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An existence result for the mean-field equation on compact surfaces in a doubly supercritical regime

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We consider a class of variational equations with exponential nonlinearities on a compact Riemannian surface, describing the mean-field equation of the equilibrium turbulence with arbitrarily signed vortices. For the first time, we consider the problem with both supercritical parameters and we give an existence result by using variational methods. In doing so, we present a new Moser–Trudinger-type inequality under suitable conditions on the centre of mass and the scale of concentration of both e^u and e^{-u}, where u is the unknown function in the equation.

1. Introduction

We consider the equation

$$-\Delta_g u = \rho_1\left(\frac{h_1(x)e^u}{\int_\Sigma h_1(x)e^u \, dV_g} - \frac{1}{|\Sigma|}\right) - \rho_2\left(\frac{h_2(x)e^{-u}}{\int_\Sigma h_2(x)e^{-u} \, dV_g} - \frac{1}{|\Sigma|}\right) \quad \text{on } \Sigma, \tag{1.1}$$

where ρ_1, ρ_2 are two non-negative parameters, $h_1, h_2 \colon \Sigma \to \mathbb{R}$ are two smooth positive functions and Σ is a compact orientable surface without boundary with Riemannian metric g and volume $|\Sigma|$.

This equation arises in mathematical physics as a mean-field equation of the equilibrium turbulence with arbitrarily signed vortices, and was obtained by Joyce and Montgomery [10] and by Pointin and Lundgren [17] from different statistical arguments. Later, many authors worked on this model; see, for example, [3,11,14,16] and the references therein.

Equation (1.1) has a variational structure, and solutions can be found as critical points of the functional

$$\begin{aligned} I_{\rho_1,\rho_2}(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 \, dV_g - \rho_1 \log \int_\Sigma h_1(x)e^u \, dV_g - \rho_2 \log \int_\Sigma h_2(x)e^{-u} \, dV_g \\ + \rho_1 \int_\Sigma u \, dV_g - \rho_2 \int_\Sigma u \, dV_g, \quad u \in H^1(\Sigma), \end{aligned} \tag{1.2}$$

where we have normalized the volume $|\Sigma|$ of Σ by $|\Sigma| = 1$. The structure of the functional I_{ρ_1,ρ_2} strongly depends on the parameters ρ_1, ρ_2. A Moser–Trudinger-type inequality relative to this functional was proved in [16], and one has that

$$\log \int_\Sigma e^{u-\bar{u}} \, dV_g + \log \int_\Sigma e^{-u+\bar{u}} \, dV_g \leqslant \frac{1}{16\pi}\int_\Sigma |\nabla_g u|^2 \, dV_g + C_\Sigma,$$

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where $\bar{u}$ denotes the average of u. By the above inequality, if we consider the case $(\rho_1, \rho_2) \in (0, 8\pi) \times (0, 8\pi)$, the functional I_{ρ_1,ρ_2} is bounded from below and coercive; hence, solutions can be found as global minima.

The value 8π, or more generally $8\pi\mathbb{N}$, is *critical* and the existence problem becomes subtler due to a loss of compactness. Even in the case $\rho_2 = 0$, namely, the Liouville-type problem

$$-\Delta_g u = \rho\left(\frac{h(x)\mathrm{e}^u}{\int_\Sigma h(x)\mathrm{e}^u \,\mathrm{d}V_g} - \frac{1}{|\Sigma|}\right) \quad \text{on } \Sigma, \tag{1.3}$$

the existence problem is a difficult one (see [1, 4, 15]). To solve (1.1) (or (1.3)) in this critical case, one always needs geometry conditions (see [4, 20]). For example, for (1.1), with $\rho_1 = 8\pi$ and $\rho_2 \in (0, 8\pi]$, in [20] Zhou gave an existence result under suitable conditions on the Gaussian curvature $K(x)$ of Σ, namely, $K(x)$ should satisfy

$$8\pi - \rho_2 - 2K(x) > 0 \quad \text{for } x \in \Sigma.$$

If $\rho_i > 8\pi$ for some $i = 1, 2$, then I_{ρ_1,ρ_2} is unbounded from below and a minimization technique is no longer possible. In general, one needs to apply variational methods to obtain the existence of critical points (generally of saddle type) for I_{ρ_1,ρ_2}.

The case with $\rho_2 = 0$ (for instance (1.3)) has been much studied in the literature. Again, the problem has a variational structure and the associated functional is given by

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 \,\mathrm{d}V_g + \rho\int_\Sigma u \,\mathrm{d}V_g - \rho\log\int_\Sigma h(x)\mathrm{e}^u \,\mathrm{d}V_g.$$

There are by now many results regarding existence, compactness of solutions, bubbling behaviour, etc. (see [5, 6, 12, 19]). In particular, we have the existence of solutions for (1.3) for $\rho \in (8k\pi, 8(k+1)\pi)$, with $k \geqslant 1$ (see, for example, [12]). This existence result is based on a detailed study of the topology of large negative sublevels of the functional I_ρ. It is indeed possible to find a homotopy equivalence between these sublevels and the so-called *space of formal baricentres* Σ_k, namely, the family of elements

$$\sum_{i=1}^k t_i\delta_{x_i} \quad \text{with } (x_i)_i \subset \Sigma \qquad \text{and} \qquad \sum_{i=1}^k t_i = 1, \quad t_i \geqslant 0.$$

Exploiting the fact that the set Σ_k is non-contractible, it is then possible to introduce a min–max scheme based on this set.

On the other hand, in the case when $\rho_2 \neq 0$ and $\rho_i > 8\pi$ for some $i = 1, 2$ there are very few results. We point some of them out here. The first is given in [9] and concerns the case $\rho_1 \in (8\pi, 16\pi)$ and $\rho_2 < 8\pi$. Via a blow-up analysis Jost *et al*. proved the existence of solutions for (1.1) on a smooth, bounded, non-simply connected domain Σ in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition. Later, in [21], Zhou generalized this result to any compact surface without boundary by using analogous variational methods to those employed in the study of (1.3). In a certain sense, one can describe the topology of negative sublevels of the functional I_{ρ_1,ρ_2} using the behaviour of the function e^u.

The general blow-up behaviour of solutions of (1.1) is not yet fully developed. However, as in the case for $\rho_2 = 0$, in [9] Jost *et al*. exhibited a volume quantization.

More precisely, they proved that the blow-up values are multiples of 8π (see the proof of theorem 2.1 for the definition of the blow-up value). In a similar way, by using a local quantization proved in [16], in § 2 we deduce a global blow-up value for the case when $\rho_1, \rho_2 \in (8\pi, 16\pi)$.

We then turn to the existence issue, and via a min–max scheme we obtain a positive result without any geometry and topology conditions. Our main theorem is the following.

THEOREM 1.1. *Assume that* $\rho_1, \rho_2 \in (8\pi, 16\pi)$. *It follows that there exists a solution to (1.1).*

The method to prove this existence result relies on a min–max scheme introduced by Malchiodi and Ruiz in [13] for the study of Toda systems. Such a scheme is based on study of the topological properties of the low sublevels of I_{ρ_1,ρ_2}.

We shall see that on low sublevels of I_{ρ_1,ρ_2} at least one of the functions e^u or e^{-u} is very concentrated around some point of Σ. Moreover, both e^u and e^{-u} can concentrate at two points that could eventually coincide, but in this case the *scale of concentration* must be different. Roughly speaking, if e^u and e^{-u} concentrate around the same point at the same rate, then I_{ρ_1,ρ_2} is bounded from below. We next make this statement more formal.

First, following the argument in [13], we define a continuous *rate of concentration* $\sigma = \sigma(f)$ of a positive function $f \in \Sigma$, normalized in L^1. Somehow, the smaller σ is, the higher the rate of concentration of f. Moreover, we define a continuous *centre of mass* $\beta = \beta(f) \in \Sigma$. This can be done when $\sigma \leqslant \delta$ for some fixed δ; therefore, we have a map $\psi\colon H^1(\Sigma) \to \bar{\Sigma}_\delta$,

$$\psi(u) = (\beta(f_1), \sigma(f_1)), \qquad \psi(-u) = (\beta(f_2), \sigma(f_2)),$$

where we have set

$$f_1 = \frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u \,\mathrm{d}V_g}, \qquad f_2 = \frac{\mathrm{e}^{-u}}{\int_\Sigma \mathrm{e}^{-u} \,\mathrm{d}V_g}.$$

Here $\bar{\Sigma}_\delta$ is the topological cone over Σ, where we make the identification to a point when $\sigma \geqslant \delta$ for some $\delta > 0$ fixed (see (2.1)).

The improvement of the Moser–Trudinger inequality discussed above is made rigorous in the following way: if $\psi(f_1) = \psi(f_2)$, then $I_{\rho_1,\rho_2}(u)$ is bounded from below (see proposition 3.6). The proof is based on local versions of the Moser–Trudinger inequality on small balls and on annuli with small internal radius. We point out that our improved inequality is scaling invariant, as opposed to those proved by Chen and Li [2] and Zhou [21].

Using this fact, for $L > 0$ large we can introduce a continuous map

$$I^{-L}_{\rho_1,\rho_2} \xrightarrow{(\psi,\psi)} X := (\bar{\Sigma}_\delta \times \bar{\Sigma}_\delta) \setminus \bar{D},$$

where $\bar{D}$ is the diagonal of $\bar{\Sigma}_\delta \times \bar{\Sigma}_\delta$ and $I^{-L}_{\rho_1,\rho_2} = \{u \in H^1(\Sigma)\colon I_{\rho_1,\rho_2}(u) < -L\}$. On the other hand, it is also possible to do the converse, namely, to map (a retraction of) the set X into appropriate sublevels of I_{ρ_1,ρ_2}. In § 4 we construct a family of new test functions parametrized on (a suitable subset of) X, on which I_{ρ_1,ρ_2} attains arbitrarily low values; see proposition 4.4. Letting

$$X \xrightarrow{\phi} I^{-L}_{\rho_1,\rho_2}$$

be the corresponding map, it turns out that the composition of these two maps is homotopic to the identity on X; see proposition 4.7.

Exploiting the fact that X is non-contractible, we can introduce a min–max argument to find a critical point of I_{ρ_1,ρ_2}. In this framework, an essential point is to use the 'monotonicity argument' introduced by Struwe in [18] together with the compactness result of solutions proved in § 2, since it is not known whether the Palais–Smale condition holds or not.

