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Phragmén–Lindelöf theorems in cylinders

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In unbounded cylindrical domains, new Phragmén–Lindelöf theorems 'at infinity' are established for solutions of Dirichlet problems for elliptic equations whose operators belong to a large class to which the existing literature does not apply.

1. Introduction

Theorems of Phragmén–Lindelöf type usually yield the asymptotic behaviour of solutions of boundary-value problems for second-order elliptic or parabolic operators at points of the boundary of the domain or at points at infinity. (Examples of other types of 'Phragmén–Lindelöf' articles (e.g. [10, 16, 17]) certainly occur in the literature.) For fixed positive integers N, m and $n = N + m$, we assume throughout this paper that Ω is an unbounded open subset of $\mathbb{R}^n$ such that, for some fixed $M > 0$, Ω is contained in the cylinder

$$C_M^m = \left\{ X = (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{m} x_{k+N}^2 < M^2 \right\}.$$

Our goal is to obtain a *general Phragmén–Lindelöf theorem at infinity* for domains contained in such cylinders.

We consider operators of the form

$$Qu(X) = \sum_{i,j=1}^{n} a_{i,j}(X, u(X), Du(X)) D_{ij} u(X) + b(X, u(X), Du(X)), \tag{1.1}$$

where $(a_{i,j}(X, t, P))$ is a positive semi-definite matrix in which each entry is in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and b is a function in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. For $\phi \in C^0(\mathbb{R}^n)$, we will consider the Dirichlet problem

$$Qf = 0 \text{ in } \Omega \quad \text{and} \quad f = \phi \text{ on } \partial\Omega. \tag{1.2}$$

We first prove a Phragmén–Lindelöf theorem at infinity for operators Q with $b \equiv 0$ satisfying (2.1) and (2.5) and solutions $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (1.2) (see theorem 2.3). We then prove a Phragmén–Lindelöf theorem for bounded solutions $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (1.2) when Q satisfies (2.1) and (2.5) and some conditions *at infinity* (see theorem 2.7). The proofs of these results use barrier functions related to those constructed in [11–15] to obtain Phragmén–Lindelöf theorems. The results of [12–14] (see also [15]) deal with solutions defined in slabs

$$S_M = \{X = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_n| < M\};$$

the primary difference between these papers and the present one involves the hypotheses on the top-order coefficients of the operator. In particular, [12, eqn (14)] and [13, eqn (2.6)] require $a_{n,n}$ to have a positive lower bound while (2.5) requires $\sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}$ and (2.16) requires $\sum_{k=1}^{m} a_{k+N,k+N}$ to have a positive lower bound; this allows us to handle degenerate operators in which $a_{n,n}$ can be zero (e.g. example 3.4). Even when $m = 1$, and so $C_M^m = S_M$, the difference between corollaries 3.2 and 4.2 of [12] illustrate that different (but related) classes of operators are involved (see example 3.3).

In certain areas of, for example, elasticity theory (e.g. [5–7]), domains contained in cylinders occupy a special place based on their occurrence in applications (e.g. [2,3, 8,19]). In the study of anti-plane shear deformations of nonlinearly elastic materials contained in cylinders, the convergence of solutions to specified limiting values (e.g. 0) may be one of the assumptions in the problem (e.g. [9]); our examples in § 3 illustrate that, for second-order elliptic operators, obtaining Phragmén–Lindelöf conclusions (and, perhaps, later obtaining 'spatial decay' estimates) may require new theorems such as ours.

2. Main results

We will now assume that the coefficients of Q have been normalized so that

$$\sum_{i=1}^{n} a_{i,i}(X, z, P) = 1 \quad \text{for } (X, z, P) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

We will write elements $X = (x_1, \ldots, x_n)$ as $(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x} = (x_1, \ldots, x_N)$ and $\boldsymbol{y} = (x_{N+1}, \ldots, x_n)$. We shall set $y_k = x_{k+N}$ for $k = 1, \ldots, m$, so that $\boldsymbol{y} = (y_1, \ldots, y_m)$.

We shall assume the following hypothesis on the behaviour of the boundary data ϕ.

ASSUMPTION 2.1. There is a function $\Phi \in C^0(S^{N-1})$ such that $\phi(r\omega, \boldsymbol{y}) \to \Phi(\omega)$ as $r \to \infty$ uniformly for $\omega \in S^{N-1}$ and $|\boldsymbol{y}| \leqslant M$.

The assumptions on the operator Q will be described by the behaviour of the following functions.

DEFINITION 2.2. For an operator Q in (1.1) satisfying (2.1), let

$$\epsilon(X, z, P) = \epsilon(\boldsymbol{x}, \boldsymbol{y}, z, P) = \sum_{i,j=1}^{n} a_{i,j}(X, z, P) P_i P_j \tag{2.2}$$

and

$$\gamma(X, z, P) = \sum_{k=1}^{m} a_{k+N,k+N}(X, z, P) \tag{2.3}$$

for $X, P \in \mathbb{R}^n$, $z \in \mathbb{R}$. For each $\nu \in S^{m-1}$, $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$ and $z, t, q \in \mathbb{R}$, let

$$\epsilon_\nu^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \frac{\sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}}{1 + \epsilon - \sum_{k=1}^{m} a_{k+N,k+N} + 4q^{-2} \sum_{i=1}^{N} \sum_{k=1}^{m} p_i \nu_k a_{i,k+N}}, \tag{2.4}$$

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where

$$a_{i,j} = a_{i,j}\left(\boldsymbol{x}, t\nu, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right), \quad 1 \leqslant i, j \leqslant n, \quad \text{and} \quad \epsilon = \epsilon\left(\boldsymbol{x}, t\nu, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right).$$

THEOREM 2.3. *Let $\Omega \subset C_M^m$. Suppose that:*

(1) *$f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1.2);*

(2) *Q satisfies (2.1) and $b \equiv 0$;*

(3) *there exist $L > 0$, and a positive continuous function σ on $[1, \infty)$ such that, for each $\nu \in S^{m-1}$,*

$$\epsilon_\nu^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + |q|^2) \tag{2.5}$$

whenever $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$ and $z, t, q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + |q|^2 \geqslant 1$, $|t| \leqslant M$ and $|q| > 0$;

(4) *ϕ satisfies assumption 2.1.*

Then

$$\lim_{j \to \infty} f(\boldsymbol{x}_j, \boldsymbol{y}_j) = \Phi(\omega) \tag{2.6}$$

uniformly for $\omega \in S^{N-1}$ and sequences $\{(\boldsymbol{x}_j, \boldsymbol{y}_j)\}$ in $\bar{\Omega}$ such that $|\boldsymbol{x}_j| \to \infty$ and $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ as $j \to \infty$.

Let $T(\Omega)$ represent the set of directions $\eta \in S^{n-1}$ at infinity of Ω; i.e.

$$T(\Omega) = \bigcap_{L=1}^{\infty} \overline{\bigcup_{r \geqslant L} \{\eta \in S^{n-1} : r\eta \in \Omega\}} \subset S^{N-1} \times \{\boldsymbol{0}\}. \tag{2.7}$$

Let $B(M) = \{\boldsymbol{y} \in \mathbb{R}^m : |\boldsymbol{y}| \leqslant M\}$. For the direction $\eta = (\omega, \boldsymbol{0}) \in T(\Omega)$, let $B_\eta(\Omega)$ denote the set of $\boldsymbol{y} \in B(M)$ for which there exists a sequence $\{\boldsymbol{x}_i\}$ in $\mathbb{R}^N$ which satisfies

$$\lim_{i \to \infty} |\boldsymbol{x}_i| = \infty, \quad \lim_{i \to \infty} \frac{\boldsymbol{x}_i}{|\boldsymbol{x}_i|} = \omega, \quad (\boldsymbol{x}_i, \boldsymbol{y}) \in \bar{\Omega} \text{ for } i = 1, 2, \dots,$$

and let $B_\eta^0(\Omega)$ denote its interior; if, for example, $\Omega = U \times V$ with $U \subset \mathbb{R}^N$, $V \subset \mathbb{R}^m$ and $\omega \in T(U)$, then $B_\eta(\Omega) = \bar{V}$ and $B_\eta^0(\Omega) = V$.

For $\eta = (\omega, \boldsymbol{0}) \in T(\Omega)$, $\omega \in S^{N-1}$, consider the following assumptions.

ASSUMPTION 2.4. For some open subset O of S^{N-1} with $\omega \in O$, there exist

$$A_{k,l} \in C^0(O \times B_\eta(\Omega) \times \mathbb{R}^{m+1}) \quad \text{and} \quad E \in C^0(O \times B_\eta(\Omega) \times \mathbb{R}^{m+1})$$

such that $A_{k,l}((\boldsymbol{x}/|\boldsymbol{x}|), \boldsymbol{y}, z, \boldsymbol{q})$ is independent of z, $E((\boldsymbol{x}/|\boldsymbol{x}|), \boldsymbol{y}, z, \boldsymbol{q})$ is non-increasing in z,

$$\frac{a_{k+N,l+N}(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})}{\gamma(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})} \to A_{k,l}(\nu, \boldsymbol{y}, z, \boldsymbol{q}) \tag{2.8}$$

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and

$$\frac{b(\boldsymbol{x},\boldsymbol{y},z,\boldsymbol{p},\boldsymbol{q})}{\gamma(\boldsymbol{x},\boldsymbol{y},z,\boldsymbol{p},\boldsymbol{q})} \to E(\nu,\boldsymbol{y},z,\boldsymbol{q}) \tag{2.9}$$

as $|\boldsymbol{x}| \to \infty$ with $\boldsymbol{x}/|\boldsymbol{x}| \to \nu$ and $|\boldsymbol{p}| \to 0$ uniformly for $\boldsymbol{y} \in B_\eta(\Omega)$, $\nu \in O$, $z \in \mathbb{R}$ and $\boldsymbol{q} \in \mathbb{R}^m$ when $1 \leqslant k,l \leqslant m$.

