, given by (3.7), and therefore the continuous line would coincide with the dotted line. For simplicity, only the dotted line is shown in this case. In the case of type II distributions, the dotted line is given by (3.7), and the continuous line is given by (3.16). We have chosen to compare the results of PIC simulation with approximated forms of the dispersion relation, instead of the exact expressions, due to the analytical dependency on parallel wavenumber and on the κ index, featured by the approximate expressions. In the region of relatively small wavenumbers, where most of the points of PIC simulations are concentrated, the difference between the approximate solutions and the exact ones is not very expressive.

Figure 9(*a,b*) shows the case of $\kappa = 3$. It is noticed that the electrostatic fluctuations in the case of the type II distribution, panel (*b*) are well concentrated in the region of small values of k (i.e. $k\lambda_{De} < 0.3$), while in the case of type I distribution, panel (*a*) the electrostatic fluctuations are significant in a larger region of k space (i.e. $k\lambda_{De} \leqslant 0.6$). However, for both cases the fluctuations follow with good precision the curve describing the dispersion relation for L waves in the case of Kappa distribution, particularly in the region of small values of k. Figure 9(*c,d*) shows the cases of $\kappa = 5$. Panel (*d*) shows that for this value of κ the dispersion relation for Kappa distribution is closer to the dispersion relation for Maxwellian distribution, in comparison with the case of $\kappa = 3$ seen in (*b*). As in (*a,b*), the fluctuations follow with good precision the theoretical dispersion relation, both for type I and for type II distributions. Finally,

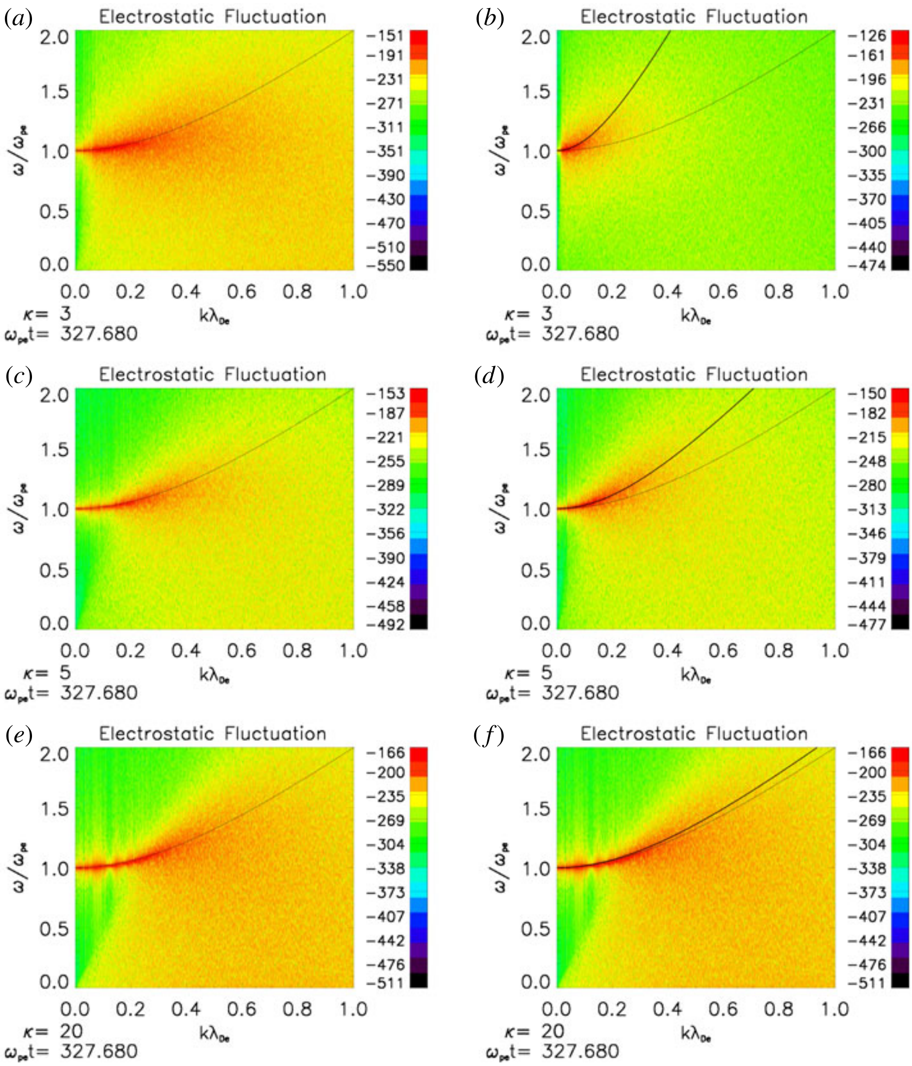

FIGURE 9. ω–k diagram obtained from the time and space evolution of the x (longitudinal) component of the electric field, E_x, in a PIC simulation code. (*a,c,e*) Results obtained with isotropic Kappa distribution of type I; (*b,d,f*) results obtained with isotropic Kappa distribution of type II. The theoretical dispersion relation for L waves in the case of isotropic Maxwellian distribution is shown by a dotted line. The dispersion relation obtained for isotropic Kappa distribution of type II is shown at the right-hand side by a continuous line. In the case of type I isotropic Kappa distributions, the dispersion relation for L waves coincides with that of a Maxwellian plasma, so that the panels (*a,c,e*) only feature the dotted line. (*a,b*) $\kappa = 3$; (*c,d*) $\kappa = 5$; (*e,f*) $\kappa = 20$.