2. Notation and preliminaries

In this section we fix our notation and recall some useful preliminary facts. Throughout the paper, Σ stands for a compact orientable surface without boundary with metric g. For simplicity, we normalize the volume $|\Sigma|$ of Σ by $|\Sigma| = 1$. We state, in particular, some variants and improvements of the Moser–Trudinger-type inequality and some of their consequences.

We write $d(x, y)$ to denote the distance between two points $x, y \in \Sigma$. In the same way, for any $p \in \Sigma$ and $\Omega, \Omega' \subseteq \Sigma$, we define

$$d(p,\Omega) = \inf\{d(p,x) \colon x \in \Omega\}, \qquad d(\Omega,\Omega') = \inf\{d(x,y) \colon x \in \Omega,\ y \in \Omega'\}.$$

Moreover, $B_p(r)$ denotes the open metric ball of radius r and centre p, while $A_p(r, R)$ stands for the open annulus of radii r and R, $r < R$. The complement of a set Ω in Σ will be denoted by Ω^{c}.

Recalling that we are assuming that $|\Sigma| = 1$, given a function $u \in L^1(\Sigma)$, we denote its average as

$$\bar{u} = \int_\Sigma u \,\mathrm{d}V_g.$$

Given $\delta > 0$, we define the topological cone

$$\bar{\Sigma}_\delta = (\Sigma \times (0,+\infty))/(\Sigma \times [\delta,+\infty)), \tag{2.1}$$

where the equivalence relation identifies $\Sigma \times [\delta, +\infty)$ to a single point.

Throughout the paper we denote by C large constants that are allowed to vary between different formulae or even within lines. When we want to stress the dependence of the constants on some parameter(s), we add subscripts to C (e.g. C_δ, etc.). Also, constants with subscripts are allowed to vary. Moreover, sometimes we shall write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$, depending on the case. We shall similarly use the symbol $O_\alpha(1)$ for bounded quantities.

We begin with a compactness result that is deduced from the blow-up theorem in [16].

Theorem 2.1. *Suppose that u_n satisfies*

$$-\Delta_g u_n = \rho_{1,n}\left(\frac{h_1(x)\mathrm{e}^{u_n}}{\int_\Sigma h_1(x)\mathrm{e}^{u_n}\,\mathrm{d}V_g} - \frac{1}{|\Sigma|}\right) - \rho_{2,n}\left(\frac{h_2(x)\mathrm{e}^{-u_n}}{\int_\Sigma h_2(x)\mathrm{e}^{-u_n}\,\mathrm{d}V_g} - \frac{1}{|\Sigma|}\right) \quad on\ \Sigma.$$

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Assume that $\rho_{1,n}, \rho_{2,n} \in (8\pi, 16\pi)$ for any $n \in \mathbb{N}$ and that $\rho_{1,n} \to \rho_1 \in (8\pi, 16\pi)$ and $\rho_{2,n} \to \rho_2 \in (8\pi, 16\pi)$. The solution sequence $(u_n)_n$ (up to adding suitable constants) is then uniformly bounded in $L^\infty(\Sigma)$ and there exist u and a subsequence $(u_{n_k})_k$ such that

$$u_{n_k} \to u,$$

where this u is a solution to (1.1) for these ρ_1 and ρ_2.

Proof. Since I_{ρ_1,ρ_2} is invariant under translation by constants in the argument, we can restrict ourselves to considering the subspace of $H^1(\Sigma)$ of functions with zero average.

Consider the blow-up sets of the sequence $(u_n)_n$ given by

$$S_1 = \{x \in \Sigma\colon \exists x_n \to x \text{ such that } u_n(x_n) \to +\infty\},$$
$$S_2 = \{x \in \Sigma\colon \exists x_n \to x \text{ such that } u_n(x_n) \to -\infty\}.$$

From the blow-up theorem in [16], it is sufficient to show that $S_1 \cap S_2 = \emptyset$. We argue by contradiction. Assume that $x_0 \in S_1 \cap S_2$. Define the blow-up values at x_0 by

$$m_1(x_0) = \lim_{r\to 0} \lim_{n\to+\infty} \int_{B_r(x_0)} \frac{\rho_{1,n} h_1(x) \mathrm{e}^{u_n}}{\int_\Sigma h_1(x) \mathrm{e}^{u_n} \, \mathrm{d}V_g} \, \mathrm{d}V_g,$$
$$m_2(x_0) = \lim_{r\to 0} \lim_{n\to+\infty} \int_{B_r(x_0)} \frac{\rho_{2,n} h_2(x) \mathrm{e}^{-u_n}}{\int_\Sigma h_2(x) \mathrm{e}^{-u_n} \, \mathrm{d}V_g} \, \mathrm{d}V_g.$$

Since $\rho_{1,n}, \rho_{2,n} \in (8\pi, 16\pi)$, from the blow-up theorem in [16], we have that

$$4\pi \leqslant m_1(x_0) < 16\pi, \qquad 4\pi \leqslant m_2(x_0) < 16\pi, \tag{2.2}$$

and

$$(m_1(x_0) - m_2(x_0))^2 = 8\pi(m_1(x_0) + m_2(x_0)). \tag{2.3}$$

By the last equality we derive

$$m_1(x_0) = m_2(x_0) + 4\pi \pm 4\sqrt{\pi m_2(x_0) + \pi^2}.$$

First, we consider the case $m_1(x_0) = m_2(x_0) + 4\pi + 4\sqrt{\pi m_2(x_0) + \pi^2}$. Using the fact that $4\pi \leqslant m_2(x_0)$, we derive that $m_1(x_0) \geqslant 16\pi$, which contradicts the first estimate in (2.2).

If, instead, we consider the case $m_1(x_0) = m_2(x_0) + 4\pi - 4\sqrt{\pi m_2(x_0) + \pi^2}$, the estimate $4\pi \leqslant m_2(x_0) < 16\pi$ implies that $m_1(x_0) < 12\pi$. By interchanging the roles of $m_1(x_0)$ and $m_2(x_0)$, we obtain the same inequality for $m_2(x_0)$. Therefore, we have that

$$4\pi \leqslant m_1(x_0) < 12\pi, \qquad 4\pi \leqslant m_2(x_0) < 12\pi. \tag{2.4}$$

On the other hand, using (2.3) together with the fact that $m_i(x_0) \geqslant 4\pi$, $i = 1, 2$, we deduce that

$$|m_1(x_0) - m_2(x_0)| \geqslant 8\pi,$$

which contradicts (2.4). □

Next, we recall some Moser–Trudinger-type inequalities by starting with the standard one, i.e. for $u \in H^1(\Sigma)$ it holds that

$$\log \int_\Sigma \mathrm{e}^{u-\bar{u}}\,\mathrm{d}V_g \leqslant \frac{1}{16\pi}\int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C_\Sigma. \tag{2.5}$$

As observed in § 1, (1.1) is the Euler–Lagrange equation of the functional I_{ρ_1,ρ_2} given in (1.2). Consider the space

$$\bar{H}^1(\Sigma) = \left\{u \in H^1(\Sigma)\colon \int_\Sigma u\,\mathrm{d}V_g = 0\right\},$$

for which the following result was proved by Ohtsuka and Suzuki in [16].

THEOREM 2.2. *The functional I_{ρ_1,ρ_2} is bounded from below on $\bar{H}^1(\Sigma)$ if and only if $\rho_i \leqslant 8\pi$, $i = 1, 2$.*

In view of this result, similarly to inequality (2.5), it is possible to also obtain a Moser–Trudinger inequality with e^u and e^{-u} simultaneously. Namely, for $u \in H^1(\Sigma)$ it holds that

$$\log \int_\Sigma \mathrm{e}^{u-\bar{u}}\,\mathrm{d}V_g + \log \int_\Sigma \mathrm{e}^{-u+\bar{u}}\,\mathrm{d}V_g \leqslant \frac{1}{16\pi}\int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C_\Sigma. \tag{2.6}$$

It is well known that an improved inequality will hold if e^u has an integral bounded from below on different regions of Σ of positive mutual distance.

PROPOSITION 2.3 (Zhou [21]). *For a fixed integer l, let $\Omega_1, \dots, \Omega_l$ be subsets of Σ satisfying $d(\Omega_i, \Omega_j) \geqslant \delta_0$ for $i \neq j$, where δ_0 is a positive real number, and let $\gamma_0 \in (0, 1/l)$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\Sigma, l, \varepsilon, \delta_0, \gamma_0)$ such that*

$$l \log \int_\Sigma \mathrm{e}^{u-\bar{u}}\,\mathrm{d}V_g + \log \int_\Sigma \mathrm{e}^{-u+\bar{u}}\,\mathrm{d}V_g \leqslant \frac{1}{16\pi - \varepsilon}\int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} \mathrm{e}^u\,\mathrm{d}V_g}{\int_\Sigma \mathrm{e}^u\,\mathrm{d}V_g} \geqslant \gamma_0 \quad \forall i \in \{1, \dots, l\}.$$

We next state a result that is a local version of (2.6), which will be of use later on.

PROPOSITION 2.4. *Fix $\delta > 0$, and let $\Omega_1 \subset \Omega_2 \subset \Sigma$ be such that $d(\Omega_1, \partial\Omega_2) \geqslant \delta$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta)$ such that, for all $u \in H^1(\Sigma)$,*

$$\log \int_{\Omega_1} \mathrm{e}^{u}\,\mathrm{d}V_g + \log \int_{\Omega_1} \mathrm{e}^{-u}\,\mathrm{d}V_g \leqslant \frac{1}{16\pi - \varepsilon}\int_{\Omega_2} |\nabla_g u|^2\,\mathrm{d}V_g + C.$$

Proof. The proof is developed exactly as in [13, proposition 2.3], with obvious modifications. Here, we just sketch the proof for the reader's convenience. First, we consider a spectral decomposition of the Laplacian on Ω_2 (with Neumann boundary conditions), in order to write u as $u = v + w$, with $v \in L^\infty(\Omega_2)$ and $w \in H^1(\Omega_2)$. We next consider a smooth cut-off function χ with values into $[0,1]$, satisfying

$$\chi(x) = \begin{cases} 1, & x \in \Omega_1, \\ 0, & d(x, \Omega) > \delta/2, \end{cases}$$

and then define $\tilde{w}(x) = \chi(x) w(x)$. We now apply the Moser–Trudinger inequality (2.6) to $\tilde{w}$ to deduce the desired inequality. □

We now give a criterion that is a first step in studying the properties of the low sublevels of I_{ρ_1,ρ_2}. We first state a lemma concerning a covering argument, which is a particular case of a more general setting in [13, lemma 2.5].

