ORBIFOLD POINTS ON PRYM–TEICHMÜLLER CURVES IN GENUS 4

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Abstract For each discriminant $D > 1$, McMullen constructed the Prym–Teichmüller curves $W_D(4)$ and $W_D(6)$ in $\mathcal{M}_3$ and $\mathcal{M}_4$, which constitute one of the few known infinite families of geometrically primitive Teichmüller curves. In the present paper, we determine for each D the number and type of orbifold points on $W_D(6)$. These results, together with a previous result of the two authors in the genus 3 case and with results of Lanneau–Nguyen and Möller, complete the topological characterisation of all Prym–Teichmüller curves and determine their genus. The study of orbifold points relies on the analysis of intersections of $W_D(6)$ with certain families of genus 4 curves with extra automorphisms. As a side product of this study, we give an explicit construction of such families and describe their Prym–Torelli images, which turn out to be isomorphic to certain products of elliptic curves. We also give a geometric description of the flat surfaces associated to these families and describe the asymptotics of the genus of $W_D(6)$ for large D.

Contents

1. Introduction

A *flat surface* is a pair (X, ω) where X is a compact Riemann surface of genus g and ω is a holomorphic differential on X. By integration, the differential endows X with a flat structure away from the zeros of ω. Consider now $\Omega\mathcal{M}_g$, the moduli space of flat surfaces which is a natural bundle over the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g. There is a natural $\mathrm{SL}_2(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ by affine shearing of the flat structure and we consider the projections of orbit closures to $\mathcal{M}_g$. In the rare case that the $\mathrm{SL}_2(\mathbb{R})$ orbit of (X, ω) projects to an (algebraic) curve in $\mathcal{M}_g$ we call this the *Teichmüller curve generated by (X, ω)* in $\mathcal{M}_g$.

Not many families of (primitive) Teichmüller curves are known; see e.g. [15] for a brief overview. Among them, McMullen constructed the *Weierstraß curves* in genus 2 [12] and generalised this construction to the *Prym–Teichmüller curves* in genus 3 and 4 [14]. Recently, Eskin, McMullen, Mukamel and Wright announced the existence of six exceptional orbit closures, two of which contain an infinite collection of Teichmüller curves. One of them is treated in [15].

Any Teichmüller curve $\mathcal{C}$ is a sub-orbifold of $\mathcal{M}_g$. Therefore, denoting by χ the (orbifold) Euler Characteristic, by h_0 the number of connected components, by C the number of cusps and by e_d the number of points of order d, these invariants determine the genus g:

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right),$$

i.e. they determine the topological type of $\mathcal{C}$.

For the Prym–Weierstraß curves, the situation is as follows. In genus 2, cusps and connected components were determined by McMullen [13], the Euler characteristic was computed by Bainbridge [1], and the number and types of orbifold points were established by Mukamel [19]. In genus 3 and 4, Möller [17] calculated the Euler characteristic and Lanneau and Nguyen [9] classified the cusps. The number of connected components in genus 3 were also determined in [9] (see also [26]) and the number and type of their orbifold points in genus 3 were established in [25]. In the case of genus 4, Lanneau has recently communicated to the authors that the Prym locus is always connected [10]. The present paper classifies the orbifold points of these curves.

Theorem 1.1. *For discriminant $D > 12$, the Prym–Teichmüller curves $W_D(6)$ have orbifold points of order 2 and 3. More precisely:*

• *the number of orbifold points of order 2 is*

$$e_2(D) = \begin{cases} 0, & \text{if } D \text{ is odd,} \\ h(-D) + h(-D/4), & \text{if } D \equiv 12 \bmod 16, \\ h(-D), & \text{if } D \equiv 0, 4, 8 \bmod 16, \end{cases}$$

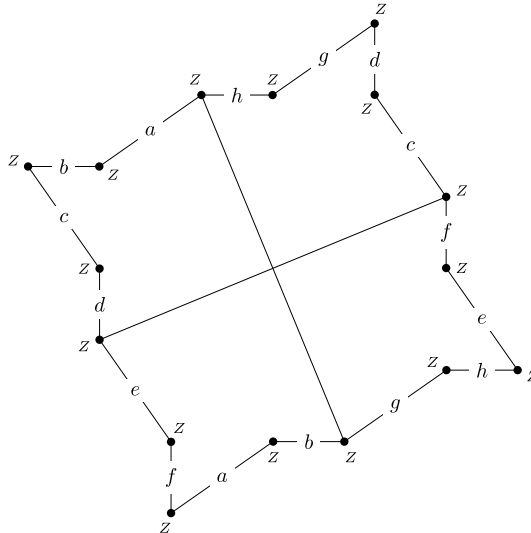*where $h(-D)$ is the class number of $\mathcal{O}_{-D}$;*

Figure 1. A C_4-eigendifferential of genus 4 with a single zero (the Turtle): the canonical 4-cover of the 4-differential on an elliptic curve pictured in Figure 3 (§ 6).

- *the number of orbifold points of order* 3 *is*

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D, \gcd(a, i, j) = 1\}/12;$$

- $W_5(6)$ *has one point of order* 3 *and one point of order* 5;
- $W_8(6)$ *has one point of order* 2 *and one point of order* 3;
- $W_{12}(6)$ *has one point of order* 2 *and one point of order* 6.

Theorem 1.1 combines the results of Theorems 3.1, 4.1, and 5.1 and thus completes the topological classification of the Prym–Weierstraß curves. The topological invariants of $W_D(6)$ for nonsquare discriminants $D \leqslant 200$ are listed in Table 3 on page 704.

Recall that the orbifold locus of $W_D(6)$ consists of flat surfaces (X, ω) where ω is not only an eigenform for the real multiplication but *also* for some (holomorphic) automorphism α of X. To describe this locus, it is therefore natural to consider instead families $\mathcal{F}$ of curves with a suitable automorphism α and consider the α-eigenspace decomposition of $\Omega\mathcal{F}$. We isolate suitable eigendifferentials ω with a single zero, and check whether the Prym part of $(X, \omega) \in \Omega\mathcal{F}$ admits real multiplication respecting ω, i.e. find the intersections of $\mathcal{F}$ with $W_D(6)$ for some D.

To be more precise, it is essentially topological considerations that not only show the possible orders d of orbifold points that can occur on a curve $W_D(6)$, but in fact determine the possibilities for the group $\text{Aut } X$, in the case that (X, ω) is an orbifold point (see § 2). It turns out that there are essentially two relevant families: curves admitting a D_8 action – giving points of order 2 – and curves admitting a $C_6 \times C_2$ action – giving points of order 3. Because of the flat picture of the single-zero differentials on these families, we call them the *Turtle family* (Figure 1) and the *Hurricane family* (Figure 2); see § 6 for details. In
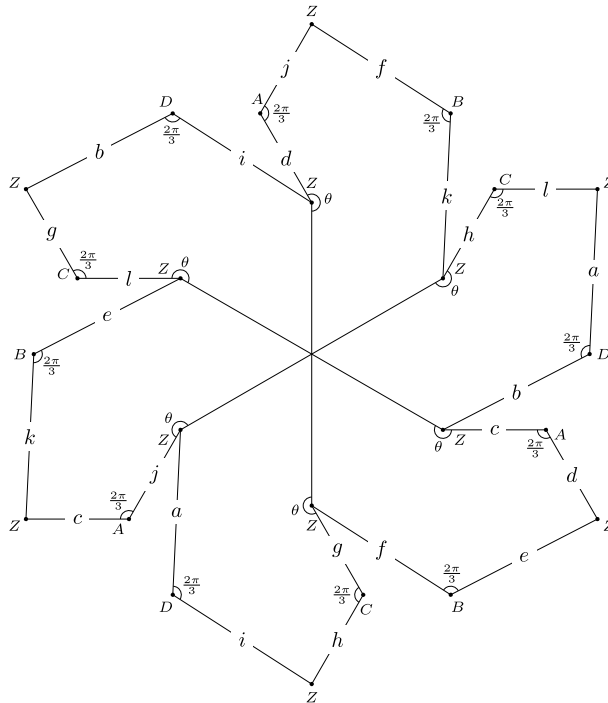
Figure 2. A C_6-eigendifferential of genus 4 with a single zero (the Hurricane): the canonical 6-cover of the 6-differential on $\mathbb{P}^1$ pictured in Figure 3 (§6).

addition, these families intersect, giving the (unique) point of order 6 on $W_{12}(6)$. Also, there is a unique point with a C_{10} action, giving the point of order 5 on $W_5(6)$. Any orbifold point on $W_D(6)$ must necessarily lie on one of these families (Proposition 2.1).

The difficulty when studying these families comes from obtaining the eigenforms in a basis where we can calculate the endomorphism ring in order to study real multiplication or, equivalently, understanding the analytic representation of suitable real multiplication in the eigenbasis of the automorphism on the Prym variety.

We begin by analysing $\mathcal{M}_4(D_8)$, the 2-dimensional locus of genus 4 curves with a specific D_8 action; see §3. The Turtle family is a 1-dimensional sub-locus of this moduli space.

As a by-product, we give an explicit description of $\mathcal{M}_4(D_8)$. For an elliptic curve E, let ϕ denote the elliptic involution.

Theorem 1.2. *The family $\mathcal{M}_4(D_8)$ is in bijection with the family*

$$\mathcal{E} = \{(E, [P]) : E \in \mathcal{M}_{1,1}, \ [P] \in (E \setminus E[2])/\phi\}$$

of elliptic curves with a distinguished base point, together with an elliptic pair.

In particular, this family is 2-dimensional; however, the sub-locus $\mathcal{X}$ of curves admitting a C_4-eigenform with a single zero is 1-dimensional and in bijection with $\mathcal{M}_{1,1}\setminus\{E_2\}$.

This bijection is induced by the construction of this family as a fibre product of two (isomorphic) families of elliptic curves over a base projective line.

To determine which points admit real multiplication with a common eigenform, we fix an eigenbasis of ΩX and consider the *Prym–Torelli map* PT, which associates the corresponding Prym variety to a Prym pair (X, ρ). We show that, in the D_8 case, the Prym variety of such a pair is always isomorphic to the product $E \times E$, where E is an elliptic curve arising as a quotient of X, and then the Prym variety admits suitable real multiplication if and only if the elliptic curve has complex multiplication, accounting for the class numbers.

The Hurricane family behaves quite differently; see § 4. We denote by E_ζ the unique elliptic curve with an automorphism ψ of order 6.

Theorem 1.3. *The Hurricane family agrees with the family*

$$\mathcal{Y}_t : y^6 = x(x-1)^2(x-t)^2$$

of cyclic covers of $\mathbb{P}^1$.

However, the Prym–Torelli image of $\mathcal{Y}$ *is the single point* $E_\zeta \times E_\zeta$.

The Hurricane family has the advantage that it is 1-dimensional and can be understood in terms of cyclic covers of $\mathbb{P}^1$. However, due to the large automorphism group, the whole family collapses to a single point under the Prym–Torelli map, which of course admits real multiplication in many different ways. Now, each fibre $\mathcal{Y}_t$ gives a different C_6-eigenbasis in $\Omega E_\zeta \times \Omega E_\zeta$ and checking when *this* basis is also an eigenbasis for some real multiplication gives the intersections of $\mathcal{Y}$ and some $W_D(6)$.

The Hurricane family can also be constructed as a family of fibre products over certain quotient curves. More precisely, all fibres $\mathcal{Y}_t$ of $\mathcal{Y}$ can be seen as a fibre product of two copies of E_ζ over a projective line quotient $\mathbb{P}^1$. However, in contrast to the D_8 case, the base of the Hurricane family will not be isomorphic to a modular curve, but it will be a dense subset inside the curve E_ζ.

More precisely, denote by E_ζ^* the curve E_ζ with the 2-torsion points and the ψ-orbit of order 3 removed and let ϕ be again the elliptic involution of E_ζ. There is a generically 6-to-1 map between the set of elliptic pairs of points E_ζ^*/ϕ and the fibres of $\mathcal{Y}$ (cf. Proposition 4.9).

Moreover, for each isomorphism class $[Y] \in \mathcal{Y}$, there exist generically 12 elements (up to scale) in $\Omega E_\zeta \times \Omega E_\zeta$ defining a C_6-eigendifferential with a single zero on the curves in $[Y]$ (cf. Proposition 4.13). This fact explains the factor of 12 in the formula for the number of orbifold points of order 3.

Using the work of Möller [17] and Lanneau and Nguyen [9], Theorem 1.1 lets us calculate the genus of the Prym–Weierstraß curves $W_D(6)$. In § 7, we describe the asymptotic growth rate of the genus, $g(W_D(6))$ with respect to the discriminant D.

Theorem 1.4. *There exist constants* $C_1, C_2 > 0$, *independent of D, such that*

$$C_1 \cdot D^{3/2} < g(W_D(6)) < C_2 \cdot D^{3/2}.$$

Moreover, $g(W_D) = 0$ *if and only if* $D \leqslant 20$.

TABLE 1. Topological invariants of the six Teichmüller curves $W_D(6)$ that have genus 0. The number of cusps is described in [9], the Euler characteristic in [17]. For larger D, see Table 3 on page 704.

