Proceedings of the Royal Society of Edinburgh, **146A**, 947–982, *2016*
DOI:10.1017/S0308210515000797

Nodal vector solutions with clustered peaks for nonlinear elliptic equations in $\mathbb{R}^3$

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(MS received 25 March 2014; accepted 24 June 2015)

We study the following coupled nonlinear Schrödinger system in $\mathbb{R}^3$:

$$-\varepsilon^2\Delta u + P(x)u = \mu_1 u^3 + \beta v^2 u, \quad x \in \mathbb{R}^3,$$
$$-\varepsilon^2\Delta v + Q(x)v = \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^3,$$

where $\mu_1 > 0$, $\mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant. Irrespective of whether the system is repulsive or attractive, we prove that it has nodal semi-classical segregated or synchronized bound states with clustered spikes for sufficiently small ε under some additional conditions on $P(x)$, $Q(x)$ and β. Moreover, the number of this type of solutions will go to infinity as $\varepsilon \to 0^+$.

Keywords: clustered peaks; nodal vector solutions; nonlinear coupled equations

2010 *Mathematics subject classification:* Primary 35J10; 35B99; 35J60

1. Introduction

In this paper, we consider the following nonlinear Schrödinger system in $\mathbb{R}^3$:

$$\left.\begin{aligned} -\varepsilon^2\Delta u + P(x)u &= \mu_1 u^3 + \beta v^2 u, \quad x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta v + Q(x)v &= \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^3, \end{aligned}\right\} \tag{1.1}$$

where we assume that $P(x)$ and $Q(x)$ are continuous bounded radial functions, $\mu_1 > 0, \mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant.

To study problem (1.1) we aim to look for standing-wave solutions for the following time-dependent coupled nonlinear Schrödinger system:

$$\left.\begin{aligned} \mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} &= -\frac{\varepsilon^2}{2m}\Delta_x\psi + P(x)\psi - \mu_1|\psi|^2\psi - \beta|\phi|^2\psi, \quad x \in \mathbb{R}^3,\ t > 0, \\ \mathrm{i}\varepsilon\frac{\partial\phi}{\partial t} &= -\frac{\varepsilon^2}{2m}\Delta_x\phi + Q(x)\phi - \mu_2|\phi|^2\phi - \beta|\psi|^2\phi, \quad x \in \mathbb{R}^3,\ t > 0, \\ \psi &= \psi(x,t) \in \mathcal{C}, \qquad \phi = \phi(x,t) \in \mathcal{C}, \end{aligned}\right\} \tag{1.2}$$

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which models a binary mixture of Bose–Einstein condensates in two different hyperfine states (see [12, 13, 19, 45]), and where ε is the Planck constant, m is the atomic mass and $P(x)$ and $Q(x)$ are the trapping potentials for two hyperfine states, respectively; the constants μ_1 and μ_2 represent the intraspecies scattering lengths and β is the interspecies scattering length. The sign of the interspecies scattering length determines whether the interaction of states is repulsive or attractive. If $\beta > 0$, the interaction is attractive, and the components of the vector of solutions are synchronized. On the other hand, if $\beta < 0$, the interaction is repulsive, leading to phase separations. These phenomena have been confirmed in experiment and in numerical simulations (see [13, 15, 19, 28] and the references therein). Problem (1.2), the system of Gross–Pitaevskii equations, arises in many applications, for example, in some problems arising in nonlinear optics, in plasma physics and in condensed matter physics. Physically, ψ and ϕ are the corresponding condensate wave functions (see [6]).

System (1.1) has been extensively investigated under various assumptions on $P(x)$, $Q(x)$ and β in recent years (see [1–5, 7, 9, 10, 12, 14–16, 18, 20, 22, 25, 28–41, 43, 44, 46, 47] and the references therein). Here mention some significant works: in [33], by using Nehari's manifold, Lin and Wei obtained least energy solutions that are independent of the sign of interspecies scattering length β for the two coupled nonlinear Schrödinger systems with trap potentials, and derived their asymptotic behaviours using techniques for the singular perturbation problem. Chen *et al*. [16] proved the existence of positive solutions with any prescribed spikes by using reduction methods. In [1], Alves was concerned with finding the existence and concentration of positive solutions by using the mountain pass theorem. Wan [46] used category theory to study the multiplicity of positive solutions and their limiting behaviour as $\varepsilon \to 0^+$. Wan and Ávila [47] also used category theory to study the relationship between the number of positive standing wave solutions for the special system (1.1) with $P(x) = Q(x)$ and $\beta = 0$ in $\mathbb{R}^N$ and the topology of the set of minimum points of potentials. Pomponio [41] proved the existence of concentrating solutions for a general system with repulsive interactions between states and that the location of the concentration points depends strictly on the potentials. In [10], Bartsch *et al*. considered the repulsive case and obtained segregated radial solutions by using global bifurcation methods for the general system (1.1), establishing the existence of infinite branches of radial solutions with the property that $\sqrt{\mu_1 - \beta}\psi - \sqrt{\mu_2 - \beta}\phi$ has exactly k nodal domains for solutions along the kth branch. Recently, Pi and Wang [40] constructed multiple solutions with arbitrary spikes and proved that these spikes will approach the local maximum point of the trapping potentials as $\varepsilon \to 0^+$.

Here we should point out that in the above results the solutions are positive. Although there is a wide literature studying the existence, multiplicity and shape of positive solutions, there are few papers dealing with the case of nodal solutions, with the exception of the single Schrödinger equation for the one-dimensional case or the radial case [8], which allows methods, such as the use of a natural constraint, that do not work in the non-radial setting considered here.

As far as we know, there are no results on the existence of nodal non-radial semiclassical bound states that have any prescribed nodal domain. In this paper, we shall present some results that remedy this gap in the literature.

In order to state our main results, we first assume that $\inf_{r\geqslant 0} P(r) > 0$ and $\inf_{r\geqslant 0} Q(r) > 0$ satisfy the following conditions.

(P) There exist constants $a \in \mathbb{R}$, $m > 1$ and $\theta > 0$ such that

$$P(r) = 1 + ar^m + O(r^{m+\theta}) \quad \text{as } r \to 0^+.$$

(Q) There exist constants $b \in \mathbb{R}$, $n > 1$ and $\delta > 0$, such that

$$Q(r) = 1 + br^n + O(r^{n+\delta}) \quad \text{as } r \to 0^+.$$

The main results of our paper are as follows.

THEOREM 1.1. *Let (P) and (Q) hold. Then, for any fixed* $k \in N^+$, *there exist a decreasing sequence* $\{\beta_l\} \subset (-\sqrt{\mu_1\mu_2}, 0)$, *with* $\beta_l \to -\sqrt{\mu_1\mu_2}$ *as* $l \to \infty$, *and* $\varepsilon_0 > 0$ *such that, for* $\beta \in (-\sqrt{\mu_1\mu_2}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, $\beta \neq \beta_l$, *and* $0 < \varepsilon < \varepsilon_0$, *(1.1) has a vector solution* $(u_\varepsilon, v_\varepsilon)$ *with* k *positive peaks and* k *negative peaks, and the peaks of the solution approach the extremal point* 0 *of* $P(x)$ *and* $Q(x)$ *provided that one of the following two conditions holds:*

(1) $m < n$, $a > 0$ *and* $b \in \mathbb{R}$ *or* $m > n$, $a \in \mathbb{R}$ *and* $b > 0$;

(2) $m = n$, $aB + bC_0 > 0$, *where* B *and* C_0 *are defined in proposition A.1.*

Furthermore,

$$\|\sqrt{|\mu_1 - \beta|}u_\varepsilon - \sqrt{|\mu_2 - \beta|}v_\varepsilon\|_{H^1} + \|\sqrt{|\mu_1 - \beta|}u_\varepsilon - \sqrt{|\mu_2 - \beta|}v_\varepsilon\|_{L^\infty} \to 0$$

as $\varepsilon \to 0^+$.

THEOREM 1.2. *Let (P) and(Q) hold. If* $m = n$, $a > 0$, $b > 0$, *then for any fixed* $k \in N^+$ *there exist constants* $\beta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that, for any* $\beta < \beta_0$ *and* $0 < \varepsilon < \varepsilon_0$, *(1.1) has a vector solution* $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ *with* k *positive peaks and* k *negative peaks that approach the local minimum point* 0 *of* $P(x)$ *and* $Q(x)$ *as* $\varepsilon \to 0^+$. *Furthermore,*

$$\|\sqrt{\mu_2}\tilde{u}_\varepsilon(\cdot) - \sqrt{\mu_1}\tilde{v}_\varepsilon(T_\varepsilon\cdot)\|_{H^1} + \|\sqrt{\mu_2}\tilde{u}_\varepsilon(\cdot) - \sqrt{\mu_1}\tilde{v}_\varepsilon(T_\varepsilon\cdot)\|_{L^\infty} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Here $T_\varepsilon \in \mathrm{SO}(3)$ *is the rotation on the* (x_1, x_2)*-plane of* π/k.

Next, we introduce some notation to be used in the proofs of the main results and formulate versions of the main results that describe the segregated or synchronized character of the solutions more precisely. In doing so, we also outline the main idea and the approaches to the proofs of theorems 1.1 and 1.2.

Define

$$\begin{aligned} H_s = \bigg\{ u \in H^1(\mathbb{R}^3) \colon & u \text{ is even in } x_h,\ h = 2, 3, \\ & u\left(r\cos\left(\theta + \frac{\pi j}{k}\right), r\sin\left(\theta + \frac{\pi j}{k}\right), x_3\right) \\ & \qquad = (-1)^j u(r\cos\theta, r\sin\theta, x_3) \bigg\}, \end{aligned} \tag{1.3}$$

https://doi.org/10.1017/S0308210515000797 Published online by Cambridge University Press

where $H^1(\mathbb{R}^3)$ is the usual Sobolev space with the norm

$$\|u\|_{\varepsilon,K}^2 = (u,u)_\varepsilon = \int_{\mathbb{R}^3} (\varepsilon^2|\nabla u|^2 + K(x)|u|^2)\,\mathrm{d}x$$

for any bounded function $K(x)$. Next, define $H = H_s \times H_s$ endowed with the following norm:

$$\|(u,v)\|_\varepsilon^2 = \|u\|_{\varepsilon,P}^2 + \|v\|_{\varepsilon,Q}^2.$$

Set

$$w_{y,\varepsilon}(x) = w\left(\frac{x-y}{\varepsilon}\right)$$

and

$$S_\varepsilon := \left[\frac{\min\{m,n\}-\delta}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}, \frac{\min\{m,n\}+\delta}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right], \tag{1.4}$$

where $\delta \in (0, (\sigma/(1+\sigma))\min\{n,m\})$, and σ is defined in proposition A.2. Define

$$x^j := \left(r\cos\frac{(j-1)\pi}{k}, r\sin\frac{(j-1)\pi}{k}, x_3\right), \quad j = 1,2,\dots,2k,\ r \in S_\varepsilon. \tag{1.5}$$

It is well known that the following problem has a unique radial solution denoted by w:

$$-\Delta u + u = u^3, \quad \max_{x\in\mathbb{R}^3} u(x) = u(0), \quad u > 0, \tag{1.6}$$

and that w satisfies the following:

$$w'(r) < 0, \quad \lim_{r\to\infty} r^{(N-1)/2}\mathrm{e}^r w(r) = C_0 > 0, \quad \lim_{r\to\infty}\frac{w'(r)}{w(r)} = -1.$$

When $-\sqrt{\mu_1\mu_2} < \beta < \min\{\mu_1,\mu_2\}$ or $\beta > \max\{\mu_1,\mu_2\}$, $(U,V) := (\alpha w, \gamma w)$ is a solution of the following system:

$$\left.\begin{aligned} -\Delta u + u &= \mu_1 u^3 + \beta v^2 u, \quad x \in \mathbb{R}^3, \\ -\Delta v + v &= \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^3, \end{aligned}\right\} \tag{1.7}$$

where

$$\alpha = \sqrt{\frac{\mu_2-\beta}{\mu_1\mu_2-\beta^2}}, \qquad \gamma = \sqrt{\frac{\mu_1-\beta}{\mu_1\mu_2-\beta^2}}.$$

We let

$$U_r(x) = \sum_{j=1}^{2k}(-1)^{j-1}U_{x^j,\varepsilon}, \qquad V_r(x) = \sum_{j=1}^{2k}(-1)^{j-1}V_{x^j,\varepsilon}.$$

We shall verify theorem 1.1 by proving the following result.

THEOREM 1.3. *Under the assumptions of theorem 1.1, there exists a positive constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ (1.1) has a solution of the form*

$$(u_\varepsilon, v_\varepsilon) = (U_r(x) + \varphi(x), V_r(x) + \psi(x)),$$

where $(\varphi(x), \psi(x)) \in H$ *and*

$$\|(\varphi(x), \psi(x))\|_\varepsilon = O(\varepsilon^{(3+\min\{m,n\}-\sigma)/2}), \quad |x^j| = O\biggl(\varepsilon \ln \frac{1}{\varepsilon}\biggr),$$

for some small constant $\sigma > 0$.

Let U_i be the unique radial solution of the following problem:

$$-\Delta u + u = \mu_i u^3, \quad \max_{x\in\mathbb{R}^3} u(x) = u(0), \quad u > 0.$$

It is well known that U_i is non-degenerate and

$$U_i'(r) < 0, \qquad \lim_{r\to\infty} r^{(N-1)/2} \mathrm{e}^r U_i(r) = C_0 > 0, \qquad \lim_{r\to\infty} \frac{U_i'(r)}{U_i(r)} = -1.$$

We shall use (U_1, U_2) to build up the approximate solutions for (1.1).

Let x^j be as defined in (1.5) and define

$$y^j := \biggl(\rho \cos \frac{(2j-1)\pi}{2k}, \rho \sin \frac{(2j-1)\pi}{2k}, x_3\biggr), \quad j = 1, 2, \dots, 2k, \tag{1.8}$$

where $\rho \in S_\varepsilon$.

Let

$$\tilde{U}_r = \sum_{j=1}^{2k} (-1)^{j-1} U_{1,x^j,\varepsilon}, \qquad \tilde{V}_\rho = \sum_{j=1}^{2k} (-1)^{j-1} U_{2,y^j,\varepsilon}. \tag{1.9}$$

To prove theorem 1.2, we need to prove the following result.

