The locus of points at which two sides of a given triangle subtend equal or supplementary angles.

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1. Let the sides AB, AC subtend equal angles λ at P.

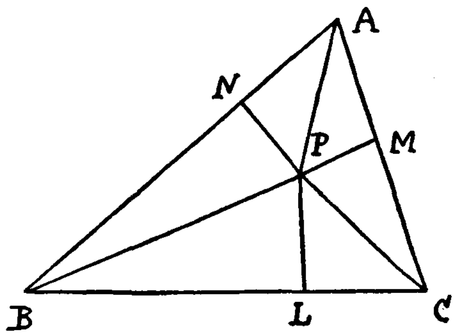

Fig. 1.

Take ABC as the triangle of reference and denote AP, BP, CP by x, y, z. Then we have, using trilinears,

$$b\beta : c\gamma = xz\sin\lambda : yx\sin\lambda = z : y$$
$$\therefore \quad \beta : \gamma = cz : by = z\sin C : y\sin B.$$

Also, $$LN = y\sin B, \; LM = z\sin C.$$
$$\therefore \quad y^2\sin^2 B = LN^2 = PL^2 + PN^2 + 2PL \,.\, PN\cos B$$
and $$z^2\sin^2 C = LM^2 = PL^2 + PM^2 + 2PL \,.\, PM\cos C.$$
Hence
$$\gamma^2 : \beta^2 = y^2\sin^2 B : z^2\sin^2 C$$
$$= PL^2 + PN^2 + 2PL \,.\, PN\cos B : PL^2 + PM^2 + 2PL \,.\, PM\cos C$$
$$= \alpha^2 + \gamma^2 + 2\alpha\gamma\cos B : \alpha^2 + \beta^2 + 2\alpha\beta\cos C.$$
Simplifying $$\alpha(\beta^2 - \gamma^2) + 2\beta\gamma(\beta\cos B - \gamma\cos C) = 0.$$

The locus of P is therefore *a cubic* passing through the vertices of the triangle of reference.

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2. It is readily seen that this cubic is identical with the locus of the points of contact of tangents from A to a system of confocals having B, C for foci.

For, if X be one of the confocals the tangent AP to it bisects the angle BPC. So that AB, AC subtend equal or supplementary angles at P.

Since AP is normal to the second confocal through P, it is clear that the cubic is also the locus of the feet of normals from A to the system of confocals.

3. The cartesian equation of the cubic is easily found.

Let (h, k) be A referred to the mid point of BC as origin, and let B, C be $(\mp\delta, 0)$. Then the equation of the confocal system is $\dfrac{x^2}{\lambda^2+\delta^2}+\dfrac{y^2}{\lambda^2}=1$ for different values of λ.

Now, the polar of A with respect to this system is $\dfrac{hx}{\lambda^2+\delta^2}+\dfrac{ky}{\lambda^2}=1$. Eliminating λ between these equations the locus required is obtained.

Thus the cartesian equation of the locus is

$$\{x(x-h)+y(y-k)\}(hy-kx)=\delta^2(x-h)(y-k).$$

4. The curve may also be defined as the locus of the vertices of triangles on a given base BC, having the bisectors of the vertical angles always passing through a fixed point A.

Some properties of the cubic.

5. It is a circular cubic having a node at A.

This may be inferred from the cartesian equation found in §3, by changing the origin to A.

It may also be seen by writing the trilinear equation of §1 in the form

$$a(\beta^2-\gamma^2)+\beta\gamma\left\{\beta\frac{a^2+c^2-b^2}{ac}-\gamma\frac{a^2+b^2-c^2}{ab}\right\}=0,$$

that is, $$\left(\frac{\beta}{c}-\frac{\gamma}{b}\right)(a\beta\gamma+b\gamma\alpha+c\alpha\beta)+\frac{c^2-b^2}{a}(a\alpha+b\beta+c\gamma)\beta\gamma=0$$

or, $u_1 S + I \,.\, u_2 = 0$, where S denotes the circumcircle and I the line at infinity. Hence, the locus is a *circular cubic* having u_1 parallel to its asymptote.*

* *Vide* Basset: *Cubic and Quartic Curves*, §121.