LEMMA 2.5. *Let $\delta_0 > 0$, $\gamma_0 > 0$ be fixed, and let $\Omega_{i,j} \subseteq \Sigma$, $i, j = 1, 2$, satisfy $d(\Omega_{i,j}, \Omega_{i,k}) \geqslant \delta_0$ for $j \neq k$. Suppose that $u \in H^1(\Sigma)$ is a function verifying*

$$\frac{\int_{\Omega_{1,j}} \mathrm{e}^u \,\mathrm{d}V_g}{\int_\Sigma \mathrm{e}^u \,\mathrm{d}V_g} \geqslant \gamma_0, \quad \frac{\int_{\Omega_{2,j}} \mathrm{e}^{-u} \,\mathrm{d}V_g}{\int_\Sigma \mathrm{e}^{-u} \,\mathrm{d}V_g} \geqslant \gamma_0, \quad j = 1, 2.$$

There then exist positive constants $\tilde{\gamma}_0$, $\tilde{\delta}_0$, depending only on γ_0, δ_0, and two sets $\tilde{\Omega}_1, \tilde{\Omega}_2 \subseteq \Sigma$, depending also on u, such that

$$d(\tilde{\Omega}_1, \tilde{\Omega}_2) \geqslant \tilde{\delta}_0, \quad \frac{\int_{\tilde{\Omega}_i} \mathrm{e}^u \,\mathrm{d}V_g}{\int_\Sigma \mathrm{e}^u \,\mathrm{d}V_g} \geqslant \tilde{\gamma}_0, \quad \frac{\int_{\tilde{\Omega}_i} \mathrm{e}^{-u} \,\mathrm{d}V_g}{\int_\Sigma \mathrm{e}^{-u} \,\mathrm{d}V_g} \geqslant \tilde{\gamma}_0, \quad i = 1, 2.$$

Using this result it is indeed possible to obtain an improvement of the constant in the Moser–Trudinger inequality (2.6).

PROPOSITION 2.6. *Let $u \in H^1(\Sigma)$ be a function satisfying the assumptions of lemma 2.5 for some positive constants δ_0, γ_0. Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$, depending on ε, δ_0 and γ_0 such that*

$$\log \int_\Sigma \mathrm{e}^{u - \bar{u}} \,\mathrm{d}V_g + \log \int_\Sigma \mathrm{e}^{-u + \bar{u}} \,\mathrm{d}V_g \leqslant \frac{1}{32\pi - \varepsilon} \int_\Sigma |\nabla_g u|^2 \,\mathrm{d}V_g + C.$$

Proof. To obtain the thesis we can argue exactly as in [13, proposition 2.6]. First we set $\tilde{\delta}_0$, $\tilde{\gamma}_0$ and $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ as in lemma 2.5. We then apply proposition 2.4 with $\tilde{\Omega}_i$ and $U_i = \{x \in \Omega \colon d(x, \tilde{\Omega}_i) < \tilde{\delta}_0/2\}$ for $i = 1, 2$. Observing that

$$\log \int_{\tilde{\Omega}_i} \mathrm{e}^u \,\mathrm{d}V_g \geqslant \log \left(\int_\Sigma \mathrm{e}^u \,\mathrm{d}V_g \right) + \log \tilde{\gamma}_0,$$
$$\log \int_{\tilde{\Omega}_i} \mathrm{e}^{-u} \,\mathrm{d}V_g \geqslant \log \left(\int_\Sigma \mathrm{e}^{-u} \,\mathrm{d}V_g \right) + \log \tilde{\gamma}_0$$

for $i = 1, 2$, and that $U_1 \cap U_2 = \emptyset$, we deduce the thesis. □

Proposition 2.6 implies that on low sublevels of the functional I_{ρ_1,ρ_2}, at least one of the components of the couple $(\mathrm{e}^u, \mathrm{e}^{-u})$ must be very concentrated around a certain point. In the following we present a more detailed description of the topology of low sublevels.

3. Improved inequality

Following the ideas presented by Malchiodi and Ruiz in [13], in this section we exhibit an improved Moser–Trudinger inequality under suitable conditions of concentration of the involved function.

First, we give continuous definitions of the *centre of mass* and *scale of concentration* of positive functions normalized in L^1. We consider the set

$$A = \left\{ f \in L^1(\Sigma) \colon f > 0 \text{ almost everywhere and } \int_\Sigma f \, \mathrm{d}V_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$. We then have the following result.

Proposition 3.1 (Malchiodi and Ruiz [13]). *We fix a constant $R > 1$. There then exist $\delta = \delta(R) > 0$ and a continuous map*

$$\psi \colon A \to \bar{\Sigma}_\delta, \qquad \psi(f) = (\beta, \sigma),$$

satisfying that for any $f \in A$ there exists $p \in \Sigma$ such that the following hold:

(a) $d(p, \beta) \leqslant C'\sigma$ *for* $C' = \max\{3R + 1, \delta^{-1} \operatorname{diam}(\Sigma)\}$,

(b)

$$\int_{B_p(\sigma)} f \, \mathrm{d}V_g > \tau, \qquad \int_{B_p(R\sigma)^{\mathrm{c}}} f \, \mathrm{d}V_g > \tau,$$

where $\tau > 0$ depends only on R and Σ.

This result is obtained in several steps, which we summarize in the following. The explicit definition of the map $\psi(f) = (\beta, \sigma)$ is given below.

First, take $R_0 = 3R$, and define $\sigma \colon A \times \Sigma \to (0, +\infty)$ such that

$$\int_{B_x(\sigma(x,f))} f \, \mathrm{d}V_g = \int_{B_x(R_0\sigma(x,f))^{\mathrm{c}}} f \, \mathrm{d}V_g. \tag{3.1}$$

The map $\sigma(x, f)$ is clearly uniquely determined and continuous. Moreover, we have the following lemma.

Lemma 3.2 (Malchiodi and Ruiz [13]). *The map σ satisfies*

$$d(x, y) \leqslant R_0 \max\{\sigma(x, f), \sigma(y, f)\} + \min\{\sigma(x, f), \sigma(y, f)\}. \tag{3.2}$$

We now define

$$T \colon A \times \Sigma \to \mathbb{R}, \qquad T(x, f) = \int_{B_x(\sigma(x,f))} f \, \mathrm{d}V_g.$$

Lemma 3.3 (Malchiodi and Ruiz [13]). *If $x_0 \in \Sigma$ exists such that*

$$T(x_0, f) = \max_{y \in \Sigma} T(y, f),$$

then we have that $\sigma(x_0, f) < 3\sigma(x, f)$ for any other $x \in \Sigma$.

As a consequence of the previous lemma, one can obtain the following.

LEMMA 3.4 (Malchiodi and Ruiz [13]). *There exists a fixed $\tau > 0$ such that*

$$\max_{x\in\Sigma} T(x,f) > \tau > 0 \quad \textit{for all } f \in A.$$

We define

$$\sigma\colon A \to \mathbb{R}, \qquad \sigma(f) = 3\min\{\sigma(x,f)\colon x \in \Sigma\},$$

which is obviously a continuous function. Given τ as in lemma 3.4, consider the set

$$S(f) = \{x \in \Sigma\colon T(x,f) > \tau,\ \sigma(x,f) < \sigma(f)\}, \tag{3.3}$$

which is a non-empty open set for any $f \in A$, by lemmas 3.3 and 3.4. Moreover, from (3.2), we have that

$$\operatorname{diam}(S(f)) \leqslant (R_0 + 1)\sigma(f). \tag{3.4}$$

By the Nash embedding theorem, we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically, $N \in \mathbb{N}$. Take an open tubular neighbourhood $\Sigma \subset U \subset \mathbb{R}^N$ of Σ, and take $\delta > 0$ small enough that

$$\operatorname{co}[B_x((R_0+1)\delta) \cap \Sigma] \subset U \quad \forall x \in \Sigma, \tag{3.5}$$

where 'co' denotes the convex hull in $\mathbb{R}^N$.

We now define

$$\eta(f) = \frac{\int_\Sigma (T(x,f)-\tau)^+(\sigma(f)-\sigma(x,f))^+ x \,\mathrm{d}V_g}{\int_\Sigma (T(x,f)-\tau)^+(\sigma(f)-\sigma(x,f))^+ \,\mathrm{d}V_g} \in \mathbb{R}^N.$$

The map η defines a sort of centre of mass in $\mathbb{R}^N$. Observe that the integrands become non-zero only on the set $S(f)$. Moreover, whenever $\sigma(f) \leqslant \delta$, (3.4) and (3.5) imply that $\eta(f) \in U$, and so we can define

$$\beta\colon \{f \in A\colon \sigma(f) \leqslant \delta\} \to \Sigma, \qquad \beta(f) = P \circ \eta(f),$$

where $P\colon U \to \Sigma$ is the orthogonal projection.

The map $\psi(f) = (\beta(f), \sigma(f))$ then satisfies the conditions given by proposition 3.1. If $\sigma(f) \geqslant \delta$, β is not defined. Observe that (a) is then satisfied for any $\beta \in \Sigma$.

REMARK 3.5. The above map $\psi(f) = (\beta, \sigma)$ gives us a centre of mass of f and its scale of concentration around that point. The identification in $\bar{\Sigma}_\delta$ is somehow natural, indeed, if σ exceeds a certain positive constant, we do not have concentration at a point and so β cannot be defined.

We next state an improved Moser–Trudinger inequality for functions $u \in H^1(\Sigma)$ such that both e^u and e^{-u} are concentrated at the same point with the same rate of concentration. In terms of proposition 3.1, we have the following result.

PROPOSITION 3.6. *Given any $\varepsilon > 0$, there exist $R = R(\varepsilon) > 1$ and ψ as given in proposition 3.1, such that, for any $u \in H^1(\Sigma)$ with*

$$\psi\biggl(\frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u \,\mathrm{d}V_g}\biggr) = \psi\biggl(\frac{\mathrm{e}^{-u}}{\int_\Sigma \mathrm{e}^{-u} \,\mathrm{d}V_g}\biggr),$$

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the following inequality holds:

$$\log \int_{\Sigma} \mathrm{e}^{u-\bar{u}}\, \mathrm{d}V_g + \log \int_{\Sigma} \mathrm{e}^{-u+\bar{u}}\, \mathrm{d}V_g \leqslant \frac{1}{32\pi - \varepsilon} \int_{\Sigma} |\nabla_g u|^2\, \mathrm{d}V_g + C$$

for some $C = C(\varepsilon)$.

Before proving the proposition, we need some preliminary lemmas concerning the Moser–Trudinger-type inequality for small balls, and also for annuli with small internal radius. The former are obtained simply by using a dilation argument.