Let Q_∞ denote the operator defined on $C^2(B^0_\eta(\Omega))$ by

$$Q_\infty u(\boldsymbol{y}) = \sum_{k,l=1}^{m} A_{k,l}(\omega,\boldsymbol{y},u,D_{\boldsymbol{y}}u)\frac{\partial^2 u}{\partial y_k \partial y_l} + E(\omega,\boldsymbol{y},u,D_{\boldsymbol{y}}u). \tag{2.10}$$

ASSUMPTION 2.5. There exists a function $k : B_\eta(\Omega) \times T(\Omega) \to \mathbb{R}$ with $k(\cdot,\omega) \in C^2(B^0_\eta(\Omega)) \cap C^0(B_\eta(\Omega))$ such that

$$\phi(\boldsymbol{x},\boldsymbol{y}) \to k(\boldsymbol{y},\omega) \tag{2.11}$$

uniformly as $|\boldsymbol{x}| \to \infty$ and $\boldsymbol{x}/|\boldsymbol{x}| \to \omega$ for $(\boldsymbol{x},\boldsymbol{y}) \in \partial\Omega$ and

$$Q_\infty(k(\cdot,\omega)) = 0 \text{ in } B^0_\eta(\Omega).$$

ASSUMPTION 2.6. For each $\alpha > 0$, there exist $\delta = \delta_{\alpha,\omega} > 0$ and functions k_1 and k_2 in $C^1(B_\eta(\Omega)) \cap C^2(B^0_\eta(\Omega))$ such that

$$|k_1(\boldsymbol{y}) - k(\boldsymbol{y},\omega)| \leqslant \alpha, \quad \boldsymbol{y} \in B_\eta(\Omega), \tag{2.12}$$

$$|k_2(\boldsymbol{y}) - k(\boldsymbol{y},\omega)| \leqslant \alpha, \quad \boldsymbol{y} \in B_\eta(\Omega), \tag{2.13}$$

$$Q_\infty(k_1)(\boldsymbol{y}) \geqslant \delta, \quad \boldsymbol{y} \in B^0_\eta(\Omega), \tag{2.14}$$

and

$$Q_\infty(k_2)(\boldsymbol{y}) \leqslant -\delta, \quad \boldsymbol{y} \in B^0_\eta(\Omega). \tag{2.15}$$

THEOREM 2.7. *Let $\Omega \subset C^m_M$ and $\eta = (\omega,\mathbf{0}) \in T(\Omega)$. Suppose that*

(1) *$f \in C^2(\Omega) \cap C^0(\bar{\Omega}) \cap L^\infty(\Omega)$ satisfies (1.2);*

(2) *assumptions 2.4, 2.5 and 2.6 are satisfied for η;*

(3) *there exist $L \geqslant 0$ and a positive continuous function σ on $[1,\infty)$ such that*

$$\gamma(\boldsymbol{x},\boldsymbol{y},z,\boldsymbol{p},\boldsymbol{q}) \geqslant \sigma(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2) \tag{2.16}$$

whenever $\boldsymbol{x},\boldsymbol{p} \in \mathbb{R}^N$, $z \in \mathbb{R}$, $\boldsymbol{y},\boldsymbol{q} \in \mathbb{R}^m$ with $|\boldsymbol{x}| \geqslant L$ and $\boldsymbol{y} \in B^0_\eta(\Omega)$;

(4) *Q satisfies (2.1).*

Then

$$\lim_{j\to\infty} |f(\boldsymbol{x}_j,\boldsymbol{y}_j) - k(\boldsymbol{y}_j,\omega)| = 0 \tag{2.17}$$

uniformly for sequences $\{(\boldsymbol{x}_j,\boldsymbol{y}_j)\}$ in $\bar{\Omega}$ with $|\boldsymbol{x}_j| \to \infty$ and $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ as $j \to \infty$.

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3. Operators and examples

Let Q be an operator given by (1.1) and satisfying (2.1). We will say that Q is in our class of operators, denoted C, if there exist $L > 0$ and a positive continuous function σ on $[1, \infty)$ such that (2.5) holds for each $\nu \in S^{m-1}$, $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$ and $z, t, q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + |q|^2 \geqslant 1$, $|t| \leqslant M$ and $|q| > 0$.

Let us define a new operator $\tilde{Q}$ associated with Q by setting

$$\tilde{Q}u(X) = \sum_{i,j=1}^{n} \tilde{a}_{i,j}(X, u, Du) D_{ij} u,$$

where $\tilde{a}_{i,j} = -a_{i,j}$ if $1 \leqslant i \leqslant N$ and $N+1 \leqslant j \leqslant n$ (or $N+1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant N$) and $\tilde{a}_{i,j} = a_{i,j}$ otherwise. We shall require the following definition.

Definition 3.1 (Bernstein [1]; Serrin [18, p. 425]). Equation (1.1) has *genre* λ if and only if it satisfies (2.1) and there are positive constants μ_1 and μ_2 such that, for $|P| \geqslant 1$, $X \in \Omega$, $t \in R$, $P \in R^n$,

$$\mu_1 |P|^{2-\lambda} \leqslant \epsilon(X, t, P) \leqslant \mu_2 |P|^{2-\lambda}. \tag{3.1}$$

We recall that uniformly elliptic operators have genre 0, while prescribed mean curvature operators have genre 2. Denote by C_1 the set of operators Q given by (1.1) which satisfy (2.1) and the conditions that $\tilde{Q}$ has genre greater than or equal to 2 and there exist $L \geqslant 0$ and a positive continuous function σ_1 on $[1, \infty)$ such that

$$\sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}\left(\boldsymbol{x}, t\nu, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right) \geqslant \sigma_1(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2) \tag{3.2}$$

holds whenever $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$, $z, t, q \in \mathbb{R}$, $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $0 \leqslant t \leqslant M$ and $|q| > 0$. Now

$$\epsilon_\nu^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \frac{\sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}}{1 + \tilde{\epsilon} - \sum_{k=1}^{m} a_{k+N,k+N}},$$

where

$$a_{i,j} = a_{i,j}\left(\boldsymbol{x}, t\nu, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right), \quad 1 \leqslant i, j \leqslant n$$

and $\tilde{\epsilon} = \tilde{\epsilon}(\boldsymbol{x}, t\nu, z, -\boldsymbol{p}/q, \nu/q)$ is the ϵ-invariant for $\tilde{Q}$. Since $\tilde{Q}$ has genre $\gamma \geqslant 2$, there exist μ_1 and μ_2 with $0 < \mu_1 \leqslant \mu_2$ such that $\mu_1 |P|^{2-\gamma} \leqslant \tilde{\epsilon}(X, z, P) \leqslant \mu_2 |P|^{2-\gamma}$ for $|P| \geqslant 1$. Then the proof of [12, corollary 4.2] shows $C_1 \subset C$. Theorem 2.3 can be applied to operators in C_1 to yield the following corollary.

Corollary 3.2. *Suppose $Q \in C_1$, ϕ satisfies assumption 2.1 and $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Qf = 0$ in Ω and $f = \phi$ on $\partial\Omega$. Then*

$$\limsup_{|\boldsymbol{x}| \to \infty, (\boldsymbol{x}, \boldsymbol{y}) \in \Omega} \left| f(\boldsymbol{x}, \boldsymbol{y}) - \Phi\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right| = 0.$$

Example 3.3. Let $m = N = 1$, $n = 2$, $\Omega = (-1, 1) \times (0, \infty)$ and Q be the 'false minimal surface operator' (e.g. [18, § 11]), normalized so (2.1) holds. Then $\tilde{Q}$ is the minimal surface operator and so $Q \in C_1$. From corollary 3.2, we see, for example, that if υ is any real constant and $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Qf = 0$ in the strip Ω and $f(x, \pm 1) \to \upsilon$ as $x \to \infty$, then, without any *a priori* assumptions about the growth of $f(x, y)$ as $x \to \infty$, we have $f(x, y) \to \upsilon$ as $x \to \infty$, uniformly for $|y| \leqslant 1$.

EXAMPLE 3.4. Let $N = 1$, $m = 2$, $M = 3$,

$$U_A = \{(x_2, x_3) \in \mathbb{R}^2 : 1 < x_2^2 + x_3^2 < 9\},$$
$$U_B = \{(x_2, x_3) \in U_A : x_2 > 0, \tfrac{1}{\sqrt{3}}x_2 < x_3 < \sqrt{3}x_2\}$$

and

$$U_C = U_A \setminus \bar{U}_B.$$

Let U be one of these three sets and define $\Omega = \mathbb{R} \times U$. Let Q be the operator given by

$$Qu(X) \stackrel{\text{def}}{=} \sum_{i,j=1}^{3} a_{i,j}(X, Du(X)) D_{i,j} u(X),$$

where

$$a_{1,1}(X, P) = \frac{1 + p_2^2 + p_3^2}{2 + 2|P|^2}, \qquad a_{1,2}(X, P) = \frac{p_1 p_2}{2 + 2|P|^2},$$
$$a_{2,2}(X, P) = \frac{x_2^2/(x_2^2 + x_3^2) + p_1^2 + p_3^2}{2 + 2|P|^2}, \qquad a_{3,3}(X, P) = \frac{x_3^2/(x_2^2 + x_3^2) + p_1^2 + p_2^2}{2 + 2|P|^2},$$
$$a_{1,3}(X, P) = \frac{p_1 p_3}{2 + 2|P|^2}, \qquad a_{2,3}(X, P) = \frac{x_2 x_3/(x_2^2 + x_3^2) - p_2 p_3}{2 + 2|P|^2},$$

for $X = (x_1, x_2, x_3) \in \bar{\Omega}$ and $P = (p_1, p_2, p_3) \in \mathbb{R}^3$. Note that Q is a quasi-linear operator which is degenerate elliptic (e.g. $a_{3,3}(x_1, \pm 2, 0, 0, 0, p_3) = 0$). Note also that

$$\epsilon(X, z, P) = \frac{(x_2 p_2 + x_3 p_3)^2/(x_2^2 + x_3^2) + p_1^2 + 4p_1^2 p_2^2 + 4p_1^2 p_3^2}{2 + 2|P|^2},$$
$$\epsilon_\nu^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \frac{p_1^2 + q^2}{2 + p_1^2 + q^2}, \qquad \gamma(X, z, P) = \frac{1 + 2p_1^2 + p_2^2 + p_3^2}{2 + 2|P|^2}$$

and

$$Q_\infty u = \sum_{k,l=1}^{2} A_{k,l}(\boldsymbol{y}, u, D_{\boldsymbol{y}} u) \frac{\partial^2 u}{\partial y_k \partial y_l},$$

where

$$A_{1,1}(\boldsymbol{y}, z, \boldsymbol{q}) = \frac{y_1^2/(y_1^2 + y_2^2) + q_2^2}{1 + q_1^2 + q_2^2},$$
$$A_{1,2}(\boldsymbol{y}, z, \boldsymbol{q}) = \frac{y_1 y_2/(y_1^2 + y_2^2) - q_1 q_2}{1 + q_1^2 + q_2^2}$$

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and

$$A_{2,2}(\boldsymbol{y}, z, \boldsymbol{q}) = \frac{y_2^2/(y_1^2 + y_2^2) + q_1^2}{1 + q_1^2 + q_2^2}.$$

Suppose firstly that $\upsilon \in \mathbb{R}$, $\phi \in C^0(\partial\Omega)$ with $\lim_{x_1\to+\infty} \phi(x_1, y_1, y_2) = \upsilon$ when $(x_1, y_1, y_2) \in \partial\Omega$ and $f \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ is any function satisfying $Qf = 0$ in Ω and $f = \phi$ on $\partial\Omega$. Then theorem 2.3 implies that

$$\lim_{x_1\to+\infty} f(x_1, y_1, y_2) = \upsilon \quad \text{when } (y_1, y_2) \in \bar{U}.$$

Suppose next that $\Omega = \mathbb{R} \times U_C$ and $\lim_{x_1\to+\infty} \phi(x_1, y_1, y_2) = k(y_1, y_2)$, where $k(y_1, y_2) = \arg(y_1 + \mathrm{i}y_2) \in (\frac{1}{4}\pi, \frac{9}{4}\pi)$. Let $f \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a bounded function satisfying $Qf = 0$ in Ω and $f = \phi$ on $\partial\Omega$. Then theorem 2.7 implies that

$$\lim_{x_1\to+\infty} f(x_1, y_1, y_2) = k(y_1, y_2) \quad \text{when } (y_1, y_2) \in \bar{U}_C.$$

(The graph of $k(y_1, y_2)$ is a piece of a helicoid and $Q_\infty k = 0$. The functions $k_1(\boldsymbol{y}) = k(\boldsymbol{y}) + \lambda|\boldsymbol{y}|^2$ and $k_2(\boldsymbol{y}) = k(\boldsymbol{y}) - \lambda|\boldsymbol{y}|^2$ for $\lambda > 0$ sufficiently small satisfy assumption 2.6.)