6. The tangents at the node are obtained by equating the coefficient of α to zero. They are $\beta^2 - \gamma^2 = 0$, which denotes the internal and external bisectors of the angle A.

7. The cubic passes through the feet of perpendiculars from A on the coordinate axes, viz., D and E.

This is obvious from §1, since AB, AC subtend equal and supplementary angles at these points.

8. The tangents at B and C are the reflections of BC in AB, AC respectively.

This follows from the geometrical property; and may be deduced analytically thus:—The tangent at B is found by equating the coefficient of β^2 to zero and is $\alpha + 2\gamma\cos B = 0$.

If this makes θ with BC,

$$\sin\theta = 2\sin(\theta - B)\cos B = \sin\theta - \sin\overline{2B - \theta}$$

$$\therefore \quad \theta = 2B.$$

Similarly for the other tangent.

9. The tangent at D bisects the line joining the mid-point of BC to the orthocentre, that is, is parallel to OA′, where A′ is the intersection of AD and the circumcircle. (Fig. 2.)

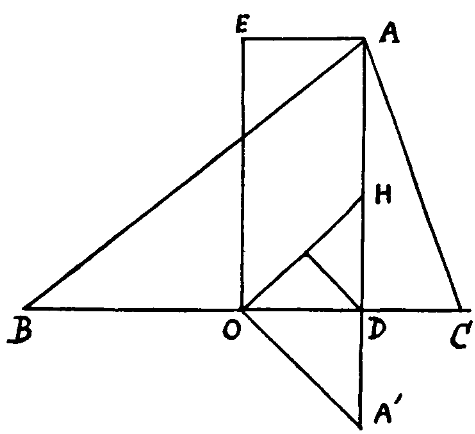

Fig. 2.

The cartesian equation of the tangent gives for its *slope* $(h^2 - \delta^2)/hk$, which is equal to $-(A'D/OD) = -\tan A'OD = -\tan HOD$.

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10. The node A is the centre of curvature, at the point of contact, of the confocal which *touches* the cubic.

For, the normals to the confocal at its intersections with the cubic always pass through A by §2. When two of the points of intersection coincide, the corresponding normals are consecutive and their intersection is the centre of curvature.

Or, the line joining A to the point of contact is normal to the cubic.

11. The inverse of the cubic with respect to the node is a *rectangular hyperbola.* Transferring the origin to A, the cartesian equation of the cubic is written

$$(x^2+y^2+hx+ky)(hy-kx)=\delta^2 . xy.$$

Therefore the inverse with respect to a circle of radius ρ is

$$\rho^2\frac{(hy-kx)}{r^2}\left\{\frac{\rho^4}{r^2}+\rho^2\frac{(hx+ky)}{r^2}\right\}=\frac{\delta^2\rho^4}{r^4}xy.$$

$$\therefore\quad (\rho^2+hx+ky)(hy-kx)=\delta^2xy$$

$$\therefore\quad hk(x^2-y^2)-(h^2-k^2-\delta^2)xy-\rho^2(hy-kx)=0,$$

which is a rectangular hyperbola (1) whose asymptotes are parallel to the bisectors of angle A, and (2) which touches the median through A.

The focal conic of the cubic.

12. Reciprocating the rectangular hyperbola with respect to A we get a parabola, since the origin is on the hyperbola.

Now, the cubic being the inverse of the hyperbola, the reciprocal parabola is the negative pedal of the circular cubic. In other words, the cubic is the *first positive pedal* of the parabola. Since the cubic may be regarded as the locus of the reflections of A in the tangent to a parabola, that parabola is its *focal parabola.* That is, the focal parabola is the envelope of lines bisecting at right angles the joins of A to different points of the cubic. Hence, the focal conic is the parabola touched by the lines bisecting at right angles AB, AC, AD, AE.