Lemma 3.7. *For any* $\varepsilon > 0$ *there exists* $C = C(\varepsilon) > 0$ *such that*

$$\log \int_{B_p(s/2)} \mathrm{e}^{u}\, \mathrm{d}V_g + \log \int_{B_p(s/2)} \mathrm{e}^{-u}\, \mathrm{d}V_g \leqslant \frac{1}{16\pi - \varepsilon} \int_{B_p(s)} |\nabla_g u|^2\, \mathrm{d}V_g + 4\log s + C$$

for any $u \in H^1(\Sigma)$, $p \in \Sigma$, $s > 0$ *small.*

Proof. Note that, as $s \to 0$, we consider quantities defined on smaller and smaller geodesic balls $B_p(\xi)$ on Σ. By considering normal geodesic coordinates at p, gradients, averages and the volume element will almost correspond to the Euclidean ones. If we assume that near p the metric of Σ is flat, we get negligible error terms, which will be omitted.

We just perform a convenient dilation of u given by

$$v(x) = u(sx + p).$$

We have the following equalities:

$$\int_{B_p(s)} |\nabla_g u|^2\, \mathrm{d}V_g = \int_{B_0(1)} |\nabla_g v|^2\, \mathrm{d}V_g,$$

$$\int_{B_p(s/2)} \mathrm{e}^{u}\, \mathrm{d}V_g = s^2 \int_{B_0(1/2)} \mathrm{e}^{v}\, \mathrm{d}V_g.$$

We then apply proposition 2.4 to the function v to deduce the desired inequality. □

Remark 3.8. Observe that in lemma 3.7 and in the results presented in the following there is no explicit dependence of the average of u, due to the fact that the average of u is cancelled by the average of $-u$.

We next deduce a Moser–Trudinger-type inequality on thick annuli. In order to do this, we use the Kelvin transform to exploit the geometric properties of the problem.

Lemma 3.9. *Given* $\varepsilon > 0$, *there exists a fixed* $r_0 > 0$ *(depending only on* Σ *and* ε*) satisfying the following property: for any* $r \in (0, r_0)$ *fixed, there exists* $C = C(r, \varepsilon) > 0$ *such that, for any* $u \in H^1(\Sigma)$ *with* $u = c \in \mathbb{R}$ *in* $\partial B_p(2r)$,

$$\log \int_{A_p(s,r)} \mathrm{e}^{u}\, \mathrm{d}V_g + \log \int_{A_p(s,r)} \mathrm{e}^{-u}\, \mathrm{d}V_g \leqslant \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla_g u|^2\, \mathrm{d}V_g - 4\log s + C,$$

with $p \in \Sigma$, $s \in (0, r)$.

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Proof. As in the proof of lemma 3.7, by taking r_0 small enough, here the metric also becomes close to the Euclidean one. We can then assume that the metric is flat around p.

We consider the Kelvin transform $K\colon A_p(s/2, 2r) \to A_p(s/2, 2r)$ given by

$$K(x) = p + rs\frac{x-p}{|x-p|^2}.$$

Observe that K maps the interior boundary of $A_p(s/2, 2r)$ onto the exterior one and vice versa. We next define the function $\tilde{u} \in H^1(B_p(2r))$ as

$$\tilde{u}(x) = \begin{cases} u(K(x)) & \text{if } |x-p| \geqslant s/2, \\ c & \text{if } |x-p| \leqslant s/2. \end{cases}$$

Our goal is to apply the local Moser–Trudinger inequality, given by proposition 2.4, to $\tilde{u}$. First of all, observe that

$$\int_{A_p(s,r)} \mathrm{e}^{\tilde{u}}\,\mathrm{d}V_g = \int_{A_p(s,r)} \mathrm{e}^{u(K(x))}\,\mathrm{d}V_g = \int_{A_p(s,r)} \mathrm{e}^{u(x)}\frac{|x-p|^4}{s^2r^2}\,\mathrm{d}V_g, \tag{3.6}$$

since the Jacobian of K is $J(K(x)) = -r^2s^2|x-p|^{-4}$. Moreover, for $|x-p| \geqslant s/2$, we have that

$$|\nabla_g\tilde{u}(x)|^2 = |\nabla_g u(K(x))|^2\frac{s^2r^2}{|x-p|^4}. \tag{3.7}$$

Therefore,

$$\begin{aligned} \log\int_{A_p(s,r)} \mathrm{e}^{u}\,\mathrm{d}V_g &+ \log\int_{A_p(s,r)} \mathrm{e}^{-u}\,\mathrm{d}V_g + 4\log s \\ &= \log\int_{A_p(s,r)} \mathrm{e}^{u}s^2\,\mathrm{d}V_g + \log\int_{A_p(s,r)} \mathrm{e}^{-u}s^2\,\mathrm{d}V_g \\ &\leqslant \log\int_{A_p(s,r)} \mathrm{e}^{u}\frac{s^2}{r^2}\,\mathrm{d}V_g + \log\int_{A_p(s,r)} \mathrm{e}^{-u}\frac{s^2}{r^2}\,\mathrm{d}V_g + C \\ &\leqslant \log\int_{A_p(s,r)} \mathrm{e}^{u}\frac{|x-p|^4}{r^2s^2}\,\mathrm{d}V_g + \log\int_{A_p(s,r)} \mathrm{e}^{-u}\frac{|x-p|^4}{r^2s^2}\,\mathrm{d}V_g + C, \end{aligned}$$

where we have used the trivial inequality $s \leqslant |x-p|$ for $x \in A_p(s,r)$. By using (3.6), applying proposition 2.4 to $\tilde{u}$ and then using (3.7), we have that

$$\begin{aligned} \log\int_{A_p(s,r)} \mathrm{e}^{u}\frac{|x-p|^4}{r^2s^2}\,\mathrm{d}V_g &+ \log\int_{A_p(s,r)} \mathrm{e}^{-u}\frac{|x-p|^4}{r^2s^2}\,\mathrm{d}V_g + C \\ &= \log\int_{A_p(s,r)} \mathrm{e}^{u(K(x))}\,\mathrm{d}V_g + \log\int_{A_p(s,r)} \mathrm{e}^{-u(K(x))}\,\mathrm{d}V_g + C \\ &\leqslant \frac{1}{16\pi-\varepsilon}\int_{B_p(2r)} |\nabla_g\tilde{u}|^2\,\mathrm{d}V_g + C \end{aligned}$$

$$= \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla_g \tilde{u}|^2 \, \mathrm{d}V_g + C$$
$$= \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla_g u(K(x))|^2 \frac{r^2 s^2}{|x-p|^4} \, \mathrm{d}V_g + C$$
$$= \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla_g u|^2 \, \mathrm{d}V_g + C.$$

This concludes the proof of the lemma. □

REMARK 3.10. We are now able to prove the improved inequality given in proposition 3.6. The basis of the proof is to jointly use lemmas 3.7 and 3.9. Indeed, assume that e^u and e^{-u} concentrate around the same point at the same rate (in the sense of proposition 3.1). If we sum the inequalities given by lemmas 3.7 and 3.9, the extra term $4 \log s$ cancels and we can deduce the improved inequality of proposition 3.6.

We have to manage the fact that when

$$\psi\left(\frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g}\right) = \psi\left(\frac{\mathrm{e}^{-u}}{\int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g}\right)$$

we do not really have concentration around the same point. Moreover, the property in lemma 3.9 of u being constant on the boundary of a ball need not be satisfied.

Proof of proposition 3.6. Fix $\varepsilon > 0$, take $R > 1$ (depending only on ε), let ψ be the continuous map given by proposition 3.1 and fix $\delta > 0$ small.

Let $u \in H^1(\Sigma)$ be a function with $\int_\Sigma u \, \mathrm{d}V_g = 0$, such that

$$\psi\left(\frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g}\right) = \psi\left(\frac{\mathrm{e}^{-u}}{\int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g}\right) = (\beta, \sigma) \in \bar{\Sigma}_\delta.$$

If $\sigma \geqslant \delta/R^2$, then, applying proposition 2.6, we get the result. Therefore, assume that $\sigma < \delta/R^2$. Proposition 3.1 implies the existence of $\tau > 0$, $p_1, p_2 \in \Sigma$ satisfying

$$\int_{B_{p_1}(\sigma)} \mathrm{e}^u \, \mathrm{d}V_g \geqslant \tau \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g, \qquad \int_{B_{p_2}(\sigma)} \mathrm{e}^{-u} \, \mathrm{d}V_g \geqslant \tau \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g \tag{3.8}$$

and

$$\int_{B_{p_1}(R\sigma)^c} \mathrm{e}^u \, \mathrm{d}V_g \geqslant \tau \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g, \quad \int_{B_{p_2}(R\sigma)^c} \mathrm{e}^{-u} \, \mathrm{d}V_g \geqslant \tau \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g, \tag{3.9}$$

with $d(p_1, p_2) \leqslant (6R + 2)\sigma$. We divide the proof into two cases.

CASE 1. Assume that

$$\int_{A_{p_1}(R\sigma,\delta)} \mathrm{e}^u \, \mathrm{d}V_g \geqslant \frac{\tau}{2} \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g, \qquad \int_{A_{p_2}(R\sigma,\delta)} \mathrm{e}^{-u} \, \mathrm{d}V_g \geqslant \frac{\tau}{2} \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g. \tag{3.10}$$

In order to satisfy the hypothesis of lemma 3.9, we need to modify our function outside a certain ball. Via a dyadic decomposition, choose $k \in \mathbb{N}$, $k \leqslant 2\varepsilon^{-1}$, such that

$$\int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla u|^2 \, \mathrm{d}V_g \leqslant \varepsilon \int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g.$$

We define $\tilde{u} \in H^1(\Sigma)$ by

$$\begin{aligned} \tilde{u}(x) &= u(x), \quad x \in B_{p_1}(2^k\delta), \\ \Delta\tilde{u}(x) &= 0, \qquad x \in A_{p_1}(2^k\delta, 2^{k+1}\delta), \\ \tilde{u}(x) &= c, \qquad x \notin B_{p_1}(2^{k+1}\delta), \end{aligned}$$

where $c \in \mathbb{R}$. Moreover, since we want to apply lemma 3.9 to $\tilde{u}$, we have to choose δ small enough that $2^{3\varepsilon^{-1}}\delta < r_0$, where r_0 is given by said lemma.

We have that

$$\int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla\tilde{u}|^2 \, \mathrm{d}V_g \leqslant C \int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla u|^2 \, \mathrm{d}V_g \leqslant C\varepsilon \int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g \tag{3.11}$$

for some universal constant $C > 0$.