D	g	e_2	e_3	e_5	e_6	C	χ
5	0	0	1	1	0	1	$-7/15$
8	0	1	1	0	0	2	$-7/6$
12	0	1	0	0	1	3	$-7/3$
13	0	0	2	0	0	3	$-7/3$
17	0	0	1	0	0	6	$-14/3$
20	0	2	1	0	0	5	$-14/3$

The topological invariants of the geometrically primitive genus 0 Prym–Teichmüller curves are summarised in Table 1.

Theorem 1.1 can be seen as the next and final step after [19, 25] in the study of orbifold points on Prym–Weierstraß curves, thus bringing closure to the topological characterisation of such curves. While the general method is similar in the genus 2, 3 and 4 cases (namely, studying the intersection of the Teichmüller curves with certain families), the specific phenomena occurring are different.

In genus 2, the situation was simpler essentially due to the fact that the Prym variety was the entire Jacobian [19]. While the relevant family also had a generic D_8 automorphism group, this was a 1-dimensional object, while in $\mathcal{M}_4$ this locus is a surface where the C_4-eigendifferentials are contained in an embedded Modular curve.

In genus 3, the defining phenomenon was the fact that the Prym variety was a $(2, 1)$-polarised abelian sub-variety of the Jacobian [25]. Also, for the first time, two 1-dimensional families occurred and in the case of C_4 curves, the Prym–Torelli image also collapsed to a point. However, both these families could be described as cyclic covers of $\mathbb{P}^1$ in which case the eigenspace decomposition of ΩX is well understood. The main technical difficulty in those cases was the explicit calculation of period matrices using Bolza's method. Moreover, the formulae obtained were of a slightly different flavour, as the C_6-family turned out to be isomorphic to the *compact* Shimura curve $\mathbb{H}/\Delta(2, 6, 6)$, giving a more general class number than in the other cases.

In contrast, in genus 4, for the first time a 2-dimensional locus plays a central role: indeed, the space $\mathcal{M}_4(D_8)$ can be seen as a cyclic cover over an elliptic curve, which makes the computation of the eigendifferentials with a single zero more difficult, cf. Theorem 3.6. On the other hand, while in almost all cases the Prym variety is isogenous to a product of elliptic curves (this is the reason for the abundance of modular curves and class numbers in the formulae), it turns out that in genus 4 the Prym varieties are actually *isomorphic* to this product. This results in a closer relationship of the endomorphism rings in this case and is the reason we obtain an exact class number of a negative discriminant order in Theorem 1.1.

In particular, the technical approach in this paper is completely different from the one in [25], since the computational aspects of Bolza's method have been replaced by a more conceptual description of the families.

TABLE 2. Families parametrising possible orbifold points on Prym–Weierstraß curves. Here, $\mathcal{X}_t$ is a general fibre of $\mathcal{X}$ and E_t is an elliptic curve that appears as a quotient of $\mathcal{X}_t$. For genus 2 see [19], for genus 3 see [25].

$g(\mathcal{X}_t)$	d	$\dim \mathcal{X}$	$\dim \mathrm{PT}(\mathcal{X})$	$\mathrm{Aut}(\mathcal{X}_t)$	$\mathrm{End}(\mathcal{P}(\mathcal{X}_t, \sigma^d))$
2	2	1	1	D_8	order in $M_2(\mathrm{End}(E_t))$
3	2	1	0	$C_2 \ltimes (C_2 \times C_4)$	order in $M_2(\mathbb{Q}[i])$
3	3	1	1	C_6	order in $\left(\frac{2, -3}{\mathbb{Q}}\right)$
4	2	2 (1)	1	D_8	$M_2(\mathrm{End}(E_t))$
4	3	1	0	$C_6 \times C_2$	$M_2(\mathbb{Z}[\zeta_6])$

The occurring positive dimensional families are summarised and compared to the families occurring in genus 2 and 3 in Table 2.

Finally, in §6, we provide the flat pictures associated to the eigendifferentials in the Turtle family and the Hurricane family.

2. The orbifold locus of W_D

The aim of this section is to describe the orbifold locus of $W_D(6)$ as the intersection with families of curves with a prescribed automorphism group in $\mathcal{M}_4$. In particular, Proposition 2.1 determines the possible orders of orbifold points that may occur.

As usual, we write $(g; n_1, \ldots, n_r)$ for orbifolds of genus g with r points of order $n_1, \ldots, n_r$. Recall that, given an automorphism α of order N on X, points of order n_i on X/α correspond to orbits of length N/n_i on X. Moreover, we denote by ζ_d a primitive dth root of unity.

Proposition 2.1. *Let* $(X, \omega) \in \Omega W_D$ *be a flat surface that parametrises an orbifold point of* W_D *of order* d. *Then there exists a holomorphic automorphism* $\alpha \in \mathrm{Aut}\, X$ *of order* $2d$ *that satisfies* $\alpha^* \omega = \zeta_{2d} \omega$ *and one of the following conditions:*

(1) *the order of* α *is* 4 *and* X/α *has signature* $(1; 4, 4)$;

(2) *the order of* α *is* 6 *and* X/α *has signature* $(0; 3, 3, 6, 6)$;

(3) *the order of* α *is* 10 *and* X/α *has signature* $(0; 5, 10, 10)$;

(4) *the order of* α *is* 12 *and* X/α *has signature* $(0; 3, 12, 12)$.

Remark. Observe that family (1) is 2-dimensional, family (2) is 1-dimensional and families (3) and (4) consist of a finite number of points in $\mathcal{M}_4$.

Before we proceed to the proof, we first recall some background and notation.

Flat surfaces and Teichmüller curves

A *flat surface* is a pair (X, ω) where X is a Riemann surface (or equivalently a smooth irreducible complex curve) of genus g and $\omega \in \Omega X$ is a holomorphic differential form on X. Note that X may be endowed with a flat structure away from the zeros of ω via integration. We denote the moduli space of flat surfaces by $\Omega \mathcal{M}_g$ and note that it

can be viewed as a bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ over the moduli space of smooth, irreducible, complex curves of genus g. The space $\Omega\mathcal{M}_g$ is naturally stratified by the distribution of the zeros of the differentials; given a partition $\mu = (\mu_1, \ldots, \mu_r)$ of $2g - 2$, denote by $\Omega\mathcal{M}_g(\mu)$ the corresponding stratum and given a family $\mathcal{F}$ of curves in $\mathcal{M}_g$ we set $\Omega\mathcal{F}(\mu) := \Omega\mathcal{M}_g(\mu) \cap \Omega\mathcal{F}$. We use exponential notation for repeated indices, so that, for instance $(1, \ldots, 1) = (1^{2g-2})$.

Recall that $\Omega\mathcal{M}_g$ admits a natural $\mathrm{GL}_2(\mathbb{R})$ action by affine shearing of the flat structures. A *Teichmüller curve* is the (projection of a) $\mathrm{GL}_2(\mathbb{R})^+$ orbit that projects to an algebraic curve in $\mathcal{M}_g$. See for instance [16] for background on Teichmüller curves and flat surfaces.

Prym–Teichmüller curves in genus 4

McMullen [14] constructed families of primitive Teichmüller curves in genus 2, 3 and 4, the *Prym–Teichmüller* (or *Prym–Weierstraß*) curves $W_D(2g - 2)$. We briefly recall the construction in the genus 4 case. For brevity, we denote the curve $W_D(6)$ by W_D in the following.

Let X be of genus 4 admitting a holomorphic involution ρ. We say that ρ is a *Prym involution* if X/ρ has genus 2. In particular, this gives a decomposition $\Omega X = \Omega X^+ \oplus \Omega X^-$ into 2-dimensional ρ-eigenspaces with eigenvalues 1 and -1 respectively. It also determines sublattices $H_1(X, \mathbb{Z})^+, H_1(X, \mathbb{Z})^- \subset H_1(X, \mathbb{Z})$ consisting of ρ-invariant and ρ-anti-invariant cycles that satisfy $H_1(X, \mathbb{Z})^\pm = H_1(X, \mathbb{Z}) \cap (\Omega X^\pm)^*$. All this implies that the *Prym variety*

$$\mathcal{P}(X, \rho) := \frac{(\Omega X^-)^*}{H_1(X, \mathbb{Z})^-} = \ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho))^0$$

is a 2-dimensional, $(2, 2)$-polarised abelian sub-variety of the Jacobian $\mathrm{Jac}(X)$ (see [17] or [3, Chapter 12] for details).

For any discriminant $D \equiv 0, 1 \bmod 4$, write $D = b^2 - 4ac$ for some $a, b, c \in \mathbb{Z}$. The (unique) quadratic order of discriminant D is defined as $\mathcal{O}_D = \mathbb{Z}[T]/(aT^2 + bT + c)$, which agrees with

$$\mathcal{O}_D = \mathbb{Z} \oplus T_D\mathbb{Z}, \quad \text{where } T_D = \begin{cases} \dfrac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \bmod 4 \text{ and} \\ \dfrac{\sqrt{D}+1}{2}, & \text{if } D \equiv 1 \bmod 4, \end{cases}$$

provided D is not a square. If $D = f^2$, the order $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 - fT)$ is isomorphic to the subring $\{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} : a \equiv b \bmod f\}$.

Now let $D > 0$ be a positive discriminant. We say that a polarised abelian surface A has *real multiplication* by $\mathcal{O}_D$ if it admits an embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}\, A$ that is self-adjoint with respect to the polarisation. We call the real multiplication by $\mathcal{O}_D$ *proper*, if the embedding cannot be extended to any quadratic order containing $\mathcal{O}_D$.

Denote by ΩW_D the locus of $(X, \omega) \in \Omega \mathcal{M}_4(6)$ such that

(1) X admits a Prym involution ρ, so that $\mathcal{P}(X, \rho)$ is 2-dimensional,

(2) the form ω has a single zero and satisfies $\rho^*\omega = -\omega$, and

(3) $\mathcal{P}(X, \rho)$ admits proper real multiplication by $\mathcal{O}_D$ with ω as an eigenform.

McMullen showed that the projection W_D of ΩW_D to $\mathcal{M}_4$ gives (a union of) Teichmüller curves for every discriminant D [14]. In fact, Lanneau has communicated to the authors that W_D is connected for all D [10].

Orbifold points on Prym–Teichmüller curves

An orbifold point of order d on W_D corresponds to a flat surface $(X, \omega) \in \Omega W_D$ such that

- there exists a holomorphic automorphism $\alpha \in \mathrm{Aut}\, X$, such that $\alpha^*\omega = \lambda\, \omega$ for some $\lambda \in \mathbb{C}^* \setminus \{\pm 1\}$;

- the element $\rho = \alpha^d$ is a Prym involution satisfying $\rho^*\omega = -\omega$;

- ω is an eigenform for real multiplication on the Prym variety $\mathcal{P}(X, \rho)$.

Note that this implies that α is of order $2d$ and must have a fixed point (at the single zero of ω). Details and background can be found in [25].

Definition. We say that (ω_1, ω_2) is an α-*eigenbasis* of ΩX^- if the ω_i are both eigenforms for the action of α^*.

To study these points, we study the locus of curves in $\mathcal{M}_4$ with an appropriate automorphism α and an eigenform with a single zero.

Proof of Proposition 2.1. Let (X, ω) correspond to an orbifold point in W_D of order d. The Prym involution ρ in genus 4 gives a genus 2 quotient with two fixed points, i.e. $X/\rho \cong (2; 2^2)$. By the argument above, the curve X must possess an automorphism α of order $2d$ that admits ω as an eigenform with eigenvalue ζ_{2d}, has at least one fixed point and satisfies $\alpha^d = \rho$.

The automorphism α descends to an automorphism of X/ρ of order d and, looking at possible orders of automorphisms on curves of genus 2, one sees that $d = 2, 3, 4, 5, 6, 8$ or 10 (see for instance [5, 22]). Now, points of odd order k on X/α (equivalently, α-orbits of length $2d/k$ on X) give unramified points on X/ρ, since they are not fixed by $\rho = \alpha^d$ (more precisely, their preimages on X are not fixed). Points of even order $2k$ on X/α (equivalently, α-orbits of length d/k on X) give d/k ramified points on X/ρ.

Since there are only two ramification points on X/ρ and at least one of them necessarily comes from a fixed point of α, the automorphism α has two fixed points and no more ramification points of even order. A case-by-case analysis using Riemann–Hurwitz yields the four options in the statement. □

Products of elliptic curves

In the analysis of orbifold points on Prym–Teichmüller curves in genus 4, Klein-four actions and products of elliptic curves will be omnipresent. The following result will be a crucial technical tool.

Proposition 2.2. *Let (X, ρ) be a genus 4 curve in the Prym locus admitting a Klein four-group of automorphisms $V_4 = \langle \rho, \beta \rangle$ such that both X/β and $X/\rho\beta$ have genus 1. Then $\mathcal{P}(X, \rho) \cong X/\beta \times X/\rho\beta$ as $(2, 2)$-polarised abelian varieties.*

Remark. Note that this is in stark contrast to the situation in genus 2 and 3. Although in those cases V_4 actions were also ubiquitous, the Prym variety was always only *isogenous* to a product of elliptic curves (cf. [19, Proposition 2.13] and [25, Theorem 1.2]) as the Prym variety is $(1, 1)$, respectively $(2, 1)$, polarised in those situations. In the genus 4 case, the result above yields an even stronger relationship between the geometry of the quotient elliptic curves and the Prym variety.