Theorem 1.4. *Under the assumptions of theorem 1.2, there exists a positive constant* ε_0 *such that for any* $0 < \varepsilon < \varepsilon_0$ *(1.1) has a solution of the form*

$$(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) = (\tilde{U}_r(x) + \tilde{\varphi}(x), \tilde{V}_\rho(x) + \tilde{\psi}(x)),$$

where $(\tilde{\varphi}(x), \tilde{\psi}(x)) \in H$ *and*

$$\|(\tilde{\varphi}(x), \tilde{\psi}(x))\|_\varepsilon = O(\varepsilon^{(3+\min\{m,n\}-\sigma)/2}), \quad |x^j| = O\biggl(\varepsilon \ln \frac{1}{\varepsilon}\biggr), \quad |y^j| = O\biggl(\varepsilon \ln \frac{1}{\varepsilon}\biggr)$$

for some small constant $\sigma > 0$.

Remark 1.5. Radial symmetries can be replaced by weaker symmetrical assumptions. After suitably rotating the coordinate system, we have the following.

(P′) $P(x) = P(x', x_3) = P(|x' - \bar{x}'|, x_3 - \bar{x}_3)$ and $P(x)$ has the following expansion:

$$P(r) = P(\bar{x}) + a|x - \bar{x}|^m + O(|x - \bar{x}|^{m+\theta}) \quad \text{as } |x - \bar{x}| \to 0,$$

where $\bar{x} \in \mathbb{R}^3$, $a \in \mathbb{R}, m > 1, \theta > 0$ and $P(\bar{x}) > 0$ are constants.

(Q′) $Q(x) = Q(x', x_3) = Q(|x' - \bar{x}'|, x_3 - \bar{x}_3)$ and $Q(x)$ has the following expansion:

$$Q(r) = Q(\bar{x}) + b|x - \bar{x}|^n + O(|x - \bar{x}|^{n+\delta}) \quad \text{as } |x - \bar{x}| \to 0,$$

where $\bar{x} \in \mathbb{R}^3$, $b \in \mathbb{R}$, $n > 1$, $\delta > 0$ and $Q(\bar{x}) > 0$ are constants.

REMARK 1.6. For $N = 2$, if we adjust the constants δ, τ, τ_2 in (1.4), then both lemma 2.4 and proposition 2.5 still hold. In order to guarantee that proposition 2.5 holds, we can find nodal synchronized solutions of (1.1) for the attractive case under the same assumptions. However, for the repulsive case, we cannot find nodal segregated solutions of (1.1), since proposition 3.4 cannot hold.

The proofs of our main results are based on the well-known Lyapunov–Schmidt reduction procedure which has been used widely (see, for example, [11, 17, 21, 23, 24, 26, 27, 29]). In particular, in order to deal with nodal clustered solutions, we perform the reduction in suitable symmetric settings in the spirit of [48], where infinitely many positive non-radial solutions for nonlinear Schrödinger equations were obtained. For the attractive case, we shall construct nodal synchronized solutions approximately as

$$\left(\sum_{j=1}^{2k} (-1)^{j-1} U_{x^j,\varepsilon}, \sum_{j=1}^{2k} (-1)^{j-1} V_{x^j,\varepsilon} \right)$$

with the points x^j located on and dividing the circle with radius $C\varepsilon \ln(1/\varepsilon)$ into $2k$ equal parts. Since the distance between two neighbouring peaks with the same sign is larger than that between two neighbouring peaks with opposite sign, the interaction between peaks with opposite sign dominates the interaction between peaks with the same sign. Hence, if the more slowly decaying function of $Q(x)$ and $P(x)$ has a local minimum at the centre of the circle, we can easily conclude that the equilibrium is achieved for a suitable configuration of the points x^j, which can be reduced to solving a minimization problem related to the energy functional. Generally speaking, the key to constructing nodal solutions by using the reduction argument is to compare the influence of the interaction between the peaks with the same sign and that between the peaks with opposite sign: the idea in [48] can help us to construct a symmetric configuration space consisting of x^j $(j = 1, \dots, 2k)$. For the repulsive case, we shall construct approximate nodal segregated solutions as

$$\left(\sum_{j=1}^{2k} (-1)^{j-1} U_{1,x^j,\varepsilon}, \sum_{j=1}^{2k} (-1)^{j-1} U_{2,y^j,\varepsilon} \right)$$

with the points x^j and y^j located on and dividing the circles with radius

$$C_1 \varepsilon \ln\left(\frac{1}{\varepsilon}\right) \quad \text{and} \quad C_2 \varepsilon \ln\left(\frac{1}{\varepsilon}\right),$$

respectively, into $2k$ equal parts, and the vector from the origin to y^j equally dividing the angle $\angle x^j o x^{j+1}$. Then, using similar methods to the attractive case, we can construct nodal segregated solutions. This idea is also effective in finding infinitely many non-radial positive solutions for semilinear elliptic problems (see [39]).

This paper is organized as follows. In § 2, we shall study the finite-dimensional reduced problem and prove theorem 1.3. In § 3 we study the existence of segregated solutions for (1.1) and prove theorem 1.4. All the technical calculations are given in the appendix.

2. Synchronized vector solutions and the proof of theorem 1.1

In this section, we consider synchronized vector solutions and prove theorem 1.1 by proving theorem 1.3. The functional corresponding to (1.1) is

$$I_\varepsilon(u,v) = \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u|^2 + P(x)u^2 + \varepsilon^2|\nabla v|^2 + Q(x)v^2)\,\mathrm{d}x \\ - \frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|u|^4 + \mu_2|v|^4)\,\mathrm{d}x - \tfrac{1}{2}\beta\int_{\mathbb{R}^3}u^2v^2\,\mathrm{d}x. \quad (2.1)$$

Then $I_\varepsilon \in C^2(H)$ and its critical points correspond to the solutions of (1.1).

Define

$$Y_j := \frac{\partial U_{x^j,\varepsilon}}{\partial r}, \quad Z_j := \frac{\partial V_{x^j,\varepsilon}}{\partial r}, \quad j = 1,2,\dots,2k,$$

where x^j is as defined in (1.5), and define

$$E = \biggl\{(u,v) \in H\colon \sum_{j=1}^{2k}\int_{\mathbb{R}^3}(U^2_{x^j,\varepsilon}Y_ju + V^2_{x^j,\varepsilon}Z_jv)\,\mathrm{d}x = 0\biggr\}. \quad (2.2)$$

Let

$$J(\varphi,\psi) = I_\varepsilon(U_r+\varphi, V_r+\psi), \quad (\varphi,\psi) \in E.$$

Expand $J(\varphi,\psi)$ as follows:

$$J(\varphi,\psi) = J(0,0) + l(\varphi,\psi) + \tfrac{1}{2}Q(\varphi,\psi) + R(\varphi,\psi), \quad (\varphi,\psi) \in E, \quad (2.3)$$

where

$$\begin{aligned} l(\varphi,\psi) = \sum_{j=1}^{2k}(-1)^{j-1}&\int_{\mathbb{R}^3}(P(x)-1)U_{x^j,\varepsilon}\varphi \\ &- \mu_1\int_{\mathbb{R}^3}\biggl(U_r^3 - \sum_{j=1}^{2k}(-1)^{j-1}U^3_{x^j,\varepsilon}\biggr)\varphi \\ &+ \sum_{j=1}^{2k}(-1)^{j-1}\int_{\mathbb{R}^3}(Q(x)-1)V_{x^j,\varepsilon}\psi \\ &- \mu_2\int_{\mathbb{R}^3}\biggl(V_r^3 - \sum_{j=1}^{2k}(-1)^{j-1}V^3_{x^j,\varepsilon}\biggr)\psi \\ &- \beta\int_{\mathbb{R}^3}\biggl(U_rV_r^2 - \sum_{j=1}^{2k}(-1)^{j-1}V^2_{x^j,\varepsilon}U_{x^j,\varepsilon}\biggr)\varphi \\ &- \beta\int_{\mathbb{R}^3}\biggl(U_r^2V_r - \sum_{j=1}^{2k}(-1)^{j-1}V_{x^j,\varepsilon}U^2_{x^j,\varepsilon}\biggr)\psi, \end{aligned}$$

$$\begin{aligned}Q(\varphi,\psi) = \int_{\mathbb{R}^3} (\varepsilon^2|\nabla\varphi|^2 + P(x)\varphi^2 - 3\mu_1 U_r^2\varphi^2) \\ + \int_{\mathbb{R}^3} (\varepsilon^2|\nabla\psi|^2 + Q(x)\psi^2 - 3\mu_2 V_r^2\psi^2) \\ - \beta\int_{\mathbb{R}^3} (U_r^2\psi^2 + 4U_rV_r\varphi\psi + V_r^2\varphi^2)\end{aligned}$$

and

$$\begin{aligned}R(\varphi,\psi) = \int_{\mathbb{R}^3} (\mu_1 U_r\varphi^3 + \mu_2 V_r\psi^3 + \tfrac{1}{4}\mu_1\varphi^4 + \tfrac{1}{4}\mu_2\psi^4) \\ - \tfrac{1}{2}\beta\int_{\mathbb{R}^3} [(U_r+\varphi)^2(V_r+\psi)^2 - U_r^2V_r^2 - 2(U_rV_r^2\varphi + U_r^2V_r\psi) \\ - (U_r^2\psi^2 + V_r^2\varphi^2 + 4U_rV_r\varphi\psi)].\end{aligned}$$

In order to find a critical point $(\varphi,\psi) \in E$ for $J(\varphi,\psi)$, we need to discuss each term in the expansion (2.3).

It is easy to check that

$$\begin{aligned}\int_{\mathbb{R}^3} (\varepsilon^2\nabla u\nabla\varphi + P(x)u\varphi - 3\mu_1 U_r^2 u\varphi) \\ + \int_{\mathbb{R}^3} (\varepsilon^2\nabla v\nabla\psi + Q(x)v\psi - 3\mu_2 V_r^2 v\psi) \\ - \beta\int_{\mathbb{R}^3} (U_r^2 v\psi + V_r^2 u\varphi + 2U_rV_r u\psi + 2U_rV_r v\varphi)\end{aligned}$$

is a bounded bi-linear functional in E. Thus, there exists a bounded linear operator L from E to E such that

$$\begin{aligned}&\langle L(u,v),(\varphi,\psi)\rangle \\ &= \int_{\mathbb{R}^3} (\varepsilon^2\nabla u\nabla\varphi + P(x)u\varphi - 3\mu_1 U_r^2 u\varphi) \\ &\quad + \int_{\mathbb{R}^3} (\varepsilon^2\nabla v\nabla\psi + Q(x)v\psi - 3\mu_2 V_r^2 v\psi) \\ &\quad - \beta\int_{\mathbb{R}^3} (U_r^2 v\psi + V_r^2 u\varphi + 2U_rV_r u\psi + 2U_rV_r v\varphi), \quad (u,v),(\varphi,\psi) \in E.\end{aligned}$$

From the above analysis, we have the following lemma.

LEMMA 2.1. *There exists a constant $C > 0$, independent of ε, such that, for any $r \in S_\varepsilon$,*

$$\|L(u,v)\| \leqslant C\|(u,v)\|_\varepsilon, \quad (u,v) \in E.$$

Next, we discuss the invertibility of L.

LEMMA 2.2. *There exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and any $r \in S_\varepsilon$,*

$$\|L(u,v)\| \geqslant C_0\|(u,v)\|_\varepsilon, \quad (u,v) \in E.$$

Proof. We argue by contradiction. Suppose that there exist $\varepsilon_n \to 0^+$, $r_n \in S_{\varepsilon_n}$ and $(u_n, v_n) \in E$ such that

$$\|L(u_n, v_n)\| = o_n(1)\|(u_n, v_n)\|_{\varepsilon_n}.$$

Since L is linear, we may also assume that

$$\|(u_n, v_n)\|_{\varepsilon_n}^2 = \varepsilon_n^3$$

and

$$\|L(u_n, v_n)\| = o_n(1)\varepsilon_n^{3/2}. \tag{2.4}$$

Then

$$\langle L(u_n, v_n), (\varphi, \psi)\rangle = o_n(1)\|(\varphi, \psi)\|_{\varepsilon_n}\varepsilon_n^{3/2} \quad \forall(\varphi, \psi) \in E.$$

That is,

$$\begin{aligned}\int_{\mathbb{R}^3} &(\varepsilon_n^2 \nabla u_n \nabla \varphi + P(x)u_n\varphi - 3\mu_1 U_{r_n}^2 u_n \varphi) \\ &+ \int_{\mathbb{R}^3} (\varepsilon_n^2 \nabla v_n \nabla \psi + Q(x)v_n\psi - 3\mu_2 V_{r_n}^2 v_n \psi) \\ &\quad - \beta \int_{\mathbb{R}^3} (U_{r_n}^2 v_n \psi + V_{r_n}^2 u_n \varphi + 2U_{r_n} V_{r_n} u_n \psi + 2U_{r_n} V_{r_n} v_n \varphi) \\ &\qquad = o_n(1)\|(\varphi, \psi)\|_{\varepsilon_n}\varepsilon_n^{3/2} \quad \forall(\varphi, \psi) \in E. \end{aligned} \tag{2.5}$$

In particular, we have

$$\begin{aligned}\int_{\mathbb{R}^3} &(\varepsilon_n^2 |\nabla u_n|^2 + P(x)|u_n|^2 - 3\mu_1 U_{r_n}^2 u_n^2) \\ &+ \int_{\mathbb{R}^3} (\varepsilon_n^2 |\nabla v_n|^2 + Q(x)|v_n|^2 - 3\mu_2 V_{r_n}^2 v_n^2) \\ &\quad - \beta \int_{\mathbb{R}^3} (U_{r_n}^2 v_n^2 + V_{r_n}^2 u_n^2 + 4U_{r_n} V_{r_n} u_n v_n) = o_n(1)\varepsilon_n^3. \end{aligned} \tag{2.6}$$

We set $\tilde{u}_n(y) = u_n(\varepsilon_n y + x^1)$ and $\tilde{v}_n(y) = v_n(\varepsilon_n y + x^1)$. Then

$$\int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + P(\varepsilon_n y + x^1)\tilde{u}_n^2 + |\nabla \tilde{v}_n|^2 + Q(\varepsilon_n y + x^1)\tilde{v}_n^2) = 1. \tag{2.7}$$

Therefore, there exist $u, v \in H^1(\mathbb{R}^3)$ such that $n \to \infty$,

$$\begin{aligned} \tilde{u}_n \to u \quad &\text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \qquad \tilde{u}_n \to u \quad \text{strongly in } \ L^2_{\text{loc}}(\mathbb{R}^3), \\ \tilde{v}_n \to v \quad &\text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \qquad \tilde{v}_n \to v \quad \text{strongly in } \ L^2_{\text{loc}}(\mathbb{R}^3). \end{aligned}$$

Since $\tilde{u}_n$ and $\tilde{v}_n$ are even in y_2 and y_3, it is easy to see that u and v are even in y_2 and y_3.