Suppose finally that $\Omega = \mathbb{R} \times U_B$ and $\lim_{x_1\to+\infty} \phi(x_1, y_1, y_2) = k(y_1, y_2)$, where $k(y_1, y_2) = \ln(y_1) - \ln(y_2)$. Let $f \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a bounded function satisfying $Qf = 0$ in Ω and $f = \phi$ on $\partial\Omega$. Then theorem 2.7 implies that

$$\lim_{x_1\to+\infty} f(x_1, y_1, y_2) = k(y_1, y_2) \quad \text{when } (y_1, y_2) \in \bar{U}_B.$$

(Note that $Q_\infty k = 0$ and the functions $k_1(\boldsymbol{y}) = k(\boldsymbol{y}) + \lambda|\boldsymbol{y}|^2$ and $k_2(\boldsymbol{y}) = k(\boldsymbol{y}) - \lambda|\boldsymbol{y}|^2$ for $\lambda > 0$ sufficiently small satisfy assumption 2.6.)

The conclusions in example 3.4 do not follow from existing results in the literature, to the best of our knowledge. Even though Q is degenerate, it has a special structure for which a ‘strong’ (i.e. without *a priori* growth or boundedness hypotheses) Phragmén–Lindelöf theorem in a cylinder holds. This type of limiting behaviour of solutions does not always hold, even for ‘better’ (i.e. non-degenerate) elliptic operators, as the following example illustrates.

Example 3.5. Let $\upsilon \in \mathbb{R}$ and consider the bounded solutions $g \equiv \upsilon$ and

$$h(x_1, x_2, x_3) = \left(1 + \frac{2}{\ln x_1}\right) \cos x_2 \cos x_3 + \upsilon$$

of Dirichlet problems for the elliptic linear operator L,

$$Lu(x_1, x_2, x_3) = \frac{(x_1 \ln x_1)^2}{2 + (x_1 \ln x_1)^2} D_{11}u + \frac{1}{2 + (x_1 \ln x_1)^2}(D_{22}u + D_{33}u),$$

in the domain $U = \{(x_1, x_2, x_3) : x_1 > e, |x_2| < \frac{1}{2}\pi, |x_3| < \frac{1}{2}\pi\}$ with $\phi = \upsilon$ on $\partial U \cap \{(x_1, x_2, x_3) : x_1 > e\}$. Clearly, $Lg = 0 = Lh$ in U, $g = \phi = h$ on $\partial U \cap \{x_1 > e\}$ and yet

$$\lim_{x_1\to+\infty} h(x_1, x_2, x_3) = \upsilon + \cos x_2 \cos x_3 \neq \upsilon = \lim_{x_1\to+\infty} g(x_1, x_2, x_3)$$

for $\max\{|x_2|, |x_3|\} < \frac{1}{2}\pi$.

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EXAMPLE 3.6. Let $N_0 \geqslant 1$, $N = N_0+1$, $m = 2$, $n = N+2$, $M > 0$, $U = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < M^2\}$, and $\Omega = \{(t, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n : \boldsymbol{y} \in U, |\boldsymbol{x}| < 1\} \subset C_M^m$. Let T be a quasi-linear elliptic second-order operator on $C^2(\mathbb{R}^{N_0})$ with trace 1 and let Q be the parabolic operator defined by

$$Qu(X) = \frac{1}{2}\left(T(u) + \cos^2(u)\frac{\partial^2 u}{\partial y_1^2} + \sin^2(u)\frac{\partial^2 u}{\partial y_2^2} - \frac{\partial u}{\partial t} - u\right)$$

for $X = (t, x_1, \dots, x_{N_0}, y_1, y_2) \in \bar{\Omega}$, where $T(u)(X)$ means $T(u(t, \cdot, \dots, \cdot, \boldsymbol{y}))(\boldsymbol{x})$. Note that $\gamma(X, z, P) = \frac{1}{2}$ and

$$Q_\infty u = \cos^2(u)\frac{\partial^2 u}{\partial y_1^2} + \sin^2(u)\frac{\partial^2 u}{\partial y_2^2} - u.$$

Let $k(\boldsymbol{y}) = \cosh(y_1 + y_2)$ and assume $\lim_{t\to+\infty} \phi(t, \boldsymbol{x}, \boldsymbol{y}) = k(\boldsymbol{y})$ when $(t, \boldsymbol{x}, \boldsymbol{y}) \in \partial\Omega$, uniformly for $\boldsymbol{x}$ satisfying $|\boldsymbol{x}| \leqslant 1$ and $\boldsymbol{y} \in \bar{U}$. Let $f \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a bounded function satisfying $Qf = 0$ in Ω and $f = \phi$ on $\partial\Omega$. Then theorem 2.7 implies that

$$\lim_{t\to+\infty} f(t, \boldsymbol{x}, \boldsymbol{y}) = k(\boldsymbol{y}) \quad \text{when } \boldsymbol{y} \in \bar{U}.$$

(Note that $Q_\infty k = 0$ and the functions $k_1(\boldsymbol{y}) = k(\boldsymbol{y}) + \lambda|\boldsymbol{y}|^2$ and $k_2(\boldsymbol{y}) = k(\boldsymbol{y}) - \lambda|\boldsymbol{y}|^2$ for $\lambda > 0$ sufficiently small satisfy assumption 2.6.)

4. Barrier functions

Assume throughout the next two sections that $b \equiv 0$ in (1.1). Let us define $\xi = \xi(t)$ as a continuous decreasing function from $[1, \infty)$ into $(0, 1]$ satisfying

$$\lim_{t\to\infty} \xi(t) = 0 \quad \text{and} \quad 0 < \xi(t) < \sigma(t).$$

We will construct upper barriers $u_{a,\boldsymbol{x}_0,\gamma,H}$ and lower barriers $v_{a,\boldsymbol{x}_0,\gamma,H}$ for the Dirichlet problem (1.2) using ideas from [12, §§ 7 and 9]. Specifically, we claim that there exist functions $A(t) > 0$ and $\chi(t) > 0$ and a domain $\Omega_{a,\boldsymbol{x}_0,H}$ such that if $K \geqslant 0$, $\gamma \in \mathbb{R}$, $a = A(H)$ and $\boldsymbol{x}_0 \in \mathbb{R}^N$ with $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$, then there exist $u = u_{a,\boldsymbol{x}_0,\gamma,H}$ and $v = v_{a,\boldsymbol{x}_0,\gamma,H}$ in $C^0(\bar{\Omega}_{a,\boldsymbol{x}_0,H}) \cap C^1(\Omega_{a,\boldsymbol{x}_0,H}) \cap C^2(\Omega^0_{a,\boldsymbol{x}_0,H})$ such that, for any constant ζ,

$$Q(u + \zeta) < 0 \quad \text{and} \quad Q(v + \zeta) > 0 \quad \text{in } \Omega^0_{a,\boldsymbol{x}_0,H} \tag{4.1}$$

$$u \geqslant \gamma \quad \text{and} \quad v \leqslant \gamma \quad \text{on } \bar{\Omega}_{a,\boldsymbol{x}_0,H}, \tag{4.2}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = +\infty \quad \text{and} \quad \frac{\partial v}{\partial \boldsymbol{n}} = -\infty \quad \text{on } \Omega \cap \partial\Omega_{a,\boldsymbol{x}_0,H}, \tag{4.3}$$

$$u(\boldsymbol{x}_0, \boldsymbol{y}) \leqslant \gamma + \frac{M}{H} \quad \text{and} \quad v(\boldsymbol{x}_0, \boldsymbol{y}) \geqslant \gamma - \frac{M}{H} \quad \text{for } |\boldsymbol{y}| \leqslant M, \tag{4.4}$$

$$\lim_{t\to 0} \frac{\partial^2 u}{\partial y_k^2}(\boldsymbol{x}, t\boldsymbol{e}_k) = -\infty, \quad 1 \leqslant k \leqslant m, \tag{4.5}$$

$$\lim_{t\to 0} \frac{\partial^2 v}{\partial y_k^2}(\boldsymbol{x}, t\boldsymbol{e}_k) = +\infty, \quad 1 \leqslant k \leqslant m, \tag{4.6}$$

where $\Omega^0_{a,\boldsymbol{x}_0,H} = \{(\boldsymbol{x},\boldsymbol{y}) \in \Omega_{a,\boldsymbol{x}_0,H} : |\boldsymbol{y}| > 0\}$ and $\boldsymbol{e}_k$ is the unit vector in the positive y_k-direction.