panels (*e,f*) show the case $\kappa = 20$. In this case the Kappa distribution is already quite close to the Maxwellian shape. This similarity is expressed also by the fact that the theoretical dispersion relations for Kappa distribution and for Maxwellian distribution become very close in panel (*f*), which shows the case of type II. It is also seen that the dispersions relations for the type II, in (*f*), approach that which is obtained for

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

the case of type I in panel (*e*). The fluctuations detected at the PIC simulation follow closely the theoretical dispersion relation, and it is seen that the results of type I and type II tend to be very similar. This is the expected result, since for very large κ both type I and type II distributions tend to the Maxwellian distribution.

6. Final remarks

We have evaluated the dispersion relations of Langmuir and ion-sound waves considering two types of isotropic Kappa distributions, two types of anisotropic distributions which are known as bi-Kappa distributions, and two types of anisotropic distributions which are known as product-bi-Kappa distributions. These different types of Kappa, BK and PBK distributions have been denominated as type I and type II, for reference in the text. It has been seen that for type I distributions the kinetic temperature is independent of the κ_β index (or indexes, in the PBK case) of the distribution, which characterizes these distributions in the same category denominated as 'case A' in Lazar *et al.* (2016). On the other hand, for type II distributions the kinetic temperature is dependent on the κ index, characterizing them as 'case B', according to Lazar *et al.* (2016). These different categories of Kappa distributions have been used by the plasma physics community in the last decades, and have appeared in Lazar *et al.* (2016) in the context of a discussion on the possible physical mechanisms which could lead to these different forms of velocity distributions. In the present paper, on the other hand, these distributions appeared in a different context. We have avoided the discussion on their generation, and have instead focused on the consequences of their use for the analysis of electrostatic waves. At this point of the paper, we summarize the six different forms of the dispersion relation which have been obtained for each type of ES waves, considering two different approaches. For this summary, we emphasize the approximated forms of the dispersion relations, because these approximated forms show more clearly the dependencies on wavenumber and on the κ index. The non-approximated forms of the dispersion relations depend on generalized plasma dispersion functions, requiring numerical analysis.

One possible approach to the problem is by the definition of a parameter T_β in the isotropic case (which is the thermodynamic temperature T for particles of the species β), or parameters $T_{\beta\parallel}$ and $T_{\beta\perp}$ in the anisotropic case, which lead to the definitions of the thermal velocities v_β, or $v_{\beta\parallel}$ and $v_{\beta\perp}$:

For the case of L waves:

(i) Isotropic Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2}\frac{k_\parallel^2 v_e^2}{\omega_{pe}^2}\right). \tag{6.1}$$

(ii) Isotropic Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2}\frac{\kappa_e}{\kappa_e - 5/2}\frac{k_\parallel^2 v_e^2}{\omega_{pe}^2}\right). \tag{6.2}$$

(iii) Bi-Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2}\frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{6.3}$$

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

(iv) Bi-Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2} \frac{\kappa_e}{\kappa_e - 5/2} \frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{6.4}$$

(v) Product-bi-Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2} \frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{6.5}$$

(vi) Product-bi-Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{3}{2} \frac{\kappa_{e\parallel}}{\kappa_{e\parallel} - 3/2} \frac{k_\parallel^2 v_{e\parallel}^2}{\omega_{pe}^2}\right). \tag{6.6}$$

For the case of S waves:

(i) Isotropic Kappa distribution of type I:

$$\omega^2 \left(1 + 2\frac{\kappa_e - 1/2}{\kappa_e - 3/2} \frac{\omega_{pe}^2}{k_\parallel^2 v_e^2}\right) = \omega_{pi}^2. \tag{6.7}$$

(ii) Isotropic Kappa distribution of type II:

$$\omega^2 \left(1 + 2\frac{\kappa_e - 3/2}{\kappa_e} \frac{\omega_{pe}^2}{k_\parallel^2 v_e^2}\right) = \omega_{pi}^2. \tag{6.8}$$

(iii) Bi-Kappa distribution of type I:

$$\omega^2 \left(1 + 2\frac{\kappa_e - 1/2}{\kappa_e - 3/2} \frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right) = \omega_{pi}^2. \tag{6.9}$$

(iv) Bi-Kappa distribution of type II:

$$\omega^2 \left(1 + 2\frac{\kappa_e - 3/2}{\kappa_e} \frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right) = \omega_{pi}^2. \tag{6.10}$$