On the other hand, from the definition of E we know that (u, v) satisfies

$$\int_{\mathbb{R}^3} \left(U^2 \frac{\partial U}{\partial x_1} u + V^2 \frac{\partial V}{\partial x_1} v\right) = 0. \tag{2.8}$$

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Now we claim that (u,v) satisfies

$$\left.\begin{aligned}-\Delta u+u-3\mu_1U^2u-\beta V^2u-2\beta UVv=0,\quad x\in\mathbb{R}^3,\\ -\Delta v+v-3\mu_2V^2v-\beta U^2v-2\beta UVu=0,\quad x\in\mathbb{R}^3.\end{aligned}\right\}\tag{2.9}$$

Define

$$\hat{E}=\biggl\{(\varphi,\psi)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\colon \int_{\mathbb{R}^3}\biggl(U^2\frac{\partial U}{\partial x_1}u+V^2\frac{\partial V}{\partial x_1}v\biggr)=0\biggr\}.$$

For any $R>0$, let $(\varphi,\psi)\in C_0^\infty(B_R(0))\times C_0^\infty(B_R(0))\cap\hat{E}$ and let (φ,ψ) be even in y_2 and y_3. Then

$$(\varphi_n(y),\psi_n(y)):=\biggl(\varphi\biggl(\frac{y-x^1}{\varepsilon_n}\biggr),\psi\biggl(\frac{y-x^1}{\varepsilon_n}\biggr)\biggr)\in C_0^\infty(B_{R\varepsilon_n}(x^1))\times C_0^\infty(B_{R\varepsilon_n}(x^1)).$$

Inserting $(\varphi_n(y),\psi_n(y))$ into (2.5), we find that

$$\begin{aligned}\int_{\mathbb{R}^3}(\nabla u\nabla\varphi+u\varphi-3\mu_1U^2u\varphi)+\int_{\mathbb{R}^3}(\nabla v\nabla\psi+v\psi-3\mu_2V^2v\psi)\\ -\beta\int_{\mathbb{R}^3}(U^2v\psi+V^2u\varphi+2UVu\psi+2UVv\varphi)=0.\end{aligned}\tag{2.10}$$

However, since u and v are even in y_2 and y_3, (2.10) holds for any function $(\varphi,\psi)\in C_0^\infty(B_R(0))\times C_0^\infty(B_R(0))$, which is odd in y_2 or y_3. Therefore, (2.10) holds for any $(\varphi,\psi)\in C_0^\infty(B_R(0))\times C_0^\infty(B_R(0))\cap\hat{E}$. By the density of $C_0^\infty(B_R(0))\times C_0^\infty(B_R(0))$ in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^3}(\nabla u\nabla\varphi+u\varphi-3\mu_1U^2u\varphi)+\int_{\mathbb{R}^3}(\nabla v\nabla\psi+v\psi-3\mu_2V^2v\psi)\\ -\beta\int_{\mathbb{R}^3}(U^2v\psi+V^2u\varphi+2UVu\psi+2UVv\varphi)=0\quad\forall(\varphi,\psi)\in\hat{E}.\end{aligned}\tag{2.11}$$

Noting that $(U,V)=(\alpha w,\gamma w)$ and w is a solution of (1.6), we can show that (2.10) holds for

$$(\varphi,\psi)=\biggl(\frac{\partial U}{\partial x_1},\frac{\partial V}{\partial x_1}\biggr).$$

Thus, (2.10) is true for any $(\varphi,\psi)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Therefore, we have verified (2.9).

From [39, proposition 2.3] know that (U,V) is non-degenerate. Since we work in the space of functions that are even in y_2 and y_3, the kernel of (U,V) is given by the one-dimensional $(\theta(\beta)(\partial w/\partial x_1),(\partial w/\partial x_1))$. So, we get

$$(u,v)=c\biggl(\frac{\partial U}{\partial x_1},\frac{\partial V}{\partial x_1}\biggr)$$

for some constant c. From (2.8), we can see $(u,v)=(0,0)$. As a result,

$$\int_{B_R(-x^1)}(u_n^2+v_n^2)=o_n(1)\varepsilon^3\quad\forall R>0.$$

By direct calculation, we get

$$\int_{\mathbb{R}^3} (U_{r_n}^2 u_n^2 + V_{r_n}^2 v_n^2) = o_n(1)\varepsilon_n^3 + o_R(1)\varepsilon_n^3.$$

As a result, one has

$$\begin{aligned} o_n(1)\varepsilon_n^3 &= \int_{\mathbb{R}^3} (\varepsilon_n^2 |\nabla u_n|^2 + P(x)|u_n|^2 - 3\mu_1 U_{r_n}^2 u_n^2) \\ &\quad + \int_{\mathbb{R}^3} (\varepsilon_n^2 |\nabla v_n|^2 + Q(x)|v_n|^2 - 3\mu_2 V_{r_n}^2 v_n^2) \\ &\quad - \beta \int_{\mathbb{R}^3} (U_{r_n}^2 v_n^2 + V_{r_n}^2 u_n^2 + 4U_{r_n} V_{r_n} u_n v_n) \\ &= (1 + o_n(1) + o_R(1))\varepsilon_n^3. \end{aligned} \tag{2.12}$$

This is a contradiction, so the proof is complete. □

LEMMA 2.3. *For any $(\varphi, \psi) \in E$, we have*

$$\begin{aligned} \|R(\varphi,\psi)\| &= O(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon^3 + \varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^4), \\ \|R'(\varphi,\psi)\| &= O(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon^2 + \varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^3), \\ \|R''(\varphi,\psi)\| &= O(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon + \varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^2). \end{aligned}$$

Proof. By direct calculation, we have, for any $(u_1, v_1), (u_2, v_2) \in E$,

$$\begin{aligned} |R(\varphi,\psi)| &= \bigg| \int_{\mathbb{R}^3} (\mu_1 U_r \varphi^3 + \mu_2 V_r \psi^3 + \tfrac{1}{4}\mu_1 \varphi^4 + \tfrac{1}{4}\mu_2 \psi^4) \\ &\qquad - \tfrac{1}{2}\beta \int_{\mathbb{R}^3} [(U_r + \varphi)^2 (V_r + \psi)^2 - U_r^2 V_r^2 - 2(U_r V_r^2 \varphi + U_r^2 V_r \psi) \\ &\qquad\qquad - (U_r^2 \psi^2 + V_r^2 \varphi^2 + 4U_r V_r \varphi\psi)] \bigg| \\ &= \bigg| \int_{\mathbb{R}^3} (\mu_1 U_r \varphi^3 + \mu_2 V_r \psi^3 + \tfrac{1}{4}\mu_1 \varphi^4 + \tfrac{1}{4}\mu_2 \psi^4) \\ &\qquad - \tfrac{1}{2}\beta \int_{\mathbb{R}^3} (\varphi^2\psi^2 + 2U_r \varphi \psi^2 + 2V_r \varphi^2 \psi) \bigg| \\ &\leqslant C \int_{\mathbb{R}^3} \bigg(\sum_{j=1}^{2k} U_{x^j,\varepsilon} |\varphi|^3 + \varphi^4 + \sum_{j=1}^{2k} V_{x^j,\varepsilon} |\psi|^3 + \psi^4 \bigg) \\ &\leqslant C(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon^3 + \varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} &|\langle R'(\varphi,\psi), (u_1, v_1)\rangle| \\ &\quad = \bigg| \int_{\mathbb{R}^3} (3\mu_1 U_r \varphi^2 u_1 + 3\mu_2 V_r \psi^2 v_1 + \mu_1 \varphi^3 u_1 + \mu_2 \psi^3 v_1) \\ &\qquad + \beta \int_{\mathbb{R}^3} (\varphi\psi^2 u_1 + \varphi^2 \psi v_1 + 2U_r \varphi\psi v_1 + 2U_r \psi^2 u_1 + 2V_r \varphi\psi u_1 + 2V_r \varphi^2 v_1) \bigg| \end{aligned}$$

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$$
\begin{aligned}
&\leqslant C\int_{\mathbb{R}^3}\bigg[\bigg(\sum_{j=1}^{2k}U_{x^j,\varepsilon}+\sum_{j=1}^{2k}V_{x^j,\varepsilon}\bigg)(\varphi^2+\psi^2)(|u_1|+|v_1|)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + (|\varphi|^3+|\psi|^3)(|v_1|+|u_1|)\bigg]\\
&\leqslant C(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon^2+\varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^3)\|(u_1,v_1)\|_\varepsilon.
\end{aligned}
$$

And by similar calculation, we get

$$
\begin{aligned}
&|\langle R''(\varphi,\psi)(u_1,v_1),(u_2,v_2)\rangle|\\
&\qquad\qquad \leqslant C(\varepsilon^{-3/2}\|(\varphi,\psi)\|_\varepsilon+\varepsilon^{-4}\|(\varphi,\psi)\|_\varepsilon^2)\|(u_1,v_1)\|_\varepsilon\|(u_2,v_2)\|_\varepsilon,
\end{aligned}
$$

and so this completes the proof. □

LEMMA 2.4. *There exists a small constant $\tau \in D$ such that*

$$
\|l\| = O\bigg(r^{\min\{n,m\}}+\exp\left(-\frac{3(1-\tau)r}{\varepsilon}\right)+\exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right)\bigg)\varepsilon^{3/2},
$$

where

$$
D=\{x\in(0,\tfrac{1}{3}) \mid (1-x)(2-x)\geqslant \tfrac{11\sqrt{2}}{10}\}.
$$

Proof. By direct computation, we have

$$
\begin{aligned}
&\sum_{j=1}^{2k}(-1)^{j-1}\int_{\mathbb{R}^3}(P(x)-1)U_{x^j,\varepsilon}\varphi+\sum_{j=1}^{2k}(-1)^{j-1}\int_{\mathbb{R}^3}(Q(x)-1)V_{x^j,\varepsilon}\psi\\
&\qquad\leqslant \sum_{j=1}^{2k}\left(\int_{\mathbb{R}^3}|(P(x)-1)|^2U_{x^j,\varepsilon}^2\right)^{1/2}\left(\int_{\mathbb{R}^3}\varphi^2\right)^{1/2}\\
&\qquad\quad+\sum_{j=1}^{2k}\left(\int_{\mathbb{R}^3}|(Q(x)-1)|^2V_{x^j,\varepsilon}^2\right)^{1/2}\left(\int_{\mathbb{R}^3}\psi^2\right)^{1/2}\\
&\qquad\leqslant C\varepsilon^{3/2}\bigg(r^m+\exp\left(-\frac{3(1-\tau)r}{\varepsilon}\right)\bigg)\|\varphi\|_{\varepsilon,P}\\
&\qquad\quad+C\varepsilon^{3/2}\bigg(r^n+\exp\left(-\frac{3(1-\tau)r}{\varepsilon}\right)\bigg)\|\psi\|_{\varepsilon,Q}\\
&\qquad\leqslant C\bigg(r^{\min\{m,n\}}+\exp\left(-\frac{3(1-\tau)r}{\varepsilon}\right)\bigg)\varepsilon^{3/2}\|(\varphi,\psi)\|_\varepsilon, \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
&\mu_1\int_{\mathbb{R}^3}\bigg(\sum_{j=1}^{2k}(-1)^{j-1}U_{x^j,\varepsilon}^3-U_r^3\bigg)\varphi+\mu_2\int_{\mathbb{R}^3}\bigg(\sum_{j=1}^{2k}(-1)^{j-1}V_{x^j,\varepsilon}^3-V_r^3\bigg)\psi\\
&\qquad\leqslant C\varepsilon^{3/2}\exp\left(-\frac{|x^1-x^2|}{\varepsilon}\right)\|(\varphi,\psi)\|_\varepsilon \qquad (2.14)
\end{aligned}
$$

and

$$\begin{aligned}\beta \int_{\mathbb{R}^3} \biggl(\sum_{j=1}^{2k} (-1)^{j-1} V_{x^j,\varepsilon}^2 U_{x^j,\varepsilon} - V_r^2 U_r\biggr)\varphi \\ + \beta \int_{\mathbb{R}^3} \biggl(\sum_{j=1}^{2k} (-1)^{j-1} U_{x^j,\varepsilon}^2 V_{x^j,\varepsilon} - U_r^2 V_r\biggr)\psi \\ \leqslant C\varepsilon^{3/2} \exp\biggl(-\frac{|x^1 - x^2|}{\varepsilon}\biggr) \|(\varphi, \psi)\|_\varepsilon. \end{aligned} \tag{2.15}$$

Combining (2.13)–(2.15) and the definition of l, we can deduce that

$$\|l\| = O\biggl(r^{\min\{n,m\}} + \exp\biggl(-\frac{3(1-\tau)r}{\varepsilon}\biggr) + \exp\biggl(-\frac{2r\sin(\pi/2k)}{\varepsilon}\biggr)\biggr)\varepsilon^{3/2}.$$

□

Proposition 2.5. *For ε sufficiently small, there exists a C^1-map (φ, ψ) from S_ε to H: $(\varphi, \psi) := (\varphi(r), \psi(r))$, $r = |x|$ satisfying $(\varphi, \psi) \in E$ and*

$$\biggl\langle \frac{\partial J(\varphi, \psi)}{\partial(\varphi, \psi)}, (g, h)\biggr\rangle = 0 \quad \forall (g, h) \in E.$$

Moreover, there exists a small constant

$$0 < \tau_2 < \min\biggl\{\frac{1}{5}, \frac{\min\{n, m\} - 1 - \sigma}{\min\{n, m\}}\biggr\}$$

such that

$$\begin{aligned}\|(\varphi, \psi)\|_\varepsilon \leqslant \biggl(r^{(1-\tau_2)\min\{m,n\}} + \exp\biggl(-\frac{3(1-\tau_2)(1-\tau)r}{\varepsilon}\biggr) \\ + \exp\biggl(-\frac{(1-\tau_2)2r\sin(\pi/2k)}{\varepsilon}\biggr)\biggr)\varepsilon^{3/2}.\end{aligned}$$

Proof. It follows from lemma 2.4 that l is a bounded linear functional in E. Thus, there exists an $l' \in E$ such that $l(\varphi, \psi) = \langle l', (\varphi, \psi)\rangle$. Thus, finding a critical point for $J(\varphi, \psi)$ is equivalent to solving

$$l' + L(\varphi, \psi) + R'(\varphi, \psi) = 0. \tag{2.16}$$

By lemma 2.2, L is invertible. Hence, (2.16) can be written as

$$(\varphi, \psi) = A(\varphi, \psi) := -L^{-1}l' - L^{-1}R'(\varphi, \psi). \tag{2.17}$$

We choose a small constant

$$0 < \tau_2 < \min\biggl\{\frac{1}{5}, \frac{\min\{n, m\} - 1 - \sigma}{\min\{n, m\}}\biggr\}$$

and set

$$S = \left\{ (\varphi, \psi) \in E \colon \|(\varphi, \psi)\|_\varepsilon \leqslant \varepsilon^{3/2} \left(r^{(1-\tau_2)\min\{m,n\}} + \exp\left(-\frac{3(1-\tau_2)(1-\tau)r}{\varepsilon} \right) + \exp\left(-\frac{(1-\tau_2)2r\sin(\pi/2k)}{\varepsilon} \right) \right) \right\}.$$

For ε sufficiently small, we have

$$\begin{aligned} \|A(\varphi, \psi)\| &\leqslant C\|l'\| + C\|R'(\varphi, \psi)\| \\ &\leqslant C\varepsilon^{3/2} \left(r^{\min\{n,m\}} + \exp\left(-\frac{3(1-\tau)r}{\varepsilon} \right) + \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon} \right) \right) \\ &\quad + C(\varepsilon^{-3/2}\|(\varphi, \psi)\|_\varepsilon^2 + \varepsilon^{-4}\|(\varphi, \psi)\|_\varepsilon^3) \\ &\leqslant \varepsilon^{3/2} \left(r^{(1-\tau_2)\min\{m,n\}} + \exp\left(-\frac{3(1-\tau_2)(1-\tau)r}{\varepsilon} \right) \right. \\ &\quad \left. + \exp\left(-\frac{(1-\tau_2)2r\sin(\pi/2k)}{\varepsilon} \right) \right) \quad \forall (\varphi, \psi) \in S, \end{aligned}$$

which implies that A is a map from S to S.