Before beginning our construction, let us introduce a new operator. Corresponding to the operator Q and to $\nu \in S^{m-1}$, we define an operator $Q^{\#}_{\nu}$ by

$$Q^{\#}_{\nu} v(\boldsymbol{x}, z) = \sum_{i,j=1}^{N+1} A^{\nu}_{ij}(\boldsymbol{x}, z, v, Dv) D_{ij} v + B^{\nu}(\boldsymbol{x}, z, v, Dv) \tag{4.7}$$

for $v = v(\boldsymbol{x}, z) \in C^2(\mathbb{R}^{N+1})$ with $\partial v/\partial z \neq 0$, where

$$\begin{aligned}
A^{\nu}_{i,j}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &= q^2 a_{i,j}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right), \quad 1 \leqslant i, j \leqslant N,\\
A^{\nu}_{i,N+1}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &= -q \sum_{k=1}^{m} \nu_k a_{i,k+N}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right)\\
&\quad - q \sum_{j=1}^{N} p_j a_{i,j}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right), \quad 1 \leqslant i \leqslant N,\\
A^{\nu}_{N+1,N+1}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &= \sum_{i,j=1}^{N} p_i p_j a_{i,j}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right)\\
&\quad + 2 \sum_{j=1}^{N} \sum_{k=1}^{m} \nu_k p_j a_{j,k+N}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right)\\
&\quad + \sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right)
\end{aligned}$$

and

$$\begin{aligned}
B^{\nu}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &= -\frac{q^2}{t} \sum_{k,l=1}^{m} \nu_k \nu_l a_{k+N,l+N}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right)\\
&\quad + \frac{q^2}{t} \sum_{k=1}^{m} a_{k+N,k+N}\left(\boldsymbol{x}, \nu t, z, -\frac{\boldsymbol{p}}{q}, \frac{\nu}{q}\right).
\end{aligned}$$

Note that Q and $Q^{\#}_{\nu}$ are related in the following manner.

If $u(\boldsymbol{x},\boldsymbol{y}) = w(\boldsymbol{x}, M - |\boldsymbol{y}|)$, $w = w(\boldsymbol{x}, t)$ is in $C^2(\mathbb{R}^{N+1})$, $g = g(\boldsymbol{x}, z)$ is in $C^2(\mathbb{R}^{N+1})$, $g_z \neq 0$ and $g(\boldsymbol{x}, w(\boldsymbol{x}, t)) = t$ for $0 \leqslant t \leqslant M$, then

$$Qu(\boldsymbol{x},\boldsymbol{y}) = -\left(\frac{\partial g}{\partial z}(\boldsymbol{x}, z)\right)^{-3} Q^{\#}_{\nu} g(\boldsymbol{x}, z) \quad \text{if } \boldsymbol{y} \neq \boldsymbol{0},\ \nu = \frac{\boldsymbol{y}}{|\boldsymbol{y}|},\ z = u(\boldsymbol{x},\boldsymbol{y}). \tag{4.8}$$

Note that

$$\epsilon^{\#}_{\nu}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \frac{\sum_{i,j=1}^{N} A^{\nu}_{i,j} p_i p_j + 2\sum_{i=1}^{N} A^{\nu}_{i,N+1} p_i q + A^{\nu}_{N+1,N+1} q^2}{\sum_{i=1}^{N+1} A^{\nu}_{i,i}} \tag{4.9}$$

for $q \neq 0$, where $A^{\nu}_{i,j} = A^{\nu}_{i,j}(\boldsymbol{x}, z, t, \boldsymbol{p}, q)$ for $1 \leqslant i, j \leqslant N+1$ were given previously, and $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$, $z, t, q \in \mathbb{R}$. (ϵ is the ϵ-invariant for Q given in [18, p. 425]; $\epsilon^{\#}_{\nu}$ is the equivalent invariant for $Q^{\#}_{\nu}$.)

4.1. The construction

Let ν represent an element of $S^{m-1} = \{\boldsymbol{y} \in \mathbb{R}^m : |\boldsymbol{y}| = 1\}$. Define $\Psi \in C^0([1, \infty))$ by $\Psi(\rho) = 1/\xi(\rho^2)$. Then

$$\int_1^{\infty} \frac{1}{\rho^3 \Psi(\rho)} \, \mathrm{d}\rho < \infty$$

and

$$\epsilon^{\#}_{\nu}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \Psi(\sqrt{|\boldsymbol{p}|^2 + q^2}) \geqslant 1 \tag{4.10}$$

for $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^{n-1}$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| > 0$. Define Ψ_1 by setting $\Psi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$ and $\Psi_1(\rho) = \Psi(\rho)$ if $1 \leqslant \rho$. Define χ by

$$\chi(\alpha) = \int_{\alpha}^{\infty} \frac{\mathrm{d}\rho}{\rho^3 \Psi_1(\rho)}$$

for $\alpha > 0$. Then it is clear that $\chi(\alpha)$ is a decreasing function with range $(0, \infty)$. Let η be the inverse of χ. Then η is a positive, decreasing function with range $(0, \infty)$.

Let $H \geqslant 1$. Since $\eta(\chi(H)) = H$ and η is decreasing, we have $\eta(\beta) > H$ for $0 < \beta < \chi(H)$. For each $a > 0$, define $h_a = h_{a,H}$ by

$$h_a(r) = \int_r^{a\mathrm{e}^{\chi(H)}} \eta\left(\ln \frac{t}{a}\right) \mathrm{d}t \quad \text{for } a \leqslant r \leqslant a\mathrm{e}^{\chi(H)}. \tag{4.11}$$

Then $h_a(a\mathrm{e}^{\chi(H)}) = 0$ and $h_a(a) = a h_1(1)$. Recalling $\Omega \subset \{(\boldsymbol{x}, \boldsymbol{y}) : |\boldsymbol{y}| \leqslant M\}$, we define a function $A(H)$ by

$$A(H) = M\left(\int_1^{\mathrm{e}^{\chi(H)}} \eta(\ln t) \, \mathrm{d}t\right)^{-1}. \tag{4.12}$$

Let $a = A(H)$ and observe that $h_a(a) = M$. Furthermore, for $a < r \leqslant a\mathrm{e}^{\chi(H)}$,

$$h'_a(r) = -\eta\left(\ln \frac{r}{a}\right) < 0, \quad |h'_a(r)| > H,$$

and

$$h''_a(r) = \frac{1}{r}\left(\eta\left(\ln \frac{r}{a}\right)\right)^3 \Psi_1\left(\eta\left(\ln \frac{r}{a}\right)\right) > 0.$$

Thus, for $a < r \leqslant a\mathrm{e}^{\chi(H)}$,

$$\frac{h''_a(r)}{(h'_a(r))^2} = -\frac{h'_a(r)}{r} \Psi_1(-h'_a(r)). \tag{4.13}$$

Consider $\boldsymbol{x}_0 \in \mathbb{R}^N$ with $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$ and a constant $\Gamma \in \mathbb{R}$. Now we define a function $g = g_{a,\boldsymbol{x}_0,\Gamma,H}$ by

$$g_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, z) = h_a(\sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2}) \tag{4.14}$$

for $a^2 \leqslant |\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2 \leqslant a^2 \mathrm{e}^{2\chi(H)}$.

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Then, for

$$r = \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}, \quad a < r \leqslant a\mathrm{e}^{\chi(H)},\ 1 \leqslant i,j \leqslant N,$$

$$\left.\begin{aligned}
\frac{\partial g}{\partial x_i} &= \frac{x_i - x_{0i}}{r} h_a'(r),\\
\frac{\partial g}{\partial z} &= \frac{z-\Gamma}{r} h_a'(r),\\
\frac{\partial^2 g}{\partial z^2} &= h_a''(r)\frac{(z-\Gamma)^2}{r^2} - h_a'(r)\frac{(z-\Gamma)^2}{r^3} + h_a'(r)\frac{1}{r},\\
\frac{\partial^2 g}{\partial x_i \partial z} &= h_a''(r)\frac{(x_i - x_{0i})(z-\Gamma)}{r^2} - h_a'(r)\frac{(x_i - x_{0i})(z-\Gamma)}{r^3},\\
\frac{\partial^2 g}{\partial x_i \partial x_j} &= h_a''(r)\frac{(x_i - x_{0i})(x_j - x_{0j})}{r^2} - h_a'(r)\frac{(x_i - x_{0i})(x_j - x_{0j})}{r^3} + \delta_{i,j} h_a'(r)\frac{1}{r},
\end{aligned}\right\} \tag{4.15}$$

where $\delta_{i,j}$ is the Kronecker delta. Note that

$$\frac{\partial g}{\partial z}(\boldsymbol{x}, z) > 0 \quad \text{if } z < \Gamma$$

$$\frac{\partial g}{\partial z}(\boldsymbol{x}, z) < 0 \quad \text{if } z > \Gamma.$$

It is clear that, for any $\nu \in S^{m-1}$ and $(\boldsymbol{x}, z)$ satisfying

$$a < \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2} \leqslant a\mathrm{e}^{\chi(H)},$$

$$\begin{aligned}
P^\nu g(\boldsymbol{x}, z) &\overset{\text{def}}{=} \sum_{i,j=1}^{N+1} A_{i,j}^\nu(\boldsymbol{x}, z, g, Dg) D_{ij} g\\
&= \left[\frac{h_a''(r)}{(h_a'(r))^2}\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg) + \frac{1}{r}h_a'(r) - \frac{1}{r h_a'(r)}\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg)\right]\\
&\quad \times \sum_{i=1}^{N+1} A_{i,i}^\nu(\boldsymbol{x}, z, g, Dg)\\
&\geqslant \left(\sum_{i=1}^{N+1} A_{i,i}^\nu(\boldsymbol{x}, z, g, Dg)\right)\left(\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg)\frac{h_a''(r)}{(h_a'(r))^2} + \frac{h_a'(r)}{r}\right)\\
&= \left(\sum_{i=1}^{N+1} A_{i,i}^\nu(\boldsymbol{x}, z, g, Dg)\right)\left(-\frac{h_a'(r)}{r}\Psi_1(-h_a'(r))\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg) + \frac{h_a'(r)}{r}\right)\\
&= \left(\sum_{i=1}^{N+1} A_{i,i}^\nu(\boldsymbol{x}, z, g, Dg)\right)\frac{h_a'(r)}{r}(1 - \Psi_1(|Dg|)\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg))\\
&> 0;
\end{aligned}$$

here we have used the definition of $\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg)$ and the fact that, if $a < r < a\mathrm{e}^{\chi(H)}$, then $h_a'(r) < 0$, $|Dg| = |h_a'(r)| \geqslant H \geqslant 1$, and

$$\Psi(|Dg|)\epsilon_\nu^{\#}(\boldsymbol{x}, z, g, Dg) > 1 \quad \text{when } |Dg| \geqslant 1.$$

Note that $B^\nu(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant 0$ for all $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^N$, $z, q \in \mathbb{R}$ and $0 < |t| \leqslant M$, and so

$$Q_\nu^{\#} g(\boldsymbol{x}, z) = P^\nu g(\boldsymbol{x}, z) + B^\nu(\boldsymbol{x}, z, g, Dg) > 0 \tag{4.16}$$

for $(\boldsymbol{x}, z) \in \mathbb{R}^{N+1}$ satisfying $a < \sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2} \leqslant a\mathrm{e}^{\chi(H)}$ and $|\boldsymbol{x} - \boldsymbol{x}_0|^2 \leqslant L_1(|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2)$.