Case 1.1. Suppose that $d(p_1, p_2) \leqslant R^{1/2}\sigma$.

We first apply lemma 3.7 to u for $p = p_1$ and $s = 2(R^{1/2}+1)\sigma$, and take into account (3.8), to obtain that

$$\begin{aligned} \frac{1}{16\pi - \varepsilon} \int_{B_p(s)} |\nabla u|^2 \, \mathrm{d}V_g &\geqslant \log \int_{B_p(s/2)} \mathrm{e}^u \, \mathrm{d}V_g + \log \int_{B_p(s/2)} \mathrm{e}^{-u} \, \mathrm{d}V_g - 4\log\sigma - C \\ &\geqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + \log \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g - 4\log\sigma - C. \end{aligned} \tag{3.12}$$

We next apply lemma 3.9 to $\tilde{u}$ for $p = p_1$, $s' = 4(R^{1/2}+1)\sigma$ and $r = 2^{k+1}\delta$:

$$\begin{aligned} &\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2, 2r)} |\nabla_g \tilde{u}|^2 \, \mathrm{d}V_g \\ &\qquad \geqslant \log \int_{A_p(s', r)} \mathrm{e}^{\tilde{u}} \, \mathrm{d}V_g + \log \int_{A_p(s', r)} \mathrm{e}^{-\tilde{u}} \, \mathrm{d}V_g + 4\log\sigma - C. \end{aligned} \tag{3.13}$$

Using the estimate (3.9), we get that

$$\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2, 2r)} |\nabla_g \tilde{u}|^2 \, \mathrm{d}V_g \geqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + \log \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g + 4\log\sigma - C. \tag{3.14}$$

Finally, combining (3.12), (3.14) and (3.11) we obtain our thesis (after conveniently renaming ε).

Case 1.2. Suppose that $d(p_1, p_2) \geqslant R^{1/2}\sigma$ and

$$\int_{B_{p_1}(R^{1/3}\sigma)} \mathrm{e}^{-u} \, \mathrm{d}V_g \geqslant \frac{\tau}{4} \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g.$$

Here we argue as in case 1.1. First, we apply lemma 3.7 to u for $p = p_1$ and $s = 2(R^{1/3}+1)\sigma$. Then, we use lemma 3.9 with $\tilde{u}$ for $p = p_1$, $s' = 4(R^{1/3}+1)\sigma$ and $r = 2^{k+1}\delta$.

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Case 1.3. Suppose that $d(p_1,p_2) \geqslant R^{1/2}\sigma$ and

$$\int_{B_{p_2}(R^{1/3}\sigma)} \mathrm{e}^u \, \mathrm{d}V_g \geqslant \frac{\tau}{4}\int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g.$$

This case can be treated as in case 1.2, just by interchanging the indices.

Case 1.4. Suppose that $d(p_1,p_2) \geqslant R^{1/2}\sigma$ and

$$\int_{B_{p_2}(R^{1/3}\sigma)} \mathrm{e}^u \, \mathrm{d}V_g \leqslant \frac{\tau}{4}\int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g, \qquad \int_{B_{p_1}(R^{1/3}\sigma)} \mathrm{e}^{-u} \, \mathrm{d}V_g \leqslant \frac{\tau}{4}\int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g.$$

Take $n \in \mathbb{N}$, $n \leqslant 2\varepsilon^{-1}$, such that

$$\sum_{i=1}^{2}\int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} |\nabla u|^2 \, \mathrm{d}V_g \leqslant \varepsilon \int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g,$$

where we have chosen R such that $2^{3\varepsilon^{-1}} < R^{1/3}$. We now define the function $v \in H^1(\Sigma)$ by

$$\begin{aligned} v(x) &= u(x), && x \in B_{p_1}(2^n\sigma) \cup B_{p_2}(2^n\sigma), \\ \Delta v(x) &= 0, && x \in A_{p_1}(2^n\sigma, 2^{n+1}\sigma) \cup A_{p_2}(2^n\sigma, 2^{n+1}\sigma), \\ v(x) &= 0, && x \notin B_{p_1}(2^{n+1}\sigma) \cup B_{p_2}(2^{n+1}\sigma). \end{aligned}$$

As before, we have that

$$\begin{aligned} \sum_{i=1}^{2}\int_{A_{p_i}(2^{n}\sigma,2^{n+1}\sigma)} |\nabla v|^2 \, \mathrm{d}V_g &\leqslant C\sum_{i=1}^{2}\int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} |\nabla u|^2 \, \mathrm{d}V_g \\ &\leqslant C\varepsilon \int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g, \end{aligned}$$

where $C > 0$ is a universal constant.

Taking into account (3.8), we now apply lemma 3.7 to v with $p = p_1$ and $s = 4(6R+2)\sigma$ to give

$$\begin{aligned} \frac{1}{16\pi - \varepsilon}\int_{B_{p_1}(2^n\sigma)\cup B_{p_2}(2^n\sigma)} |\nabla u|^2 \, \mathrm{d}V_g &+ C\varepsilon\int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g \\ &\geqslant \frac{1}{16\pi-\varepsilon}\int_{B_p(s)} |\nabla v|^2 \, \mathrm{d}V_g \\ &\geqslant \log\int_{B_p(s/2)} \mathrm{e}^v \, \mathrm{d}V_g + \log\int_{B_p(s/2)} \mathrm{e}^{-v} \, \mathrm{d}V_g - 4\log\sigma - C \\ &\geqslant \log\int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + \log\int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g - 4\log\sigma - C. \end{aligned} \tag{3.15}$$

Next, we define $w \in H^1(\Sigma)$ by

$$\begin{aligned} w(x) &= 0, && x \in B_{p_1}(2^n\sigma) \cup B_{p_2}(2^n\sigma), \\ \Delta w(x) &= 0, && x \in A_{p_1}(2^n\sigma, 2^{n+1}\sigma) \cup A_{p_2}(2^n\sigma, 2^{n+1}\sigma), \\ w(x) &= \tilde{u}(x), && x \notin B_{p_1}(2^{n+1}\sigma) \cup B_{p_2}(2^{n+1}\sigma). \end{aligned}$$

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Again we have that

$$\sum_{i=1}^{2}\int_{A_{p_i}(2^n\sigma,2^{n+1}\sigma)}|\nabla w|^2\,\mathrm{d}V_g \leqslant C\sum_{i=1}^{2}\int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)}|\nabla u|^2\,\mathrm{d}V_g \leqslant C\varepsilon\int_{\Sigma}|\nabla u|^2\,\mathrm{d}V_g,$$

where here C is also a universal constant.

We apply lemma 3.9 to w for any point p' such that $d(p',p_1)=\frac{1}{2}R^{1/3}\sigma$, $s'=\sigma$ and $r=2^{k+1}\delta$, to obtain that

$$\begin{aligned}\frac{1}{16\pi-\varepsilon}\int_{(B_{p_1}(2^{n+1}\sigma)\cup B_{p_2}(2^{n+1}\sigma))^{\mathrm{c}}}|\nabla u|^2\,\mathrm{d}V_g + C\varepsilon\int_{\Sigma}|\nabla u|^2\,\mathrm{d}V_g \\ \geqslant \frac{1}{16\pi-\varepsilon}\int_{A_{p'}(s'/2,2r)}|\nabla w|^2\,\mathrm{d}V_g \\ \geqslant \log\int_{A_{p'}(s',r)}\mathrm{e}^{w}\,\mathrm{d}V_g+\log\int_{A_{p'}(s',r)}\mathrm{e}^{-w}\,\mathrm{d}V_g+4\log\sigma - C.\end{aligned}$$

We now use (3.10) and the hypothesis of case 1.4 to conclude that

$$\begin{aligned}\frac{1}{16\pi-\varepsilon}\int_{(B_{p_1}(2^{n}\sigma)\cup B_{p_2}(2^{n}\sigma))^{\mathrm{c}}}|\nabla u|^2\,\mathrm{d}V_g + C\varepsilon\int_{\Sigma}|\nabla u|^2\,\mathrm{d}V_g \\ \geqslant \log\int_{\Sigma}\mathrm{e}^{u}\,\mathrm{d}V_g+\log\int_{\Sigma}\mathrm{e}^{-u}\,\mathrm{d}V_g+4\log\sigma - C. \quad (3.16)\end{aligned}$$

The inequality (3.16) together with (3.15) implies our result (after properly renaming ε).

Case 2. Assume that

$$\int_{B_{p_1}(\delta)^{\mathrm{c}}}\mathrm{e}^{u}\,\mathrm{d}V_g \geqslant \frac{\tau}{2}\int_{\Sigma}\mathrm{e}^{u}\,\mathrm{d}V_g \quad \text{or} \quad \int_{B_{p_2}(\delta)^{\mathrm{c}}}\mathrm{e}^{-u}\,\mathrm{d}V_g \geqslant \frac{\tau}{2}\int_{\Sigma}\mathrm{e}^{-u}\,\mathrm{d}V_g.$$

Without loss of generality, suppose that the first alternative holds true. Now let $\delta'=\delta/2^{3/\varepsilon}$. Moreover, if

$$\int_{B_{p_2}(\delta')^{\mathrm{c}}}\mathrm{e}^{-u}\,\mathrm{d}V_g \geqslant \frac{\tau}{2}\int_{\Sigma}\mathrm{e}^{-u}\,\mathrm{d}V_g,$$

then we can apply proposition 2.6 to deduce the thesis. Therefore, we can assume that

$$\int_{A_{p_2}(R\sigma,\delta')}\mathrm{e}^{-u}\,\mathrm{d}V_g \geqslant \frac{\tau}{2}\int_{\Sigma}\mathrm{e}^{-u}\,\mathrm{d}V_g. \quad (3.17)$$

We can apply the whole procedure of case 1 to u, just replacing δ with δ'. In fact, as in case 1.1, we would get the inequalities (3.12) and (3.13). However, in this case we have to manage the fact that we do not know whether

$$\int_{A_p(s',r)}\mathrm{e}^{u}\,\mathrm{d}V_g \geqslant \alpha\int_{\Sigma}\mathrm{e}^{u}\,\mathrm{d}V_g$$