On the other hand, for ε sufficiently small, we have

$$\begin{aligned} &|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)| \\ &\quad \leqslant C|R'(\varphi_1, \psi_1) - R'(\varphi_2, \psi_2)| \\ &\quad \leqslant C\|R''(\lambda(\varphi_1, \psi_1) + (1-\lambda)(\varphi_2, \psi_2))\|\|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\|_\varepsilon \\ &\quad \leqslant \tfrac{1}{2}\|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\|_\varepsilon. \end{aligned}$$

Thus, for ε sufficiently small, A is a contraction map. Therefore, we have proved that, when ε is sufficiently small, A is a contraction map from S to S. The result therefore follows from the contraction mapping theorem. This completes the proof. □

Now we are ready to prove theorem 1.1. Let $(\varphi_r, \psi_r) = (\varphi(r), \psi(r))$ be the map obtained in proposition 2.5. Define

$$F(r) = I_\varepsilon(U_r + \varphi_r, V_r + \psi_r), \quad r \in S_\varepsilon.$$

With the same argument as in [14, 42], we can easily check that if r is a critical point of $F(r)$, then $(U_r + \varphi_r, V_r + \psi_r)$ is a critical point of I_ε.

Proof of theorem 1.3. It follows from lemmas 2.1 and 2.3 that

$$\|L(\varphi_r, \psi_r)\| \leqslant C\|(\varphi_r, \psi_r)\|_\varepsilon, \qquad \|R(\varphi, \psi)\| \leqslant C(\varepsilon^{-3/2}\|(\varphi, \psi)\|_\varepsilon^3 + \varepsilon^{-4}\|(\varphi, \psi)\|_\varepsilon^4).$$

So, from lemma 2.4 and proposition A.2, we obtain

$$F(r) = 2k\varepsilon^3 \left[A + aBr^m + bC_0r^n + C(\tfrac{1}{2}\mu_1\alpha^4 + \tfrac{1}{2}\mu_2\gamma^4 + \beta\alpha^2\gamma^2) \times \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon} \right) + O(r^{\min\{m-1,n-1\}}\varepsilon) \right].$$

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Without loss of generality, we may also assume that $n > m$. Therefore,

$$F(r) = 2k\varepsilon^3\left[A + aBr^m + C\exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right) + O(r^{m-1}\varepsilon)\right],$$

where A, B, C are fixed positive constants.

Consider $\min\{F(r)\colon r \in S_\varepsilon\}$, where S_ε is defined in (1.4).

Let

$$f(r) := aBr^m + C\exp\left(\frac{-2r\sin(\pi/2k)}{\varepsilon}\right).$$

By the assumption, we know that $a > 0$. So, by direct calculation, we can show that $f(r)$ has a local minimum point

$$\bar{r} = \frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}.$$

So there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ there exists $r_0 \in S_\varepsilon$ such that $f'(r_0) = 0$.

By direct computation, we can obtain

$$\begin{aligned}
F(\bar{r}) &= 2k\varepsilon^3\biggl[A + \left(\frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\right)^m aB\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m \\
&\qquad\qquad + \frac{maB}{2\sin(\pi/2k)}r^{m-1}\varepsilon + O(r^{m-1}\varepsilon)\biggr] \\
&= 2k\varepsilon^3\left[A + \left(aB\left(\frac{m}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m\right].
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&F\left(\frac{m-\delta}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) \\
&\quad = 2k\varepsilon^3\left[A + aB\left(\frac{m-\delta}{2\sin(\pi/2k)}\right)^m\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m + C\varepsilon^{m-\delta} + O(r^{m-1}\varepsilon)\right] \\
&\quad \geqslant 2k\varepsilon^3(A + C\varepsilon^{m-\delta})
\end{aligned}$$

and

$$\begin{aligned}
&F\left(\frac{m+\delta}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) \\
&\quad = 2k\varepsilon^3\left[A + aB\left(\frac{m+\delta}{2\sin(\pi/2k)}\right)^m\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m + C\varepsilon^{m+\delta} + O(r^{m-1}\varepsilon)\right] \\
&\quad = 2k\varepsilon^3\left[A + \left(aB\left(\frac{m+\delta}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m\right].
\end{aligned}$$

Hence, $F(r)$ has a local minimum point r_ε in S_ε, and r_ε is an interior point of S_ε. Thus, r_ε is a critical point of $F(r)$. As a result, $(U_{r_\varepsilon} + \varphi_{r_\varepsilon}, V_{r_\varepsilon} + \psi_{r_\varepsilon})$ is a solution of (1.1).

We can prove the case $m > n$ by using a similar argument to that above.

For the case $m = n$,

$$F(r) = 2k\varepsilon^3\left[A + (aB + bC_0)r^m + C\exp\left(\frac{-2r\sin(\pi/2k)}{\varepsilon}\right) + O(r^{m-1}\varepsilon)\right].$$

Letting

$$f(r) = (aB + bC_0)r^m + C\exp\left(\frac{-2r\sin(\pi/2k)}{\varepsilon}\right),$$

and using the above methods, we can prove the result. This completes the proof. □

3. Segregated vector solutions and the proof of theorem 1.2

In this section we consider segregated vector solutions and prove theorem 1.2 by proving theorem 1.4.

Let

$$\tilde{Y}_j = \frac{\partial U_{1,x^j,\varepsilon}}{\partial r}, \quad \tilde{Z}_j = \frac{\partial U_{2,y^j,\varepsilon}}{\partial \rho}, \quad j = 1, 2, \dots, 2k,$$

where x^j and y^j are defined in (1.5) and (1.8) respectively.

For simplicity of notation, in the following we replace $U_{x^j,\varepsilon}$ and $V_{x^j,\varepsilon}$ by $U_{1,x^j,\varepsilon}$ and $U_{2,y^j,\varepsilon}$, respectively. In this section, we assume

$$(r, \rho) \in S_\varepsilon \times S_\varepsilon. \tag{3.1}$$

Define

$$\tilde{E} = \left\{(\varphi, \psi) \in H\colon \sum_{j=1}^{2k}\int_{\mathbb{R}^3} U_{1,x^j,\varepsilon}^2 \tilde{Y}_j\varphi = 0,\ \sum_{j=1}^{2k}\int_{\mathbb{R}^3} U_{2,y^j,\varepsilon}^2 \tilde{Z}_j\psi = 0\right\}. \tag{3.2}$$

Let

$$\tilde{J}(\tilde{\varphi}, \tilde{\psi}) = I_\varepsilon(\tilde{U}_r + \tilde{\varphi}, \tilde{V}_\rho + \tilde{\psi}), \quad (\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}.$$

Then, similarly to (2.3), $\tilde{J}(\tilde{\varphi}, \tilde{\psi})$ has the following expansion:

$$\tilde{J}(\tilde{\varphi}, \tilde{\psi}) = \tilde{J}(0, 0) + \tilde{l}(\tilde{\varphi}, \tilde{\psi}) + \tfrac{1}{2}\tilde{Q}(\tilde{\varphi}, \tilde{\psi}) + \tilde{R}(\tilde{\varphi}, \tilde{\psi}), \quad (\tilde{\varphi}, \tilde{\psi}) \in \tilde{E},$$

where $\tilde{Q}(\tilde{\varphi}, \tilde{\psi})$ and $\tilde{R}(\tilde{\varphi}, \tilde{\psi})$ are the same as $Q(\varphi, \psi)$ and $R(\varphi, \psi)$ in § 2 if $U_{x^j,\varepsilon}$, $V_{x^j,\varepsilon}$, φ and ψ are replaced by $U_{1,x^j,\varepsilon}$, $U_{2,y^j,\varepsilon}$, $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. We note that there exists a bounded linear operator, $\tilde{B}_\varepsilon\colon \tilde{E} \to \tilde{E}$, corresponding to $\tilde{Q}(\tilde{\varphi}, \tilde{\psi})$.

Note that $\tilde{l}(\tilde{\varphi}, \tilde{\psi})$ has the following form:

$$\begin{aligned}
\tilde{l}(\tilde{\varphi}, \tilde{\psi})
&= \sum_{j=1}^{2k}(-1)^{j-1}\int_{\mathbb{R}^3}(P(|x|) - 1)U_{1,x^j,\varepsilon}\tilde{\varphi} - \mu_1\int_{\mathbb{R}^3}\left(\tilde{U}_r^3 - \sum_{j=1}^{2k}(-1)^{j-1}U_{1,x^j,\varepsilon}^3\right)\tilde{\varphi}\\
&\quad + \sum_{j=1}^{2k}(-1)^{j-1}\int_{\mathbb{R}^3}(Q(|x|) - 1)U_{2,y^j,\varepsilon}\tilde{\psi} - \mu_2\int_{\mathbb{R}^3}\left(\tilde{V_\rho}^3 - \sum_{j=1}^{2k}(-1)^{j-1}U_{2,y^j,\varepsilon}^3\right)\tilde{\psi}\\
&\quad - \beta\int_{\mathbb{R}^3}(\tilde{U}_r\tilde{V}_\rho^2\tilde{\varphi} + \tilde{U}_r^2\tilde{V}_\rho\tilde{\psi}).
\end{aligned}$$

From the above analysis, we have the following lemma.

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LEMMA 3.1. *There exists a constant $C > 0$, independent of ε, such that, for any $(r,\rho) \in S_\varepsilon \times S_\varepsilon$,*

$$\|\tilde{B}_\varepsilon(\varphi,\psi)\| \leqslant C\|(\varphi,\psi)\|_\varepsilon, \quad (\varphi,\psi) \in \tilde{E}.$$

LEMMA 3.2. *There exist $\varepsilon_0 > 0, \beta_0 > 0$ and $C_0 > 0$ such that for any $\beta < \beta_0$ and any $\varepsilon \in (0,\varepsilon_0)$, $(r,\rho) \in S_\varepsilon \times S_\varepsilon$, we have*

$$\|\tilde{B}_\varepsilon(\varphi,\psi)\| \geqslant C_0\|(\varphi,\psi)\|_\varepsilon, \quad (\varphi,\psi) \in \tilde{E}.$$

Proof. The argument is similar to lemma 2.2. We argue by contradiction. Suppose that there are $\varepsilon_n \to 0^+, (r_n,\rho_n) \in S_{\varepsilon_n} \times S_{\varepsilon_n}$ and $(\varphi_n,\psi_n) \in \tilde{E}$ with $\|(\varphi_n,\psi_n)\|_{\varepsilon_n}^2 = \varepsilon_n^3$ satisfying

$$\langle \tilde{B}_\varepsilon(\varphi_n,\psi_n),(g,h)\rangle = o_n(1)\|(\varphi_n,\psi_n)\|_{\varepsilon_n}\|(g,h)\|_{\varepsilon_n} \quad \forall (g,h) \in \tilde{E}. \tag{3.3}$$

That is,

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\varepsilon_n^2\nabla\varphi_n\nabla g + P(x)\varphi_n g - 3\mu_1\tilde{U}_r^2\varphi_n g) \\
&\quad + \int_{\mathbb{R}^3}(\varepsilon_n^2\nabla\psi_n\nabla h + Q(x)\psi_n h - 3\mu_2\tilde{V}_\rho^2\psi_n h) \\
&\quad - \beta\int_{\mathbb{R}^3}(\tilde{U}_r^2\psi_n h + \tilde{V}_\rho^2\varphi_n g + 2\tilde{U}_r\tilde{V}_\rho\varphi_n h + 2\tilde{U}_r\tilde{V}_\rho\psi_n g) \\
&\qquad\qquad = o_n(1)\|(\varphi_n,\psi_n)\|_{\varepsilon_n}\|(g,h)\|_{\varepsilon_n} \quad \forall (g,h) \in \tilde{E}.
\end{aligned} \tag{3.4}$$

In particular, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\varepsilon_n^2|\nabla\varphi_n|^2 + P(x)|\varphi_n|^2 - 3\mu_1\tilde{U}_r^2\varphi_n^2) \\
&\qquad + \int_{\mathbb{R}^3}(\varepsilon_n^2|\nabla\psi_n|^2 + Q(x)\psi_n^2 - 3\mu_2\tilde{V}_\rho^2\psi_n^2) \\
&\qquad\qquad - \beta\int_{\mathbb{R}^3}(\tilde{U}_r^2\psi_n^2 + \tilde{V}_\rho^2\varphi_n^2 + 4\tilde{U}_r\tilde{V}_\rho\varphi_n\psi_n) = o_n(1)\varepsilon_n^3
\end{aligned} \tag{3.5}$$

and

$$\int_{\mathbb{R}^3}(\varepsilon_n^2|\nabla\varphi_n|^2 + P(x)|\varphi_n|^2 + \varepsilon_n^2|\nabla\psi_n|^2 + Q(x)\psi_n^2) = \varepsilon_n^3.$$

We set $\tilde{u}_n(x) = \varphi_n(\varepsilon_n x + x^1)$, $\tilde{v}_n(x) = \psi_n(\varepsilon_n x + y^1)$. Then we have

$$\int_{\mathbb{R}^3}(|\nabla\tilde{u}_n(x)|^2 + P(\varepsilon_n x + x^1)|\tilde{u}_n(x)|^2 + |\nabla\tilde{v}_n(x)|^2 + Q(\varepsilon_n x + y^1)|\tilde{v}_n(x)|^2) = 1.$$

Upon passing to a subsequence, we may also assume that there exist $u, v \in H^1(\mathbb{R}^3)$ such that, as $n \to +\infty$,

$$\begin{aligned}
&\tilde{u}_n(x) \to u \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \qquad \tilde{u}_n(x) \to u \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3), \\
&\tilde{v}_n(x) \to v \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \qquad \tilde{v}_n(x) \to v \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).
\end{aligned}$$

Moreover, u and v satisfy

$$\int_{\mathbb{R}^3}\left(\nabla\frac{\partial U_1}{\partial x_1}\nabla u + \frac{\partial U_1}{\partial x_1}u\right) = 0, \qquad \int_{\mathbb{R}^3}\left(\nabla\frac{\partial U_2}{\partial x_1}\nabla v + \frac{\partial U_2}{\partial x_1}v\right) = 0.$$

We claim that u and v satisfy

$$-\Delta u + u - 3\mu_1 U_1^2 u = 0, \qquad -\Delta v + v - 3\mu_2 U_2^2 v = 0.$$

Let $\tilde{\varphi}(x) \in C_0^\infty(B_R(0))$ and be even in x_2 and x_3. Define

$$\tilde{\varphi}_n(x) := \tilde{\varphi}\left(\frac{x - x^1}{\varepsilon_n}\right) \in C_0^\infty(B_{\varepsilon_n R}(x^1)).$$

Then, inserting $(\tilde{\varphi}_n(x), 0)$ into (3.4) and preceding as in lemma 2.2, we can see that u satisfies

$$-\Delta u + u - 3\mu_1 U_1^2 u = 0 \quad \text{in } \mathbb{R}^3.$$

Also, by the non-degeneracy of U_1, we find that $u = 0$. In the same way, we also find that $v = 0$.