Let us first recall some general facts about elliptic curves. An elliptic curve $E := (E, O) \in \mathcal{M}_{1,1}$ is a smooth genus 1 curve together with a chosen base point $O \in E$. It always admits the structure of a group variety with neutral element O. The set of 2-torsion points with respect to this group law consists of four elements and is usually denoted by $E[2]$.

Every elliptic curve is isomorphic to $E_\lambda := \{v^2 = u(u-1)(u-\lambda)\}$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$, where we choose the base point O to be the point at infinity. Permuting $\{0, 1, \infty\}$ gives an isomorphism between the elliptic curves corresponding to

$$\lambda, \quad 1-\lambda, \quad \frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda-1}{\lambda}, \quad \frac{\lambda}{\lambda-1}.$$

By the uniformisation theorem, every elliptic curve can also be represented as the quotient of $\mathbb{C}$ by a lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some τ in the upper half-plane $\mathbb{H}$. Points in the same $\mathrm{SL}_2\,\mathbb{Z}$-orbit yield isomorphic elliptic curves, and therefore one can realise the moduli space of elliptic curves $\mathcal{M}_{1,1}$ as the quotient $\mathbb{H}/\mathrm{SL}_2\,\mathbb{Z}$. The relationship between τ and λ is given by the modular λ-function.

Each elliptic curve carries a natural *elliptic involution* ϕ, the set of fixed points of which agrees with $E[2] = \mathrm{Fix}(\phi)$. In the model E_λ, the elliptic involution is given by $(u, v) \mapsto (u, -v)$ and one has $E_\lambda[2] = \{(0, 0), (1, 0), (\lambda, 0), \infty\}$. The quotient by the elliptic involution is isomorphic to $\mathbb{P}^1$.

The general element of $\mathcal{M}_{1,1}$ has no further automorphisms fixing the base point. The only exceptions, which correspond to the orbifold points of $\mathbb{H}/\mathrm{SL}_2\,\mathbb{Z}$, are E_2 (corresponding to $\tau = \mathrm{i}$ in the upper half-plane) with a cyclic automorphism group of order 4, and E_{ζ_6} (corresponding to $\tau = \zeta_6$ in the upper half-plane), where $\zeta_6 = e^{2\pi\mathrm{i}/6}$, with a cyclic automorphism group of order 6.

Proof of Proposition 2.2. Consider the quotients $p_1 \colon X \to X/\beta$ and $p_2 \colon X \to X/\rho\beta$. Since X/V_4 has genus 0, the images of the pullbacks $p_1^* \colon \Omega(X/\beta) \to \Omega X$ and $p_2^* \colon \Omega(X/\rho\beta) \to \Omega X$ must both lie in ΩX^-, the -1-eigenspace of ρ and in fact generate ΩX^-. Therefore, denoting also by p_i^* the induced map between Jacobians and identifying the elliptic curves with their Jacobians, one has

$$
\begin{array}{ccc}
\Omega(X/\beta) \times \Omega(X/\rho\beta) & \xrightarrow{\ \cong\ } \Omega X^- \lhook\joinrel\longrightarrow & \Omega X \\
\downarrow & \downarrow & \downarrow \\
X/\beta \times X/\rho\beta & \xrightarrow{(p_1^*,\, p_2^*)} \mathcal{P}(X, \rho) \lhook\joinrel\longrightarrow & \mathrm{Jac}\, X
\end{array}
$$

Since the polarisations induced from $\mathrm{Jac}\, X$ on $X/\beta \times X/\rho\beta$ and on $\mathcal{P}(X, \rho)$ are both of type $(2, 2)$, the map (p_1^*, p_2^*) is necessarily an isomorphism of polarised abelian varieties.
□

In particular, the proof shows that, in the above situation, we have a natural decomposition

$$\Omega X^- = \Omega X_\beta^+ \oplus \Omega X_{\rho\beta}^+$$

into a β and $\rho\beta$ invariant subspace consisting of the differential forms that arise as pullbacks from the two quotient elliptic curves.

Definition. Let X be a genus 4 curve with a V_4 action. We say that (η_1, η_2) is a *product basis* of ΩX^- if $\eta_1 \in \Omega X_\beta^+$ and $\eta_2 \in \Omega X_{\rho\beta}^+$.

Note that any product basis is a β-eigenbasis. More precisely, we have

$$\beta^*\eta_1 = \eta_1 \quad \text{and} \quad \beta^*\eta_2 = -\eta_2, \tag{1}$$

as η_2 is ρ-anti-invariant.

3. Points of order 2 and 6

The aim of this section is to prove the following formula describing the number of points of order 2 on each Teichmüller curve W_D and the unique point of order 6 on W_{12} (cf. Theorem 1.1). Let $h(-C)$ denote the class number of the imaginary quadratic order $\mathcal{O}_{-C}$.

Theorem 3.1. *Let $D \neq 12$ be a positive discriminant.*
- *If $D \equiv 1 \bmod 4$ then W_D has no orbifold points of order 2.*
- *If $D \equiv 12 \bmod 16$ then W_D has $h(-D) + h(-\frac{D}{4})$ orbifold points of order 2.*
- *Otherwise, W_D has $h(-D)$ orbifold points of order 2.*

Moreover, W_{12} has one point of order 2 and one point of order 6.

To prove this theorem, we begin by a careful analysis of genus 4 curves admitting an automorphism of order 4 with two fixed points.

Curves admitting an automorphism of order 4

By Proposition 2.1, for (X, ω) to parametrise a point of order 2 on W_D, the curve X must necessarily lie in the locus of curves with an automorphism of order 4 with two fixed points. In fact, all such curves admit a faithful D_8 action.

Lemma 3.2. *Let X be a curve of genus 4 and $\alpha \in \mathrm{Aut}\, X$ an automorphism of order 4 with two fixed points.*
Then X/α^2 is of genus 2 and there exists an involution $\beta \in \mathrm{Aut}\, X$ such that $\alpha\beta = \beta\alpha^{-1}$, i.e. $\langle \alpha, \beta \rangle \leqslant \mathrm{Aut}\, X$ is a D_8.

Proof. Since the quotient X/α by such an automorphism yields a curve of genus 1 with two orbifold points of order 4, this is just case $N2$ in [6]. The proof of Bujalance and Conder relies on a previous result by Singerman [24, Theorem 1] stating that every

Fuchsian group with signature $(1; t, t)$ is included in a Fuchsian group with signature $(0; 2^4, t)$. This group corresponds to the quotient $X/\langle \alpha, \beta \rangle$. $\qquad \square$

In terms of the corresponding curves, the situation is the following. The automorphism α descends to an involution $\overline{\alpha}$ of the genus 2 curve X/α^2 different from the hyperelliptic involution $\overline{\beta}$. The hyperelliptic involution lifts to an involution β on X which, together with α, generates the dihedral group. We denote by $p_1 : X \to X/\beta$ and $p_2 : X \to X/\alpha^2 \beta$ the corresponding projections.

Definition. We denote by $\mathcal{M}_4(D_8)$ the family of genus 4 curves admitting an automorphism of order 4 with two fixed points.

Remark. Note that by Lemma 3.2, this family agrees with the moduli space of Riemann surfaces of genus 4 with D_8-symmetry, where we fix the topological action as in the lemma. Moreover, moduli spaces of curves with automorphisms have been studied intensively, see e.g. [8] for background and notations.

It turns out that such a curve is essentially determined by its genus 1 quotients.

Proposition 3.3. *The family* $\mathcal{M}_4(D_8)$ *is in bijection with the family*

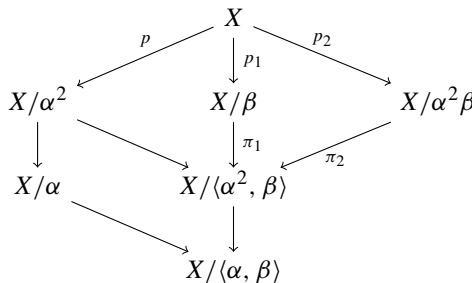$$\mathcal{E} = \left\{ (E, [P]) \, : \, E \in \mathcal{M}_{1,1}, [P] \in (E \backslash E[2]) / \{\pm 1\} \right\}$$

of elliptic curves with a distinguished base point, together with an elliptic pair.

The bijection is given by $X \mapsto (X/\beta, [p_1(\mathrm{Fix}(\alpha\beta))])$, *where the origin of the elliptic curve is chosen to be the point* $p_1(\mathrm{Fix}(\alpha))$.

Remark. Note that this is a 2-dimensional locus inside $\mathcal{M}_4$. However, we show in Theorem 3.6 that the sub-locus $\mathcal{X}$ where X admits an α-eigenform in ΩX^- with a single zero is in fact 1-dimensional.

This classification is obtained by a careful analysis of the ramification data of $D_8 \cong \langle \alpha, \beta \rangle \leqslant \mathrm{Aut}\, X$.

Consider the following diagram of ramified covers:

$$
\begin{array}{ccccc}
& & X & & \\
& {\scriptstyle p} \swarrow & {\scriptstyle p_1} \downarrow & \searrow {\scriptstyle p_2} & \\
X/\alpha^2 & & X/\beta & & X/\alpha^2\beta \\
\downarrow & \searrow & {\scriptstyle \pi_1} \downarrow & \swarrow {\scriptstyle \pi_2} & \\
X/\alpha & & X/\langle \alpha^2, \beta \rangle & & \\
& \searrow & \downarrow & & \\
& & X/\langle \alpha, \beta \rangle & &
\end{array}
$$

Observe that all maps in the diagram are of degree 2.

The involutions β and $\alpha^2\beta$ each have 6 fixed points on X. Together they form three orbits of length 4 under $\langle \alpha, \beta \rangle$. Similarly, $\alpha\beta$ and $\alpha^{-1}\beta$ have 2 fixed points each, forming a whole orbit under $\langle \alpha, \beta \rangle$ of length 4. Now, the four points of order 2 in $X/\langle \alpha, \beta \rangle$ correspond

to the (three) orbits of the fixed points of β and $\alpha^2\beta$ plus the orbit of the fixed points of $\alpha\beta$ and $\alpha^{-1}\beta$.

Looking at the ramification data of α and β, one sees that the quotients X/β and $X/\alpha^2\beta$ by β and $\alpha^2\beta = \alpha\beta\alpha^{-1}$ respectively correspond to curves of genus 1. Choosing the image of $\mathrm{Fix}(\alpha)$ as an origin on each quotient, they are in fact isomorphic as elliptic curves, since β and $\alpha^2\beta$ are conjugate.

Also, the above-described action of $\alpha^2\beta$ and β may be described purely in terms of the quotient maps: the six branch points of p_1 are mapped via p_2 to the three 2-torsion points on $X/\alpha^2\beta$, while p_1 maps the six branch points of p_2 to the three 2-torsion points on X/β.

Proof of Proposition 3.3. Denote by ϕ the elliptic involution on E and let $\varphi\colon E \to \mathbb{P}^1$ be the corresponding quotient map, which we normalise such that $\varphi(O) = \infty$ and $\varphi(P) = 0$. We define $X = X_{(E,[P])}$ as the fibre product of the diagram

$$E \xrightarrow{\varphi} \mathbb{P}^1 \xleftarrow{-\varphi} E.$$

Note that, although there is a degree of freedom in choosing φ, this does not affect the construction.

It is obvious from the ramification data that $X_{(E,[P])}$ has genus 4, and the automorphisms $(Q_1, Q_2) \mapsto (Q_2, \phi(Q_1))$ and $(Q_1, Q_2) \mapsto (Q_1, \phi(Q_2))$ of $E \times E$ restrict to automorphisms α and β of $X_{(E,[P])}$ generating a D_8. It is straightforward to check that the map $(E, [P]) \mapsto X_{(E,[P])}$ thus defined is inverse to $X \mapsto (X/\beta, [p_1(\mathrm{Fix}(\alpha\beta))])$. $\qquad\square$

In particular, these curves satisfy the assumptions of Proposition 2.2, and therefore their Prym varieties are isomorphic to a product of elliptic curves.

Corollary 3.4. *Let $X \in \mathcal{M}_4(D_8)$. Then $\mathcal{P}(X, \alpha^2) \cong E \times E$ as polarised abelian varieties, where $E \cong X/\beta \cong X/\alpha^2\beta$.*

As the quotient elliptic curves are isomorphic, we pick some differential form η_E on E and denote by

$$\eta_i = p_i^* \eta_E, \quad \text{for } i = 1, 2, \tag{2}$$

the corresponding product basis. Using the explicit description of X as a fibre product and the expression of α used in the proof of Proposition 3.3, one can easily describe the action of α^* on these differentials to see that

$$\alpha^* \eta_1 = -\eta_2 \quad \text{and} \quad \alpha^* \eta_2 = \eta_1.$$

In particular, α interchanges the spaces ΩX_β^+ and $\Omega X_{\rho\beta}^+$.

The eigenspace decomposition

For a D_8 curve to parametrise an orbifold point, it must necessarily admit an α-eigenform with a single (6-fold) zero. To determine the possible eigenforms, we must analyse the decomposition of ΩX into α-eigenspaces. We denote, as usual, by ΩX^- and ΩX^+ the -1- and $+1$-eigenspaces of ΩX with respect to the (Prym) involution α^2.