Note that the functions $w^+ = w^+_{a,\boldsymbol{x}_0,\Gamma,H}$ and $w^- = w^-_{a,\boldsymbol{x}_0,\Gamma,H}$ defined by

$$w^+_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, t) = \Gamma - \sqrt{(h_a^{-1}(t))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2} \tag{4.17}$$

and

$$w^-_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, t) = \Gamma + \sqrt{(h_a^{-1}(t))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2} \tag{4.18}$$

satisfy the equation

$$g_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, w^\pm_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, t)) = t. \tag{4.19}$$

Set $\Gamma^+ = \gamma + a\mathrm{e}^{\chi(H)}$ and $\Gamma^- = \gamma - a\mathrm{e}^{\chi(H)}$. Let us define $u = u_{a,\boldsymbol{x}_0,\gamma,H}$ and $v = v_{a,\boldsymbol{x}_0,\gamma,H}$ by

$$u_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, \boldsymbol{y}) = w^+_{a,\boldsymbol{x}_0,\Gamma^+,H}(\boldsymbol{x}, M - |\boldsymbol{y}|)$$

and

$$v_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, \boldsymbol{y}) = w^-_{a,\boldsymbol{x}_0,\Gamma^-,H}(\boldsymbol{x}, |\boldsymbol{y}|);$$

then

$$u_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, \boldsymbol{y}) = \gamma + a\mathrm{e}^{\chi(H)} - \sqrt{(h_a^{-1}(M - |\boldsymbol{y}|))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2} \tag{4.20}$$

and

$$v_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, \boldsymbol{y}) = \gamma - a\mathrm{e}^{\chi(H)} + \sqrt{(h_a^{-1}(|\boldsymbol{y}|))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2} \tag{4.21}$$

for $(\boldsymbol{x}, \boldsymbol{y}) \in \bar{\Omega}_{a,\boldsymbol{x}_0,H}$, where

$$\Omega_{a,\boldsymbol{x}_0,H} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^m : |\boldsymbol{y}| < M,\ |\boldsymbol{x} - \boldsymbol{x}_0| < h_a^{-1}(M - |\boldsymbol{y}|)\}. \tag{4.22}$$

Set

$$\Omega^0_{a,\boldsymbol{x}_0,H} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{a,\boldsymbol{x}_0,H} : |\boldsymbol{y}| > 0\} \tag{4.23}$$

and observe that $u, v \in C^2(\Omega^0_{a,\boldsymbol{x}_0,H})$. Since $w^+(\boldsymbol{x}, z) < \Gamma^+$ and $w^-(\boldsymbol{x}, z) > \Gamma^-$, we see from (4.8) that, for any constant ζ,

$$Q(u_{a,\boldsymbol{x}_0,\gamma,H} + \zeta) < 0 \quad \text{and} \quad Q(v_{a,\boldsymbol{x}_0,\gamma,H} + \zeta) > 0 \quad \text{on } \Omega^0_{a,\boldsymbol{x}_0,H}.$$

We will verify properties (4.1)–(4.5) for u. Now

$$Du(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\sqrt{(h_a^{-1}(M - |\boldsymbol{y}|))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} \left(\boldsymbol{x} - \boldsymbol{x}_0, \frac{h_a^{-1}(M - |\boldsymbol{y}|)}{h_a'(h_a^{-1}(M - |\boldsymbol{y}|))} \frac{\boldsymbol{y}}{|\boldsymbol{y}|} \right).$$

Recall that

$$\lim_{|\boldsymbol{y}|\to 0} h_a^{-1}(M-|\boldsymbol{y}|) = h_a^{-1}(M) = a, \quad h_a'(r) = -\eta(\ln(r/a)) \quad \text{and} \quad \lim_{t\to 0^+} \eta(t) = +\infty$$

and note that, when $|\boldsymbol{x}-\boldsymbol{x}_0| < a$,

$$\lim_{|\boldsymbol{y}|\to 0} Du(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\sqrt{a^2-|\boldsymbol{x}-\boldsymbol{x}_0|^2}}(\boldsymbol{x}-\boldsymbol{x}_0,\boldsymbol{0}).$$

In particular, this shows that $u \in C^1(\Omega_{a,\boldsymbol{x}_0,H})$ and

$$Du(\boldsymbol{x},\boldsymbol{0}) = \biggl(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\sqrt{a^2-|\boldsymbol{x}-\boldsymbol{x}_0|^2}},\boldsymbol{0}\biggr).$$

Furthermore,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x},\boldsymbol{y}) = \frac{\delta_{ij}[(h_a^{-1}(M-|\boldsymbol{y}|))^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2] + (x_i - x_{0i})(x_j - x_{0j})}{((h_a^{-1}(M-|\boldsymbol{y}|))^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2)^{3/2}}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial y_k}(\boldsymbol{x},\boldsymbol{y}) = \frac{y_k(x_i - x_{0i})h_a^{-1}(M-|\boldsymbol{y}|)}{|\boldsymbol{y}|h_a'(h_a^{-1}(M-|\boldsymbol{y}|))((h_a^{-1}(M-|\boldsymbol{y}|))^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2)^{3/2}};$$

hence

$$\frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \frac{\partial^2 u}{\partial x_i \partial y_k} \in C^0(\Omega_{a,\boldsymbol{x}_0,H}) \quad \text{for } 1 \leqslant k \leqslant m,\ 1 \leqslant i,j \leqslant N.$$

Let us abbreviate our notation by setting

$$S = \sqrt{(h_a^{-1}(M-|\boldsymbol{y}|))^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2}, \quad T = h_a^{-1}(M-|\boldsymbol{y}|),$$
$$U = \Psi_1(\eta(\ln(h_a^{-1}(M-|\boldsymbol{y}|)/a))) \quad \text{and} \quad V = h_a'(h_a^{-1}(M-|\boldsymbol{y}|)).$$

Then

$$\frac{\partial^2 u}{\partial y_k \partial y_l}(\boldsymbol{x},\boldsymbol{y}) = -\frac{y_k y_l}{|\boldsymbol{y}|^2 V^2 S} + \frac{(\delta_{k,l} - (y_k y_l/|\boldsymbol{y}|^2))T}{|\boldsymbol{y}|VS} - \frac{y_k y_l TU}{|\boldsymbol{y}|^2 S} + \frac{y_k y_l T^2}{|\boldsymbol{y}|^2 V^2 S^3}.$$

Since $\lim_{t\to\infty} \Psi_1(t) = +\infty$, $\lim_{|\boldsymbol{y}|\to 0} U = +\infty$. In particular, note that

$$\frac{\partial^2 u}{\partial y_k^2}(\boldsymbol{x}, y_k\boldsymbol{e}_k) = -\frac{1}{V^2 S} - \frac{TU}{S} + \frac{T^2}{V^2 S^3}$$

(with $\boldsymbol{e}_k$ the unit y_kth-coordinate vector) and so

$$\lim_{y_k\to 0} \frac{\partial^2 u}{\partial y_k^2}(\boldsymbol{x}, y_k\boldsymbol{e}_k) = -\infty \tag{4.24}$$

if $1 \leqslant k \leqslant m$ and $|\boldsymbol{x}-\boldsymbol{x}_0| < a$. Hence (4.5) holds.

To verify (4.2), we use the fact that $u(\boldsymbol{x},\boldsymbol{y}) \geqslant \Gamma - h_a^{-1}(M-|\boldsymbol{y}|)$. Since $h_a(r)$ is a decreasing function, h_a^{-1} is also a decreasing function. Thus

$$h_a^{-1}(M-|\boldsymbol{y}|) \leqslant h_a^{-1}(0) = a\mathrm{e}^{\chi(H)} \quad \text{for } |\boldsymbol{y}| \leqslant M.$$

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Hence

$$u(\boldsymbol{x},\boldsymbol{y}) \geqslant \Gamma - h_a^{-1}(0) = \gamma + a\mathrm{e}^{\chi(H)} - a\mathrm{e}^{\chi(H)} = \gamma.$$

To verify (4.3), note that $\Omega \cap \partial\Omega_{a,\boldsymbol{x}_0,H}$ is (a portion of) the M-level surface $h_a(|\boldsymbol{x}-\boldsymbol{x}_0|)+|\boldsymbol{y}| = M$ and its outer unit normal at $(\boldsymbol{x},\boldsymbol{y}) \in \Omega\cap\partial\Omega_{a,\boldsymbol{x}_0,H}$ is (recall $h_a' < 0$)

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{x},\boldsymbol{y}) &= \frac{-1}{\sqrt{1+(h_a'(h_a^{-1}(M-|\boldsymbol{y}|)))^2}}\biggl(h_a'(h_a^{-1}(M-|\boldsymbol{y}|))\frac{\boldsymbol{x}-\boldsymbol{x}_0}{|\boldsymbol{x}-\boldsymbol{x}_0|},\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\biggr)\\ &= \frac{-1}{\sqrt{1+(h_a'(|\boldsymbol{x}-\boldsymbol{x}_0|))^2}}\biggl(h_a'(|\boldsymbol{x}-\boldsymbol{x}_0|)\frac{\boldsymbol{x}-\boldsymbol{x}_0}{|\boldsymbol{x}-\boldsymbol{x}_0|},\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\biggr).\end{aligned}$$

Consider $(\boldsymbol{x},\boldsymbol{y}) \in \Omega\cap\partial\Omega_{a,\boldsymbol{x}_0,H}$ and note that, if $(\boldsymbol{x}_c,\boldsymbol{y}_c)$ satisfies $h_a^{-1}(M-|\boldsymbol{y}_c|) > |\boldsymbol{x}_c-\boldsymbol{x}_0|$, then, with $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x},\boldsymbol{y})$,

$$\begin{aligned}&Du(\boldsymbol{x}_c,\boldsymbol{y}_c)\cdot\boldsymbol{n}\\ &\quad= \frac{-1}{\sqrt{(h_a^{-1}(M-|\boldsymbol{y}_c|))^2-|\boldsymbol{x}_c-\boldsymbol{x}_0|^2}\sqrt{1+(h_a'(|\boldsymbol{x}-\boldsymbol{x}_0|))^2}}\\ &\qquad\times\biggl(\frac{(\boldsymbol{x}_c-\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{|\boldsymbol{x}-\boldsymbol{x}_0|}h_a'(|\boldsymbol{x}-\boldsymbol{x}_0|)+\frac{\boldsymbol{y}_c\cdot\boldsymbol{y}}{|\boldsymbol{y}_c||\boldsymbol{y}|}\frac{h_a^{-1}(M-|\boldsymbol{y}_c|)}{h_a'(h_a^{-1}(M-|\boldsymbol{y}_c|))}\biggr);\end{aligned}$$

by letting $(\boldsymbol{x}_c,\boldsymbol{y}_c) \to (\boldsymbol{x},\boldsymbol{y})$, so that $h_a^{-1}(M-|\boldsymbol{y}_c|) \to |\boldsymbol{x}-\boldsymbol{x}_0|$, we see that

$$\frac{\partial u}{\partial \boldsymbol{n}} = +\infty \quad \text{on } \Omega\cap\partial\Omega_{a,\boldsymbol{x}_0,H}. \tag{4.25}$$