(v) Product-bi-Kappa distribution of type I:

$$\omega^2 \left(1 + 2\frac{\kappa_{e\parallel} + 1/2}{\kappa_{e\parallel} - 1/2} \frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right) = \omega_{pi}^2. \tag{6.11}$$

(vi) Product-bi-Kappa distribution of type II:

$$\omega^2 \left(1 + 2\frac{\kappa_{e\parallel} - 1/2}{\kappa_{e\parallel}} \frac{\omega_{pe}^2}{k_\parallel^2 v_{e\parallel}^2}\right) = \omega_{pi}^2. \tag{6.12}$$

It is seen that, given the temperature parameter T_β, distributions of type I, either isotropic or anisotropic, lead to the same form of the dispersion relation for Langmuir waves as the Maxwellian or bi-Maxwellian distributions. Dispersion relations for L waves obtained with distributions of type II feature a dependence on wavenumber which is dependent on κ_β (or $\kappa_{\beta\|}$), and it is seen that the difference relative to the Maxwellian case increases for small values of κ.

In the case of S waves, it is seen that both forms of Kappa distributions which have been considered, type I and type II, lead to dispersion relations which are different from those obtained in the case of Maxwellian or bi-Maxwellian plasmas.

The second approach to the presentation of the results obtained is by emphasis on the averaged squared velocity instead of the T parameter. This second approach seems indicated for comparison with experimental data or with data generated in PIC simulations, since the average squared velocity of the particles can be directly obtained from the observations or from the numerical data generated in the simulations. In terms of the average value of the squared velocity, the dispersion relation which we have obtained can be written as follows.

For the case of L waves:

(i) Isotropic Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.13}$$

(ii) Isotropic Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.14}$$

(iii) Bi-Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.15}$$

(iv) Bi-Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.16}$$

(v) Product-bi-Kappa distribution of type I:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.17}$$

(vi) Product-bi-Kappa distribution of type II:

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + 3\frac{k_\|^2 \langle v_\|^2 \rangle_e}{\omega_{pe}^2}\right). \tag{6.18}$$

For the case of S waves:

(i) Isotropic Kappa distribution of type I:

$$\omega^2\left(1+\frac{\kappa_e-1/2}{\kappa_e-3/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.19}$$

(ii) Isotropic Kappa distribution of type II:

$$\omega^2\left(1+\frac{\kappa_e-3/2}{\kappa_e-5/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.20}$$

(iii) Bi-Kappa distribution of type I:

$$\omega^2\left(1+\frac{\kappa_e-1/2}{\kappa_e-3/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.21}$$

(iv) Bi-Kappa distribution of type II:

$$\omega^2\left(1+\frac{\kappa_e-3/2}{\kappa_e-5/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.22}$$

(v) Product-bi-Kappa distribution of type I:

$$\omega^2\left(1+\frac{\kappa_{e\parallel}+1/2}{\kappa_{e\parallel}-1/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.23}$$

(vi) Product-bi-Kappa distribution of type II:

$$\omega^2\left(1+\frac{\kappa_{e\parallel}-1/2}{\kappa_{e\parallel}-3/2}\frac{\omega_{pe}^2}{k_\parallel^2\langle v_\parallel^2\rangle_e}\right)=\omega_{pi}^2. \tag{6.24}$$

Both for L waves and for S waves, we have taken into account that for isotropic distributions $\langle v^2\rangle=3\langle v_\parallel^2\rangle$.

According to this second approach, it is seen that both distributions of type I and type II lead to behaviour of the dispersion relation for L waves which is similar to that obtained in the Maxwellian case, with the dependence on wavenumber proportional to the average kinetic energy along parallel direction, without explicit dependence on the κ indexes. On the other hand, the dispersion relation for S waves exhibits dependence of the κ indexes both for type I and type II distributions, in the isotropic as well as in the anisotropic case. It is noticed that isotropic Kappa distributions and bi-Kappa distributions of a given type lead to the same form of dispersion relation, but the dispersion relations obtained for type I distributions are different from those obtained for type II distributions. However, the product-bi-Kappa distributions which we have utilized lead to the dispersion relations for S waves which are different for type I and type II distributions, and also different from the dispersion relations obtained for isotropic Kappa or bi-Kappa distributions.

In addition to these findings, which can be observed from the approximated analytical expressions, numerical analysis have shown that the actual and exact numerical solutions of the dispersion relation for L waves show significant difference relative to the approximate analytical solution, and that this difference increases with the decrease of the kappa indexes. Moreover, the numerical results obtained have shown that the dispersion relations for L waves are significantly different, for two different representations of the Kappa distributions which are pervasively used in the literature in recent years.

The dispersion relations for S waves also are significantly different, for the two representations of Kappa distributions which have been considered in the analysis. For the S waves there is also significant difference between the results obtained with the approximate analytical solution and with the exact dispersion relation. However, contrary to what is observed for L waves, the difference between the exact and the expanded dispersion relation for S waves decreases with the decrease of κ.