As a result,

$$\int_{B_R(-x^1)} \varphi_n^2 = o_n(1)\varepsilon_n^3, \quad \int_{B_R(-y^1)} \psi_n^2 = o_n(1)\varepsilon_n^3 \quad \forall R > 0.$$

Thus, it follows from (3.5) and lemma A.3 that

$$\begin{aligned} o_n(1)\varepsilon_n^3 &= \int_{\mathbb{R}^3} (\varepsilon_n^2 |\nabla \varphi_n|^2 + P(x)|\varphi_n|^2 - 3\mu_1 \tilde{U}_r^2 \varphi_n^2) \\ &\quad + \int_{\mathbb{R}^3} (\varepsilon_n^2 |\nabla \psi_n|^2 + Q(x)\psi_n^2 - 3\mu_2 \tilde{V}_\rho^2 \psi_n^2) \\ &\quad - \beta \int_{\mathbb{R}^3} (\tilde{U}_r^2 \psi_n^2 + \tilde{V}_\rho^2 \varphi_n^2 + 4\tilde{U}_r \tilde{V}_\rho \varphi_n \psi_n) \\ &\geqslant \|(\tilde{\varphi}_n, \tilde{\psi}_n)\|_{\varepsilon_n}^2 - C\beta \|(\tilde{\varphi}_n, \tilde{\psi}_n)\|_{\varepsilon_n}^2 + \varepsilon_n^3 (o_n(1) + o_R(1)). \end{aligned} \tag{3.6}$$

If $\beta < \beta_0 := 1/C$, and for large n and large R, we get a contradiction. So the result in this lemma is true. This completes the proof. □

From (2.13), (2.14) and lemma A.3, we can get the following lemma.

Lemma 3.3. *There exists a small constant $\tilde{\tau}_1 \in D$ such that*

$$\begin{aligned} \|\tilde{l}\| = \varepsilon^{3/2} O\Bigg(& r^m + \rho^n + \exp\left(-\frac{3(1-\tilde{\tau}_1)r}{\varepsilon}\right) + \exp\left(-\frac{3(1-\tilde{\tau}_1)\rho}{\varepsilon}\right) \\ & + \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right) + \exp\left(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\right) \\ & + \frac{\beta}{(\ln(1/\varepsilon))^{1/6}} \exp\left(-\frac{\sqrt{(\rho - r\cos(\pi/2k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right)\Bigg), \end{aligned}$$

where D is as defined in lemma 2.4.

Proposition 3.4. *For $\varepsilon > 0$ sufficiently small, there exists a C^1-map $(\tilde{\varphi}, \tilde{\psi})$ from $S_\varepsilon \times S_\varepsilon$ to H: $(\tilde{\varphi}, \tilde{\psi}) = (\tilde{\varphi}(r,\rho), \tilde{\psi}(r,\rho))$ $(r = |x^1|,\ \rho = |y^1|)$ satisfying $(\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}$ and*

$$\left\langle \frac{\partial \tilde{J}(\tilde{\varphi}, \tilde{\psi})}{\partial(\tilde{\varphi}, \tilde{\psi})}, (g,h) \right\rangle = 0 \quad \forall (g,h) \in \tilde{E}.$$

https://doi.org/10.1017/S0308210515000797 Published online by Cambridge University Press

Moreover, there exist a small constant

$$0 < \tilde{\tau}_2 < \min\left\{\frac{1}{5}, \frac{\min\{n,m\} - 1 - \sigma}{\min\{n,m\}}\right\}$$

and a constant $\tilde{C}$ *such that*

$$\begin{aligned}
&\|(\tilde{\varphi}, \tilde{\psi})\|_\varepsilon \\
&\quad \leqslant \varepsilon^{3/2}\biggl(r^{(1-\tilde{\tau}_2)m} + \rho^{(1-\tilde{\tau}_2)n} + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)r}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)\rho}{\varepsilon}\right) + \exp\left(-\frac{(1-\tilde{\tau}_2)2r\sin(\pi/2k)}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{(1-\tilde{\tau}_2)2\rho\sin(\pi/2k)}{\varepsilon}\right) \\
&\qquad + \tilde{C}\frac{\beta}{(\ln(1/\varepsilon))^{1/6}}\exp\left(-\frac{\sqrt{(\rho - r\cos(\pi/2k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right)\biggr).
\end{aligned}$$

Proof. From the definition of $\tilde{l}(\tilde{\varphi}, \tilde{\psi})$, we know that $\tilde{l}(\tilde{\varphi}, \tilde{\psi})$ is a bounded linear functional in $\tilde{E}$. Thus, it follows from the Riesz representation theorem that there is an $\tilde{l}' \in \tilde{E}$ such that

$$\tilde{l}(\tilde{\varphi}, \tilde{\psi}) = \langle \tilde{l}', (\tilde{\varphi}, \tilde{\psi})\rangle.$$

So, finding a critical point of $\tilde{J}(\tilde{\varphi}, \tilde{\psi})$ is equivalent to solving

$$\tilde{l}' + \tilde{B}_\varepsilon(\tilde{\varphi}, \tilde{\psi}) + \tilde{R}'(\tilde{\varphi}, \tilde{\psi}) = 0. \tag{3.7}$$

By lemma 3.2, $\tilde{B}_\varepsilon$ is invertible. Hence, (3.7) can be written as

$$(\tilde{\varphi}, \tilde{\psi}) = \tilde{A}(\tilde{\varphi}, \tilde{\psi}) := -\tilde{B}_\varepsilon^{-1}\tilde{l}' - \tilde{B}_\varepsilon^{-1}\tilde{R}'(\tilde{\varphi}, \tilde{\psi}).$$

We choose a small constant

$$0 < \tilde{\tau}_2 < \min\left\{\frac{1}{5}, \frac{\min\{n,m\} - 1 - \sigma}{\min\{n,m\}}\right\}$$

and a sufficiently large constant $\tilde{C}$, and set

$$\begin{aligned}
\tilde{S} = \biggl\{&(\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}\colon \\
&\|(\tilde{\varphi}, \tilde{\psi})\|_\varepsilon \leqslant \varepsilon^{3/2}\biggl(r^{(1-\tilde{\tau}_2)m} + \rho^{(1-\tilde{\tau}_2)n} + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)r}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)\rho}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{(1-\tilde{\tau}_2)2r\sin(\pi/2k)}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{(1-\tilde{\tau}_2)2\rho\sin(\pi/2k)}{\varepsilon}\right) + \tilde{C}\frac{\beta}{(\ln(1/\varepsilon))^{1/6}} \\
&\qquad \times \exp\left(-\frac{\sqrt{(\rho - r\cos(\pi/2k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right)\biggr)\biggr\}.
\end{aligned}$$

For ε sufficiently small, we have

$$
\begin{aligned}
&\|\tilde{A}(\tilde{\varphi},\tilde{\psi})\| \\
&\quad \leqslant C\|\tilde{l}_k\| + C\|\tilde{R}'(\tilde{\varphi},\tilde{\psi})\| \\
&\quad \leqslant C\biggl(r^m + \rho^n + \exp\left(-\frac{3(1-\tilde{\tau}_1)r}{\varepsilon}\right) + \exp\left(-\frac{3(1-\tilde{\tau}_1)\rho}{\varepsilon}\right) \\
&\qquad\quad + \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right) + \exp\left(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\right) \\
&\qquad\quad + \frac{\beta}{(\ln(1/\varepsilon))^{1/6}}\exp\left(-\frac{\sqrt{(\rho - r\cos(\pi/2k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right)\biggr)\varepsilon^{3/2} \\
&\qquad\quad + C(\varepsilon^{-3/2}\|(\tilde{\varphi},\tilde{\psi})\|_\varepsilon^2 + \varepsilon^{-4}\|(\tilde{\varphi},\tilde{\psi})\|_\varepsilon^3) \\
&\quad \leqslant \biggl(r^{(1-\tilde{\tau}_2)m} + \rho^{(1-\tilde{\tau}_2)n} + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)r}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{3(1-\tilde{\tau}_2)(1-\tilde{\tau}_1)\rho}{\varepsilon}\right) + \exp\left(-\frac{(1-\tilde{\tau}_2)2r\sin(\pi/2k)}{\varepsilon}\right) \\
&\qquad + \exp\left(-\frac{(1-\tilde{\tau}_2)2\rho\sin(\pi/2k)}{\varepsilon}\right) + \tilde{C}\frac{\beta}{(\ln(1/\varepsilon))^{1/6}} \\
&\qquad \times \exp\left(-\frac{\sqrt{(\rho - r\cos(\pi/2k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right)\biggr)\varepsilon^{3/2} \quad \forall(\tilde{\varphi},\tilde{\psi}) \in \tilde{S},
\end{aligned}
$$

which implies that $\tilde{A}$ is a map from $\tilde{S}$ to $\tilde{S}$.

On the other hand, for ε sufficiently small, we get

$$
\begin{aligned}
&|\tilde{A}(\tilde{\varphi}_1,\tilde{\psi}_1) - \tilde{A}(\tilde{\varphi}_2,\tilde{\psi}_2)| \\
&\quad \leqslant C|\tilde{R}'(\tilde{\varphi}_1,\tilde{\psi}_1) - \tilde{R}'(\tilde{\varphi}_2,\tilde{\psi}_2)| \\
&\quad \leqslant C\|\tilde{R}''(\lambda(\tilde{\varphi}_1,\tilde{\psi}_1) + (1-\lambda)(\tilde{\varphi}_2,\tilde{\psi}_2))\|\|(\tilde{\varphi}_1,\tilde{\psi}_1) - (\tilde{\varphi}_2,\tilde{\psi}_2)\|_\varepsilon \\
&\quad \leqslant C[\varepsilon^{-3/2}(\|(\tilde{\varphi}_1,\tilde{\psi}_1)\|_\varepsilon + \|(\tilde{\varphi}_2,\tilde{\psi}_2)\|_\varepsilon) + \varepsilon^{-4}(\|(\tilde{\varphi}_1,\tilde{\psi}_1)\|_\varepsilon^2 + \|(\tilde{\varphi}_2,\tilde{\psi}_2)\|_\varepsilon^2)] \\
&\qquad \times \|(\tilde{\varphi}_1,\tilde{\psi}_1) - (\tilde{\varphi}_2,\tilde{\psi}_2)\|_\varepsilon \\
&\quad \leqslant \tfrac{1}{2}\|(\tilde{\varphi}_1,\tilde{\psi}_1) - (\tilde{\varphi}_2,\tilde{\psi}_2)\|_\varepsilon.
\end{aligned}
$$

Thus, for ε sufficiently small, $\tilde{A}$ is a contraction map. Therefore, we have proved that when ε is sufficiently small $\tilde{A}$ is a contraction map from $\tilde{S}$ to $\tilde{S}$. So the result follows from the contraction mapping theorem. This completes the proof. □

Now we are ready to prove theorem 1.2. Let $(\tilde{\varphi}(r,\rho),\tilde{\psi}(r,\rho))$ be the map obtained in proposition 3.4. Define

$$
\tilde{F}(r,\rho) = I_\varepsilon(\tilde{U}_r + \tilde{\varphi}(r,\rho), \tilde{V}_\rho + \tilde{\psi}(r,\rho)), \quad (r,\rho) \in S_\varepsilon \times S_\varepsilon.
$$

We can check that, for ε sufficiently small, if (r,ρ) is a critical point of $\tilde{F}(r,\rho)$, then $(\tilde{U}_r + \tilde{\varphi}(r,\rho), \tilde{V}_\rho + \tilde{\psi}(r,\rho))$ is a critical point of I_ε.

Proof of theorem 1.4. From lemmas 2.3 and 3.3 and propositions 3.4 and A.5, we have

$$\begin{aligned}\tilde{F}(r,\rho) = 2k\varepsilon^3\biggl[&\tilde{A} + a\tilde{B}r^m + b\tilde{C}\rho^n \\ &+ B_1 \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right) + B_2 \exp\left(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\right) \\ &+ o_\varepsilon(1)\exp\left(-\frac{2\sqrt{(\rho - r\cos(\pi/k))^2 + (r\sin(\pi/2k))^2}}{\varepsilon}\right) \\ &+ O(r^{m-1}\varepsilon + \rho^{n-1}\varepsilon)\biggr].\end{aligned}$$

Consider the minimization problem

$$\min\{\tilde{F}(r,\rho)\colon (r,\rho) \in S_\varepsilon \times S_\varepsilon\}.$$

Since $\tilde{F}(r,\rho)$ is defined in a closed domain, the minimization can be attained. So we may assume that

$$\tilde{F}(r_1,\rho_1) = \min\{\tilde{F}(r,\rho)\colon (r,\rho) \in S_\varepsilon \times S_\varepsilon\}.$$

We claim that (r_1,ρ_1) is an interior point of $S_\varepsilon \times S_\varepsilon$.