Similarly, we see that

$$\frac{\partial v}{\partial \boldsymbol{n}} = -\infty \quad \text{on } \Omega\cap\partial\Omega_{a,\boldsymbol{x}_0,H}.$$

To verify (4.4), recall that $u(\boldsymbol{x}_0,\boldsymbol{y}) = \Gamma^+ - h_a^{-1}(M-|\boldsymbol{y}|)$ and so, for $1 \leqslant k \leqslant m$,

$$\frac{\partial u}{\partial y_k}(\boldsymbol{x}_0,\boldsymbol{y}) = \frac{y_k}{|\boldsymbol{y}|h_a'(h_a^{-1}(M-|\boldsymbol{y}|))} = \frac{-y_k}{|\boldsymbol{y}|\eta(\ln(a^{-1}h_a^{-1}(M-|\boldsymbol{y}|)))}.$$

Using the fact that $h_a^{-1}(r)$ is a decreasing function again, we have

$$\ln\frac{h_a^{-1}(M-|\boldsymbol{y}|)}{a} \leqslant \ln\mathrm{e}^{\chi(H)} = \chi(H) \quad \text{for } |\boldsymbol{y}| \leqslant M.$$

Since η is also decreasing, we have

$$\Bigl|\frac{\partial u}{\partial y_k}(\boldsymbol{x}_0,\boldsymbol{y})\Bigr| \leqslant \frac{1}{\eta(\chi(H))} = \frac{1}{H} \quad \text{for } |\boldsymbol{y}| \leqslant M,\ 1 \leqslant k \leqslant m.$$

Then (4.4) follows from this and the fact that, if $|\boldsymbol{y}| = M$, then

$$u(\boldsymbol{x}_0,\boldsymbol{y}) = \Gamma - h_a^{-1}(0) = \Gamma - a\mathrm{e}^{\chi(H)} = \gamma.$$

Properties (4.1)–(4.4) and (4.6) of $v_{a,\boldsymbol{x}_0,\gamma,H}$ can be established in a similar manner.

Remark 4.1. Before continuing, it seems advisable to compare the techniques, proofs and conclusions in this paper with those of [12–14]. Here the discussion involving equation (4.8), which leads us to the use of barriers of the form $u(\boldsymbol{x},\boldsymbol{y}) = w(\boldsymbol{x}, M-|\boldsymbol{y}|)$ with $\boldsymbol{x} \in \mathbb{R}^N = \mathbb{R}^{n-m}$, $\boldsymbol{y} \in \mathbb{R}^m$ and $|\boldsymbol{y}| \leqslant M$, is crucial to theorem 2.3. Now theorem 2.3 is related to [12, theorem 2.4], and corollary 3.2 is related to the results in [12, § 4] (see also [15]). However, the results of [12] follow from the use of barriers of the form $u(\boldsymbol{x}, y) = w(\boldsymbol{x}, y)$ with $\boldsymbol{x} \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$ and $|y| \leqslant M$, for which the discussion involving [12, eqn (2.6)] is crucial. The proof of theorem 2.3 will involve one additional case (when $\boldsymbol{y} = \boldsymbol{0}$) not present in [12]. The set of operators Q to which both corollary 3.2 and [12, corollary 4.2], apply is probably empty, even when $m = 1$ and $C^m_M = S_M$.

Theorem 2.7 is related primarily to [13, theorem 2.2] and [14, theorem 2.6], and secondarily to [12, theorem 2.5]. The results of [13, 14] require $a_{n,n}$ in [13] or $\min_{k=1,\dots,m} a^{n,n}_k$ in [14] to have a positive lower bound (e.g. [13, eqn (14)], [14, eqn (2.6)]), while theorem 2.7 requires only the weaker condition (2.16); when $m = 1$ (so that $C^m_M = S_M$), theorem 2.7 and [13, theorem 2.2] are essentially the same. The proof of theorem 2.7 represents a refinement of that in [13] in which an inessential dependence of the barrier function on y is eliminated.

5. Proof of theorem 2.3

For any $\epsilon > 0$, by the assumption on $\phi(\boldsymbol{x}, y)$ and the continuity of $\Phi(\omega)$, there exist $\delta > 0$ and $R > 0$ such that, if $(\boldsymbol{x},\boldsymbol{y}) \in \partial\Omega$, $|\boldsymbol{x}| \geqslant R$, $|\boldsymbol{y}| \leqslant M$, $\omega \in S^{N-1}$ and $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| < \delta$, we have

$$|\phi(\boldsymbol{x},\boldsymbol{y}) - \Phi(\omega)| < \epsilon. \tag{5.1}$$

Fix $\omega \in S^{N-1}$ and set $\gamma = \Phi(\omega) + 2\epsilon$. We choose H such that $H \geqslant 1$ and $M/H < \epsilon$ and set $a = A(H)$. Let $u(\boldsymbol{x},\boldsymbol{y}) = u_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x},\boldsymbol{y})$ be the upper barrier given by (4.20) with properties (4.1)–(4.5). We choose a large number $R_1 > R + L + A(H)\mathrm{e}^{\chi(H)}$ and a small number $0 < \delta_1 < \delta$ such that, if $|\boldsymbol{x}| > R_1$, $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| < \delta_1$, we have

$$\left|\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - \omega\right| < \delta \quad \text{for all } \boldsymbol{v} \text{ with } |\boldsymbol{v} - \boldsymbol{x}| \leqslant A(H)\mathrm{e}^{\chi(H)}.$$

Now u is defined on the domain $\Omega_{a,\boldsymbol{x}_0,H}$; we shall compare the functions f and u on the domain $\Omega_1 \equiv \Omega_{a,\boldsymbol{x}_0,H} \cap \Omega$.

Set

$$W = \left\{\boldsymbol{x} \;\middle|\; |\boldsymbol{x}| > R_1, \left|\frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \omega\right| < \delta_1\right\}.$$

We claim that, if $(\boldsymbol{x}_0,\boldsymbol{y}) \in \bar{\Omega}$ and $\boldsymbol{x}_0 \in W$, then

$$f(\boldsymbol{x}_0,\boldsymbol{y}) < \Phi(\omega) + 3\epsilon. \tag{5.2}$$

If $(\boldsymbol{x},\boldsymbol{y}) \in \partial\Omega \cap \partial\Omega_1$, from the definition of W, (4.2) and (5.1), we have

$$f(\boldsymbol{x},\boldsymbol{y}) = \phi(\boldsymbol{x},\boldsymbol{y}) < \Phi(\omega) + 2\epsilon = \gamma \leqslant u(\boldsymbol{x},\boldsymbol{y}). \tag{5.3}$$

Thus,

$$f - u < 0 \quad \text{on } \partial\Omega \cap \partial\Omega_1. \tag{5.4}$$

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Note also that

$$\frac{\partial u}{\partial \boldsymbol{n}} = +\infty \quad \text{and} \quad \frac{\partial v}{\partial \boldsymbol{n}} = -\infty \quad \text{on } \Omega \cap \partial\Omega_1. \tag{5.5}$$

We claim that

$$f(\boldsymbol{x},\boldsymbol{y}) - u(\boldsymbol{x},\boldsymbol{y}) < 0 \quad \text{on } \Omega_1. \tag{5.6}$$

Suppose that (5.6) is not true. Then there exists a $\zeta \geqslant 0$ such that

$$f \leqslant u + \zeta \text{ on } \Omega_1 \quad \text{and} \quad f(\boldsymbol{x}_1,\boldsymbol{y}_1) = u(\boldsymbol{x}_1,\boldsymbol{y}_1) + \zeta \quad \text{for some } (\boldsymbol{x}_1,\boldsymbol{y}_1) \in \Omega_1. \tag{5.7}$$

Note that $Df(\boldsymbol{x}_1,\boldsymbol{y}_1) = Du(\boldsymbol{x}_1,\boldsymbol{y}_1)$.

Suppose first that $\boldsymbol{y}_1 = \boldsymbol{0}$. Consider the path $\boldsymbol{\sigma}(t) = (\boldsymbol{x}_1, t, 0, \dots, 0)$ in Ω_1 and the functions $\alpha(t) = f(\boldsymbol{\sigma}(t))$ and $\beta(t) = u(\boldsymbol{\sigma}(t)) + \zeta$. Note that $\alpha(t) \leqslant \beta(t)$, $\alpha(\cdot), \beta(\cdot) \in C^1$, $\alpha(0) = \beta(0)$, $\alpha'(0) = \beta'(0) = 0$ and, since $f \in C^2(\Omega)$, $\alpha(\cdot) \in C^2$. From (4.24), we see that $\lim_{t\to 0} \beta''(t) = -\infty$. Simple integration shows that, for every $c > 0$, there exists $\delta > 0$ such that $\beta(t) \leqslant \beta(0) - \frac{1}{2}ct^2$ if $|t| \leqslant \delta$. By choosing c sufficiently large (e.g. $c = 1 + \max_{|t|\leqslant\delta} |\alpha''(t)|$), we obtain $\beta(t) < \alpha(t)$ for $0 < |t| \leqslant \delta$, in contradiction of (5.7). Hence $\boldsymbol{y}_1 \neq \boldsymbol{0}$.

The remainder of the argument required to establish (5.6) is standard; we shall present it for the sake of completeness. Suppose next that $(\boldsymbol{x}_1,\boldsymbol{y}_1) \in \Omega^0_{a,\boldsymbol{x}_0,H}$. Now, from (4.1) we have

$$\sum_{i,j=1}^{n} a_{i,j}(\boldsymbol{x},\boldsymbol{y},u(\boldsymbol{x},\boldsymbol{y}) + \zeta, Du(\boldsymbol{x},\boldsymbol{y}))D_{ij}u(\boldsymbol{x},\boldsymbol{y}) < 0 \quad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \Omega_1 \cap \Omega^0_{a,\boldsymbol{x}_0,H}.$$

In particular,

$$\sum_{i,j=1}^{n} a_{i,j}(\boldsymbol{x}_1,\boldsymbol{y}_1, f(\boldsymbol{x}_1,y_1), Df(\boldsymbol{x}_1,\boldsymbol{y}_1))D_{ij}u(\boldsymbol{x}_1,y_1) < 0.$$

A standard argument (e.g. the proof of [4, theorem 3.1]) yields a contradiction; hence $(\boldsymbol{x}_1,\boldsymbol{y}_1) \notin \Omega^0_{a,\boldsymbol{x}_0,H}$. Consider the last two cases, in which $(\boldsymbol{x}_1,\boldsymbol{y}_1)$ is in $\partial\Omega \cap \partial\Omega_1$ or $\Omega \cap \partial\Omega_1$. Note that (5.4) rules out the first of these cases. In the second of these cases, (5.5) rules out the possibility that $\zeta > 0$ and the only possibilities which remain are that $\zeta = 0$ and $(\boldsymbol{x}_1,\boldsymbol{y}_1) \in \Omega \cap \partial\Omega_1$ or that (5.7) is impossible. Suppose $(\boldsymbol{x}_1,\boldsymbol{y}_1) \in \Omega \cap \partial\Omega_1$. Then a standard argument using (5.5) shows that $f(\boldsymbol{\sigma}(t)) > u(\boldsymbol{\sigma}(t)) + \zeta$ for $t > 0$ small if $\boldsymbol{\sigma}(t) = (\boldsymbol{x}_1,\boldsymbol{y}_1) - t\boldsymbol{\eta} \in \Omega_1$, in contradiction of (5.7). Thus $(\boldsymbol{x}_1,\boldsymbol{y}_1) \notin \Omega \cap \partial\Omega_1$. Hence (5.6) holds.