Acknowledgements

L.F.Z. acknowledges support from CNPq (Brazil), grant no. 304363/2014-6. R.G. acknowledges support from CNPq (Brazil), grants no. 478728/2012-3 and 307626/2015-6. F.J.R.S. acknowledges support from CNPq (Brazil), grant no. 473931/2013-3.

Appendix A. Evaluation of the integrals $J(n, m, h; f_\beta)$ and I in the case of Kappa distribution functions

Here we present details of the evaluation of the integrals which appear in the components of the dispersion relation, for the case in which the distribution functions of plasma particles are given by (2.1). We consider the generic case of a magnetized plasma, and write (1.2) in terms of dimensionless variables,

$$J(n, m, h; f_\beta) = z \int \mathrm{d}^3u \, \frac{u_\parallel^h u_\perp^{2(m-1)} u_\perp L(f_\beta)}{z - nr_\beta - q_\parallel u_\parallel}, \tag{A 1}$$

where the distribution function is also written in terms of dimensionless variables, normalized such that $\int \mathrm{d}^3u f_\beta = 1$. For simplicity of notation, here we will write κ and α instead of κ_β and α_β.

Using the differential operators,

$$\begin{aligned} \mathcal{L}(f_\beta) &= \left(u_\parallel \frac{\partial}{\partial u_\perp} - u_\perp \frac{\partial}{\partial u_\parallel} \right) f_\beta = 0, &\text{(A 2)} \\ L(f_\beta) &= \left[\left(1 - \frac{q_\parallel}{z} u_\parallel \right) \frac{\partial}{\partial u_\perp} + \frac{q_\parallel}{z} u_\perp \frac{\partial}{\partial u_\parallel} \right] f_\beta \\ &= - \left[\left(1 - \frac{q_\parallel}{z} u_\parallel \right) \frac{\kappa+\alpha}{\kappa} \frac{2u_\perp}{u_{\beta,\kappa}^2} \left(1 + \frac{u^2}{\kappa u_{\beta,\kappa}^2} \right)^{-1} \right. \\ &\quad \left. + \frac{q_\parallel}{z} u_\perp \frac{\kappa+\alpha}{\kappa} \frac{2u_\parallel}{u_{\beta,\kappa}^2} \left(1 + \frac{u^2}{\kappa u_{\beta,\kappa}^2} \right)^{-1} \right] f_\beta, &\text{(A 3)} \end{aligned}$$

which leads to

$$L(f_\beta) = - \frac{2u_\perp}{u_{\beta,\kappa}^2} \frac{\kappa+\alpha}{\kappa} \left(1 + \frac{u^2}{\kappa u_{\beta,\kappa}^2} \right)^{-1} f_\beta. \tag{A 4}$$

Therefore, the integral containing operator $\mathcal{L}$ vanishes, and the expression for the $J(n, m, h; f_\beta)$ becomes

$$\frac{2\omega}{u_{\beta,\kappa}^2 v_* k_\parallel}\frac{(2\pi)n_{\beta 0}}{\pi^{3/2}\kappa^{3/2}u_{\beta,\kappa}^3}\frac{\kappa+\alpha}{\kappa}\frac{\Gamma(\kappa+\alpha)}{\Gamma(\kappa+\alpha-3/2)}\int_0^\infty \mathrm{d}u_\perp\, u_\perp^{2m+1} \times\int_{-\infty}^{\infty}\mathrm{d}u_\parallel \frac{u_\parallel^h}{u_\parallel - u_{\parallel,res}}\left(1+\frac{u_\parallel^2}{\kappa u_{\beta,\kappa}^2}+\frac{u_\perp^2}{\kappa u_{\beta,\kappa}^2}\right)^{-(\kappa+\alpha+1)}, \tag{A 5}$$

where $u_{\parallel,res} = (\omega - n\Omega_\beta)/(k_\parallel v_*)$.

By defining $\alpha_\parallel = 1 + u_\parallel^2/(\kappa u_{\beta,\kappa}^2)$, we can write this expression as follows,

$$\frac{2\omega}{u_{\beta,\kappa}^2 v_* k_\parallel}\frac{(2\pi)n_{\beta 0}}{\pi^{3/2}\kappa^{3/2}u_{\beta,\kappa}^3}\frac{\kappa+\alpha}{\kappa}\frac{\Gamma(\kappa+\alpha)}{\Gamma(\kappa+\alpha-3/2)} \times\int_{-\infty}^{\infty}\mathrm{d}u_\parallel \frac{u_\parallel^h}{u_\parallel - u_{\parallel,res}}\left(1+\frac{u_\parallel^2}{\alpha_\parallel\kappa u_{\beta,\kappa}^2}\right)^{-(\kappa+\alpha+1)} \times\int_0^\infty \mathrm{d}u_\perp\, u_\perp^{2m+1}\left(1+\frac{u_\perp^2}{\alpha_\parallel\kappa u_{\beta,\kappa}^2}\right)^{-(\kappa+\alpha+1)}. \tag{A 6}$$