We assume that

$$\tilde{g}_1(r) = a\tilde{B}r^m + B_1 \exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right)$$

and

$$\tilde{h}_1(\rho) = b\tilde{C}\rho^m + B_2 \exp\left(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\right).$$

By direct computation, we see that $\tilde{g}_1(r)$ attains local minimization at

$$\bar{r} = \frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\varepsilon \ln\frac{1}{\varepsilon}.$$

We have

$$\tilde{g}_1(\bar{r}) = \left(a\tilde{B}\left(\frac{m}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m,$$

$$\tilde{g}_1\left(\frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = C\varepsilon^{m-\tilde{\delta}}$$

and

$$\tilde{g}_1\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = \left(a\tilde{B}\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m.$$

Similarly, $\tilde{h}_1(\rho)$ also attains local minimization at

$$\bar{\rho} = \frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\varepsilon \ln\frac{1}{\varepsilon},$$

and we have

$$\tilde{h}_1(\bar{\rho}) = \left(b\tilde{C}\left(\frac{m}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m,$$

$$\tilde{h}_1\left(\frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = C\varepsilon^{m-\tilde{\delta}}$$

and

$$\tilde{h}_1\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = \left(b\tilde{C}\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m.$$

We may also assume that

$$\tilde{g}_2(r) = a\tilde{B}r^m + (B_1 + o_\varepsilon(1))\exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right)$$

and

$$\tilde{h}_2(\rho) = b\tilde{C}\rho^m + B_2\exp\left(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\right).$$

By direct computation, we see that $\tilde{g}_2(r)$ attains local minimization at

$$\bar{r} = \frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon},$$

and we have

$$\tilde{g}_2(\bar{r}) = \left(a\tilde{B}\left(\frac{m}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m,$$

$$\tilde{g}_2\left(\frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = C\varepsilon^{m-\tilde{\delta}}$$

and

$$\tilde{g}_2\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = \left(a\tilde{B}\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m.$$

Similarly, $\tilde{h}_2(\rho)$ also attains local minimization at

$$\bar{\rho} = \frac{m + o_\varepsilon(1)}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon},$$

and we have

$$\tilde{h}_2(\bar{\rho}) = \left(b\tilde{C}\left(\frac{m}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m,$$

$$\tilde{h}_2\left(\frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = C\varepsilon^{m-\tilde{\delta}}$$

and

$$\tilde{h}_2\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}\right) = \left(b\tilde{C}\left(\frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\right)^m + o_\varepsilon(1)\right)\left(\varepsilon\ln\frac{1}{\varepsilon}\right)^m.$$

If $o_\varepsilon(1) > 0$, then

$$\begin{aligned}
&\tilde{F}(r_1, \rho_1) \\
&\quad \leqslant 2k\varepsilon^3[\tilde{A} + \min_{(r,\rho)\in S_\varepsilon\times S_\varepsilon}\{\tilde{g}_2(r) + \tilde{h}_2(\rho) + O(r^{m-1}\varepsilon + \rho^{n-1}\varepsilon)\}] \\
&\quad \leqslant 2k\varepsilon^3\biggl[\tilde{A} + \biggl(a\tilde{B}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned}$$

If $o_\varepsilon(1) \leqslant 0$, then

$$\begin{aligned}
&\tilde{F}(r_1, \rho_1) \\
&\quad \leqslant 2k\varepsilon^3[\tilde{A} + \min_{(r,\rho)\in S_\varepsilon\times S_\varepsilon}\{\tilde{g}_1(r) + \tilde{h}_1(\rho) + O(r^{m-1}\varepsilon + \rho^{n-1}\varepsilon)\}] \\
&\quad \leqslant 2k\varepsilon^3\biggl[\tilde{A} + \biggl(a\tilde{B}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\tilde{F}(r_1, \rho_1) \\
&\quad \leqslant 2k\varepsilon^3\biggl[\tilde{A} + \biggl(a\tilde{B}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned} \tag{3.8}$$

For convenience, we define

$$\begin{aligned}
r_l &:= \frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}, \qquad r_r := \frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}, \\
\rho_l &:= \frac{m-\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}, \qquad \rho_r := \frac{m+\tilde{\delta}}{2\sin(\pi/2k)}\varepsilon\ln\frac{1}{\varepsilon}.
\end{aligned}$$

If $o_\varepsilon(1) > 0$, then

$$\begin{aligned}
\tilde{F}(r_l, \rho) &\geqslant 2k\varepsilon^3[\tilde{A} + \tilde{g}_1(r_l) + \tilde{h}_1(\rho) + O(r_l^{m-1}\varepsilon + \rho^{n-1}\varepsilon)] \\
&\geqslant 2k\varepsilon^3\biggl[\tilde{A} + C\varepsilon^{m-\tilde{\delta}} + \biggl(b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned}$$

If $o_\varepsilon(1) \leqslant 0$, then

$$\begin{aligned}
\tilde{F}(r_l, \rho) &= 2k\varepsilon^3[\tilde{A} + \tilde{g}_2(r_l) + \tilde{h}_2(\rho) + O(r_l^{m-1}\varepsilon + \rho^{n-1}\varepsilon)] \\
&\geqslant 2k\varepsilon^3\biggl[\tilde{A} + C\varepsilon^{m-\tilde{\delta}} + \biggl(b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned}$$

Therefore, we have

$$\tilde{F}(r_l, \rho) \geqslant 2k\varepsilon^3\biggl[\tilde{A} + C\varepsilon^{m-\tilde{\delta}} + \biggl(b\tilde{C}\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m + o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr]. \tag{3.9}$$

Similarly, we have

$$
\begin{aligned}
&\tilde F(r_r,\rho)\\
&\quad\geqslant 2k\varepsilon^3\biggl[\tilde A+\biggl(a\tilde B\biggl(\frac{m+\tilde\delta}{2\sin(\pi/2k)}\biggr)^m+b\tilde C\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m+o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr],
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\tilde F(r,\rho_l)\\
&\quad\geqslant 2k\varepsilon^3\biggl[\tilde A+C\varepsilon^{m-\tilde\delta}+\biggl(a\tilde B\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m+o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr]
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
&\tilde F(r,\rho_r)\\
&\quad\geqslant 2k\varepsilon^3\biggl[\tilde A+\biggl(a\tilde B\biggl(\frac{m}{2\sin(\pi/2k)}\biggr)^m+b\tilde C\biggl(\frac{m+\tilde\delta}{2\sin(\pi/2k)}\biggr)^m+o_\varepsilon(1)\biggr)\biggl(\varepsilon\ln\frac{1}{\varepsilon}\biggr)^m\biggr].
\end{aligned}
\tag{3.12}
$$

From (3.8)–(3.12), we can see that when ε is sufficiently small the local minimization of $\tilde F(r,\rho)$ cannot be obtained at the boundary of $S_\varepsilon\times S_\varepsilon$. That is, (r_1,ρ_1) is an interior point of $S_\varepsilon\times S_\varepsilon$. Thus, (r_1,ρ_1) is a critical point of $\tilde F(r,\rho)$. So $(\tilde U_{r_1}+\tilde\varphi(r_1,\rho_1),\tilde V_{\rho_1}+\tilde\psi(r_1,\rho_1))$ is a solution of (1.1). This completes the proof. $\square$

Appendix A. Energy estimate

In this section, we give some energy estimates of the approximate solutions. Recall that

$$
\begin{aligned}
x^j&:=\biggl(r\cos\frac{(j-1)\pi}{k},r\sin\frac{(j-1)\pi}{k},x_3\biggr),\qquad j=1,2,\dots,2k,\\
y^j&:=\biggl(\rho\cos\frac{(2j-1)\pi}{2k},\rho\sin\frac{(2j-1)\pi}{2k},x_3\biggr),\quad j=1,2,\dots,2k,\\
U_r(x)&=\sum_{j=1}^{2k}(-1)^{j-1}U_{x^j,\varepsilon},\qquad V_r(x)=\sum_{j=1}^{2k}(-1)^{j-1}V_{x^j,\varepsilon},\\
\tilde U_r&=\sum_{j=1}^{2k}(-1)^{j-1}U_{1,x^j,\varepsilon},\qquad \tilde V_\rho=\sum_{j=1}^{2k}(-1)^{j-1}U_{2,y^j,\varepsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
I_\varepsilon(u,v)=\frac12\int_{\mathbb R^3}(\varepsilon^2|\nabla u|^2+P(x)u^2+\varepsilon^2|\nabla v|^2+Q(x)v^2)&\\
-\frac14\int_{\mathbb R^3}(\mu_1|u|^4+\mu_2|v|^4)-\tfrac12\beta\int_{\mathbb R^3}u^2v^2.&
\end{aligned}
$$

PROPOSITION A.1. *Assume (P) and (Q) hold. Then we get the following energy estimate:*

$$\begin{aligned} &I_\varepsilon(U_{x^j,\varepsilon}, V_{x^j,\varepsilon}) \\ &\quad = \varepsilon^3 \left[A + aBr^m + bC_0 r^n + O\left(r^{m-1}\varepsilon + r^{n-1}\varepsilon + \exp\left(-\frac{(2-\tau)(1-\tau)r}{\varepsilon} \right) \right) \right], \end{aligned}$$

where a, b are as given in (P) and (Q), τ is determined in lemma 2.4,

$$A = \tfrac{1}{4}(\mu_1\alpha^4 + \mu_2\gamma^4 + 2\beta\alpha^2\gamma^2) \int_{\mathbb{R}^3} w^4 \,\mathrm{d}x,$$
$$B = \tfrac{1}{2}\alpha^2 \int_{\mathbb{R}^3} w^2 \,\mathrm{d}x \quad \textit{and} \quad C_0 = \tfrac{1}{2}\gamma^2 \int_{\mathbb{R}^3} w^2 \,\mathrm{d}x.$$

Proof. By direct computation, we have

$$\begin{aligned} I_\varepsilon(U_{x^j,\varepsilon}, V_{x^j,\varepsilon}) &= \frac{1}{2}\int_{\mathbb{R}^3} (\varepsilon^2|\nabla U_{x^j,\varepsilon}|^2 + U_{x^j,\varepsilon}^2 + \varepsilon^2|\nabla V_{x^j,\varepsilon}|^2 + V_{x^j,\varepsilon}^2) \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} (\mu_1|U_{x^j,\varepsilon}|^4 + \mu_2|V_{x^j,\varepsilon}|^4) - \tfrac{1}{2}\beta \int_{\mathbb{R}^3} U_{x^j,\varepsilon}^2 V_{x^j,\varepsilon}^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} [(P(x)-1)U_{x^j,\varepsilon}^2 + (Q(x)-1)V_{x^j,\varepsilon}^2] \\ &= \frac{1}{4}\int_{\mathbb{R}^3} (\mu_1|U_{x^j,\varepsilon}|^4 + \mu_2|V_{x^j,\varepsilon}|^4) + \tfrac{1}{2}\beta \int_{\mathbb{R}^3} U_{x^j,\varepsilon}^2 V_{x^j,\varepsilon}^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} [(P(x)-1)U_{x^j,\varepsilon}^2 + (Q(x)-1)V_{x^j,\varepsilon}^2]. \end{aligned} \tag{A 1}$$

But

$$\begin{aligned} \frac{1}{4}\int_{\mathbb{R}^3} (\mu_1|U_{x^j,\varepsilon}|^4 + \mu_2|V_{x^j,\varepsilon}|^4) &= \tfrac{1}{4}\varepsilon^3 \int_{\mathbb{R}^3} (\mu_1 U^4 + \mu_2 V^4) \\ &= \tfrac{1}{4}\varepsilon^3(\mu_1\alpha^4 + \mu_2\gamma^4) \int_{\mathbb{R}^3} w^4 \end{aligned} \tag{A 2}$$

and

$$\tfrac{1}{2}\beta \int_{\mathbb{R}^3} U_{x^j,\varepsilon}^2 V_{x^j,\varepsilon}^2 = \tfrac{1}{2}\beta\varepsilon^3 \int_{\mathbb{R}^3} U^2V^2 = \tfrac{1}{2}\beta\varepsilon^3\alpha^2\gamma^2 \int_{\mathbb{R}^3} w^4. \tag{A 3}$$

For any $m > 1$ and any $0 < d < 1$, we have

$$|\varepsilon y + x^j|^m = |x^j|^m \left(1 + O\left(\frac{|\varepsilon y|}{|x^j|} \right) \right), \quad y \in B_{dr/\varepsilon}(0).$$

Since

$$P(r) = 1 + ar^m + O(r^{m+\theta}) \quad \text{as } r \to 0^+,$$

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we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}(P(x)-1)U_{x^j,\varepsilon}^2 \\
&\quad = \tfrac{1}{2}\varepsilon^3\int_{\mathbb{R}^3}(P(\varepsilon y+x^j)-1)U^2 \\
&\quad = \tfrac{1}{2}\varepsilon^3\biggl[\int_{B_{(1-\tau)r/\varepsilon}(0)}(P(\varepsilon y+x^j)-1)U^2+\int_{B^c_{(1-\tau)r/\varepsilon}(0)}(P(\varepsilon y+x^j)-1)U^2\biggr] \\
&\quad = \tfrac{1}{2}\varepsilon^3\biggl[\int_{B_{(1-\tau)r/\varepsilon}(0)}(a|\varepsilon y+x^j|^m+O(|\varepsilon y+x^j|^{m+\theta}))U^2 \\
&\qquad\qquad +O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr] \\
&\quad = \tfrac{1}{2}\varepsilon^3\biggl[\int_{B_{(1-\tau)r/\varepsilon}(0)}\biggl(a|x^j|^m\biggl(1+O\biggl(\frac{|\varepsilon y|}{|x^j|}\biggr)\biggr) \\
&\qquad\qquad +O\biggl(|x^j|^{m+\theta}\biggl(1+O\biggl(\frac{|\varepsilon y|}{|x^j|}\biggr)\biggr)\biggr)\biggr)U^2 \\
&\qquad\qquad +O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr] \\
&\quad = \tfrac{1}{2}\varepsilon^3\biggl[\int_{B_{(1-\tau)r/\varepsilon}(0)}ar^mU^2+O\biggl(\int_{B_{(1-\tau)r/\varepsilon}(0)}r^{m-1}\varepsilon|y|U^2\biggr) \\
&\qquad\qquad +O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr] \\
&\quad = \tfrac{1}{2}\varepsilon^3\biggl[\int_{\mathbb{R}^3}ar^mU^2-\int_{B^c_{(1-\tau)r/\varepsilon}(0)}ar^mU^2+O(r^{m-1}\varepsilon) \\
&\qquad\qquad +O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr] \\
&\quad = \varepsilon^3\biggl[aBr^m+O(r^{m-1}\varepsilon)+O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr],
\end{aligned}
\tag{A 4}
$$

where τ is a small positive constant. Noting that

$$Q(r)=1+br^n+O(r^{n+\delta})\quad \text{as } r\to 0^+,$$

by the same argument as above we get

$$\frac{1}{2}\int_{\mathbb{R}^3}(Q(x)-1)V_{x^j,\varepsilon}^2=\varepsilon^3\biggl[bC_0r^n+O(r^{n-1}\varepsilon)+O\biggl(\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr]. \tag{A 5}$$

So, combining (A 1)–(A 5), we get

$$
\begin{aligned}
&I_\varepsilon(U_{x^j,\varepsilon},V_{x^j,\varepsilon}) \\
&\quad =\varepsilon^3\biggl[A+aBr^m+bC_0r^n+O\biggl(r^{m-1}\varepsilon+r^{n-1}\varepsilon+\exp\biggl(-\frac{(2-\tau)(1-\tau)r}{\varepsilon}\biggr)\biggr)\biggr].
\end{aligned}
$$

□

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PROPOSITION A.2. *Assume that (P) and (Q) hold. Then there exist a small constant $0<\sigma<\min\{\frac{1}{10},\min\{m,n\}-1\}$ and a positive constant C such that*

$$\begin{aligned}I_\varepsilon(U_r,V_r)=2k\varepsilon^3\bigg[&A+aBr^m+bC_0r^n+C(\tfrac12\mu_1\alpha^4+\tfrac12\mu_2\gamma^4+\beta\alpha^2\gamma^2)\\&\times\exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right)\\&+O\bigg(r^{m-1}\varepsilon+r^{n-1}\varepsilon+\exp\left(-\frac{(1-\tau)(2-\tau)r}{\varepsilon}\right)\\&\qquad+\exp\left(-\frac{(1+\sigma)2r\sin(\pi/2k)}{\varepsilon}\right)\bigg)\bigg],\end{aligned}$$

where τ is defined in lemma 2.4.