Now, from (5.6), we have $f(\boldsymbol{x}_0,\boldsymbol{y}) \leqslant u(\boldsymbol{x}_0,\boldsymbol{y})$ for $(\boldsymbol{x}_0,\boldsymbol{y})$ in $\overline{\Omega_1}$. Thus (4.4) and the choices of γ and H yield

$$f(\boldsymbol{x}_0,\boldsymbol{y}) \leqslant \gamma + \frac{M}{H} \leqslant \Phi(\omega) + 3\epsilon \quad \text{for } (\boldsymbol{x}_0,\boldsymbol{y}) \in \Omega_1.$$

Since $(\boldsymbol{x}_0,\boldsymbol{y}) \in \Omega_1$ implies $(\boldsymbol{x}_0,\boldsymbol{y}) \in \Omega$ (from the definition of $\Omega_{a,\boldsymbol{x}_0,H}$), the claim (5.2) is proven. A similar argument using our lower bounds shows that $f(\boldsymbol{x}_0,\boldsymbol{y}) > \Phi(\omega) - 3\epsilon$.

From this, we can conclude that $|f(\boldsymbol{x}_0,\boldsymbol{y}) - \Phi(\omega)| \leqslant 3\epsilon$ for $(\boldsymbol{x}_0,\boldsymbol{y}) \in \Omega$. Since $\boldsymbol{x}_0 \in W$ is arbitrary, we finally have

$$|f(\boldsymbol{x},\boldsymbol{y}) - \Phi(\omega)| \leqslant 3\epsilon \quad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \Omega \text{ with } \boldsymbol{x} \in W. \tag{5.8}$$

Now if $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ as $j \to \infty$, there exists $L > 0$ such that $\boldsymbol{x}_j \in W$ when $j \geqslant L$. Then from (5.8), for $(\boldsymbol{x}_j, \boldsymbol{y}_j) \in \Omega$, we have

$$|f(\boldsymbol{x}_j, \boldsymbol{y}_j) - \Phi(\omega)| \leqslant 3\epsilon \quad \text{if } j \geqslant L.$$

Since $\epsilon > 0$ is arbitrary, (2.6) follows.

6. Proof of theorem 2.7

We may assume that the set O mentioned in assumption 2.4 is all of S^{N-1}. Let $\omega \in T$, $\epsilon > 0$ and $\alpha = \epsilon$. Let $\delta = \delta_{\alpha,\omega}$, k_1 and k_2 be as given in assumption 2.6. From assumption 2.5 and the continuity of $k(\boldsymbol{y}, \omega)$, we see that there exist $\delta_1 > 0$ and R_1 such that, if $(\boldsymbol{x}, \boldsymbol{y}) \in \partial\Omega$, $|\boldsymbol{x}| \geqslant R_1$, $|\boldsymbol{y}| \leqslant M$, and $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| < \delta_1$, we have

$$|\phi(\boldsymbol{x}, \boldsymbol{y}) - k(\boldsymbol{y}, \omega)| < \epsilon. \tag{6.1}$$

Assumption 2.4 implies there exist $\delta_2 > 0$ and R_2 such that

$$\left|\frac{a_{k+N,l+N}(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})}{\gamma(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})} - A_{k,l}(\omega, \boldsymbol{y}, z, \boldsymbol{q})\right| \leqslant \frac{\delta}{4m^2\|D^2_{\boldsymbol{y}} k_2\|_\infty} \tag{6.2}$$

for $1 \leqslant k, l \leqslant m$ and

$$\left|\frac{b(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})}{\gamma(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q})} - E(\omega, \boldsymbol{y}, z, \boldsymbol{q})\right| \leqslant \tfrac{1}{4}\delta \tag{6.3}$$

for $z \in \mathbb{R}$, $\boldsymbol{y}, \boldsymbol{q} \in \mathbb{R}^m$ with $|\boldsymbol{y}| \leqslant M$ if $|\boldsymbol{x}| \geqslant R_2$, $|\boldsymbol{p}| \leqslant \delta_2$ and $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| \leqslant 2\delta_2$. Consider the compact set

$$K = \{(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^n : |\boldsymbol{p}|^2 + |\boldsymbol{q}|^2 \leqslant 1 + \|D_{\boldsymbol{y}} k_2\|^2_\infty\}.$$

From (2.16), we see that there exists $\mu(K) > 0$ such that

$$\gamma(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{p}, \boldsymbol{q}) \geqslant \mu(K)$$

if $(\boldsymbol{p}, \boldsymbol{q}) \in K$, $\boldsymbol{x} \in \mathbb{R}^N$, $\boldsymbol{y} \in \mathbb{R}^m$ and $z \in \mathbb{R}$ with $|\boldsymbol{y}| \leqslant M$. Set $\mu = \mu(K)$, $J = \|f - k_2\|_\infty$ and $\delta_0 = \min\{1, \frac{1}{2}\delta_1, \frac{1}{2}\delta_2\}$.

Choose $a > \min\{1, J, 2/\mu\delta\}$ so that

$$(2a - J)J < \min\left\{\frac{\delta_0^2}{1 + \delta_0^2}a^2, a^2 - \left(\frac{2a^2}{\mu\delta}\right)^{2/3}\right\}$$

and pick $H > 0$ such that

$$(2a - J)J < H^2 < \min\left\{\frac{\delta_0^2}{1 + \delta_0^2}a^2, a^2 - \left(\frac{2a^2}{\mu\delta}\right)^{2/3}\right\}.$$

Note that

$$\frac{H}{\sqrt{a^2 - H^2}} < \delta_0, \qquad a - \sqrt{a^2 - H^2} > J, \qquad \frac{a^2}{(a^2 - H^2)^{3/2}} < \frac{\mu\delta}{2}. \tag{6.4}$$

There exists $R_3 > 0$ such that, if $|\boldsymbol{x}_0| \geqslant R_3$, $|\boldsymbol{x} - \boldsymbol{x}_0| \leqslant H$ and $|\boldsymbol{x}_0/|\boldsymbol{x}_0| - \omega| < \delta_0$, then $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| < 2\delta_0$. Set $R_0 = \max\{R_1, R_2, R_3\} + H$.

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Now define

$$W = \left\{ \boldsymbol{x} \in \mathbb{R}^N \;\middle|\; |\boldsymbol{x}| > R_0,\ \left| \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \omega \right| < \delta_0 \right\}.$$

We claim that if $(\boldsymbol{x}_0, \boldsymbol{y}) \in \bar{\Omega}$ and $\boldsymbol{x}_0 \in W$, then

$$f(\boldsymbol{x}_0, \boldsymbol{y}) < k(\boldsymbol{y}, \omega) + 2\epsilon. \tag{6.5}$$

Throughout the remainder of this proof, let $\boldsymbol{x}_0$ represent a point in W such that $(\boldsymbol{x}_0, \boldsymbol{y}) \in \bar{\Omega}$ for some $\boldsymbol{y} \in \overline{B(M)}$. Let

$$w(\boldsymbol{x}) = w_{a,\boldsymbol{x}_0}(\boldsymbol{x}) = a - \sqrt{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2} + 2\epsilon$$

and note that $w \geqslant 2\epsilon$. Now set

$$\Omega_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \Omega_{a,\boldsymbol{x}_0,H} \cap \Omega : |\boldsymbol{x} - \boldsymbol{x}_0| < H\} \tag{6.6}$$

and define $u_2 \in C^1(\overline{\Omega_1}) \cap C^2(\Omega_1)$ by

$$u_2(\boldsymbol{x}, \boldsymbol{y}) = w(\boldsymbol{x}) + k_2(\boldsymbol{y}).$$

Note that, if $(\boldsymbol{x}, \boldsymbol{y}) \in \Omega_1$, then $|\boldsymbol{x}| \geqslant \max\{R_1, R_2, R_3\}$ and $|\boldsymbol{x}/|\boldsymbol{x}| - \omega| < 2\delta_0$.

Let $\zeta \geqslant 0$. We claim that

$$Q(u_2 + \zeta) < 0 \text{ in } \Omega_1. \tag{6.7}$$

Note that

$$\frac{\partial w}{\partial x_i}(\boldsymbol{x}) = \frac{x_i - x_i^{(0)}}{\sqrt{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} \quad \text{for } 1 \leqslant i \leqslant N$$

and

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(\boldsymbol{x}) = \frac{\delta_{ij}(a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2) + (x_i - x_i^{(0)})(x_j - x_j^{(0)})}{(a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2)^{3/2}} \quad \text{for } 1 \leqslant i, j \leqslant N,$$

where $\boldsymbol{x}_0 = (x_1^{(0)}, \dots, x_{n-1}^{(0)})$. If we set

$$\xi_i = \frac{x_i - x_i^{(0)}}{\sqrt{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} \quad \text{for } 1 \leqslant i \leqslant N$$

and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$, then

$$\begin{aligned}
\sum_{i,j=1}^{N} a_{i,j} D_{ij} w(\boldsymbol{x}) &= \frac{1}{\sqrt{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} \left(\sum_{i=1}^{N} a_{i,i} + \sum_{i,j=1}^{N} a_{i,j} \xi_i \xi_j \right) \\
&\leqslant \frac{1}{\sqrt{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} (1 + |\boldsymbol{\xi}|^2) \leqslant \frac{1}{\sqrt{a^2 - H^2}} \left(1 + \frac{H^2}{a^2 - H^2} \right) \\
&= \frac{a^2}{(a^2 - H^2)^{3/2}} \\
&< \tfrac{1}{2}\mu\delta,
\end{aligned}$$

where $a_{i,j} = a_{i,j}(\boldsymbol{x}, \boldsymbol{y}, u_2 + \zeta, Du_2)$, $1 \leqslant i, j \leqslant N$.