Changing variable in the integrals over $u_\perp$, $t = u_\perp^2/(\alpha_\parallel \kappa u_{\beta,\kappa}^2)$, the integral over $u_\perp$ becomes

$$\frac{1}{2}\alpha_\parallel^{m+1}\kappa^{m+1}u_{\beta,\kappa}^{2(m+1)}\int_0^\infty \mathrm{d}t\frac{t^m}{(1+t)^{\kappa+\alpha+1}}. \tag{A 7}$$

It is seen that we can use the following integral, with appropriate values of z and w,

$$\int_0^\infty \mathrm{d}t\frac{t^{z-1}}{(1+t)^{w+z}} = \frac{\Gamma(z)\Gamma(w)}{\Gamma(w+z)}, \quad (\mathrm{Re}\, z > 0, \mathrm{Re}\, w > 0), \tag{A 8}$$

which leads to the following for the $u_\perp$ integral,

$$\frac{1}{2}\alpha_\parallel^{m+1}\kappa^{m+1}u_{\beta,\kappa}^{2(m+1)}\frac{\Gamma(m+1)\Gamma(\kappa+\alpha-m)}{\Gamma(\kappa+\alpha+1)}. \tag{A 9}$$

Using this result,

$$J(n, m, h; f_\beta) = \frac{z}{q_\parallel}\frac{(2\pi)n_{\beta 0}}{\pi^{3/2}}\kappa^{m-3/2}u_{\beta,\kappa}^{2m-3+h}(m!) \times\frac{\Gamma(\kappa+\alpha-m)}{\Gamma(\kappa+\alpha-3/2)}\int_{-\infty}^{\infty}\mathrm{d}s\,\frac{s^h}{s-\zeta_\beta^n}\left(1+\frac{s^2}{\kappa}\right)^{m-\kappa-\alpha}, \tag{A 10}$$

where $s = u_\parallel/u_{\beta,\kappa}$ and $\zeta_\beta^n = (z - nr_\beta)/(q_\parallel u_{\beta,\kappa})$.

We now introduce the plasma dispersion function of order m, for Kappa distributions,

$$Z_\kappa^{(m)}(\xi) = \frac{1}{\pi^{1/2}}\frac{\Gamma(\kappa)}{\kappa^{1/2}\Gamma(\kappa-1/2)}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{(s-\xi)(1+s^2/\kappa)^{\kappa+m}}, \tag{A 11}$$

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

which can be written in terms of the Gauss hypergeometric function ${}_2F_1(a, b, c, z)$, as in (2.3), for $\kappa > -m - 1/2$.

Let us now consider in some detail the evaluation of $J(n, m, h; f_\beta)$ in the case of $h = 0$,

$$J(n, m, 0; f_\beta) = \frac{z}{q_\parallel}\frac{(2\pi)n_{\beta 0}}{\pi^{3/2}}\kappa^{m-3/2}u_{\beta,\kappa}^{2m-3}(m!) \times \frac{\Gamma(\kappa+\alpha-m)}{\Gamma(\kappa+\alpha-3/2)}\int_{-\infty}^{\infty}\mathrm{d}s\frac{1}{s-\zeta_\beta^n}\left(1+\frac{s^2}{\kappa}\right)^{-(\kappa+\alpha-m)}. \quad (A\,12)$$

The integrand diverges for $\kappa + 1 - m < 0$. For $m < \kappa + 1$, the integral can be evaluated using (A 11) and (2.3), leading to

$$J(n, m, 0; f_\beta) = (2)n_{\beta 0}(\kappa u_{\beta,\kappa}^2)^{m-1}(u_{\beta,\kappa})^0(m!) \times \frac{\Gamma(\kappa+\alpha-m)\Gamma(\kappa-1/2)}{\Gamma(\kappa+\alpha-3/2)\Gamma(\kappa)}\zeta_\beta^0 Z_\kappa^{(\alpha-m)}(\zeta_\beta^n), \quad (A\,13)$$

where $\zeta_\beta^0 = z/(q_\parallel u_{\beta,\kappa})$ and $\zeta_\beta^n = (z - nr_\beta)/(q_\parallel u_{\beta,\kappa})$.