Proposition 3.5. *Let $X \in \mathcal{M}_4(D_8)$. There is a natural splitting*

$$\Omega X^- = \Omega X_\alpha^{\mathrm{i}} \oplus \Omega X_\alpha^{-\mathrm{i}}$$

into $\pm\mathrm{i}$-eigenspaces of α. The spaces $\Omega X_\alpha^{\pm\mathrm{i}}$ are interchanged by β.

Proof. The quotient X/α has genus 1, so it is obvious that ΩX^+ decomposes into α-eigenspaces of dimension 1 with eigenvalue $+1$ and -1.

On the other hand, since $\alpha\beta = \beta\alpha^{-1}$, if $\alpha^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}$, clearly $\alpha^*(\beta^*\omega) = \lambda^{-1}\beta^*\omega$. In particular, the eigenvalues of α^* on ΩX^- can only be $\pm\mathrm{i}$, therefore the space necessarily decomposes as the sum of the α^* i-eigenspace and the $-\mathrm{i}$-eigenspace. $\square$

Note that any α-eigenbasis (ω_1, ω_2) of ΩX^- will satisfy $\omega_1 \in \Omega X_\alpha^{\mathrm{i}}$ and $\omega_2 \in \Omega X_\alpha^{-\mathrm{i}}$, up to renumbering. Moreover, any product basis (η_1, η_2) as in (2) gives rise to an α-eigenbasis

$$\omega_1 = \eta_1 + \mathrm{i}\eta_2, \quad \omega_2 = \eta_1 - \mathrm{i}\eta_2. \tag{3}$$

Now, while the family $\mathcal{M}_4(D_8)$ of curves admitting a D_8 action is 2-dimensional, it turns out that requiring an α-eigenform with a single zero reduces the dimension of the locus we are interested in by one. Let us define

$$\mathcal{X} = \left\{ X \in \mathcal{M}_4(D_8) : \exists\, \omega \in \Omega X^-\ \alpha\text{-eigenform with a single zero} \right\}.$$

Because of the flat picture of the elements (X, ω) in $\Omega\mathcal{X}^-(6)$, we call $\mathcal{X}$ the *Turtle family* (see §6).

Theorem 3.6. *The map*

$$\begin{aligned} \mathcal{X} &\longrightarrow \mathcal{M}_{1,1} \\ X &\longmapsto X/\beta, \end{aligned}$$

where the origin of X/β is chosen to be $p_1(\mathrm{Fix}(\alpha))$, induces a bijection between $\mathcal{X}$ and $\mathcal{M}_{1,1}\backslash\{E_2\}$.

The only curve in $\mathcal{X}$ where α is extended by an automorphism of order 12 is the one corresponding to $X/\beta \cong E_{\zeta_6}$. It agrees with family (4) in Proposition 2.1.

Proof. By (3) α-eigenforms in ΩX^+ are given, up to scale, by $\omega_1 = p_1^*\eta_E + \mathrm{i}p_2^*\eta_E$ and $\omega_2 = p_1^*\eta_E - \mathrm{i}p_2^*\eta_E$. We proceed in several steps.

Step 1. The α-eigenforms in ΩX^- can have a zero at most at one of the (two) fixed points of α.

Otherwise, every differential in ΩX^- would vanish at both fixed points of α. In particular, so would $p_1^*\eta_E$ and $p_2^*\eta_E$. But the maps p_i are unramified at $\mathrm{Fix}(\alpha)$ and we know that η_E has no zeroes in E. Note that, since zeroes of α-eigenforms outside $\mathrm{Fix}(\alpha)$ must be permuted by α, this immediately implies that the differentials ω_1 and ω_2 lie either in $\Omega\mathcal{X}(1^4, 2)$ or in $\Omega\mathcal{X}(6)$. Hence it remains to show that $\omega_1, \omega_2 \in \Omega\mathcal{X}(6)$.

Step 2. Note that $p_i^*\eta_E$ vanishes only at the six branch points of p_i.

In particular neither ω_1 nor ω_2 vanish at $\mathrm{Fix}(\beta) \cup \mathrm{Fix}(\alpha^2\beta)$, as the two sets of fixed points are disjoint.

Step 3. Choose λ so that $E \cong E_\lambda = \left\{ y^2 = x(x-1)(x-\lambda) \right\}$ with the point at infinity as a distinguished point, and let $[P] = (A, \pm B)$ in these coordinates. Then we claim that $\omega_1, \omega_2 \in \Omega\mathcal{X}(6)$ if and only if $3A - 1 - \lambda = 0$.

In fact note that, in this case, the map $\varphi \colon E_\lambda \to \mathbb{P}^1$ in the proof of Proposition 3.3 can be chosen to be $(x, y) \mapsto x - A$, and the points in $X_{(E_\lambda, [P])}$ outside of the branch loci of the maps p_i can be seen as pairs of points

$$Q = \left((x, \varepsilon_1\sqrt{x(x-1)(x-\lambda)}), (-x+2A, \varepsilon_2\mathrm{i}\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)}) \right)$$
$$\in E_\lambda \times E_\lambda,$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Normalising $\eta_E = \mathrm{d}x/y$ and evaluating a local expression around Q yields

$$\omega_1(Q) = \frac{\varepsilon_1}{\sqrt{x(x-1)(x-\lambda)}} + \frac{\varepsilon_2}{\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)}}$$
$$= \frac{\varepsilon_1\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)} \pm \varepsilon_2\sqrt{x(x-1)(x-\lambda)}}{\varepsilon_1\varepsilon_2\sqrt{x(x-1)(x-\lambda)(x-2A)(x-2A+1)(x-2A+\lambda)}},$$

and similarly for ω_2.

Now, comparing the addends in the numerator and taking squares one sees that the differential ω_1 will vanish at Q, for (exactly) two choices $(\varepsilon_1, \varepsilon_2)$ and $(-\varepsilon_1, -\varepsilon_2)$, whenever

$$2(3A - \lambda - 1)x^2 - 4A(3A - \lambda - 1)x + 2A(2A - \lambda)(2A - 1) = 0.$$

In particular, if (and only if) $3A - \lambda - 1 = 0$ the differentials do not vanish in the affine part of E_λ, hence the zeroes of ω_1 and ω_2 must be at infinity, i.e. in $\mathrm{Fix}(\alpha)$, and Step 1 implies that there is only a single zero on $X_{(E_\lambda, [P])}$.

Step 4. The point $P = (A, B) \in E_\lambda$ can be uniquely chosen as a non-2-torsion point subject to the condition $A = (\lambda + 1)/3$ from above if and only if $\lambda \neq -1, 1/2, 2$.

Note that these three values of λ give rise to the same elliptic curve, namely the square torus E_2. In particular, for all $P \in E_2 \setminus E_2[2]$ the curve $X_{(E_2, [P])}$ has no α-eigenform with a single zero.

Therefore, for any $(E, O) \in \mathcal{M}_{1,1} \setminus \{E_2\}$, there is a unique choice of $[P]$, such that the fibre product $X_{(E, [P])}$ admits a D_8 action together with an α-eigenform that has a single 6-fold zero.

It remains to check when α can be extended, i.e. when there exists an $\alpha' \in \mathrm{Aut}\, X_{(E, [P])}$ that satisfies $\alpha \in \langle \alpha' \rangle$.

However, the proof of Proposition 2.1 shows that this can happen only if α' is of order 12. In this case, $(\alpha')^6 = \alpha^2$ commutes with β, hence descends to an automorphism of order 6 on the elliptic curve X/β which must therefore be isomorphic to E_{ζ_6}.

On the other hand, denote by $\psi \in \mathrm{Aut}\, E_{\zeta_6}$ the automorphism of order 6 on E_{ζ_6}. It is easy to see that the automorphism $(Q_1, Q_2) \mapsto (\psi(Q_2), \psi^4(Q_1))$ on $E_{\zeta_6} \times E_{\zeta_6}$ restricts to an automorphism of order 12, extending α, on the curve of $\mathcal{X}$ corresponding to this elliptic curve. In fact, the corresponding fibre product is the curve $y^6 = x(x+1)^2(x-1)^2$, see also §4. $\qquad\square$

Moreover, we have the following corollary.

Corollary 3.7. *Let X be a genus 4 curve admitting an automorphism α of order 4 with two fixed points. If additionally $(X, \omega) \in \Omega W_D$ then $X \in \mathcal{X}$ and $\omega = \omega_1$ or ω_2. In particular, ω is an α-eigenform and (X, ω) is a point of order 2 on W_D.*

To check which $(X, \omega_i) \in \Omega \mathcal{X}$ are on W_D, we need to check when $\mathcal{P}(X, \alpha^2)$ admits real multiplication with ω_i as an eigenform. Note that β^* interchanges ω_1 and ω_2, and therefore it is enough to focus on one of the two eigenforms.

First, we need the following explicit description of the endomorphism ring of the Prym variety. Recall that the endomorphism ring $\mathrm{End}(E)$ of an elliptic curve is either $\mathbb{Z}$ or an order in an imaginary quadratic field.

Lemma 3.8. *Let (η_1, η_2) be the product basis of ΩX^- as in (2). Then*

$$\mathrm{End}(\mathcal{P}(X, \alpha^2)) = M_2(\mathrm{End}(E)),$$

where $E \cong X/\beta$. Self-adjoint endomorphisms correspond to matrices satisfying $M^T = M^\sigma$, where M^σ denotes conjugation by the non-trivial Galois automorphism of $\mathrm{End}(E)$ on each entry.

Moreover, ω_1 corresponds to the η-representation $(1, \, i)$ and ω_2 to the representation $(1, \, -i)$.

Proof. The first part of the lemma follows immediately from Corollary 3.4. The claim about the eigenforms follows from (3). $\qquad\square$

We now have all the ingredients assembled to prove the formula for the points of order 2.

Proof of Theorem 3.1. Recall that X classifies an orbifold point of order 2 on W_D if and only if $\mathcal{P}(X, \alpha^2)$ admits proper self-adjoint real multiplication with the α-eigenforms ω_1 or ω_2 as an eigenform. Since these are interchanged by β, it is enough to focus our attention on ω_1.

Again, we set $E = X/\beta$. Note that, by Theorem 3.6, E must not be isomorphic to E_{ζ_6}.

Assume that D is not a square. Now, in the η-basis of $E \times E$, the form ω_1 has the representation $(1, \, i)$ (cf. Lemma 3.8). In other words, $(X, \omega_1) \in \Omega \mathcal{X}$ is an orbifold point on W_D if and only if there exists $T_D \in M_2(\mathrm{End}(E))$, where

$$T_D = \begin{cases} \begin{pmatrix} \dfrac{1}{2} & -\dfrac{\sqrt{-D}}{2} \\[2mm] \dfrac{\sqrt{-D}}{2} & \dfrac{1}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4, \text{ and} \\[6mm] \begin{pmatrix} 0 & -\dfrac{\sqrt{-D}}{2} \\[2mm] \dfrac{\sqrt{-D}}{2} & 0 \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \end{cases}$$

while there is no $T_{D'} \in M_2(\mathrm{End}(E))$ for $D = f^2 D'$.

As $\mathrm{End}(E)$ is integral over $\mathbb{Z}$ and $1/2$ is not, the case $D \equiv 1 \bmod 4$ can never occur. The other case occurs whenever $\sqrt{-D}/2 \in \mathrm{End}(E)$, and this happens if and only if E has complex multiplication by the order $\mathcal{O}_{-D} \subset \mathbb{Q}(\sqrt{-D})$.

To determine precisely which orders $\mathcal{O}_{-C}$ contain such a maximal T_D, note that, by definition, $\sqrt{-D}/2 \in \mathcal{O}_{-C}$ if and only if $D = b^2 C$ for some integer b. Moreover, C must be congruent with 0 or $3 \bmod 4$ so that $-C$ is a discriminant.

For $b > 2$ the action is never proper, and therefore we can assume $b = 1$ or 2.

The case $b = 1$ implies that elliptic curves E not isomorphic to E_{ζ_6} admitting complex multiplication by $\mathcal{O}_{-D}$ always determine an orbifold point of order 2 on W_D.

As for $b = 2$, there are several options. If $D/4 \equiv 1 \bmod 4$, then $-D/4 \equiv 3 \bmod 4$ is not a discriminant. If, however, $C = D/4 \equiv 3 \bmod 4$, then $-C$ is a discriminant and complex multiplication by $\mathcal{O}_{-C} = \mathcal{O}_{-D/4}$ also gives proper real multiplication by $\mathcal{O}_D$ on the Prym part. Finally, if $C = D/4 \equiv 0 \bmod 4$, then $-C$ is a discriminant but the Prym then admits real multiplication by $\mathcal{O}_C$, hence the real multiplication by $\mathcal{O}_D$ is not proper in these cases.

Moreover, observe that $E \cong E_{\zeta_6}$ if and only if $C = 3$, i.e. $D = 12$. On the other hand, if $D = 12$, there exists precisely one elliptic curve with proper complex multiplication by $\mathcal{O}_{-12}$ and hence $W_{12}(6)$ admits one point of order 2 and one point of order 6.

Finally, as it is well known that there are $h(-C)$ elliptic curves admitting complex multiplication by $\mathcal{O}_{-C}$, this proves the result.