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holds for some fixed $\alpha > 0$. This property is needed in (3.13) to get the estimate

$$\log \int_{A_p(s',r)} \mathrm{e}^{\tilde{u}} \, \mathrm{d}V_g \geqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g - C,$$

which allows us to deduce (3.14). To do this, we first apply both the Jensen and Poincaré–Wirtinger inequalities, to obtain

$$\begin{aligned} \log \int_{A_p(s',r)} \mathrm{e}^{\tilde{u}} \, \mathrm{d}V_g &\geqslant \log \int_{A_p(r/8,r/4)} \mathrm{e}^u \, \mathrm{d}V_g \\ &\geqslant \log \fint_{A_{p_1}(r/8,r/4)} \mathrm{e}^u \, \mathrm{d}V_g - C \\ &\geqslant \fint_{A_{p_1}(r/8,r/4)} u \, \mathrm{d}V_g - C \\ &\geqslant -\varepsilon \int_\Sigma |\nabla u|^2 \, \mathrm{d}V_g - C. \end{aligned}$$

Therefore, taking into account (3.17) and the last inequality, from (3.13) we obtain (after properly renaming ε) that

$$\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2,2r)} |\nabla \tilde{u}|^2 \, \mathrm{d}V_g \geqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + 4 \log \sigma - C. \tag{3.18}$$

Next, we apply proposition 2.4, to get that

$$\frac{1}{16\pi - \varepsilon} \int_{B_{p_1}(\delta/2)^c} |\nabla u|^2 \, \mathrm{d}V_g \geqslant \log \int_{B_{p_1}(\delta)^c} \mathrm{e}^u \, \mathrm{d}V_g + \log \int_{B_{p_1}(\delta)^c} \mathrm{e}^{-u} \, \mathrm{d}V_g.$$

Reasoning as above and using the hypothesis of case 2, we can deduce that

$$\frac{1}{16\pi - \varepsilon} \int_{B_{p_1}(\delta)^c} |\nabla u|^2 \, \mathrm{d}V_g \geqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + 4 \log \sigma - C. \tag{3.19}$$

Finally, we obtain our result by combining (3.19), (3.18) and (3.12).

If we are under the conditions of cases 1.2–1.4, the thesis follows by arguing in the same way. □

REMARK 3.11. Our goal is to use proposition 3.6 to obtain a lower bound of the functional I_{ρ_1,ρ_2} under suitable conditions. The presence of the two functions h_1 and h_2 in I_{ρ_1,ρ_2} is not so relevant because of the estimates

$$\begin{aligned} \log \int_\Sigma h_1(x) \mathrm{e}^u \, \mathrm{d}V_g &\leqslant \log \int_\Sigma \mathrm{e}^u \, \mathrm{d}V_g + \log \|h_1\|_\infty, \\ \log \int_\Sigma h_2(x) \mathrm{e}^{-u} \, \mathrm{d}V_g &\leqslant \log \int_\Sigma \mathrm{e}^{-u} \, \mathrm{d}V_g + \log \|h_2\|_\infty. \end{aligned}$$

4. Min–max scheme

Let $\bar{\Sigma}_\delta$ be the topological cone over Σ defined in (2.1), and set

$$\begin{aligned} \bar{D}_\delta &= \mathrm{diag}(\bar{\Sigma}_\delta \times \bar{\Sigma}_\delta) = \{(\vartheta_1, \vartheta_2) \in \bar{\Sigma}_\delta \times \bar{\Sigma}_\delta \colon \vartheta_1 = \vartheta_2\}, \\ X &= (\bar{\Sigma}_\delta \times \bar{\Sigma}_\delta) \setminus \bar{D}_\delta. \end{aligned}$$

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Let $\varepsilon > 0$ be sufficiently small and let R, δ, ψ be as in proposition 3.1. Consider then the map Ψ defined by

$$\Psi(u) = \left(\psi\left(\frac{e^u}{\int_\Sigma e^u \, dV_g}\right), \psi\left(\frac{e^{-u}}{\int_\Sigma e^{-u} \, dV_g}\right)\right). \tag{4.1}$$

By proposition 3.6 and remark 3.11, we have a lower bound of the functional I_{ρ_1,ρ_2} on functions u such that $u \in \bar{D}_\delta$. Therefore, there exists a large $L > 0$ such that if $I_{\rho_1,\rho_2}(u) \leqslant -L$, then it follows that $\Psi(u) \in X$.

In [13] Malchiodi and Ruiz proved that even though the set X is non-compact, it retracts to some compact subset $\mathcal{X}_\nu$. Indeed, we have the following lemma.

Lemma 4.1. *For $\nu \ll \delta$, define*

$$\begin{aligned}
\mathcal{X}_{\nu,1} &= \{((x_1,t_1),(x_2,t_2)) \in X \colon |t_1 - t_2|^2 + d(x_1,x_2)^2 \geqslant \delta^4, \\
&\qquad\qquad \max\{t_1,t_2\} < \delta,\ \min\{t_1,t_2\} \in [\nu^2,\nu]\}, \\
\mathcal{X}_{\nu,2} &= \{((x_1,t_1),(x_2,t_2)) \in X \colon \max\{t_1,t_2\} = \delta,\ \min\{t_1,t_2\} \in [\nu^2,\nu]\},
\end{aligned}$$

and set

$$\mathcal{X}_\nu = (\mathcal{X}_{\nu,1} \cup \mathcal{X}_{\nu,2}) \subseteq X.$$

There then exists a retraction R_ν of X onto $\mathcal{X}_\nu$.

Our next goal is to introduce a family of test functions labelled on the set $\mathcal{X}_\nu$, on which the functional I_{ρ_1,ρ_2} attains large negative values. For $(\vartheta_1,\vartheta_2) = ((x_1,t_1),(x_2,t_2)) \in \mathcal{X}_\nu$ define

$$\varphi(y) = \varphi_{(\vartheta_1,\vartheta_2)}(y) = \log \frac{(1 + \tilde{t}_2^2 d(x_2,y)^2)^2}{(1 + \tilde{t}_1^2 d(x_1,y)^2)^2}, \tag{4.2}$$

where

$$\tilde{t}_i = \tilde{t}_i(t_i) = \begin{cases} \dfrac{1}{t_i} & \text{for } t_i \leqslant \dfrac{\delta}{2}, \\ -\dfrac{4}{\delta^2}(t_i - \delta) & \text{for } t_i \geqslant \dfrac{\delta}{2} \end{cases}$$

for $i = 1, 2$.

We start by proving the following estimate.

Lemma 4.2. *For ν sufficiently small, and for $(\vartheta_1,\vartheta_2) \in \mathcal{X}_\nu$, there exists a constant $C = C(\delta,\Sigma) > 0$, depending only on Σ and δ, such that*

$$\frac{1}{C}\frac{t_1^2}{t_2^4} \leqslant \int_\Sigma e^\varphi \, dV_g \leqslant C\frac{t_1^2}{t_2^4}. \tag{4.3}$$

Proof. First, observe that the following equality holds true for some fixed positive constant C_0:

$$\int_{\mathbb{R}^2} \frac{1}{(1 + \lambda^2|x|^2)^2} \, dx = \frac{C_0}{\lambda^2}, \quad \lambda > 0. \tag{4.4}$$

To prove the lemma, we distinguish the two cases

$$|t_1 - t_2| \geqslant \delta^3 \quad \text{and} \quad |t_1 - t_2| < \delta^3,$$

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in order to exploit the properties of $\mathcal{X}_\nu$. Starting with the first alternative, by the definition of $\mathcal{X}_\nu$ and by the fact that $\nu \ll \delta$, it turns out that one of the t_i belongs to $[\nu^2, \nu]$, while the other is greater than or equal to $\delta^3/2$.

If $t_1 \in [\nu^2, \nu]$ and if $t_2 \geqslant \delta^3/2$, then the function $1 + \tilde{t}_2^2 d(x_2, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ. Therefore, using (4.4) we get that

$$\frac{t_1^2}{C} = \frac{1}{C\tilde{t}_1^2} \leqslant \int_\Sigma \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \leqslant \frac{C}{\tilde{t}_1^2} = Ct_1^2.$$

On the other hand, if $t_2 \in [\nu^2, \nu]$ and if $t_1 \geqslant \delta^3/2$, then the function $1 + \tilde{t}_1^2 d(x_1, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ; hence,

$$\int_\Sigma \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \geqslant \frac{1}{C}\int_\Sigma (1 + \tilde{t}_2^2 d(x_2, y)^2)^2\,\mathrm{d}V_g(y) \geqslant \frac{\tilde{t}_2^4}{C} = \frac{1}{Ct_2^4},$$

and, similarly,

$$\int_\Sigma \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \leqslant C\int_\Sigma (1 + \tilde{t}_2^2 d(x_2, y)^2)^2\,\mathrm{d}V_g(y) \leqslant C\tilde{t}_2^4 = \frac{C}{t_2^4}.$$

In both the last two cases we then obtain the conclusion.

Suppose now that we are in the second alternative, i.e. that $|t_1 - t_2| < \delta^3$. Then, by the definition of $\mathcal{X}_\nu$ we have that $d(x_1, x_2) \geqslant \delta^2/2$ and that $t_1, t_2 \leqslant \nu + \delta^3$. Using (4.4) we obtain that

$$\int_\Sigma \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \geqslant \int_{B_{x_1}(\delta^3)} \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \geqslant \frac{1}{C}\frac{(1 + \tilde{t}_2^2 d(x_1, x_2)^2)^2}{\tilde{t}_1^2} \geqslant \frac{1}{C}\frac{t_1^2}{t_2^4}.$$

In an analogous way, we derive

$$\int_{B_{x_1}(\delta^3)} \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \leqslant C\frac{(1 + \tilde{t}_2^2 d(x_1, x_2)^2)^2}{\tilde{t}_1^2} \leqslant C\frac{t_1^2}{t_2^4}.$$

Finally, by the estimate

$$\int_{(B_{x_1}(\delta^3))^c} \mathrm{e}^{\varphi(y)}\,\mathrm{d}V_g(y) \leqslant \frac{C}{\tilde{t}_1^4}\int_{(B_{x_1}(\delta^3))^c} (1 + \tilde{t}_2^2 d(x_2, y)^2)^2\,\mathrm{d}V_g(y) \leqslant C\frac{t_1^4}{t_2^4},$$

we are done. □

Remark 4.3. Note that for $\mathrm{e}^{-\varphi}$ the same result holds true if we just exchange the indices of t_1 and t_2.

Proposition 4.4. *For $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$, let $\varphi_{(\vartheta_1,\vartheta_2)}$ be defined as in (4.2). Then,*

$$I_{\rho_1,\rho_2}(\varphi_{(\vartheta_1,\vartheta_2)}) \to -\infty \quad \text{as } \nu \to 0$$

uniformly for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$.