13. From the reciprocal properties we see

(i) that the *focal parabola* touches the bisectors of angle A. Thus A is on the *directrix* of the parabola.

(ii) Again, since the median touches the hyperbola the *axis* of the parabola is $\perp^r$ to it. That is, the median is the directrix of the parabola.

(iii) The *focus* of the parabola is the point *corresponding* to the *centre* of the hyperbola.

14. The *focus* of the parabola is the middle point of the symmedian chord of the circumcircle.

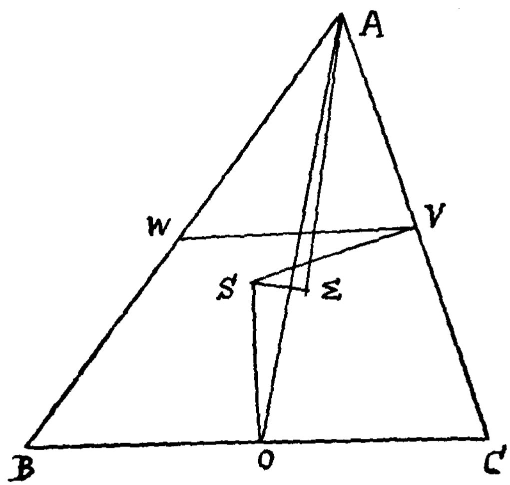

Fig. 3.

For in (Fig. 3) if AΣ be the symmedian through A, AO, AΣ are equally inclined to the bisectors of A, that is, to the tangents to the parabola. Hence, the focus lies on AΣ.

Again, if S be the circumcentre and V, W the mid points of the sides, the parabola touches SV, SW, VW by §12. Therefore the focus lies on the circle SVW, which passes through A. Therefore the focus Σ is the foot of the $\perp^r$ from S on AΣ.

15. The *asymptote* of the cubic is parallel to the median AO, that is, to the directrix of the focal parabola, and is as far behind it as the focus is in front of it.

This may be deduced from the cartesian equation of the cubic.

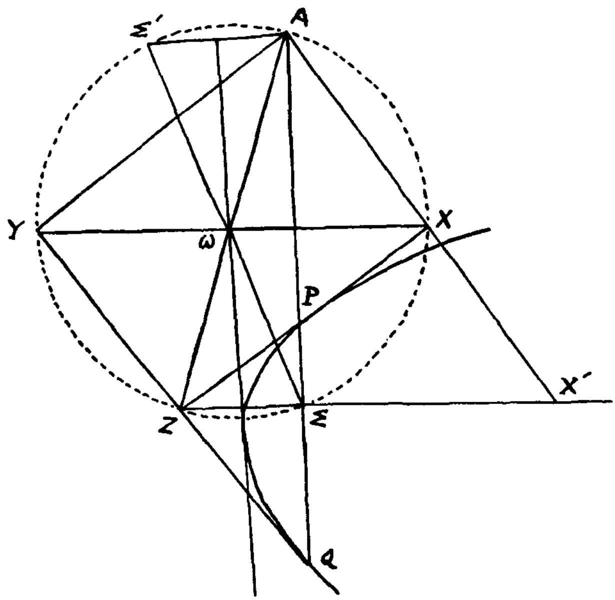

Fig. 4.

16. In Fig. 4 let the tangents at the ends of the focal chord APΣQ intersect at Z on the directrix. Then the circle AΣZ being drawn, Σ' (the point diametrically opposite to Σ) is the point in which the asymptote cuts the cubic.

Drop $\perp^{rs}$AX, AY on the tangents at P and Q and let ω be the centre of the circle. Then, since ω lies on the directrix, the bisector of angle ΣωZ touches the parabola and is evidently $\perp^{r}$ to AΣ'. Hence, Σ' is the reflection of A in this tangent, and therefore lies on the cubic. It also lies on the asymptote from §15. Thus Σ' is the intersection of the asymptote and the cubic.