Proof. We know that

$$\begin{aligned}I_\varepsilon(U_r,V_r)=\sum_{j=1}^{2k}&I_\varepsilon(U_{x^j,\varepsilon},V_{x^j,\varepsilon})\\&-\tfrac14\mu_1\int_{\mathbb{R}^3}\bigg[U_r^4-\sum_{j=1}^{2k}U_{x^j,\varepsilon}^4-2\sum_{i\neq j}(-1)^{i+j}U_{x^j,\varepsilon}^3U_{x^i,\varepsilon}\bigg]\\&-\tfrac14\mu_2\int_{\mathbb{R}^3}\bigg[V_r^4-\sum_{j=1}^{2k}V_{x^j,\varepsilon}^4-2\sum_{i\neq j}(-1)^{i+j}V_{x^j,\varepsilon}^3V_{x^i,\varepsilon}\bigg]\\&-\tfrac12\beta\int_{\mathbb{R}^3}\bigg[U_r^2V_r^2-\sum_{j=1}^{2k}U_{x^j,\varepsilon}^2V_{x^j,\varepsilon}^2-\sum_{i\neq j}(-1)^{i+j}V_{x^j,\varepsilon}^2U_{x^j,\varepsilon}U_{x^i,\varepsilon}\\&\qquad-\sum_{i\neq j}(-1)^{i+j}U_{x^j,\varepsilon}^2V_{x^i\varepsilon}V_{x_i,\varepsilon}\bigg]\\&+\frac12\sum_{i\neq j}(-1)^{i+j}\int_{\mathbb{R}^3}[(P(x)-1)U_{x^j,\varepsilon}U_{x^i,\varepsilon}+(Q(x)-1)V_{x^j,\varepsilon}V_{x^i,\varepsilon}].\end{aligned}\tag{A 6}$$

But there exists a small positive $0<\sigma<\min\{\frac{1}{10},\min\{m,n\}-1\}$ such that

$$\begin{aligned}&-\tfrac14\mu_1\int_{\mathbb{R}^3}\bigg[U_r^4-\sum_{j=1}^{2k}U_{x^j,\varepsilon}^4-2\sum_{i\neq j}(-1)^{i+j}U_{x^j,\varepsilon}^3U_{x^i,\varepsilon}\bigg]\\&\quad=\tfrac12\mu_1\int_{\mathbb{R}^3}\bigg[\sum_{\substack{|i-j|=1\text{ or}\\2k-1}}U_{x^j,\varepsilon}^3U_{x^i,\varepsilon}+O\bigg(\sum_{1<|i-j|<2k-1}U_{x^j,\varepsilon}^3U_{x^i,\varepsilon}+\sum_{i\neq j}U_{x^j,\varepsilon}^2U_{x^i,\varepsilon}^2\bigg)\bigg]\\&\quad=\tfrac12\mu_1\alpha^4\int_{\mathbb{R}^3}\sum_{\substack{|i-j|=1\text{ or}\\2k-1}}w_{x^j,\varepsilon}^3w_{x^i,\varepsilon}+\varepsilon^3O\bigg(\exp\left(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\right)\bigg)\\&\quad=\varepsilon^3\bigg(C\frac{\mu_1\alpha^4}{2}\exp\left(-\frac{2r\sin(\pi/2k)}{\varepsilon}\right)+O\bigg(\exp\left(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\right)\bigg)\bigg).\end{aligned}\tag{A 7}$$

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Similarly, we have

$$
\begin{aligned}
&-\tfrac{1}{4}\mu_2\int_{\mathbb{R}^3}\bigg[V_r^4-\sum_{j=1}^{2k}V_{x^j,\varepsilon}^4-2\sum_{i\neq j}(-1)^{i+j}V_{x^j,\varepsilon}^3V_{x^i,\varepsilon}\bigg]\\
&\quad=\tfrac{1}{2}\mu_2\gamma^4\int_{\mathbb{R}^3}\sum_{|i-j|=1\text{ or }2k-1}w_{x^j,\varepsilon}^3w_{x^i,\varepsilon}+\varepsilon^3O\bigg(\exp\bigg(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\bigg)\bigg),\\
&\quad=\varepsilon^3\bigg(C\frac{\mu_2\gamma^4}{2}\exp\bigg(-\frac{2r\sin(\pi/2k)}{\varepsilon}\bigg)+O\bigg(\exp\bigg(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\bigg)\bigg)\bigg)
\end{aligned}
\tag{A 8}
$$

and

$$
\begin{aligned}
&-\tfrac{1}{2}\beta\int_{\mathbb{R}^3}\bigg[U_r^2V_r^2-\sum_{j=1}^{2k}U_{x^j,\varepsilon}^2V_{x^j,\varepsilon}^2-\sum_{i\neq j}(-1)^{i+j}V_{x^j,\varepsilon}^2U_{x^j,\varepsilon}U_{x^i,\varepsilon}\\
&\qquad\qquad-\sum_{i\neq j}(-1)^{i+j}U_{x^j,\varepsilon}^2V_{x^j,\varepsilon}V_{x^i,\varepsilon}\bigg]\\
&=\beta\alpha^2\gamma^2\int_{\mathbb{R}^3}\sum_{|i-j|=1\text{ or }2k-1}w_{x^j,\varepsilon}^3w_{x^i,\varepsilon}+\varepsilon^3O\bigg(\exp\bigg(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\bigg)\bigg)\\
&=\varepsilon^3\bigg(C\beta\alpha^2\gamma^2\exp\bigg(-\frac{2r\sin(\pi/2k)}{\varepsilon}\bigg)+O\bigg(\exp\bigg(-\frac{(1+\sigma)|x^1-x^2|}{\varepsilon}\bigg)\bigg)\bigg).
\end{aligned}
\tag{A 9}
$$

Combining (A 6)–(A 9) and proposition A.1 yields

$$
\begin{aligned}
I_\varepsilon(U_r,V_r)=2k\varepsilon^3\bigg[&A+aBr^m+bC_0r^n\\
&+C\bigg(\frac{\mu_1\alpha^4}{2}+\frac{\mu_2\gamma^4}{2}+\beta\alpha^2\gamma^2\bigg)\exp\bigg(-\frac{2r\sin(\pi/2k)}{\varepsilon}\bigg)\\
&+O\bigg(r^{m-1}\varepsilon+r^{n-1}\varepsilon+\exp\bigg(-\frac{(1-\tau)(2-\tau)r}{\varepsilon}\bigg)\\
&\qquad+\exp\bigg(-\frac{(1+\sigma)2r\sin(\pi/2k)}{\varepsilon}\bigg)\bigg)\bigg].
\end{aligned}
\tag{A 10}
$$

This completes the proof. □

LEMMA A.3.

$$
\int_{\mathbb{R}^3}U_{1,x^i,\varepsilon}^2U_{2,y^j,\varepsilon}^2=\varepsilon^3o_\varepsilon(1)\exp\bigg(-\frac{2|x^i-y^j|}{\varepsilon}\bigg).
$$

Proof. Define

$$
\begin{aligned}
\Omega_1&=\{x\in\mathbb{R}^3\colon |x-y^j|\geqslant|x-x^i|\},\qquad \Omega_2=\{x\in\mathbb{R}^3\colon |x-y^j|\leqslant|x-x^i|\},\\
\omega_1&=\{x\in\mathbb{R}^3\colon |x^i-y^j|\geqslant|x-y^j|\},\qquad \omega_2=\{x\in\mathbb{R}^3\colon |x^i-y^j|\leqslant|x-y^j|\}
\end{aligned}
$$

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and

$$\omega_1' = \left\{x \in \omega_1 \colon |x-y^j| \leqslant \varepsilon\left(\ln\frac{1}{\varepsilon}\right)^{1/3}\right\}, \qquad \omega_1'' = \left\{x \in \omega_1 \colon |x-y^j| \geqslant \varepsilon\left(\ln\frac{1}{\varepsilon}\right)^{1/3}\right\}.$$

Then we have

$$\int_{\mathbb{R}^3} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} = \int_{\Omega_1} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} + \int_{\Omega_2} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}.$$

Since we can estimate the term $\int_{\Omega_1} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}$ similarly, here we only estimate

$$\int_{\Omega_2} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}.$$

By the definition of Ω_2, we can conclude that

$$|x-x^i| \geqslant \tfrac{1}{2}|x^i-y^j| \quad \forall x \in \Omega_2.$$

Then we have

$$\begin{aligned}
\int_{\Omega_2\cap\omega_2} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} &\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\int_{\Omega_2\cap\omega_2}\exp\left(-\frac{2|x-x^i|}{\varepsilon}\right)\\
&\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\\
&\qquad\times\int_{\Omega_2\cap\omega_2}\exp\left(-\frac{|x-x^i|}{\varepsilon}\right)\exp\left(-\frac{|x^i-y^j|}{2\varepsilon}\right)\\
&\leqslant C\varepsilon^3\exp\left(-\frac{5|x^i-y^j|}{2\varepsilon}\right),
\end{aligned} \tag{A 11}$$

$$\int_{\Omega_2\cap\omega_1} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} = \int_{\Omega_2\cap\omega_1'} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} + \int_{\Omega_2\cap\omega_1''} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon},$$

$$\begin{aligned}
\int_{\Omega_2\cap\omega_1'} U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} &\leqslant C\frac{1}{|(x^i-y^j)/\varepsilon|^2}\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\\
&\qquad\times\int_{\Omega_2\cap\omega_1'}\frac{1}{|(x-y^j)/\varepsilon|^2}\\
&\qquad\qquad\times\exp\left(-\frac{2(|x-x^i|+|x-y^j|-|x^i-y^j|)}{\varepsilon}\right)\\
&\leqslant C\frac{1}{|(x^i-y^j)/\varepsilon|^2}\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\\
&\qquad\times\int_{|x|\leqslant(\ln(1/\varepsilon))^{1/3}}\frac{1}{|x|^2}\\
&\qquad\qquad\times\exp\left(-2\left(|x|+\left|x-\frac{(x^i-y^j)}{\varepsilon}\right|-\left|\frac{x^i-y^j}{\varepsilon}\right|\right)\right)
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C\frac{1}{|(x^i-y^j)/\varepsilon|^2}\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\int_{|x|\leqslant(\ln(1/\varepsilon))^{1/3}}\frac{1}{|x|^2}\\
&\leqslant C\frac{1}{|(x^i-y^j)/\varepsilon|^2}\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\left(\ln\frac{1}{\varepsilon}\right)^{1/3}\\
&\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\frac{1}{(\ln(1/\varepsilon))^{1/3}}
\end{aligned}
\tag{A 12}
$$

and

$$
\begin{aligned}
\int_{\Omega_2\cap\omega_1''}U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon} &\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\\
&\quad\times\int_{\Omega_2\cap\omega_1''}\frac{1}{|(x-y^j)/\varepsilon|^4}\\
&\qquad\times\exp\left(-\frac{2(|x-x^i|+|x-y^j|-|x^i-y^j|)}{\varepsilon}\right)\\
&\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\\
&\quad\times\int_{|x|\geqslant(\ln(1/\varepsilon))^{1/3}}\frac{1}{|x|^4}\\
&\qquad\times\exp\left(-2\left(|x|+\left|x-\frac{(x^i-y^j)}{\varepsilon}\right|-\left|\frac{x^i-y^j}{\varepsilon}\right|\right)\right)\\
&\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\int_{|x|\geqslant(\ln(1/\varepsilon))^{1/3}}\frac{1}{|x|^4}\\
&\leqslant C\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3\frac{1}{(\ln(1/\varepsilon))^{1/3}}.
\end{aligned}
\tag{A 13}
$$

From (A 12) and (A 13), we can easily get

$$
\int_{\Omega_2\cap\omega_1}U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}=o_\varepsilon(1)\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3. \tag{A 14}
$$

Combining (A 11) and (A 14), we obtain

$$
\int_{\Omega_2}U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}=o_\varepsilon(1)\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3.
$$

and, similarly, we can also obtain

$$
\int_{\Omega_1}U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}=o_\varepsilon(1)\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3.
$$

So

$$
\int_{\mathbb{R}^3}U^2_{1,x^i,\varepsilon}U^2_{2,y^j,\varepsilon}=o_\varepsilon(1)\exp\left(-\frac{2|x^i-y^j|}{\varepsilon}\right)\varepsilon^3.
$$

The proof is complete. □

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Using lemma A.3, which is similar to proposition A.1, we can obtain the following proposition.