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Proof. We start by showing the following estimates:

$$\int_{\Sigma} \varphi \,\mathrm{d}V_g = 4(1+o_\delta(1))\log t_1 - 4(1+o_\delta(1))\log t_2, \tag{4.5}$$

$$\frac{1}{2}\int_{\Sigma} |\nabla_g \varphi|^2 \,\mathrm{d}V_g \leqslant 16\pi(1+o_\delta(1))\log\frac{1}{t_1} + 16\pi(1+o_\delta(1))\log\frac{1}{t_2}. \tag{4.6}$$

We begin by proving (4.5). It is convenient to divide Σ into the two subsets

$$A_1 = B_{x_1}(\delta) \cup B_{x_2}(\delta) \quad \text{and} \quad A_2 = \Sigma \setminus \mathcal{A}_1.$$

Moreover, we write that

$$\varphi(y) = 2\log(1+\tilde{t}_2^2 d(x_2,y)^2) - 2\log(1+\tilde{t}_1^2 d(x_1,y)^2).$$

For $y \in A_2$ we clearly have that

$$\frac{1}{C_{\delta,\Sigma} t_1^2} \leqslant 1+\tilde{t}_1^2 d(x_1,y)^2 \leqslant \frac{C_{\delta,\Sigma}}{t_1^2}, \qquad \frac{1}{C_{\delta,\Sigma} t_2^2} \leqslant 1+\tilde{t}_2^2 d(x_2,y)^2 \leqslant \frac{C_{\delta,\Sigma}}{t_2^2};$$

therefore, we derive

$$\int_{A_2} \varphi \,\mathrm{d}V_g = 4(1+o_\delta(1))\log t_1 - 4(1+o_\delta(1))\log t_2.$$

Moreover, working in normal geodesic coordinates at x_i one also finds that

$$\int_{B_\delta(x_i)} \log(1+\tilde{t}_i^2 d(x_i,y)^2)\,\mathrm{d}V_g = o_\delta(1)\log t_i.$$

Using the last two inequalities jointly, we obtain (4.5).

We now prove (4.6). We have that

$$\begin{aligned}\nabla_g\varphi(y) &= 2\nabla_g \log(1+\tilde{t}_2^2 d(x_2,y)^2) - 2\nabla_g\log(1+\tilde{t}_1^2 d(x_1,y)^2)\\ &= \frac{4\tilde{t}_2^2 d(x_2,y)\nabla_g d(x_2,y)}{1+\tilde{t}_2^2 d(x_2,y)^2} - \frac{4\tilde{t}_1^2 d(x_1,y)\nabla_g d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2}.\end{aligned}$$

From now on we assume, without loss of generality, that $t_1 \leqslant t_2$. We distinguish between the case $t_2 \geqslant \delta^3$ and the case $t_2 \leqslant \delta^3$.

In the first case the function $1+\tilde{t}_2^2 d(x_2,y)^2$ is uniformly Lipschitz with bounds depending only on δ, and, therefore, we have that

$$\nabla_g\varphi(y) = -\frac{4\tilde{t}_1^2 d(x_1,y)\nabla_g d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} + O_\delta(1).$$

We fix a large constant $C_1 > 0$ and consider the subdivision of the surface Σ into the three domains

$$B_1 = B_{x_1}(C_1 t_1), \qquad B_2 = B_{x_2}(C_1 t_2), \qquad B_3 = \Sigma \setminus (B_1 \cup B_2).$$

In B_1 we have that $|\nabla_g\varphi| \leqslant C\tilde{t}_1$, while

$$\frac{\tilde{t}_1^2 d(x_1,y)\nabla_g d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} = (1+o_{C_1}(1))\frac{\nabla_g d(x_1,y)}{d(x_1,y)} \quad \text{in } \Sigma \setminus B_1. \tag{4.7}$$

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These estimates imply that

$$\frac{1}{2}\int_{\Sigma}|\nabla_g\varphi|^2\,\mathrm{d}V_g = \int_{\Sigma\setminus B_1}|\nabla_g\varphi|^2\,\mathrm{d}V_g + o_\delta(1)\log\frac{1}{t_1} + O_\delta(1)$$
$$= 16\pi\int_{C_1t_1}^{1}\frac{\mathrm{d}t}{t} + o_\delta(1)\log\frac{1}{t_1} + O_\delta(1)$$
$$= 16\pi(1+o_\delta(1))\log\frac{1}{t_1} + 16\pi(1+o_\delta(1))\log\frac{1}{t_2} + O_\delta(1),$$

recalling that $t_2 \geqslant \delta^3$.

If, instead, $t_2 \leqslant \delta_3$, by the definition of $\mathcal{X}_\nu$ we have that $d(x_1,x_2) \geqslant \delta^2/2$, and, therefore, $B_1 \cap B_2 = \emptyset$. Similarly to (4.7) we get that

$$\left.\begin{aligned}\frac{\tilde{t}_1^2 d(x_1,y)\nabla_g d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} &= (1+o_{C_1}(1))\frac{\nabla_g d(x_1,y)}{d(x_1,y)}\\ \frac{\tilde{t}_2^2 d(x_2,y)\nabla_g d(x_2,y)}{1+\tilde{t}_2^2 d(x_2,y)^2} &= (1+o_{C_1}(1))\frac{\nabla_g d(x_2,y)}{d(x_2,y)}\end{aligned}\right\}\text{ in } B_3.$$

Moreover, we have

$$|\nabla_g\varphi| \leqslant C\tilde{t}_i \quad \text{in } B_i,\ i=1,2.$$

Therefore, we find that

$$\frac{1}{2}\int_{\Sigma}|\nabla_g\varphi|^2\,\mathrm{d}V_g = \int_{B_3}|\nabla_g\varphi|^2\,\mathrm{d}V_g + o_\delta(1)\log\frac{1}{t_1} + o_\delta(1)\log\frac{1}{t_2} + O_\delta(1)$$
$$= 16\pi(1+o_\delta(1))\log\frac{1}{t_1} + 16\pi(1+o_\delta(1))\log\frac{1}{t_2} + O_\delta(1)$$

for $t_2 \leqslant \delta^3$. This concludes the proof of (4.6).

Finally, the estimates (4.5) and (4.5), together with (4.3) and remark 3.11, yield the inequality

$$I_{\rho_1,\rho_2}(\varphi) \leqslant (2\rho_1 - 16\pi + o_\delta(1))\log t_1 + (2\rho_2 - 16\pi + o_\delta(1))\log t_2 \to -\infty$$

as $\nu \to 0$, uniformly for $(\vartheta_1,\vartheta_2) \in \mathcal{X}_\nu$, since $\rho_1,\rho_2 > 8\pi$. □

We next state a technical lemma that will be of use later in the paper.

Lemma 4.5. *Let $\varphi_{(\vartheta_1,\vartheta_2)}$ be as in (4.2). Then, for some $C = C(\delta,\Sigma) > 0$, the following estimates hold uniformly in $(\vartheta_1,\vartheta_2) \in \mathcal{X}_\nu$:*

$$\sup_{x\in\Sigma}\int_{B_x(rt_1)} \mathrm{e}^{\varphi}\,\mathrm{d}V_g \leqslant Cr^2\frac{t_1^2}{t_2^4} \quad \forall r > 0. \tag{4.8}$$

Moreover, given any $\varepsilon > 0$ there exists $C = C(\varepsilon,\delta,\Sigma)$, depending only on ε, δ and Σ (but not on ν), such that

$$\int_{B_{x_1}(Ct_1)} \mathrm{e}^{\varphi}\,\mathrm{d}V_g \geqslant (1-\varepsilon)\int_{\Sigma}\mathrm{e}^{\varphi}\,\mathrm{d}V_g \tag{4.9}$$

uniformly in $(\vartheta_1,\vartheta_2) \in \mathcal{X}_\nu$.

Proof. By the elementary inequalities $(1+\tilde{t}_2^2 d(x_2,y)^2)^2 \leqslant C/t_2^4$ and $1+\tilde{t}_1^2 d(x_1,y)^2 \geqslant 1$, we have that

$$\begin{aligned}\int_{B_x(t_1 r)} \mathrm{e}^{\varphi(y)}\, \mathrm{d}V_g(y) &\leqslant \frac{C}{t_2^4}\int_{B_x(t_1 r)} \frac{1}{(1+\tilde{t}_1^2 d(x_1,y)^2)^2}\, \mathrm{d}V_g(y)\\ &\leqslant Cr^2\frac{t_1^2}{t_2^4} \quad \text{for all } x \in \Sigma,\end{aligned}$$

which gives (4.8).

We now prove (4.9). Using again that $(1+\tilde{t}_2^2 d(x_2,y)^2)^2 \leqslant C/t_2^4$ we have that

$$\int_{\Sigma\setminus B_{x_1}(Rt_1)} \mathrm{e}^{\varphi(y)}\, \mathrm{d}V_g(y) \leqslant \frac{C}{t_2^4}\int_{\Sigma\setminus B_{x_1}(Rt_1)} \frac{1}{(1+\tilde{t}_1^2 d(x_1,y)^2)^2}\, \mathrm{d}V_g(y). \tag{4.10}$$

Finally, using normal geodesic coordinates centred at x_1 and (4.4) with a change of variable, we find that

$$\lim_{t_1\to 0^+} t_1^{-2}\int_{\Sigma\setminus B_{x_1}(Rt_1)} \frac{1}{(1+\tilde{t}_1^2 d(x_1,y)^2)^2}\, \mathrm{d}V_g = o_R(1) \quad \text{as } R \to +\infty.$$

This fact and (4.10), with (4.3), conclude the proof of (4.9), by choosing R sufficiently large, depending on ε, δ and Σ. □

Remark 4.6. The same result holds if we consider $\mathrm{e}^{-\varphi}$, interchanging the indices of t_1 and t_2.

We next present a crucial step in describing the topology of low sublevels, which will allow us to find a min–max scheme later on.

Proposition 4.7. *Let $L > 0$ be so large that $\Psi(\{I_{\rho_1,\rho_2} \leqslant -L\}) \in X$, and let ν be so small that $I_{\rho_1,\rho_2}(\varphi_{(\vartheta_1,\vartheta_2)}) < -L$ for $(\vartheta_1,\vartheta_2) \in \mathcal{X}_\nu$. Let R_ν be the retraction given in lemma 4.1. Then, the map $T_\nu\colon \mathcal{X}_\nu \to \mathcal{X}_\nu$ defined as*

$$T_\nu((\vartheta_1,\vartheta_2)) = R_\nu(\Psi(\varphi_{(\vartheta_1,\vartheta_2)}))$$

is homotopic to the identity on $\mathcal{X}_\nu$.