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17. Again, if X', Y' be the reflections of A in the tangents at P and Q, the tangents to the cubic at X', Y' are easily seen to be parallel to the asymptote. For the angle between AX' and the tangent at $X' = \angle APX = \angle ZP\Sigma = ZAX$, since $\angle \Sigma z\rho = \angle PZA$. Therefore, the tangent at X' is parallel to AZ. Similarly, for the tangent at Y'. Also, evidently, X', Y' lie on $Z\Sigma$ produced both ways.

Hence X', Y', two centres of inversion, are the intersections of $Z\Sigma$ and the cubic, two other centres coinciding with A.*

18. $\Sigma\Sigma'$ *is the tangent at* Σ *to the cubic.*

This follows readily from the fact that Σ is the reflection of A in XY, which touches the parabola.

19. *The point of inflexion of the cubic.*

The point on the focal parabola corresponding to the point of inflexion may be found as follows:—

Let accented letters refer to the cubic, A being the origin. Then

$$\begin{aligned} \rho' &= 2r' . dr'/dp' \\ &= 2p . dp/(2p . dp/r - p^2 dr/r^2) \\ &= 2r/(2 - p\rho/r^2). \end{aligned}$$

Hence ρ' is infinite when $2r^2 = p\rho$.

Now, let ρ_1 denote the radius of the circle passing through A, Σ and touching the parabola. Then $2\rho_1 . p = r^2$, where p, r refer to the point of contact. Also $\rho = 4\rho_1$, since the focal chord of curvature is four times the focal distance of the point of contact. Thus, $2r^2 = p\rho$.

In other words, the point on the focal parabola corresponding to the point of inflexion is the point of contact of the circle through A and Σ.

20. From §§ 11, 12, it follows that when tangents are drawn from a given point A to a system of confocals (foci B, C) the corresponding normals envelope a *parabola*, and the inverses of the points of contact lie on a rectangular hyperbola passing through A. Also, if normals be drawn from A to a system of confocals the corresponding tangents envelope a parabola, and the inverses of the feet of normals lie on a rectangular hyperbola through A.

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* *Vide* **Basset** : *Cubic and Quartic Curves*, §235.

21. The form of the cubic when ABC is an acute-angled triangle is shown in Fig. 5.

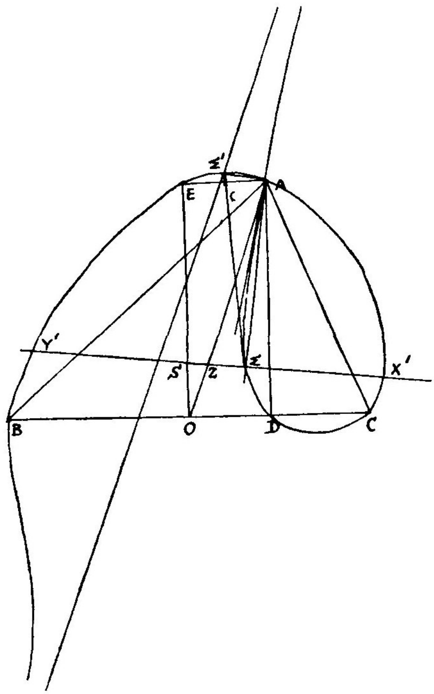

Fig. 5.

Postscript.—If the tangents at B and C meet at T, A is obviously the centre of a circle touching the sides of BTC. Hence AB, AC subtend equal angles at T, and therefore T is a point on the cubic. Also, AT meets the circumcircle of TBC at S, such that SA = SB = SC. That is, AT passes through the circumcentre S of ABC. Further, from the property of a *satellite*, it follows that the tangents at D and T intersect on the curve—properties kindly brought to my notice by Professor K. J. Sanjana, M.A., of Bhavnagar.

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