PROPOSITION A.4. *Assume (P) and (Q) hold. Then we get the following energy estimate:*

$$
\begin{aligned}
I_\varepsilon(U_{1,x^j,\varepsilon}, U_{2,y^j,\varepsilon}) = \varepsilon^3\Bigg[&\tilde{A} + a\tilde{B}r^m + b\tilde{C}\rho^n - o_\varepsilon(1)\\
&\times \exp\left(-\frac{2}{\varepsilon}\sqrt{\left(\rho - r\cos\frac{\pi}{2k}\right)^2 + \left(r\sin\frac{\pi}{2k}\right)^2}\right)\\
&+ O\Bigg(\exp\left(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)r}{\varepsilon}\right)\\
&\quad + \exp\left(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)\rho}{\varepsilon}\right) + \rho^{n-1}\varepsilon + r^{m-1}\varepsilon\Bigg)\Bigg],
\end{aligned}
$$

where a, b are given in (P) and (Q), $\tilde{\tau}_1$ has been determined in lemma 3.3 and

$$\tilde{A} = \frac{1}{4}\int_{\mathbb{R}^3}(\mu_1 U_1^4 + \mu_2 U_2^4)\,\mathrm{d}x, \quad \tilde{B} = \frac{1}{2}\int_{\mathbb{R}^3} U_1^2\,\mathrm{d}x \quad \textit{and} \quad \tilde{C} = \frac{1}{2}\int_{\mathbb{R}^3} U_2^2\,\mathrm{d}x.$$

Proof. We know that

$$
\begin{aligned}
I_\varepsilon(U_{1,x^j,\varepsilon}, U_{2,y^j,\varepsilon}) = {}&\frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla U_{1,x^j,\varepsilon}|^2 + U_{1,x^j,\varepsilon}^2 + \varepsilon^2|\nabla U_{2,y^j,\varepsilon}|^2 + U_{2,y^j,\varepsilon}^2)\\
&- \frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|U_{1,x^j,\varepsilon}|^4 + \mu_2|U_{2,y^j,\varepsilon}|^4) - \tfrac{1}{2}\beta\int_{\mathbb{R}^3} U_{1,x^j,\varepsilon}^2 U_{2,y^j,\varepsilon}^2\\
&+ \frac{1}{2}\int_{\mathbb{R}^3}[(P(x)-1)U_{1,x^j,\varepsilon}^2 + (Q(x)-1)U_{2,y^j,\varepsilon}^2].
\end{aligned}
$$

Since U_i are the unique radial solutions of the problem

$$-\Delta u + u = \mu_i u^3, \quad \max_{x\in\mathbb{R}^3} u = u(0), \quad u > 0,$$

we have

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3}&(\varepsilon^2|\nabla U_{1,x^j,\varepsilon}|^2 + U_{1,x^j,\varepsilon}^2 + \varepsilon^2|\nabla U_{2,y^j,\varepsilon}|^2 + U_{2,y^j,\varepsilon}^2)\\
&- \frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|U_{1,x^j,\varepsilon}|^4 + \mu_2|U_{2,y^j,\varepsilon}|^4)\\
&\quad = \frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|U_{1,x^j,\varepsilon}|^4 + \mu_2|U_{2,y^j,\varepsilon}|^4)\\
&\quad = \tfrac{1}{4}\varepsilon^3\int_{\mathbb{R}^3}(\mu_1 U_1^4 + \mu_2 U_2^4).
\end{aligned}
$$

Similarly to (A 4), noting that

$$P(r) = 1 + ar^m + O(r^{m+\theta}) \quad \text{as } r \to 0^+$$

and

$$Q(r) = 1 + br^n + O(r^{n+\delta}) \quad \text{as } r \to 0^+,$$

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we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3}(P(x)-1)U_{1,x^j,\varepsilon}^2\,\mathrm{d}x = \varepsilon^3\biggl[a\tilde{B}r^m + O(r^{m-1}\varepsilon) + O\biggl(\exp\biggl(-\frac{(2-\tilde{\tau}_1)(1-\tilde{\tau}_1)r}{\varepsilon}\biggr)\biggr)\biggr] \tag{A 15}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3}(Q(x)-1)U_{2,y^j,\varepsilon}^2\,\mathrm{d}x = \varepsilon^3\biggl[b\tilde{C}\rho^n + O(\rho^{n-1}\varepsilon) + O\biggl(\exp\biggl(-\frac{(2-\tilde{\tau}_1)(1-\tilde{\tau}_1)\rho}{\varepsilon}\biggr)\biggr)\biggr], \tag{A 16}$$

where $\tilde{\tau}_1 > 0$ is a constant.

From lemma A.3, we have

$$\tfrac{1}{2}\beta\int_{\mathbb{R}^3}U_{1,x^j,\varepsilon}^2U_{2,y^j,\varepsilon}^2\,\mathrm{d}x = \beta\varepsilon^3 o_\varepsilon(1)\exp\biggl(-\frac{2|x^1-y^1|}{\varepsilon}\biggr).$$

Therefore,

$$\begin{aligned}
I_\varepsilon(U_{1,x^j,\varepsilon},U_{2,y^j,\varepsilon}) &= \varepsilon^3\biggl[\tilde{A} + a\tilde{B}r^m + b\tilde{C}\rho^n - o_\varepsilon(1)\exp\biggl(-\frac{2}{\varepsilon}\sqrt{\biggl(\rho - r\cos\frac{\pi}{2k}\biggr)^2 + \biggl(r\sin\frac{\pi}{2k}\biggr)^2}\biggr) \\
&\quad + O\biggl(\exp\biggl(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)r}{\varepsilon}\biggr) + \exp\biggl(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)\rho}{\varepsilon}\biggr) \\
&\quad + \rho^{n-1}\varepsilon + r^{m-1}\varepsilon\biggr)\biggr].
\end{aligned}$$

This completes the proof. □

PROPOSITION A.5. *Assume that (P) and (Q) hold. Then there exist positive constants B_1 and B_2 such that*

$$\begin{aligned}
I_\varepsilon(\tilde{U}_r,\tilde{V}_\rho) &= 2k\varepsilon^3\biggl[\tilde{A} + a\tilde{B}r^m + b\tilde{C}\rho^n + B_1\exp\biggl(-\frac{2r\sin(\pi/2k)}{\varepsilon}\biggr) \\
&\quad + B_2\exp\biggl(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\biggr) \\
&\quad + o_\varepsilon(1)\exp\biggl(-\frac{2}{\varepsilon}\sqrt{\biggl(\rho - r\cos\frac{\pi}{2k}\biggr)^2 + \biggl(r\sin\frac{\pi}{2k}\biggr)^2}\biggr) \\
&\quad + O\biggl(\exp\biggl(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)r}{\varepsilon}\biggr) + \exp\biggl(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)\rho}{\varepsilon}\biggr) \\
&\quad + \rho^{n-1}\varepsilon + r^{m-1}\varepsilon + \exp\biggl(-\frac{(1+\sigma)2r\sin(\pi/2k)}{\varepsilon}\biggr) \\
&\quad + \exp\biggl(-\frac{(1+\sigma)2\rho\sin(\pi/2k)}{\varepsilon}\biggr)\biggr)\biggr],
\end{aligned}$$

where σ can be determined using proposition A.2.

Proof. We can obtain

$$
\begin{aligned}
I_\varepsilon(\tilde{U}_r, \tilde{V}_\rho) \\
&= \sum_{j=1}^{2k} I_\varepsilon(U_{1,x^j,\varepsilon}, U_{2,y^j,\varepsilon}) \\
&\quad - \tfrac{1}{4}\mu_1 \int_{\mathbb{R}^3} \biggl(|\tilde{U}_r|^4 - \sum_{j=1}^{2k} U_{1,x^j,\varepsilon}^4 - 2\sum_{i \neq j} (-1)^{i+j} U_{1,x^i,\varepsilon}^3 U_{1,x^j,\varepsilon} \biggr) \\
&\quad - \tfrac{1}{4}\mu_2 \int_{\mathbb{R}^3} \biggl(|\tilde{V}_\rho|^4 - \sum_{j=1}^{2k} U_{2,y^j,\varepsilon}^4 - 2\sum_{i \neq j} (-1)^{i+j} U_{2,y^i,\varepsilon}^3 U_{2,y^j,\varepsilon} \biggr) \\
&\quad - \tfrac{1}{2}\beta \int_{\mathbb{R}^3} \biggl(|\tilde{U}_r|^2 |\tilde{V}_\rho|^2 - \sum_{j=1}^{2k} U_{1,x^j,\varepsilon}^2 U_{2,y^j,\varepsilon}^2 \biggr) \\
&\quad + \frac{1}{2} \sum_{i \neq j} (-1)^{i+j} \int_{\mathbb{R}^3} [(P(x) - 1) U_{1,x^i,\varepsilon} U_{1,x^j,\varepsilon} + (Q(x) - 1) U_{2,y^i,\varepsilon} U_{2,y^j,\varepsilon}].
\end{aligned}
\tag{A 17}
$$

Similarly to (A 7), we can prove that there exist positive constants B_1 and B_2 such that

$$
\begin{aligned}
&- \tfrac{1}{4}\mu_1 \int_{\mathbb{R}^3} \biggl(|\tilde{U}_r|^4 - \sum_{j=1}^{2k} U_{1,x^j,\varepsilon}^4 - 2\sum_{i \neq j} (-1)^{i+j} U_{1,x^i,\varepsilon}^3 U_{1,x^j,\varepsilon} \biggr) \\
&\qquad = B_1 \exp\biggl(-\frac{2r\sin(\pi/2k)}{\varepsilon} \biggr) \varepsilon^3 + O\biggl(\exp\biggl(-\frac{(1+\sigma)2r\sin(\pi/2k)}{\varepsilon} \biggr) \biggr) \varepsilon^3
\end{aligned}
\tag{A 18}
$$

and

$$
\begin{aligned}
&- \tfrac{1}{4}\mu_2 \int_{\mathbb{R}^3} \biggl(|\tilde{V}_\rho|^4 - \sum_{j=1}^{2k} U_{2,y^j,\varepsilon}^4 - 2\sum_{i \neq j} (-1)^{i+j} U_{2,y^i,\varepsilon}^3 U_{2,y^j,\varepsilon} \biggr) \\
&\qquad = B_2 \exp\biggl(-\frac{2\rho\sin(\pi/2k)}{\varepsilon} \biggr) \varepsilon^3 + O\biggl(\exp\biggl(-\frac{(1+\sigma)2\rho\sin(\pi/2k)}{\varepsilon} \biggr) \biggr) \varepsilon^3.
\end{aligned}
\tag{A 19}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^3} (P(x) - 1) U_{1,x^i,\varepsilon} U_{1,x^j,\varepsilon} \\
&\qquad = \frac{1}{2} \int_{B_{4r}(0)} (P(x) - 1) U_{1,x^i,\varepsilon} U_{1,x^j,\varepsilon} + \frac{1}{2} \int_{B_{4r}^c(0)} (P(x) - 1) U_{1,x^i,\varepsilon} U_{1,x^j,\varepsilon} \\
&\qquad \leqslant Cr^m \int_{\mathbb{R}^3} U_{1,x^i,\varepsilon} U_{1,x^j,\varepsilon} + \tfrac{1}{2} C \int_{B_{4r}^c(0)} (U_{1,x^i,\varepsilon}^2 + U_{1,x^j,\varepsilon}^2)
\end{aligned}
$$

$$\leqslant C\varepsilon^3\bigg(r^m \exp\bigg(-\frac{2r\sin(\pi/2k)}{\varepsilon}\bigg) + \exp\bigg(-\frac{3(2-\tilde{\tau}_1)r}{\varepsilon}\bigg)\bigg)$$
$$= \varepsilon^3 O\bigg(r^{2m} + \exp\bigg(-\frac{4r\sin(\pi/2k)}{\varepsilon}\bigg) + \exp\bigg(-\frac{3(2-\tilde{\tau}_1)r}{\varepsilon}\bigg)\bigg). \qquad \text{(A 20)}$$

Similarly, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3}(Q(x)-1)U_{2,y^i,\varepsilon}U_{2,y^j,\varepsilon}$$
$$= \varepsilon^3 O\bigg(\rho^{2n} + \exp\bigg(-\frac{4\rho\sin(\pi/2k)}{\varepsilon}\bigg) + \exp\bigg(-\frac{3(2-\tilde{\tau}_1)\rho}{\varepsilon}\bigg)\bigg). \qquad \text{(A 21)}$$

Then

$$\bigg|\tfrac{1}{2}\beta\int_{\mathbb{R}^3}\bigg(|\tilde{U}_r|^2|\tilde{V}_\rho|^2 - \sum_{j=1}^{2k}U^2_{1,x^j,\varepsilon}U^2_{2,y^j,\varepsilon}\bigg)\bigg|$$
$$\leqslant C\sum_{j=1}^{2k}\int_{\mathbb{R}^3}U^2_{1,x^j,\varepsilon}U^2_{2,y^j,\varepsilon}$$
$$= \varepsilon^3 o_\varepsilon(1)\exp\bigg(-\frac{2}{\varepsilon}\sqrt{\bigg(\rho - r\cos\frac{\pi}{2k}\bigg)^2 + \bigg(r\sin\frac{\pi}{2k}\bigg)^2}\bigg). \qquad \text{(A 22)}$$

By (A 17)–(A 22) and proposition A.4, we can easily show that

$$\begin{aligned}I_\varepsilon(\tilde{U}_r,\tilde{V}_\rho) = 2k\varepsilon^3\bigg[&\tilde{A} + a\tilde{B}r^m + b\tilde{C}\rho^n + B_1\exp\bigg(-\frac{2r\sin(\pi/2k)}{\varepsilon}\bigg)\\
&+ B_2\exp\bigg(-\frac{2\rho\sin(\pi/2k)}{\varepsilon}\bigg)\\
&+ o_\varepsilon(1)\exp\bigg(-\frac{2}{\varepsilon}\sqrt{\bigg(\rho - r\cos\frac{\pi}{2k}\bigg)^2 + \bigg(r\sin\frac{\pi}{2k}\bigg)^2}\bigg)\\
&+ O\bigg(\exp\bigg(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)r}{\varepsilon}\bigg) + \exp\bigg(-\frac{(1-\tilde{\tau}_1)(2-\tilde{\tau}_1)\rho}{\varepsilon}\bigg)\\
&\quad + \rho^{n-1}\varepsilon + r^{m-1}\varepsilon + \exp\bigg(-\frac{(1+\sigma)2r\sin(\pi/2k)}{\varepsilon}\bigg)\\
&\quad + \exp\bigg(-\frac{(1+\sigma)2\rho\sin(\pi/2k)}{\varepsilon}\bigg)\bigg)\bigg].\end{aligned}$$

This completes the proof. □

Acknowledgements

The authors sincerely thank Professor S. Peng for helpful discussions and suggestions. This work was partly supported by the NSFC (Grant nos 11301204 and 11371159), self-determined research funds of CCNU from the colleges' basic research and operation of MOE (Grant no. CCNU14A05036) and the PhD launch scientific research projects of Guizhou Normal University (Grant no. 2014).

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