For other values of h the evaluation is similar. For instance, for the cases of $h = 1$ and $h = 2$, one obtains

$$J(n, m, 1; f_\beta) = (2)n_{\beta 0}(\kappa u_{\beta,\kappa}^2)^{m-1}u_{\beta,\kappa}(m!)\zeta_\beta^0\frac{\Gamma(\kappa+\alpha-m)\Gamma(\kappa-1/2)}{\Gamma(\kappa+\alpha-3/2)\Gamma(\kappa)} \times\left(\frac{\Gamma(\kappa)\Gamma(\kappa+\alpha-m-1/2)}{\Gamma(\kappa-1/2)\Gamma(\kappa+\alpha-m)}+\zeta_\beta^n Z_\kappa^{(\alpha-m)}(\zeta_\beta^n)\right), \quad (A\,14)$$

$$J(n, m, 2; f_\beta) = (2)n_{\beta 0}(\kappa u_{\beta,\kappa}^2)^{m-1}u_{\beta,\kappa}^2(m!)\zeta_\beta^0\zeta_\beta^n\frac{\Gamma(\kappa+\alpha-m)\Gamma(\kappa-1/2)}{\Gamma(\kappa+\alpha-3/2)\Gamma(\kappa)} \times\left(\frac{\Gamma(\kappa)\Gamma(\kappa+\alpha-m-1/2)}{\Gamma(\kappa-1/2)\Gamma(\kappa+\alpha-m)}+\zeta_\beta^n Z_\kappa^{(\alpha-m)}(\zeta_\beta^n)\right). \quad (A\,15)$$

We can obtain approximated expressions which are valid when $|\zeta_\beta^n| \gg s$ in the region which is relevant for the integration. We proceed from the equation which appears before (A 11), and then expand the denominator, obtaining

$$J(n, m, h; f_\beta) = -\frac{z}{q_\parallel}\frac{(2\pi)n_{\beta 0}}{\pi^{3/2}}\kappa^{m-3/2}u_{\beta,\kappa}^{2m-3+h}(m!)\frac{\Gamma(\kappa+\alpha-m)}{\Gamma(\kappa+\alpha-3/2)}\frac{1}{\zeta_\beta^n} \times\int_{-\infty}^{\infty}\mathrm{d}s\, s^h\left(1+\frac{s}{\zeta_\beta^n}+\frac{s^2}{(\zeta_\beta^n)^2}+\cdots\right)\left(1+\frac{s^2}{\kappa}\right)^{m-\kappa-\alpha}. \quad (A\,16)$$

Neglecting terms of order $(s/\zeta)^3$, taking into account the parity of the integrands and using (A 8) with $t = s^2/\kappa$, we obtain, for the cases $h = 0$, $h = 1$ and $h = 2$,

$$J(n, m, 0; f_\beta) = -2n_{\beta 0}\kappa^{m-1}u_{\beta,\kappa}^{2m-2}(m!) \times\frac{\zeta_\beta^0}{\zeta_\beta^n}\frac{\Gamma(\kappa+\alpha-m-3/2)}{\Gamma(\kappa+\alpha-3/2)}\left(\kappa+\alpha-m-\frac{3}{2}+\frac{1}{(\zeta_\beta^n)^2}\frac{\kappa}{2}\right), \quad (A\,17)$$

https://doi.org/10.1017/S0022377817000733 Published online by Cambridge University Press

$$J(n,m,1;f_\beta) = -n_{\beta0}\kappa^m u_{\beta,\kappa}^{2m-1}(m!)\frac{\zeta_\beta^0}{(\zeta_\beta^n)^2}\frac{\Gamma(\kappa+\alpha-m-3/2)}{\Gamma(\kappa+\alpha-3/2)}, \tag{A 18}$$

$$J(n,m,2;f_\beta) = -n_{\beta0}\kappa^m u_{\beta,\kappa}^{2m}(m!) \times\frac{\zeta_\beta^0}{\zeta_\beta^n}\frac{\Gamma(\kappa+\alpha-m-5/2)}{\Gamma(\kappa+\alpha-3/2)}\left(\kappa+\alpha-m-\frac{5}{2}+\frac{1}{(\zeta_\beta^n)^2}\frac{3\kappa}{2}\right). \tag{A 19}$$

An approximation valid for small values of $|\zeta_\beta^n|$ can be obtained by the following expansion of the integrand in the expression which appears before (A 11)

$$J(n,m,h;f_\beta) = \frac{z}{q_\parallel}\frac{(2\pi)n_{\beta0}}{\pi^{3/2}}\kappa^{m-3/2}u_\beta^{2m-3+h}(m!)\frac{\Gamma(\kappa+\alpha-m)}{\Gamma(\kappa+\alpha-3/2)} \times\int_{-\infty}^{\infty}\mathrm{d}s\, s^{h-1}\left(1+\frac{s^2}{\kappa}\right)^{m-\kappa-\alpha}\left(1+\frac{\hat{\zeta}_\beta^n}{s}+\frac{(\hat{\zeta}_\beta^n)^2}{s^2}+\cdots\right). \tag{A 20}$$