For the square discriminant case $D = f^2$, one can follow the same reasoning as above and use the fact that $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 - fT)$ to deduce that the generator $T \in M_2(\mathrm{End}(E))$ must agree with

$$T = \begin{pmatrix} \dfrac{f}{2} & -\mathrm{i}\dfrac{f}{2} \\ \mathrm{i}\dfrac{f}{2} & \dfrac{f}{2} \end{pmatrix}$$

and an analysis similar to the one above proves the theorem. $\qquad\square$

4. Points of order 3

In this section we prove the formula for the orbifold points of order 3 on W_D.

Recall the numbers

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D, \gcd(a, i, j) = 1\}/12.$$

We have the following description of the orbifold points of order 3.

Theorem 4.1. *Let $D \neq 12$ be a positive discriminant. Then W_D has $e_3(D)$ orbifold points of order 3.*

To describe the points of order 3 on W_D, we again describe the intersection with the locus of curves with a fixed type of automorphism.

Curves admitting an automorphism of order 6

By Proposition 2.1, for an (X, ω) to parametrise a point of order 3 on W_D the curve X must necessarily admit an automorphism α of order six with two fixed points and two

orbits of length 2 admitting ω as an eigenform. Note that in particular X/α has genus 0. Cyclic covers of the projective line have been thoroughly studied by several authors (see for example [4, 21], see also [25] for a brief summary of the facts required here).

Now, there are two families of cyclic covers of $\mathbb{P}^1$ of degree 6 with the given branching data, namely:

$$\mathcal{Y}_t \colon \; y^6 = x(x-1)^2(x-t)^2, \quad t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$$

and

$$\mathcal{Z}_t \colon \; y^6 = x(x-1)^2(x-t)^4, \quad t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}.$$

Denote by $\alpha = \alpha_t$ the automorphisms $(x, y) \mapsto (x, \zeta_6 y)$ of order 6 on $\mathcal{Y}_t$ and on $\mathcal{Z}_t$. Note that both $\mathcal{Y}_t/\alpha^3$ and $\mathcal{Z}_t/\alpha^3$ have genus 2, so $\rho = \alpha^3$ is actually a Prym involution.

The following proposition shows immediately that no member of the $\mathcal{Z}$ family can belong to a Teichmüller curve W_D.

Lemma 4.2. *The space $\Omega\mathcal{Z}^-$ is disjoint from the minimal stratum $\Omega\mathcal{M}_4(6)$.*

Proof. It is easy to check (see for example [4]) that for each t the space $\Omega\mathcal{Z}_t^-$ is generated by the differentials

$$\xi_1 = \frac{y\,\mathrm{d}x}{x(x-1)(x-t)} \quad \text{and} \quad \xi_2 = \frac{\mathrm{d}x}{y}.$$

They both lie in the stratum $\Omega\mathcal{M}_4(1^2, 4)$. In fact, a local calculation shows that

$$\operatorname{div}\xi_1 = 4P_1 + R'_1 + R'_2 \quad \text{and}$$
$$\operatorname{div}\xi_2 = 4P_2 + R''_1 + R''_2,$$

where the P_i are the two fixed points of α and $\{R'_i\}$ and $\{R''_i\}$ are the α-orbits of length 2.

Now, any element of $\Omega\mathcal{Z}_t^-$ different from the generators can be written as a linear combination $\xi = a\,\xi_1 + b\,\xi_2$. But, since the ξ_i vanish at different points, such a differential can never have a zero at any point of $\operatorname{Fix}(\alpha)$, nor at any point in the two α-orbits of length 2. As a consequence $\xi \in \Omega\mathcal{Z}_t^-(1^6)$ and the result follows. $\qquad\square$

The following lemma detects which fibres of the $\mathcal{Y}$ family are isomorphic, together with the special fibre having a larger automorphism group.

Lemma 4.3. *The isomorphism $z \mapsto 1/z$ of $\mathbb{P}^1$ lifts to an isomorphism $\mathcal{Y}_t \cong \mathcal{Y}_{1/t}$ for each $t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$.*

In particular, at the fixed point, the automorphism α_{-1} of the curve $\mathcal{Y}_{-1}$ extends to an automorphism $\gamma \colon (x, y) \mapsto (1/x, y/x)$ of order 12.

Proof. As the curve is given in coordinates explicitly as a cyclic cover of $\mathbb{P}^1$, this is a straightforward calculation. $\qquad\square$

The intersections of $\mathcal{Y}$ and W_D will give the orbifold points of order 3 on W_D. To make this statement more precise, we begin by the following observation.

Proposition 4.4. *For each t the space $\Omega\mathcal{Y}_t^-$ is generated by the α-eigenforms*

$$\omega_1 = \frac{y\,\mathrm{d}x}{x(x-1)(x-t)} \quad and \quad \omega_2 = \frac{-y\,\mathrm{d}x}{\sqrt{t}(x-1)(x-t)}.$$

Up to scale, the only differentials in $\Omega\mathcal{Y}_t^-(6)$ are ω_1 and ω_2.

Proof. The local expressions show that these differentials are holomorphic for all t. They obviously span the α-eigenspace of eigenvalue ζ_6 and therefore generate $\Omega\mathcal{Y}_t^-$ (cf. [4]).

It is easy to see that ω_1 (resp. ω_2) has a single zero at the single point at infinity (has a single zero at $(0,0)$). Now, for every $a \neq 0$ the zeroes of the differential

$$\omega_a := \omega_1 + a\omega_2 = \frac{y\,\mathrm{d}x}{(x-1)(x-t)}\left(\frac{1}{x} + a\right)$$

are located at the points with x-coordinate $-1/a$. They are either six simple zeroes if $a \neq -1, -1/t$, or three zeroes of order 2 otherwise. $\qquad\square$

Remark. Note that, in contrast to the family $\mathcal{X}$ of curves with a D_8 action, the α-eigenspace inside $\Omega\mathcal{Y}_t^-$ is in fact 2-dimensional. However, we are only be interested in the two 1-dimensional subspaces of eigenforms with a single zero.

Because of the flat picture of the differentials $(\mathcal{Y}_t, \omega_i)$, we call $\mathcal{Y}$ the *Hurricane family* (see § 6). Note that (ω_1, ω_2) yields an α-eigenbasis of $\Omega\mathcal{Y}_t^-$. The following is a consequence of Lemma 4.2 and Proposition 4.4.

Corollary 4.5. *Let X be a genus 4 curve admitting an automorphism α of order 6 with two fixed points and two orbits of length 2. If $(X, \omega) \in \Omega X^-(6)$ then $X \in \mathcal{Y}$ and $\omega = \omega_1$ or ω_2. In particular, ω is an α-eigenform.*

Corollary 4.6. *A flat surface (X, ω) parametrising a point on W_D corresponds to an orbifold point of order 3 if and only if there is some $t \in \mathbb{P}^1\setminus\{0, 1, -1, \infty\}$ such that $X \cong \mathcal{Y}_t$ and $[\omega] = [\omega_1]$ or $[\omega] = [\omega_2]$.*

It corresponds to an orbifold point of order 6 if and only if $X \cong \mathcal{Y}_{-1}$ and $[\omega] = [\omega_1]$ or $[\omega] = [\omega_2]$.

Proof. This is a consequence of Propositions 2.1, 4.4 and Lemma 4.2. $\qquad\square$

We must therefore analyse when the Prym part of $\mathcal{Y}_t$ admits real multiplication. Recall that the elliptic curve E_ζ, where $\zeta := \exp(2\pi\mathrm{i}/6)$, is the only elliptic curve admitting an automorphism of order 6 fixing the base point. It corresponds to the hexagonal lattice, i.e.

$$E_\zeta \cong \mathbb{C}/\Lambda_\zeta, \quad \text{with} \quad \Lambda_\zeta = \mathbb{Z} \oplus \zeta\mathbb{Z}.$$

Next, we collect some useful observations.

Lemma 4.7. *Any curve $\mathcal{Y}_t$ admits an involution β commuting with α, i.e. such that $\langle \alpha, \beta \rangle \cong C_6 \times C_2$. Moreover, one has $\mathcal{P}(\mathcal{Y}_t, \rho) \cong E_\zeta \times E_\zeta$.*

The general member $\mathcal{Z}_t$ of the $\mathcal{Z}$ family has an automorphism group equal to C_6.

Proof. By [24, Theorems 1 and 2] there is only one Fuchsian group containing a generic Fuchsian group of signature $(0; 3, 3, 6, 6)$. The signature of such supergroup is $(0; 2, 2, 3, 6)$, and the inclusion is of index 2 and therefore normal. As a consequence, the automorphism group of any general fibre in the $\mathcal{Y}$ family or the $\mathcal{Z}$ family is at most an extension of index two of C_6.

In the case of $\mathcal{Y}_t$, the inclusion induces an extra automorphism $\beta := \beta_t$, given by $(x, y) \mapsto (x/t, \sqrt{t}\,y/x)$.

In particular α^3 and β generate a Klein four-group such that the quotients $\mathcal{Y}_t/\beta$ and $\mathcal{Y}_t/\alpha^3\beta$ have genus 1. Therefore, they satisfy the conditions of Proposition 2.2 and $\mathcal{P}(\mathcal{Y}_t, \rho) \cong \mathcal{Y}_t/\beta \times \mathcal{Y}_t/\alpha^3\beta$. Since α induces an automorphism ψ of order 6 on both $\mathcal{Y}_t/\beta$ and $\mathcal{Y}_t/\alpha^3\beta$, they are necessarily isomorphic to the elliptic curve E_ζ.

As for the $\mathcal{Z}$ family, any such automorphism would induce an automorphism of $\mathcal{Z}_t/\alpha \cong \mathbb{P}^1$ permuting orbifold points of the same order. Since the exponents at 0 and ∞ and at 1 and t are different, there cannot be such an automorphism.   □

We write again $p_1 \colon \mathcal{Y}_t \to \mathcal{Y}_t/\beta$ and $p_2 \colon \mathcal{Y}_t \to \mathcal{Y}_t/\alpha^3\beta$ for the corresponding projections. The following lemma gives an explicit formula for these two maps that will be needed later to compute the explicit pullbacks of the differentials on E_ζ.

Lemma 4.8. *Consider the Weierstraß equation $\{v^2 = u^3 - 1\}$ defining E_ζ. In this model, the maps p_1 and p_2 are given by*

$$p_1 \colon \mathcal{Y}_t \to E_\zeta$$
$$(x, y) \mapsto \left(\frac{-1}{(1+\sqrt{t})^{2/3}} \frac{(x-1)(x-t)}{y^2}, \; \frac{\mathrm{i}}{(1+\sqrt{t})} \frac{(x-1)(x-t)(x+\sqrt{t})}{y^3} \right),$$

$$p_2 \colon \mathcal{Y}_t \to E_\zeta$$
$$(x, y) \mapsto \left(\frac{-1}{(1-\sqrt{t})^{2/3}} \frac{(x-1)(x-t)}{y^2}, \; \frac{\mathrm{i}}{(1-\sqrt{t})} \frac{(x-1)(x-t)(x-\sqrt{t})}{y^3} \right).$$

These maps are only unique up to composition with (a power of) α.

Proof. The map $\mathcal{Y}_t \to \mathcal{Y}_t/\beta$ induces an isomorphism between the function field $\mathbb{C}(\mathcal{Y}_t/\beta)$ and the subfield $\mathbb{C}(\mathcal{Y}_t)^{\langle\beta\rangle} \subset \mathbb{C}(\mathcal{Y}_t)$ fixed by β^*. This subfield is generated by the rational functions

$$\widetilde{u} := x + \beta(x) + 2\sqrt{t} = \frac{(x+\sqrt{t})^2}{x}, \quad \widetilde{v} := y + \beta(y) = y\frac{x+\sqrt{t}}{x}.$$

Using the equation of $\mathcal{Y}_t$ it is easy to check that the generating functions $\widetilde{u}$ and $\widetilde{v}$ satisfy the relation $\widetilde{v}^6 = \widetilde{u}^3(\widetilde{u} - c)^2$, where $c = (1+\sqrt{t})^2$. One can then check the ramification points of the degree 6 function $(\widetilde{u}, \widetilde{v}) \mapsto \widetilde{u}$ and easily deduce the isomorphism

$$\widetilde{E} \; : \; \widetilde{v}^6 = \widetilde{u}^3(\widetilde{u} - c)^2 \; \to \; E_\zeta \; : \; v^2 = u^3 - 1$$
$$(\widetilde{u}, \widetilde{v}) \qquad\qquad \mapsto \; (u, v) = \left(\frac{-\widetilde{u}(\widetilde{u} - c)}{c^{1/3}\,\widetilde{v}^2}, \; \frac{\mathrm{i}\,\widetilde{v}^3}{c^{1/2}\,\widetilde{u}(\widetilde{u} - c)} \right).$$

Finally, replacing $\widetilde{u}$ and $\widetilde{v}$ by their values in terms of the coordinates x and y, one gets the formula for p_1.