Proof. We define $\vartheta_i = (x_i, t_i)$ and

$$f_1 = \frac{\mathrm{e}^{\varphi(\vartheta_1,\vartheta_2)}}{\int_\Sigma \mathrm{e}^{\varphi(\vartheta_1,\vartheta_2)}\, \mathrm{d}V_g}, \qquad \psi(f_1) = (\beta_1, \sigma_1),$$

$$f_2 = \frac{\mathrm{e}^{-\varphi(\vartheta_1,\vartheta_2)}}{\int_\Sigma \mathrm{e}^{-\varphi(\vartheta_1,\vartheta_2)}\, \mathrm{d}V_g}, \qquad \psi(f_2) = (\beta_2, \sigma_2),$$

where ψ is given in proposition 3.1. First, observe that we have the following relations:

$$\frac{1}{C} \leqslant \frac{\sigma_i}{t_i} \leqslant C, \qquad d(\beta_i, x_i) \leqslant Ct_i \tag{4.11}$$

for some constant $C = C(\delta, \Sigma) > 0$, depending only on Σ and δ. Indeed, by (4.9), we have that

$$\sigma(x_i, f_i) \leqslant Ct_i,$$

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where $\sigma(x, f)$ is the continuous map defined in (3.1). From that, we get that $\sigma_i \leqslant Ct_i$. Moreover, by (4.8), we get the relation $t_i \leqslant C\sigma_i$.

Next, by (3.2) and using again the fact that $\sigma(x_i, f) \leqslant Ct_i$, we obtain that

$$d(x_i, S(f_i)) \leqslant Ct_i,$$

where $S(f)$ is the set defined in (3.3). But since we have the inequality

$$d(\beta_i, S(f_i)) \leqslant C\sigma_i,$$

we can conclude the proof of (4.11).

We are now able to prove the proposition. The proof will follow by taking into account a composition of three homotopies. The first deformation H_1 is defined in the following way:

$$\left(\begin{pmatrix} (\beta_1, \sigma_1) \\ (\beta_2, \sigma_2) \end{pmatrix}, s \right) \overset{H_1}{\longmapsto} \begin{pmatrix} (\beta_1, (1-s)\sigma_1 + s\kappa_1) \\ (\beta_2, (1-s)\sigma_2 + s\kappa_2) \end{pmatrix},$$

where $\kappa_i = \min\{\delta, \sigma_i/\sqrt{\nu}\}$.

We now introduce a second deformation H_2, given by

$$\left(\begin{pmatrix} (\beta_1, \kappa_1) \\ (\beta_2, \kappa_2) \end{pmatrix}, s \right) \overset{H_2}{\longmapsto} \begin{pmatrix} ((1-s)\beta_1 + sx_1, \kappa_1) \\ ((1-s)\beta_2 + sx_2, \kappa_2) \end{pmatrix},$$

where $(1-s)\beta_i + sx_i$ stands for the geodesic joining β_i and x_i in unit time. Observe that if $\kappa_i < \delta$, then we have that $\sigma_i < \sqrt{\nu}\delta$. Therefore, by choosing ν small enough, we have that β_i and x_i are close to each other, by (4.11). Instead, if $\kappa_i = \delta$, the equivalence relation in $\bar{\Sigma}_\delta$ makes the above deformation a trivial identification.

We perform a third deformation H_3 defined by

$$\left(\begin{pmatrix} (x_1, \kappa_1) \\ (x_2, \kappa_2) \end{pmatrix}, s \right) \overset{H_3}{\longmapsto} \begin{pmatrix} (x_1, (1-s)\kappa_1 + st_1) \\ (x_2, (1-s)\kappa_2 + st_2) \end{pmatrix}.$$

Finally, we define H as the composition of these three homotopies. Then,

$$((\vartheta_1, \vartheta_2), s) \quad \mapsto \quad R_\nu \circ H(\Psi(\varphi_{(\vartheta_1, \vartheta_2)}), s)$$

gives us the desired homotopy to the identity. Indeed, we observe that, since $\nu \ll \delta$, $H(\Psi(\varphi_{(\vartheta_1, \vartheta_2)}), s)$ always belongs to X, so R_ν can be applied. □

We now introduce the min–max scheme that provides existence of solutions for (1.1). The argument follows the ideas of [5], which have been used extensively (see, for example, [6, 7, 21]).

Let $\bar{\mathcal{X}}_\nu$ be the topological cone over $\mathcal{X}_\nu$, which can be represented as

$$\bar{\mathcal{X}}_\nu = (\mathcal{X}_\nu \times [0, 1])/(\mathcal{X}_\nu \times \{1\}),$$

where the equivalence relation identifies all the points in $\mathcal{X}_\nu \times \{1\}$. We choose $L > 0$ so large that $I_{\rho_1,\rho_2}(u) \leqslant -L$ implies that $\Psi(u) \in X$, and we choose ν so small that

$$I_{\rho_1,\rho_2}(\varphi_{(\vartheta_1,\vartheta_2)}) \leqslant -4L$$

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uniformly for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$. The existence of such ν is guaranteed by proposition 4.4. Fixing this value of ν, we define the following class:

$$\mathscr{H} = \{h\colon \bar{\mathcal{X}}_\nu \to H^1(\Sigma)\colon h \text{ is continuous and } h(\cdot \times \{0\}) = \varphi_{(\vartheta_1,\vartheta_2)} \text{ on } \mathcal{X}_\nu\}. \tag{4.12}$$

We then have the following properties.

Lemma 4.8. *The set $\mathscr{H}$ is non-empty and, moreover, letting*

$$c_{\rho_1,\rho_2} = \inf_{h\in\mathscr{H}} \sup_{m\in\bar{\mathcal{X}}_\nu} I_{\rho_1,\rho_2}(h(m)),$$

one has that $c_{\rho_1,\rho_2} > -2L$.

Proof. To prove that $\mathscr{H} \neq \emptyset$, we just note that the map

$$\bar{h}(\vartheta, s) = s\varphi_{(\vartheta_1,\vartheta_2)}, \quad (\vartheta, s) \in \bar{\mathcal{X}}_\nu, \tag{4.13}$$

belongs to $\mathscr{H}$. Assuming, by contradiction, that $c_{\rho_1,\rho_2} \leqslant -2L$, there would exist a map $h \in \mathscr{H}$ with $\sup_{m\in\bar{\mathcal{X}}_\nu} I_{\rho_1,\rho_2}(h(m)) \leqslant -L$. Then, since proposition 4.7 applies, writing $m = (\vartheta, t)$, with $\vartheta \in \mathcal{X}_\nu$, the map

$$t \mapsto R_\nu \circ \Psi \circ h(\cdot, t)$$

would be a homotopy in $\mathcal{X}_\nu$ between $R_\nu \circ \Psi \circ \varphi_{(\vartheta_1,\vartheta_2)}$ and a constant map. But this is impossible since $\mathcal{X}_\nu$ is non-contractible (see remark 4.9 and the fact that $\mathcal{X}_\nu$ is a retract of X) and since $R_\nu \circ \Psi \circ \varphi_{(\vartheta_1,\vartheta_2)}$ is homotopic to the identity on $\mathcal{X}_\nu$. Therefore, we deduce the proof of the lemma. □

Remark 4.9. In [13] Malchiodi and Ruiz proved that the set $X = \bar{\Sigma}_\delta \times \bar{\Sigma}_\delta \setminus \bar{D}_\delta$ is non-contractible. Indeed, if $\Sigma = \mathbb{S}^2$, then $\bar{\Sigma}_\delta$ can be identified with $B_0(1) \subset \mathbb{R}^3$ and it turns out that $X \simeq \mathbb{S}^2$, where $\simeq$ denotes homotopical equivalence. The case of positive genus is not so easy. However, Malchiodi and Ruiz proved that X is non-contractible by showing that its cohomology group $H^4(X)$ is non-trivial.

From lemma 4.8, the functional I_{ρ_1,ρ_2} has a min–max structure. By classical arguments, such a structure yields a Palais–Smale sequence. However, we cannot directly conclude the existence of a critical point, since it is not known whether the Palais–Smale condition holds or not. To bypass this problem and get the conclusion, we need a different argument, usually taking the name 'monotonicity argument'. This technique was first introduced by Struwe in [18], and then used in more general settings (see, for example, [5, 8]).

We take $\mu > 0$ such that $\Lambda_i := [\rho_i - \mu, \rho_i + \mu]$ is contained in $(8\pi, 16\pi)$ for both $i = 1, 2$. We then consider $\tilde{\rho}_i \in \Lambda_i$ and the functional $I_{\tilde{\rho}_1,\tilde{\rho}_2}$ corresponding to these values of the parameters.

It is easy to check that the above min–max scheme applies uniformly for $\tilde{\rho}_i \in \Lambda_i$ for ν sufficiently small. More precisely, given any large number $L > 0$, there exists ν so small that for $\tilde{\rho}_i \in \Lambda_i$ we have the gap

$$\sup_{m\in\partial\bar{\mathcal{X}}_\nu} I_{\tilde{\rho}_1,\tilde{\rho}_2}(m) < -4L, \qquad c_{\tilde{\rho}_1,\tilde{\rho}_2} := \inf_{h\in\mathscr{H}} \sup_{m\in\bar{\mathcal{X}}_\nu} I_{\tilde{\rho}_1,\tilde{\rho}_2}(h(m)) > -2L, \tag{4.14}$$

where $\mathscr{H}$ is defined in (4.12). Moreover, using, for example, the test map (4.13), one shows that for μ sufficiently small there exists a large constant $\bar{L}$ such that, for $\tilde{\rho}_i \in \Lambda_i$,

$$c_{\tilde{\rho}_1,\tilde{\rho}_2} \leqslant \bar{L}.$$

Under these conditions, the following proposition is well known.

PROPOSITION 4.10. *Let ν be so small that (4.14) holds. Then, the functional $I_{t\rho_1,t\rho_2}$ possesses a bounded Palais–Smale sequence $(u_n)_n$ at level $c_{t\rho_1,t\rho_2}$ for almost every*

$$t \in \Gamma := \left[1 - \frac{\mu}{16\pi}, 1 + \frac{\mu}{16\pi}\right].$$

Using the above result we are now able to prove theorem 1.1.

Proof of theorem 1.1. The existence of a bounded Palais–Smale sequence for the functional $I_{t\rho_1,t\rho_2}$ implies, by standard arguments, that the functional possesses a critical point. Now consider $t_j \to 1$, $t_j \in \Gamma$, and let $(u_j)_j$ denote the corresponding solutions. It is then sufficient to apply the compactness result in theorem 2.1, which yields convergence of $(u_j)_j$ to a solution u of (1.1), by the fact that ρ_1, ρ_2 are not multiples of 8π. □

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