Neglecting terms of order $(\zeta/s)^3$, taking into account the parity of the integrands and using (A 8) with $t=s^2/\kappa$, we obtain, for the cases $h=0$, $h=1$ and $h=2$,

$$J(n,m,0;f_\beta) = -(4)n_{\beta0}\kappa^{m-2}u_\beta^{2m-2}(m!)\frac{\Gamma(\kappa+\alpha-m+1/2)}{\Gamma(\kappa+\alpha-3/2)}\zeta_\beta^0\hat{\zeta}_\beta^n \tag{A 21}$$

$$J(n,m,1;f_\beta) = (2)n_{\beta0}\kappa^{m-1/2}u_\beta^{2m-1}(m!)\frac{\Gamma(\kappa+\alpha-m-1/2)}{\Gamma(\kappa+\alpha-3/2)} \times\zeta_\beta^0\left[1-2(\hat{\zeta}_\beta^n)^2\frac{(\kappa+\alpha-m-1/2)}{\kappa}\right] \tag{A 22}$$

$$J(n,m,2;f_\beta) = (2)n_{\beta0}\kappa^{m-1}u_\beta^{2m}(m!)\frac{\Gamma(\kappa+\alpha-m-1/2)}{\Gamma(\kappa+\alpha-3/2)}\zeta_\beta^0\hat{\zeta}_\beta^n. \tag{A 23}$$

REFERENCES

ABDUL, R. F. & MACE, R. L. 2014 A method to generate kappa distributed random deviates for particle-in-cell simulations. *Comput. Phys. Commun.* **185** (10), 2383–2386.

GAELZER, R. & ZIEBELL, L. F. 2014 The dispersion relations of dispersive Alfvén waves in superthermal plasmas. *J. Geophys. Res.* **119** (12), 9334–9356.

GAELZER, R. & ZIEBELL, L. F. 2016 Obliquely propagating electromagnetic waves in magnetized kappa plasmas. *Phys. Plasmas* **23** (2), 022110.

GAELZER, R., ZIEBELL, L. F. & MENESES, A. R. 2016 The general dielectric tensor for bi-kappa magnetized plasmas. *Phys. Plasmas* **23** (6), 062108.

HAPGOOD, M., PERRY, C., DAVIES, J. & DENTON, M. 2011 The role of suprathermal particle measurements in CrossScale studies of collisionless plasma processes. *Planet. Space Sci.* **59**, 618–629.

HASEGAWA, A., MIMA, K. & DUONGVAN, M. 1985 Plasma distribution function in a superthermal radiation-field. *Phys. Rev. Lett.* **54** (24), 2608–2610.

HAU, L. N. & FU, W. Z. 2007 Mathematical and physical aspects of Kappa velocity distribution. *Phys. Plasmas* **14** (11), 110702.

HAU, L. N., FU, W. Z. & CHUANG, S. H. 2009 Response to 'Comment on "Mathematical and physical aspects of Kappa velocity distribution"' [Phys. Plasmas 16, 094701 (2009)]. *Phys. Plasmas* **16** (9), 094702.

HELLBERG, M. & MACE, R. 2002 Generalized plasma dispersion function for a plasma with a kappa-Maxwellian velocity distribution. *Phys. Plasmas* **9** (5, 1), 1495–1504.

HELLBERG, M. A., MACE, R. L., BALUKU, T. K., KOURAKIS, I. & SAINI, N. S. 2009 Comment on 'Mathematical and physical aspects of Kappa velocity distribution' [Phys. Plasmas 14, 110702 (2007)]. *Phys. Plasmas* **16** (9), 094701.

HELLBERG, M., MACE, R. & CATTAERT, T. 2006 Effects of superthermal particles on waves in magnetized space plasmas. *Space Sci. Rev.* **121**, 127–139.

LAZAR, M. 2012 The electromagnetic ion-cyclotron instability in bi-Kappa distributed plasmas. *Astron. Astrophys.* **547**, A94.

LAZAR, M., FICHTNER, H. & YOON, P. H. 2016 On the interpretation and applicability of κ-distributions. *Astron. Astrophys.* **589**, A39.

LAZAR, M., PIERRARD, V., POEDTS, S. & SCHLICKEISER, R. 2012 Modeling space plasma dynamics with anisotropic Kappa distributions. *Astrophys. Space Sci. Proc.* **33**, 97–107.

LAZAR, M. & POEDTS, S. 2009*a* Firehose instability in space plasmas with bi-kappa distributions. *Astron. Astrophys.* **494**, 311–315.

LAZAR, M. & POEDTS, S. 2009*b* Limits for the firehose instability in space plasmas. *Solar Phys.* **258**, 119–128.

LAZAR, M. & POEDTS, S. 2014 Instability of the parallel electromagnetic modes in Kappa distributed plasmas – II. Electromagnetic ion-cyclotron modes. *Mon. Not. R. Astron. Soc.* **437** (1), 641–648.

LAZAR, M., POEDTS, S. & SCHLICKEISER, R. 2011 Proton firehose instability in bi-Kappa distributed plasmas. *Astron. Astrophys.* **534**, A116.

LEUBNER, M. P. 2002 A nonextensive entropy approach to Kappa-distributions. *Astrophys. Space Sci.* **282** (3), 573–579.

LEUBNER, M. P. 2004 Core-halo distribution functions: a natural equilibrium state in generalized thermostatistics. *Astrophys. J.* **604**, 469–478.