The same argument replacing β by $\alpha^3\beta$ yields the result for p_2.   □

Fibre products

Similarly to the case of the D_8 family, one can also construct the Hurricane family $\mathcal{Y}$ of genus 4 curves with a $C_6 \times C_2$ action as a certain family of fibre products over two

isomorphic elliptic curves. In order to do so, let ψ denote the automorphism of order 6 on E_ζ and consider the following diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\ p_1\ } & X/\beta \cong E_\zeta \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
X/\alpha^3 & \longrightarrow & X/\langle \alpha^3, \beta\rangle \cong \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \cong X/\alpha & \longrightarrow & X/\langle \alpha, \beta\rangle \cong \mathbb{P}^1
\end{array}
$$

Clearly, $\overline{\beta}$ is the hyperelliptic involution on X/α^3. On $X/\alpha \cong \mathbb{P}^1$ the involution $\overline{\beta}$ has two fixed points and the preimages of these points give the six Weierstraß points on X/α^3. Moreover, $X \to X/\alpha^3$ is ramified only over the two fixed points of α, while the map $X/\alpha^3 \to X/\alpha$ also branches at R' and R'', the preimages (on X) being $\{R'_1, R'_2\}$ and $\{R''_1, R''_2\}$, respectively.

Now, α and β have no common fixed points, hence the image of the (two) fixed points of α on X gives the (unique) fixed point O of $\psi = \overline{\alpha}$ on E_ζ. In addition, $\overline{\beta}$ interchanges R' and R'', hence we may name the fibres such that the images R_1 of $\{R'_1, R''_1\}$ and R_2 of $\{R'_2, R''_2\}$ form the unique ψ-orbit of order 3 on E_ζ.

On the other hand, the six Weierstraß points of X/α^3 have 12 preimages on X with β acting on each fibre. Three fibres form the six fixed points of β on X, i.e. the branch points of p_1, while the other three give the fixed points of $\alpha^3\beta$, which are equivalently the fixed points of the elliptic involution $\phi = \overline{\alpha}^3$ on X/β, i.e. the three 2-torsion points. The situation is exactly reversed for the projection $p_2 \colon X \to X/\alpha^3\beta \cong E_\zeta$.

Finally, note that in Lemma 4.8 the coordinates on E_ζ were chosen such that the projection $\varphi \colon E_\zeta \to \mathbb{P}^1 \cong E_\zeta/\phi$ to the quotient by the elliptic involution maps O to ∞ and both R_1 and R_2 to 0. Observe that ψ then descends to an automorphism of order 3 on the quotient that fixes 0 and ∞. In particular, we can assume $\varphi(\psi(S)) = \zeta_6^2\varphi(S)$, for each $S \in E_\zeta$.

Now, for each $P \in E_\zeta^* := E_\zeta \setminus (E_\zeta[2] \cup \{R_1, R_2\})$ consider the map $\varphi_P \colon E_\zeta \to \mathbb{P}^1$, $Q \mapsto \varphi(P) \cdot \varphi(Q)$. We define $Y = Y_P$ as the fibre product of the diagram

$$
E_\zeta \xrightarrow{\ \varphi\ } \mathbb{P}^1 \xleftarrow{\ \varphi_P\ } E_\zeta.
$$

This fibre product admits a group of automorphisms isomorphic to $C_6 \times C_2$ given by the restriction of the following automorphisms of $E_\zeta \times E_\zeta$:

$$
\alpha(Q_1, Q_2) = (\psi(Q_1), \psi(Q_2)), \quad \beta(Q_1, Q_2) = (Q_1, \psi^3(Q_2)).
$$

By Lemma 4.7, every Y_P is therefore a fibre of the $\mathcal{Y}$ family.

Proposition 4.9. *The map $P \mapsto Y_P$ gives a 6-to-1 map between the set of elliptic pairs of points E_ζ^*/ϕ and the fibres of $\mathcal{Y}$.*

It descends to a 2-to-1 map between the set E_ζ^*/ψ of regular orbits of ψ and the fibres of $\mathcal{Y}$. Moreover, the only ramification value of this map corresponds to the curve $\mathcal{Y}_{-1}$ admitting an automorphism of order 12.

Proof. Let $P \in E_\zeta^*$. Note that the construction does not depend on the choice of $\{P, \phi(P)\}$. In fact, even for any choice of a different point in the orbit $\{\psi^j(P)\}_{j=0}^5$ the automorphism of $E_\zeta \times E_\zeta$ given by $(Q_1, Q_2) \mapsto (Q_1, \psi^{-j}(Q_2))$ induces an isomorphism between Y_P and $Y_{\psi^j(P)}$.

Now, for the point $P' \in E_\zeta^*$ such that $\phi(P') = 1/\phi(P)$, the automorphism $(Q_1, Q_2) \mapsto (Q_2, Q_1)$ induces an isomorphism between Y_P and $Y_{P'}$.

On the other hand, for any $Y \in \mathcal{Y}$ take $x \in \mathrm{Fix}(\beta)$ and write $P = [x] \in Y/\beta \cong E_\zeta$ for its image in the quotient. It is straightforward to check that $Y \cong Y_P$. Any other choice of $x \in \mathrm{Fix}(\beta)$ or $x \in \mathrm{Fix}(\alpha^3\beta)$ determines different points in $\{\psi^j(P), \psi^j(P')\}_{j=0}^5$, defining the same fibre product. $\qquad\square$

Remark. Note that the action of ψ on the point P corresponds to the action of α on the maps p_i mentioned in Lemma 4.8. The remaining factor of 2 comes from the (generic) identification of $\mathcal{Y}_t$ with $\mathcal{Y}_{1/t}$

Eigenforms with a single zero

By Lemma 4.7, all Prym varieties in the $\mathcal{Y}$ family are isomorphic. To understand $\mathrm{End}(\mathcal{P}(\mathcal{Y}_t, \rho))$, where $\rho = \alpha^3$, denote by (η_1, η_2) again the product basis of $\Omega\mathcal{Y}_t^-$ given by

$$\eta_i = p_i^* \eta_E, \quad \text{for } i = 1, 2. \tag{4}$$

It is well known that $\mathcal{O}_\zeta := \mathrm{End}(E_\zeta) = \mathbb{Z} \oplus \mathbb{Z}\zeta_6^2$ are the *Eisenstein integers*.

Lemma 4.10. *Let (η_1, η_2) be the product basis of $\Omega\mathcal{Y}_t^-$ from (4). Then*

$$\mathrm{End}(\mathcal{P}(\mathcal{Y}_t, \alpha^3)) = M_2(\mathrm{End}(E_\zeta)) = M_2(\mathcal{O}_\zeta).$$

Self-adjoint endomorphisms correspond to matrices $M^T = M^\sigma$, where M^σ denotes conjugation by the non-trivial Galois automorphism of $\mathcal{O}_\zeta$ on each entry.

Proof. This is an immediate consequence of Lemma 4.7. $\qquad\square$

While the product basis gives an easy understanding of the endomorphism ring, and while in fact any differential in $\Omega\mathcal{Y}_t^-$ is an α-eigendifferential, we are interested in α-eigendifferentials *with a single zero* that are also eigenforms for real multiplication of the Prym variety. By Proposition 4.4, these are precisely the differentials ω_1 and ω_2 on $\mathcal{Y}_t$.

To check whether ω_1 or ω_2 are eigenforms for real multiplication, we must therefore keep track of these differentials in the product basis. For this, we set

$$\mu := \mu_t := \left(\frac{1 - \sqrt{t}}{1 + \sqrt{t}} \right)^{1/3}.$$

The relationship between the α-eigenbasis and the product basis can be summarised as follows:

Lemma 4.11. *Denote by (η_1, η_2) the product basis. Then*

$$[\omega_1] = [-\mu_t\,\eta_1 + \eta_2], \quad [\omega_2] = [\mu_t\,\eta_1 + \eta_2]$$

gives an α-eigenbasis with ω_i having each a single zero on $\mathcal{Y}_t$.

In particular, for each isomorphism class of curves $[Y] \in \mathcal{Y}$, with $[Y] \neq [\mathcal{Y}_{-1}]$, there exist 12 elements $\mu_t \in \mathbb{C}^*$ such that $\mu_t \eta_1 + \eta_2$ are precisely the α-eigendifferentials with a single zero on $[Y]$.

On the curves $[\mathcal{Y}_{-1}]$ there are six different values of μ giving eigendifferentials with a single zero.

Proof. The differential $\omega_1 + \omega_2$ (resp. $\omega_1 - \omega_2$) is β-invariant (resp. $\alpha^3\beta$-invariant). Therefore, there exist k_1, k_2 such that $\omega_1 + \omega_2 = k_1 \eta_1$ and $\omega_1 - \omega_2 = k_2 \eta_2$, where η_E is a fixed differential on E_ζ.

In particular

$$\frac{-x}{\sqrt{t}} = \frac{\omega_2}{\omega_1} = \frac{k_1 \eta_1 - k_2 \eta_2}{k_1 \eta_1 + k_2 \eta_2}.$$

One can solve for k_1/k_2 to get

$$\frac{k_1}{k_2} = -\frac{(x - \sqrt{t})\eta_2}{(x + \sqrt{t})\eta_1},$$

and then, using Lemma 4.8 and choosing $\eta_E = \mathrm{d}u/v$ in that model,

$$\frac{k_1}{k_2} = -\left(\frac{1 - \sqrt{t}}{1 + \sqrt{t}}\right)^{1/3} = -\mu.$$

Now, solving t in terms of μ gives

$$t = \left(\frac{\mu^3 - 1}{\mu^3 + 1}\right)^2,$$

and Lemma 4.3 implies the rest of the claims. $\qquad\square$

Note that every value of μ gives two eigendifferentials with single zeros on (generically) two different fibres of $\mathcal{Y}$, which are identified by six different isomorphisms.

Lemma 4.12. *For $t \neq -1$ we have that in $\mathbb{P}\Omega\mathcal{Y}(6)$*

$$(\mathcal{Y}_t, \omega_1) \cong (\mathcal{Y}_t, \omega_2) \cong (\mathcal{Y}_{1/t}, \omega_1) \cong (\mathcal{Y}_{1/t}, \omega_2)$$

as flat surfaces and $(\mathcal{Y}_t, \omega_1) \not\cong (X, \omega)$ for all other $(X, \omega) \in \mathbb{P}\Omega\mathcal{Y}(6)$.

Proof. This is clear by Proposition 4.4, Lemma 4.3 and the fact that β interchanges the classes of ω_1 and ω_2. $\qquad\square$

In particular, we do not have to distinguish between the classes of ω_1 and ω_2. This relationship becomes more explicit when expressed in the fibre product construction.

Proposition 4.13. *Let $(\mu \eta_E, \eta_E) \in \Omega E_\zeta \times \Omega E_\zeta$, $\mu \neq 0$ and let $P = (A, B) \in E_\zeta^*$. The corresponding α-eigendifferential $\mu \eta_1 + \eta_2$ on Y_P has a single zero at a fixed point of α if and only if $A = \mu^2$.*

In particular, this induces a 12-to-1 map

$$\mathbb{C}^* \to \mathbb{P}\,\Omega\mathcal{Y}(6)$$
$$\mu \mapsto [(Y_P, [\mu\,\eta_1 + \eta_2])],$$

where $[P] = (\mu^2, \pm\sqrt{\mu^6 - 1})$ *is an elliptic pair on* E_ζ.

Proof. For each $P \in E_\zeta^*$ we consider the differentials $\eta_1 = p_1^*\eta_E$ and $\eta_2 = p_2^*\eta_E$ on Y_P. The proof of this theorem will proceed in a similar way to the proof of Theorem 3.6 up until Step 3.

Step 1. The α-eigenforms in ΩY_P^- can have zeroes at most at one of the fixed points of α.

Otherwise, every differential in ΩY_P^- would vanish at both fixed points of α. In particular, so would η_1 and η_2, but the maps p_i are unramified at $\mathrm{Fix}(\alpha)$ and we know that η_E has no zeroes in E.

Again, zeroes of α-eigenforms must be permuted by α, the orbits of which have length 1, 2 or 6. This immediately implies that α-eigenforms lie either in $\Omega Y_P(1^6)$ if the zeroes are located at regular points, in $\Omega Y_P(6)$ if it only has zeroes at a fixed point of α, or in $\Omega Y_P(3^2)$ if it has zeroes at the two points of the orbit of length 2 (see the proof of Proposition 4.4). Again, we just need to prove that $\omega_1, \omega_2 \in \Omega Y_P(6)$.

Step 2. Again, $p_i^*\eta_E$ vanishes only at the six branch points of p_i. In particular both η_1 and η_2 lie in $\Omega Y_P(1^6)$.

Step 3. Let $E_\zeta \cong \left\{v^2 = (u^3 - 1)\right\}$ with the point at infinity as a distinguished point, and let $(\mu\,\eta_E, \eta_E) \in \Omega E_\zeta \times \Omega E_\zeta$. We claim that, given a point $P = (A, B)$ in these coordinates, the differential $\mu\,\eta_1 + \eta_2$ on Y_P has a single zero if and only if $A = \mu^2$.