Since

$$|Du_2(\boldsymbol{x},\boldsymbol{y})|^2 = |Dw(\boldsymbol{x})|^2 + |Dk_2(\boldsymbol{y})|^2 < \delta_0^2 + |Dk_2(\boldsymbol{y})|^2 \leqslant 1 + \|D_{\boldsymbol{y}}k_2\|_\infty^2$$

when $(\boldsymbol{x},\boldsymbol{y}) \in \Omega_1$, we have $Du_2(\boldsymbol{x},\boldsymbol{y}) \in K$ and so $\gamma(\boldsymbol{x},\boldsymbol{y},u_2+\zeta,Du_2) \geqslant \mu(K)$ if $(\boldsymbol{x},\boldsymbol{y}) \in \Omega_1$. Note that, for $1 \leqslant k,l \leqslant N$,

$$A_{k,l}(\omega,\boldsymbol{y},u_2+\zeta,\boldsymbol{q}) = A_{k,l}(\omega,\boldsymbol{y},u_2,\boldsymbol{q}) = A_{k,l}(\omega,\boldsymbol{y},k_2,\boldsymbol{q}) \tag{6.8}$$

and

$$E(\omega,\boldsymbol{y},u_2+\zeta,\boldsymbol{q}) \leqslant E(\omega,\boldsymbol{y},u_2,\boldsymbol{q}) \leqslant E(\omega,\boldsymbol{y},k_2,\boldsymbol{q}) \tag{6.9}$$

for all $\boldsymbol{y} \in B(M)$ and $\boldsymbol{q} \in \mathbb{R}^m$, since $\zeta \geqslant 0$ and $u_2 = w + k_2 \geqslant 2\epsilon + k_2 > k_2$. Using (6.2), (6.3), (6.8) and (6.9), we find

$$\begin{aligned}
Q(u_2+\zeta) &= \sum_{i,j=1}^{N} a_{i,j}D_{ij}w + \sum_{k,l=1}^{N} a_{k+N,l+N}\frac{\partial^2 k_2}{\partial y_k \partial y_l} + b \\
&< \tfrac{1}{2}\mu\delta + \biggl(\sum_{k,l=1}^{m} \frac{a_{k+N,l+N}}{\gamma}\frac{\partial^2 k_2}{\partial y_k \partial y_l} + \frac{b}{\gamma}\biggr)\gamma \\
&= \tfrac{1}{2}\mu\delta + \biggl[\frac{b}{\gamma} - E(u_2+\zeta, D_{\boldsymbol{y}}u_2) + E(u_2+\zeta, D_{\boldsymbol{y}}u_2) \\
&\qquad\qquad - E(k_2, D_{\boldsymbol{y}}u_2) + E(k_2, D_{\boldsymbol{y}}k_2) \\
&\qquad + \sum_{k,l=1}^{m} A_{k,l}(k_2, D_{\boldsymbol{y}}k_2)\frac{\partial^2 k_2}{\partial y_k \partial y_l} \\
&\qquad - \sum_{k,l=1}^{m} A_{k,l}(k_2, D_{\boldsymbol{y}}k_2)\frac{\partial^2 k_2}{\partial y_k \partial y_l} \\
&\qquad + \sum_{k,l=1}^{m} A_{k,l}(u_2+\zeta, D_{\boldsymbol{y}}u_2)\frac{\partial^2 k_2}{\partial y_k \partial y_l} \\
&\qquad - \sum_{k,l=1}^{m} \biggl(A_{k,l}(u_2+\zeta, D_{\boldsymbol{y}}u_2) + \frac{a_{k+N,l+N}}{\gamma}\biggr)\frac{\partial^2 k_2}{\partial y_k \partial y_l}\biggr]\gamma \\
&\leqslant \tfrac{1}{2}\mu\delta + [\tfrac{1}{4}\delta + 0 - \delta + 0 + \tfrac{1}{4}\delta]\gamma \\
&\leqslant \tfrac{1}{2}\mu\delta - \tfrac{1}{2}\mu\delta \\
&= 0,
\end{aligned}$$

where $\gamma = \gamma(\boldsymbol{x},\boldsymbol{y},u_2+\zeta,Du_2)$, $a_{k+N,l+N} = a_{k+N,l+N}(\boldsymbol{x},\boldsymbol{y},u_2+\zeta,Du_2)$ $(1 \leqslant k,l \leqslant m)$, $b = b(\boldsymbol{x},\boldsymbol{y},u_2+\zeta,Du_2)$, $E(u_2+\zeta,D_{\boldsymbol{y}}u_2) = E(u_2+\zeta,D_{\boldsymbol{y}}k_2) = E(\omega,\boldsymbol{y},u_2+\zeta,D_{\boldsymbol{y}}k_2)$, $E(k_2,D_{\boldsymbol{y}}k_2) = E(\omega,\boldsymbol{y},k_2,D_{\boldsymbol{y}}k_2)$, $A_{k,l}(u_2+\zeta,D_{\boldsymbol{y}}u_2) = A_{k,l}(\omega,\boldsymbol{y},u_2+\zeta,D_{\boldsymbol{y}}u_2)$ and $A_{k,l}(k_2,D_{\boldsymbol{y}}k_2) = A_{k,l}(\omega,\boldsymbol{y},k_2,D_{\boldsymbol{y}}k_2)$. (Note that $D_{\boldsymbol{y}}u_2 = D_{\boldsymbol{y}}k_2$.)

If $(\boldsymbol{x},y) \in \partial\Omega \cap \partial\Omega_1$, from (6.1) we have

$$f(\boldsymbol{x},\boldsymbol{y}) = \phi(\boldsymbol{x},\boldsymbol{y}) < k(\boldsymbol{y},\omega) + \epsilon \leqslant k_2(\boldsymbol{y}) + 2\epsilon \leqslant w(\boldsymbol{x},\boldsymbol{y}) + k_2(\boldsymbol{y}).$$

Thus

$$f(\boldsymbol{x},\boldsymbol{y}) - u_2(\boldsymbol{x},\boldsymbol{y}) < 0 \quad \text{on } \partial\Omega \cap \partial\Omega_1.$$

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If $(\boldsymbol{x},\boldsymbol{y})\in\Omega\cap\partial\Omega_1$, then $|\boldsymbol{x}-\boldsymbol{x}_0|=H$ and so (6.4) implies that

$$w(\boldsymbol{x},\boldsymbol{y})=a-\sqrt{a^2-H^2}+2\epsilon>J.$$

Hence

$$f(\boldsymbol{x},\boldsymbol{y})-k_2(\boldsymbol{y})\leqslant\|f-k_2\|_\infty=J<w(\boldsymbol{x},\boldsymbol{y})$$

and so $f(\boldsymbol{x},y)<u_2(\boldsymbol{x},y)$ for $(\boldsymbol{x},y)\in\Omega\cap\partial\Omega_1$.

Let $U_0=\{(\boldsymbol{x},\boldsymbol{y})\in\Omega_1 : f(\boldsymbol{x},\boldsymbol{y})>u_2(\boldsymbol{x},\boldsymbol{y})\}$. Since $f<u_2$ on $\partial\Omega_1$, U_0 is a relatively compact subset of Ω_1 and $f=u_2$ on $\Omega_1\cap\partial U_0$. Now define

$$Ru(\boldsymbol{x},\boldsymbol{y})=\sum_{ij=1}^{n}\bar{a}_{i,j}(\boldsymbol{x},\boldsymbol{y},Du)D_{ij}u(\boldsymbol{x},\boldsymbol{y})+\bar{b}(\boldsymbol{x},\boldsymbol{y},Du)$$

by setting $\bar{a}_{i,j}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})=a_{i,j}(\boldsymbol{x},\boldsymbol{y},f(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{q})$ and $\bar{b}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})=b(\boldsymbol{x},\boldsymbol{y},f(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{q})$. Let $(\boldsymbol{x}_1,\boldsymbol{y}_1)$ be an arbitrary point in U_0 and set $\zeta=f(\boldsymbol{x}_1,\boldsymbol{y}_1)-u_2(\boldsymbol{x}_1,\boldsymbol{y}_1)>0$. Since $Q(u_2+\zeta)<0$ on Ω_1, we have

$$Ru_2(\boldsymbol{x}_1,\boldsymbol{y}_1)=Q(u_2+\zeta)(\boldsymbol{x}_1,\boldsymbol{y}_1)<0.$$

Since $(\boldsymbol{x}_1,\boldsymbol{y}_1)$ is an arbitrary point in U_0, we have $Ru_2<0$ in U_0. Recalling that the ellipticity of R is not needed in [4, theorem 10.1] (as noted in the proof of [4, theorem 3.1]), we see that $f\leqslant u_2$ on U_0. Hence $U_0=\emptyset$ and so

$$f(\boldsymbol{x},\boldsymbol{y})\leqslant u_2(\boldsymbol{x},\boldsymbol{y})\quad\text{on }\Omega_1.$$

Therefore,

$$f(\boldsymbol{x}_0,\boldsymbol{y})\leqslant w(\boldsymbol{x}_0,\boldsymbol{y})+k_2(\boldsymbol{y})\leqslant\frac{2M}{H}+k_2(\boldsymbol{y})<2\epsilon+k(\boldsymbol{y},\omega)$$

or $f(\boldsymbol{x}_0,\boldsymbol{y})-k(\boldsymbol{y},\omega)<2\epsilon$.

Together with a similar argument using lower barriers and $k_1(\boldsymbol{y})$ (i.e. $u_1(\boldsymbol{x},\boldsymbol{y})=l_a(\boldsymbol{x},\boldsymbol{y})+k_1(\boldsymbol{y})$ with $\Psi(\rho)=1$), we then find that

$$|f(\boldsymbol{x}_0,\boldsymbol{y})-k(\boldsymbol{y},\omega)|<2\epsilon.$$

Since $\boldsymbol{x}_0\in W$ is arbitrary, we finally have

$$|f(\boldsymbol{x},\boldsymbol{y})-k(\boldsymbol{y},\omega)|\leqslant 2\epsilon\quad\text{for }(\boldsymbol{x},\boldsymbol{y})\in\Omega\text{ with }\boldsymbol{x}\in W.\tag{6.10}$$

Now, if $\boldsymbol{x}_j/|\boldsymbol{x}_j|\to\omega$ as $j\to\infty$, there exists $N>0$ such that $\boldsymbol{x}_j\in W$ when $j\geqslant N$. Then from (6.10), for $(\boldsymbol{x}_j,\boldsymbol{y}_j)\in\Omega$, we have

$$|f(\boldsymbol{x}_j,\boldsymbol{y}_j)-k(\boldsymbol{y}_j,\omega)|\leqslant 2\epsilon\quad\text{if }j\geqslant N.$$

Since $\epsilon>0$ is arbitrary, the conclusion of theorem 2.7 follows.

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