LEUBNER, M. & SCHUPFER, N. 2000 Mirror instability thresholds in suprathermal space plasmas. *J. Geophys. Res.* **105** (A12), 27387–27391.

LEUBNER, M. & SCHUPFER, N. 2001 A general kinetic mirror instability criterion for space applications. *J. Geophys. Res.* **106** (A7), 12993–12998.

LI, B. & CAIRNS, I. H. 2014 Fundamental emission of type III bursts produced in non-Maxwellian coronal plasmas with Kappa-distributed background particles. *Solar Phys.* **289**, 951–976.

LIVADIOTIS, G. 2015 Introduction to special section on origins and properties of Kappa distributions: statistical background and properties of Kappa distributions in space plasmas. *J. Geophys. Res.* **120** (3), 1607–1619.

LIVADIOTIS, G. & MCCOMAS, D. J. 2009 Beyond kappa distributions: exploiting Tsallis statistical mechanics in space plasmas. *J. Geophys. Res.* **114**, A11105.

LIVADIOTIS, G. & MCCOMAS, D. J. 2011 Invariant Kappa distribution in space plasmas out of equilibrium. *Astrophys. J.* **741** (2), 88.

LIVADIOTIS, G. & MCCOMAS, D. J. 2013 Understanding Kappa distributions: a toolbox for space science and astrophysics. *Space Sci. Rev.* **175**, 183–214.

MACE, R. L. 2003 A Gordeyev integral for electrostatic waves in a magnetized plasma with a kappa velocity distribution. *Phys. Plasmas* **10**, 2181.

MACE, R. L. & HELLBERG, M. A. 1995 A dispersion function for plasmas containing superthermal particles. *Phys. Plasmas* **2** (6), 2098–2109.

MACE, R. L. & HELLBERG, M. A. 2009 A new formulation and simplified derivation of the dispersion function for a plasma with a kappa velocity distribution. *Phys. Plasmas* **16** (7), 072113.

MACE, R. & HELLBERG, M. 2003 Generalized Langmuir waves in a magnetized plasma with a Maxwellian–Lorentzian distribution. *Phys. Plasmas* **10** (1), 21–28.

MAKSIMOVIC, M., PIERRARD, V. & RILEY, P. 1997 Ulysses electron distributions fitted with Kappa functions. *Geophys. Res. Lett.* **24** (9), 1151–1154.

OLBERT, S. 1968 Summary of experimental results from MIT detector on IMP-1. In *Physics of the Magnetosphere: Based Upon the Proceedings of the Conference held at Boston College June 19–28, 1967* (ed. R. L. Carovillano, J. F. McClay & H. R. Radoski), pp. 641–659. Springer.

OMURA, Y. & MATSUMOTO, H. 1993 *Computer Space Plasma Physics: Simulation Techniques and Software*. Terra Scientific Publishing Company.

PODESTA, J. J. 2015 Small-amplitude Langmuir pulse excited by a planar grid electrode in a flowing plasma. *J. Plasma Phys.* **8** (5), 905810503.

DOS SANTOS, M. S., ZIEBELL, L. F. & GAELZER, R. 2014 Ion firehose instability in plasmas with plasma particles described by product bi-kappa distributions. *Phys. Plasmas* **21**, 112102.

DOS SANTOS, M. S., ZIEBELL, L. F. & GAELZER, R. 2015 Ion-cyclotron instability in plasmas described by product-bi-kappa distributions. *Phys. Plasmas* **22** (12), 122107.

DOS SANTOS, M. S., ZIEBELL, L. F. & GAELZER, R. 2016 Ion firehose instability in a dusty plasma considering product-bi-kappa distributions for the plasma particles. *Phys. Plasmas* **23** (1), 013705.

SILVA, R., PLASTINO, A. & LIMA, J. 1998 A Maxwellian path to the q-nonextensive velocity distribution function. *Phys. Lett.* A **249** (5–6), 401–408.

SUMMERS, D. & THORNE, R. M. 1991 The modified plasma dispersion function. *Phys. Fluids* B **3** (8), 1835–1847.

THORNE, R. M. & SUMMERS, D. 1991 Landau damping in space plasmas. *Phys. Fluids* B **3** (8), 2117–2123.

TSALLIS, C. 1988 Possible generalization of Boltzmann–Gibbs statistics. *J. Stat. Phys.* **52** (1–2), 479–487.

TSALLIS, C., MENDES, R. & PLASTINO, A. 1998 The role of constraints within generalized nonextensive statistics. *Phys.* A **261** (3–4), 534–554.

VASYLIUNAS, V. M. 1968 A survey of low-energy electrons in evening sector of magnetosphere with OGO 1 and OGO 3. *J. Geophys. Res.* **73** (9), 2839.

ZIEBELL, L. F. & GAELZER, R. 2017 On the influence of the shape of kappa distributions of ions and electrons on the ion-cyclotron instability. *Phys. Plasmas* **24** (9), 102108.