Note that we can normalise $\varphi : E_\zeta \to \mathbb{P}^1$ to be $(u, v) \mapsto u$. By construction of Y_P as the fibre product of the maps $\varphi, \varphi_P : E_\zeta \to \mathbb{P}^1$, points in Y_P outside of the branch loci of the maps p_i can then be seen as pairs

$$Q = \left(\left(u, \varepsilon_1\sqrt{u^3 - 1}\right), \left(\frac{u}{A}, \varepsilon_2\sqrt{\frac{u^3 - A^3}{A^3}}\right) \right) \in E_\zeta \times E_\zeta,$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Normalising $\eta_E = \mathrm{d}x/y$, and evaluating locally around Q yields

$$\mu\,\eta_1 + \eta_2(Q) = \frac{\varepsilon_1\mu}{\sqrt{u^3 - 1}} + \frac{\varepsilon_2\sqrt{A^3}}{A\sqrt{u^3 - A^3}} = \frac{\varepsilon_1\mu A\sqrt{u^3 - A^3} + \varepsilon_2\sqrt{A^3(u^3 - 1)}}{A\sqrt{(u^3 - 1)(u^3 - A^3)}}.$$

Comparing again the addends in the numerator and taking squares, one sees that this differential vanishes at Q (for two choices $(\varepsilon_1, \varepsilon_2)$ and $(-\varepsilon_1, -\varepsilon_2)$) whenever

$$u^3 = A \cdot \frac{\mu^2 A^2 - 1}{\mu^2 - A}.$$

In particular, whenever the right-hand side is different from 0, 1 and ∞ one has that the differential $\mu\,\eta_1 + \eta_2$ necessarily has 6 simple zeroes. The case $u^3 = 1$ corresponds to $\mu = 0$, which has been treated in Step 2. The case $u = 0$ corresponds to $A = \pm 1/\mu$ and yields the differentials with zeroes at the two points of the α-orbit of length 2.

Finally, if $A = \mu^2$ the zeroes of the differential must be in $\mathrm{Fix}(\alpha)$, and Step 1 then implies that there is a single zero.

As a consequence, to each $\mu \in \mathbb{C}^*$ we can associate the elliptic pair of points

$$[P] = (\mu^2, \pm\sqrt{\mu^6 - 1}) \in E_\zeta^*,$$

defining the curve Y_P together with the differential with a single zero $\mu\,\eta_1 + \eta_2$. By Proposition 4.9 and the fact that $\pm\mu$ give the same elliptic pair and, by Lemma 4.12, the same class of flat surfaces, the association is 12-to-1. □

We are now finally in a position to prove the formula for $e_3(D)$.

Proof of Theorem 4.1. First, let D be a nonsquare discriminant and recall the order $\mathcal{O}_D = \mathbb{Z} \oplus T_D\mathbb{Z}$ associated to D, where

$$T_D = \begin{cases} \dfrac{\sqrt{D}}{2}, & D \equiv 0 \bmod 4, \\[2mm] \dfrac{\sqrt{D}+1}{2}, & D \equiv 1 \bmod 4. \end{cases}$$

Then, for $i = 1, 2$, $(\mathcal{Y}_t, \omega_i)$ lies on W_D if and only if $\mathcal{P}(\mathcal{Y}_t, \rho)$ admits real multiplication with ω_i as an eigenform. By Lemmas 4.10 and 4.11 this is equivalent to the existence of some self-adjoint matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_\zeta) \quad \text{such that} \quad A \cdot \begin{pmatrix} \pm\mu \\ 1 \end{pmatrix} = T \cdot \begin{pmatrix} \pm\mu \\ 1 \end{pmatrix}.$$

By Lemma 4.12, it suffices to consider $+\mu$. Moreover, by self-adjointness, we have $c = b^\sigma$, the Galois conjugate in $\mathcal{O}_\zeta$, and $a, d \in \mathbb{Z}$. The eigenform condition then yields

$$(a - T_D)\mu + b = 0 \quad \text{and} \quad b^\sigma\mu + d - T_D = 0.$$

The first equation gives

$$\mu = \frac{b}{T_D - a}$$

and substituting this into the second equation yields

$$bb^\sigma - ad = T_D^2 - (a+d)T_D.$$

First, we consider the case $D \equiv 0 \bmod 4$. Then this gives

$$bb^\sigma - ad = \frac{D}{4} - (a+d)\frac{\sqrt{D}}{2}.$$

As the right side of the equation must be an integer, we find $a = -d$ and hence

$$D = 4bb^\sigma + (2a)^2, \quad \text{for } D \equiv 0 \bmod 4.$$

Similarly, for $D \equiv 1 \bmod 4$, we obtain $d = a - 1$ and thus

$$D = 4bb^\sigma + (2a-1)^2, \quad \text{for } D \equiv 1 \bmod 4.$$

It is well known that the norm squared of an element in $\mathcal{O}_\zeta$ is given by

$$bb^\sigma = i^2 - ij + j^2 = \frac{3j^2 + (2i-j)^2}{4}, \quad \text{for } b = i + \zeta_6^2 j.$$

Hence, $\mathcal{P}(\mathcal{Y}_t, \rho)$ admits a real multiplication by $\mathcal{O}_D$ with ω_i as an eigenform for every $a, i, j \in \mathbb{Z}$ such that

$$a^2 + 3j^2 + (2i - j)^2 = D.$$

Clearly, this real multiplication is proper if and only if $\gcd(a, i, j) = 1$.

By Lemma 4.11 or equivalently Proposition 4.13, this gives 12 times the cardinality of points of order 3.

A similar analysis in the square discriminant case $D = f^2$ yields, with the same notation as above, $d = f - a$ and $bb^\sigma - a(f - a) = 0$. Multiplying by 4 and adding f^2 to both sides of the equation one gets

$$D = 4bb^\sigma + (2a - f)^2,$$

and the same argument as above proves the result. $\qquad\square$

5. Points of order 5

In this section we find the orbifold points of order 5 on the Teichmüller curves W_D.

Theorem 5.1. *The Teichmüller curve W_5 has one orbifold point of order 5. For any other discriminant, W_D has no orbifold points of order 5.*

Curves admitting an automorphism of order 10

By Proposition 2.1, flat surfaces (X, ω) parametrising a point of order 5 on W_D will correspond to cyclic covers of degree 10 of $\mathbb{P}^1$ ramified over three points with ramification order 5, 10 and 10. There are two such curves:

$$\mathcal{V} : \ y^{10} = x(x-1)^2, \quad \text{and} \quad \mathcal{U} : \ y^{10} = x(x-1)^8.$$

Calculations similar to the ones in the proof of Lemma 4.2 and Proposition 4.4 give us the differentials with a single zero on these curves.

Proposition 5.2. *The space $\Omega\mathcal{V}^-$ is generated by the α-eigenforms*

$$\omega_1 = \frac{y\,dx}{x(x-1)}, \quad \omega_2 = \frac{y^3\,dx}{x(x-1)}.$$

Up to scale, the only differential in $\Omega\mathcal{V}^-(6)$ is ω_1.

The space $\Omega\mathcal{U}^-$ is disjoint from the minimal stratum $\Omega\mathcal{M}_4(6)$.

In particular one has the following corollary.

Corollary 5.3. *Let X be a genus 4 curve admitting an automorphism α of order 10 with two fixed points and an orbit of length 2. If $(X, \omega) \in \Omega X^-(6)$ then $X = \mathcal{V}$ and, up to scale, $\omega = \omega_1$. In particular, ω is an α-eigenform.*
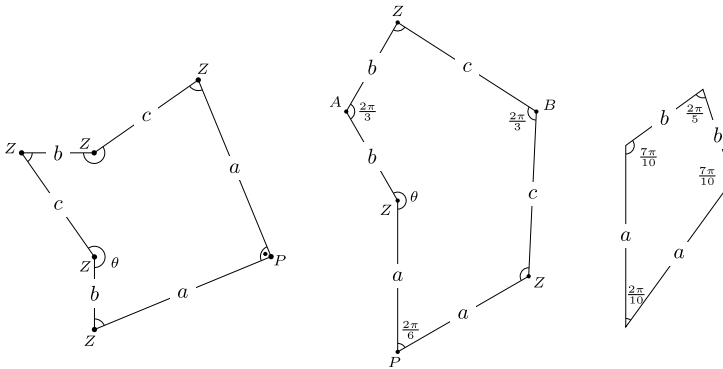
Figure 3. LEFT: A 4-differential of genus 1 with a zero and pole, each of order 3. The length of c and the angle θ give a complex parameter. CENTRE: A 6-differential of genus 0 with a single zero of order 1, a pole of order 5 and two poles of order 4 each. The length of b and the angle θ give a complex parameter. RIGHT: The unique (up to scale) 10-differential on $\mathbb{P}^1$ with poles of order 3, 8 and 9.

The action of α on $\mathcal{P}(\mathcal{V}, \alpha^5)$ induces an embedding $\mathbb{Q}(\zeta_{10}) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{P}(\mathcal{V}, \alpha^5))$ and, in particular, determines an element $T_5 = \alpha + \alpha^{-1} = (\sqrt{5} + 1)/2$ for which ω_1 is an eigenform.

Proof of Theorem 5.1. By the argument above and the maximality of $\mathcal{O}_5$, the Prym variety $\mathcal{P}(\mathcal{V}, \alpha^5)$ admits proper real multiplication by $\mathcal{O}_5$ with ω_1 as an eigenform.

Now Teichmüller curves ΩW_D and ΩW_E are disjoint for different discriminants D and E. Therefore, as ω_1 is, up to scale, the only differential with a single zero on $\mathcal{V}$, there can be no other W_D with a point of order 5. □

6. Flat geometry of orbifold points

In this section we describe, up to scale, the translation surfaces corresponding to the Turtle family $\mathcal{X}$, the Hurricane family $\mathcal{Y}$ and the curve $\mathcal{V}$. We use the notion of k-differentials and $(1/k)$-translation structures, cf. [2, §§ 2.1, 2.3].

Note that, whereas in the first two cases we have a 1-dimensional family of flat surfaces, in the case of $\mathcal{V}$ the construction is unique (cf. Corollary 5.3). The case of a flat surface with a symmetry of order twelve, also unique, is given by $\mathcal{X} \cap \mathcal{Y}$, the intersection of the Turtle family and the Hurricane family (cf. Theorem 3.6).

Points of order 2

We briefly describe the construction of flat surfaces $(\mathcal{X}_\kappa, \eta_\kappa) \in \Omega\mathcal{X}(6)$ (that is curves X_κ with a four-fold symmetry α together with a differential η_κ with a six-fold zero) in terms of a parameter $\kappa = \kappa(c, \theta)$.

By Proposition 2.1, the quotient X_κ/α is of genus 1 with two fixed points. Therefore, a 4-differential ξ of genus 1 with a zero and a pole, each of order 3, at the two fixed points will have (X_κ, η_κ) as a canonical cover, i.e. $\eta_\kappa^4 = \pi^*\xi$, cf. [2]. The polygon corresponding to ξ is given in Figure 3 with an angle of $2\pi/4$ at the pole and $7 \cdot 2\pi/4$ at the zero. Note
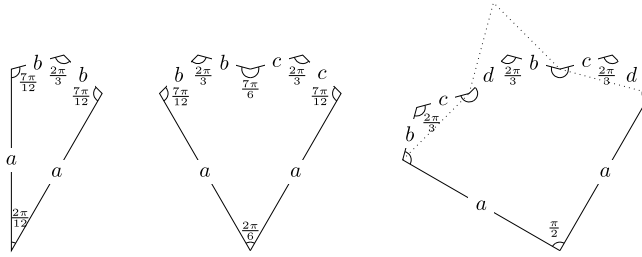
Figure 4. LEFT: A 12-differential of genus 0 with a pole of order 5 that pulls back to the zero, and poles of order 11 and 8. Note that this is unique up to scaling. CENTRE: By taking the canonical double cover, we obtain a 6-differential on $\mathbb{P}^1$ as in Figure 3. RIGHT: By taking a triple cover, we obtain a 4-differential on the elliptic curve E_ζ with an automorphism of order 6. If we cut and re-glue as indicated, we obtain a polygon as in Figure 3.

that the three pairs of sides are identified by translation and rotation by angle $\pi/2$ and that the side c can be chosen as a complex parameter (i.e. the length of c and the angle θ). The 'unfolded' canonical cover, resembling a turtle, is pictured in Figure 1 (§1).

Points of order 3

Similarly, we can construct flat surfaces $(Y_\tau, \eta_\tau) \in \Omega\mathcal{Y}(6)$ admitting a six-fold symmetry α and a six-fold zero in terms of a parameter $\tau = \tau(b, \theta)$.

By Proposition 2.1, the quotient X/α is of genus 0 with two fully ramified points and two points that are fully ramified over an intermediate cover of degree 3. For the flat picture, this implies that we have a zero with angle $7 \cdot 2\pi/6$, a pole with angle $2\pi/6$ and two poles with angles $2\pi/3$, see Figure 3 where the sides are identified by translation and rotation of multiplies of $2\pi/6$ to give a surface of genus 0. Equivalently, this is a 6-differential on $\mathbb{P}^1$ with a single zero of order 1, a pole of order 5 and two poles of order 4, admitting a canonical cover with only a single zero, cf. [2, §2]. The 'unfolded' canonical cover, resembling a hurricane, is pictured in Figure 2 (§1).

Point of order 6

By Theorem 3.1, there is a unique Prym differential (X, ω) with a symmetry of order 12 situated on $W_{12}(6)$.

By Proposition 2.1, we may picture this as a degree 12 cyclic cover of $\mathbb{P}^1$ with two points of order 12 and one point of order 3. Hence, by [2], (X, ω) is the canonical cover of a 12-differential on $\mathbb{P}^1$ with a pole of order 5 that pulls back to the zero, and poles of order 11 and 8 at the totally ramified point and the point of order 3, respectively. Equivalently, we may glue a quadrilateral with two angles of $7\pi/12$ each and angles of $2\pi/12$ and $2\pi/3$ to give a surface of genus 0, see Figure 4. By 'unfolding' once, i.e. taking the canonical 2-cover, we obtain the 6-differential on $\mathbb{P}^1$ that exhibits (X, ω) as a fibre in the Hurricane family (see Figure 4). Taking the canonical degree 3 cover, we can cut and re-glue as indicated in Figure 4 to obtain the 4-differential that is a C_{12}-eigenform on the elliptic curve with an automorphism of order 6 in the shape of the Turtle family (see Figure 3).
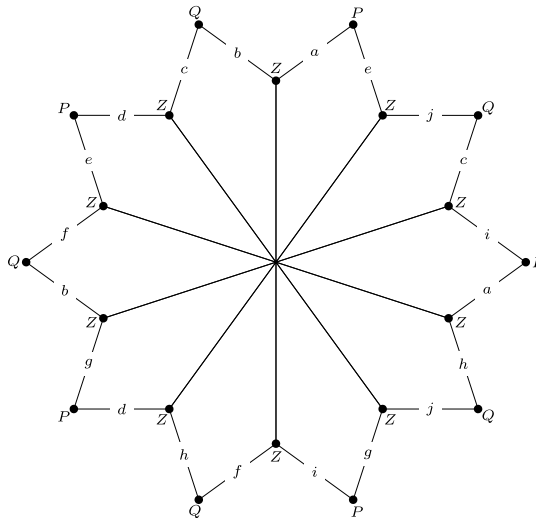
Figure 5. The canonical 10-cover of the 10-differential in Figure 3: a C_{10}-eigendifferential of genus 4 with a single zero Z.

Point of order 5

Finally, by Theorem 5.1 there is a unique point of order 5, i.e. an (X, ω) with a symmetry of order 10 and a six-fold-zero differential.

More precisely, X is a degree 10 cyclic cover of $\mathbb{P}^1$ ramified over three points, two of order 10 and one of order 5. Hence, (X, ω) is the canonical cover of a 10-differential ξ on $\mathbb{P}^1$ with a pole of order 3 (that pulls back to the single zero on X) and poles of order 9 and 8 at the second fixed point and the point of order 5 respectively, cf. [2, Proposition 2.4]. Equivalently, the flat picture has angles of size $2\pi/10$, $2\pi/5$ and two angles of size $7 \cdot 2\pi/10$ each, see Figure 3 where the sides are identified by translation and rotation of multiplies of $2\pi/10$ to give a surface of genus 0. Note that this differential is unique up to scaling. The 'unfolded' canonical cover is shown in Figure 5.

7. Genus asymptotics

The aim of this section is to describe the asymptotic behaviour of the genus $g(W_D)$ of W_D with respect to D.

As additional boundary components make the calculation of the Euler characteristic for $D = d^2$ more tedious (cf. [1, § 13]), we assume throughout this section that W_D is primitive, i.e. that D is not a square.

Theorem 7.1. *There exist constants* $C_1, C_2 > 0$, *independent of* D, *such that*

$$C_1 \cdot D^{3/2} < g(W_D) < C_2 \cdot D^{3/2}.$$

More precisely, we give the following explicit upper bound on the genus.

Proposition 7.2. *The genus of W_D satisfies $g(W_D) < 1 + D^{3/2} \cdot \frac{35}{48\pi^2}$.*

We also give an explicit lower bound.

Proposition 7.3. *The genus of W_D satisfies*

$$g(W_D) \geqslant \frac{3}{200} D^{3/2} - \frac{D}{6} - D^{3/4} - 150.$$

Corollary 7.4. *The only curves W_D with $g(W_D) = 0$ are the loci for $D \leqslant 20$.*

Proof. By Proposition 7.3, $g(W_D) > 0$ whenever $D > 1050$. The smaller values of D were checked by computer. $\qquad\square$

Recall that the genus of W_D, $g(W_D)$, is given by ($D > 12$)

$$g(W_D) = h_0(W_D) - \frac{\chi(W_D)}{2} - \frac{C(W_D)}{2} - \frac{e_2(W_D)}{4} - \frac{e_3(W_D)}{3},$$

where $\chi(W_D)$ is the (orbifold) Euler characteristic of W_D, $C(W_D)$ denotes the number of cusps and $e_d(W_D)$ the number of points of order d on W_D. Moreover, by [10], $h_0(W_D) = 1$ and by [17, Theorem 4.1],

$$\chi(W_D) = -7\chi(X_D),$$

where X_D is the Hilbert modular surface of discriminant D. Moreover, $\chi(X_D)$ was calculated, for fundamental D, by Siegel in terms of the Dedekind zeta function ζ_D of $\mathbb{Q}(\sqrt{D})$. For non-fundamental D we write $D = f^2 D_0$, where f is the conductor of D and $\left(\frac{D_0}{p}\right)$ for the Legendre symbol, if p is a prime. Furthermore, we set

$$F(D) = \prod_{p|f} \left(1 - \left(\frac{D_0}{p}\right) p^{-2}\right),$$

where the product runs over all prime divisors p of f, and thus have

$$\chi(X_D) = \chi(X_{f^2 D_0}) = 2f^3 \zeta_{D_0}(-1) F(D) = D^{3/2} \zeta_{D_0}(2) \frac{F(D)}{2\pi^4},$$

using the functional equation of ζ_{D_0}, cf. [1, §2.3]. Finally, using Euler products, we obtain the classical bounds

$$\zeta(2)^2 = \frac{\pi^4}{36} > \zeta_{D_0}(2) > \zeta(4) = \frac{\pi^4}{90}$$

and

$$\frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2} > F(D) > \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

We can now give an upper bound on $g(W_D)$.

Proof of Proposition 7.2. As $C(W_D), e_2(W_D), e_3(W_D) > 0$, these terms may be neglected yielding

$$g(W_D) \leqslant 1 + \frac{7}{2}\chi(X_D) < 1 + D^{3/2} \cdot \frac{35}{48\pi^2},$$

using the bounds given above. $\qquad\square$

Obtaining a lower bound is slightly more involved, as it involves bounding the number of cusps and orbifold points from above. In general, the cusps are hardest to control, but by [9, 13], we have

$$C(W_D(6)) = C(W_D(2)),$$

i.e. the Teichmüller curves of discriminant D in $\mathcal{M}_2$ and $\mathcal{M}_4$ have the same number of cusps. Moreover, denote by P_D the *product locus* in $\mathcal{A}_2$, i.e. abelian surfaces that are polarised products of elliptic curves. This is a union of modular curves and, again by [13],

$$C(W_D) = C(P_D).$$

To bound the cusps we may therefore proceed in complete analogy to [19, §6].

Lemma 7.5. *The cusps are bounded from above by*

$$\frac{C(W_D)}{2} \leqslant D^{3/4} + 150 + \frac{5}{4}\chi(X_D).$$

Proof. By [1, Theorem 2.22], $\chi(P_D) = -\frac{5}{2}\chi(X_D)$. Moreover, by [19, Proposition 6.5], the number of connected components of P_D can be bounded by

$$h_0(P_D) \leqslant D^{3/4} + 150.$$

Therefore, we may write

$$-\frac{C(W_D)}{2} = -\frac{C(P_D)}{2} = g(P_D) - h_0(P_D) + \frac{\chi(P_D)}{2} + \sum_d \left(1 - \frac{1}{d}\right) e_d(P_D)$$

$$\geqslant -h_0(P_D) + \frac{\chi(P_D)}{2} \geqslant -D^{3/4} - 150 - \frac{5}{4}\chi(X_D),$$

which yields the claim. $\qquad\square$

Next, we must bound the number of orbifold points.

Lemma 7.6. *The number of points of order* 2 *satisfies* $e_2(D) < \frac{D}{2}$.

Proof. By Theorem 3.1, we have $e_2(D) \leqslant h(-D) + h(-\frac{D}{4})$. Now, it is well known that class numbers of imaginary quadratic fields may be computed by counting reduced quadratic forms (cf. e.g. [7, §5.3]), giving $h(-D) < \frac{D}{3}$ and thus proving the claim. $\quad\square$

Lemma 7.7. *The number of points of order* 3 *satisfies* $e_3(D) < \frac{D}{6}$.

Proof. By Theorem 4.1,

$$e_3(D) \leqslant \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D\}/12.$$

The integers a and j essentially determine i. Moreover, a must have the same parity as D giving at most $\sqrt{D}/2$ choices (up to sign) and j ranges (again up to sign) over at most $\sqrt{D}$ possibilities. Accounting for sign choices and dividing by 12 yields the claim. $\quad\square$

TABLE 3. Topological invariants of the Teichmüller curves $W_D(6)$ for nonsquare discriminant. The number of cusps is described in [9], the Euler characteristic in [17].

D	g	χ	C	e_2	e_3	e_5	e_6	D	g	χ	C	e_2	e_3
5	0	−7/15	1	0	1	1		108	21	−63	21	4	0
8	0	−7/6	2	1	1			109	18	−63	25	0	6
12	0	−7/3	3	1	0		1	112	22	−224/3	29	2	4
13	0	−7/3	3	0	2			113	26	−84	32	0	3
17	0	−14/3	6	0	1			116	21	−70	25	6	3
20	0	−14/3	5	2	1			117	21	−56	16	0	0
21	1	−14/3	4	0	1			120	29	−238/3	20	4	2
24	1	−7	6	2	0			124	31	−280/3	29	6	2
28	1	−28/3	7	2	2			125	21	−175/3	15	0	5
29	1	−7	5	0	3			128	25	−224/3	22	4	4
32	1	−28/3	7	2	2			129	37	−350/3	44	0	1
33	2	−14	12	0	0			132	29	−84	26	4	0
37	1	−35/3	9	0	4			133	27	−238/3	22	0	8
40	2	−49/3	12	2	2			136	35	−322/3	36	4	2
41	3	−56/3	14	0	1			137	37	−112	38	0	3
44	3	−49/3	9	4	2			140	33	−266/3	18	8	4
45	4	−14	8	0	0			141	34	−84	18	0	0
48	4	−56/3	11	2	1			145	46	−448/3	58	0	2
52	4	−70/3	15	2	2			148	39	−350/3	37	2	4
53	4	−49/3	7	0	5			149	30	−245/3	19	0	7
56	6	−70/3	10	4	2			152	37	−287/3	18	6	4
57	7	−98/3	20	0	1			153	45	−140	52	0	0
60	8	−28	12	4	0			156	46	−364/3	26	8	2
61	6	−77/3	13	0	4			157	36	−301/3	25	0	8
65	8	−112/3	22	0	2			160	44	−392/3	40	4	4
68	6	−28	14	4	3			161	55	−448/3	40	0	2
69	10	−28	10	0	0			164	37	−112	34	8	3
72	10	−35	16	2	0			165	42	−308/3	18	0	4
73	10	−154/3	32	0	2			168	51	−126	24	4	0
76	11	−133/3	21	4	2			172	53	−147	37	4	6
77	9	−28	8	0	6			173	37	−91	13	0	9
80	10	−112/3	16	4	2			176	49	−392/3	29	6	4
84	14	−140/3	18	4	1			177	66	−182	52	0	0
85	12	−42	16	0	6			180	52	−140	36	4	0
88	15	−161/3	22	2	4			181	49	−133	33	0	6
89	17	−182/3	28	0	1			184	66	−518/3	38	4	4
92	15	−140/3	13	6	4			185	66	−532/3	46	0	2
93	15	−42	12	0	3			188	53	−392/3	19	10	4
96	18	−56	20	4	0			189	51	−126	26	0	0
97	21	−238/3	38	0	2			192	57	−448/3	34	4	2
101	14	−133/3	15	0	5			193	77	−686/3	74	0	4
104	18	−175/3	20	6	2			197	44	−343/3	21	0	11
105	27	−84	32	0	0			200	56	−455/3	36	6	4

Remark. The bound from Lemma 7.7 can be improved. Indeed, by the theory of modular forms of half-integral weight, integral solutions of positive definite quadratic forms can always be realised as coefficients of a suitable modular form, see [23] or e.g. [11] for the concrete case at hand. In particular, the integral solutions of $a^2 + 3j^2 + (2i - j)^2$ are coefficients of a modular form of weight 3/2, level 12 and Kronecker character 12. Hence,

$$e_3(D) < C \cdot D^{3/4}$$

for some constant C that is independent of D (cf. e.g. [18, Theorem 2.1] for growth rates of coefficients of modular forms).

This permits us to also give a lower bound for $g(W_D)$, proving Theorem 7.1.

Proof of Proposition 7.3. By the above bounds, we have

$$g(W_D) = h_0(W_D) - \frac{\chi(W_D)}{2} - \frac{C(W_D)}{2} - \frac{e_2(W_D)}{4} - \frac{e_3(W_D)}{3}$$
$$> \frac{9}{4}\chi(X_D) - \frac{D}{6} - D^{3/4} - 150,$$

which yields the claim by the above bounds on $\chi(X_D)$. $\qquad\square$

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