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Estimates for the energy of the solutions to elliptic Dirichlet problems on convex domains

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We provide an estimate of the energy of the solutions to the Poisson equation with constant data and Dirichlet boundary conditions in a convex domain $\Omega \subset \mathbb{R}^n$. This estimate is obtained by restricting the variational formulation of the problem to the space of functions depending only on the distance from the boundary of Ω. The main tool in the proof is an isoperimetric inequality for convex domains, which is a consequence of the Brunn–Minkowski theorem.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a convex domain (i.e. a non-empty open bounded convex set) and let $u_\Omega \in H_0^1(\Omega)$ be the solution of the elliptic Dirichlet problem

$$\left.\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

that is, the unique function $u_\Omega \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_\Omega \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}x \quad \forall \varphi \in H_0^1(\Omega). \tag{1.2}$$

We recall that u_Ω is also the unique solution to the minimum problem

$$\min_{u \in H_0^1(\Omega)} J(u;\Omega), \qquad J(u;\Omega) := \int_\Omega (\tfrac{1}{2}|\nabla u|^2 - u)\,\mathrm{d}x. \tag{1.3}$$

Choosing $\varphi = u_\Omega$ in (1.2), we have

$$\int_\Omega |\nabla u_\Omega|^2 \,\mathrm{d}x = \int_\Omega u_\Omega \,\mathrm{d}x,$$

and hence the minimum value of $J(\cdot;\Omega)$ on $H_0^1(\Omega)$ satisfies

$$J(u_\Omega;\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 \,\mathrm{d}x = -\frac{1}{2}\int_\Omega u_\Omega \,\mathrm{d}x. \tag{1.4}$$

The aim of this paper is to provide estimates for the energy $J(u_\Omega;\Omega)$ of u_Ω by restricting the variational formulation (1.3) of (1.1) to the class of the functions that depend only on the distance from $\partial\Omega$.

Before describing in more detail our approach, it will be of interest to discuss the physical motivation for this problem (see [2,13,18,22]). Consider a long cylindrical beam of uniform cross-section $\Omega \subset \mathbb{R}^2$. The state of stress in the interior of the beam is determined by a warping function $u_\Omega(x)$, $x \in \Omega$, satisfying (1.1). The torsional rigidity of the beam is the torque required for unit angle of twist per unit length when the shear modulus is equal to one, and can be expressed by the Dirichlet integral $\int_\Omega |\nabla u_\Omega|^2 \, dx$. We stress that its exact value can be computed explicitly only in few particular cases (for example, when Ω is a ball). Therefore, in order to have an estimate for the torsional rigidity of the beam, it is of interest to provide an estimate for $J(u_\Omega; \Omega)$.

As mentioned above, this estimate will be obtained by considering the following minimization problem,

$$\min_{u \in \mathcal{W}(\Omega)} J(u; \Omega), \tag{1.5}$$

where $\mathcal{W}(\Omega)$ is the class of *web functions*, that is, the functions $u \in H_0^1(\Omega)$ that depend only on the distance from $\partial\Omega$. This kind of approximation was proposed by Makai and Pólya [14,17] in the bidimensional case in order to obtain a lower bound for the Dirichlet integral. Notice that, when Ω is a ball centred at the origin, then $\mathcal{W}(\Omega)$ is the subset of radially symmetric functions. The existence and uniqueness of solutions to minimization problems in $\mathcal{W}(\Omega)$ for more general functionals than J was proved in [5,11]. The relative error one makes approximating (1.3) with (1.5) can be expressed by the ratio

$$\mathcal{E}(\Omega) := \frac{\min_{u \in \mathcal{W}(\Omega)} J(u; \Omega)}{\min_{u \in H_0^1(\Omega)} J(u; \Omega)}. \tag{1.6}$$

We remark that this ratio is well defined, since $J(u_\Omega; \Omega) < 0$, and $0 < \mathcal{E}(\Omega) \leqslant 1$ for every convex domain Ω, since $\mathcal{W}(\Omega) \subset H_0^1(\Omega)$. Moreover, $\mathcal{E}(\Omega) = 1$ if and only if Ω is a ball (see proposition 4.1 in § 4). It is also worth remarking that $\mathcal{E}$ is invariant under dilations, that is, $\mathcal{E}(\lambda\Omega) = \mathcal{E}(\Omega)$ for every $\lambda > 0$.

Our main result is the following sharp lower bound for $\mathcal{E}$.

THEOREM 1.1. *For every convex domain $\Omega \subset \mathbb{R}^n$, we have*

$$\mathcal{E}(\Omega) > \inf \mathcal{E} = \frac{n+1}{2n},$$

where the infimum is taken over all convex domains of $\mathbb{R}^n$.

We remark that the infimum of $\mathcal{E}$ is not attained. Some remarks about minimizing sequences are addressed at the end of § 4.

In previous papers [6,7], some estimates from below of $\mathcal{E}$ were proved for more general functionals. The bidimensional case of theorem 1.1 was considered in [8]. The main tool for extending the result for $n > 2$ is a new isoperimetric inequality, proved in § 3. This inequality is a consequence of the Brunn–Minkowski theorem on the volume of concave families of convex bodies (see, for example, [4,20]). It extends the well-known classical isoperimetric inequality and, in our setting, previous known results such as Diskant's inequality [9]. We remark that only theorem 3.1 and corollary 3.3 in § 3 are necessary for the proof of theorem 1.1, but we believe that the more general results proved in that section are of interest by themselves.

Section 4 is devoted to the proof of theorem 1.1. In that section we prove estimates from above for $\min_{\mathcal{W}(\Omega)} J(\cdot;\Omega)$ and from below for $\min_{H_0^1(\Omega)} J(\cdot;\Omega)$ in terms of the same comparing functional $\Lambda(\Omega)$ defined in (4.2) below. Combining these two estimates, we obtain the lower bound for $\mathcal{E}$ given in theorem 1.1. Some extra work is required in order to show that this lower bound is not attained. Finally, we exhibit a minimizing sequence, completing the proof of theorem 1.1.

For the sake of completeness, in the appendix we give a short self-contained proof of a known result concerning the convergence of solutions to the Dirichlet problem (1.1) in varying domains.

2. Notation

The standard scalar product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$, and $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$.

A non-empty compact convex subset of $\mathbb{R}^n$ is called a *convex body*. By $\mathcal{K}^n$ we denote the class of all convex bodies in $\mathbb{R}^n$, and by $\mathcal{K}_0^n$ the subset of convex bodies with interior points. For every $\Omega \in \mathcal{K}^n$, we denote by r_Ω its *inradius*, namely, the supremum of the radii of the open balls contained in Ω. Moreover, we denote by (Ω_t), $t \in [0, r_\Omega]$, the family of *inner parallel bodies* of Ω, namely,

$$\Omega_t := \{x \in \Omega;\ d(x, \partial\Omega) \geqslant t\},$$

where $d(x, \partial\Omega)$ denotes the distance of a point $x \in \Omega$ from the boundary of Ω. It will be convenient to extend $d(\cdot, \partial\Omega)$ to all $\mathbb{R}^n$ by setting

$$d(x, \partial\Omega) := \begin{cases} \min\limits_{y \in \partial\Omega} |x - y| & \text{if } x \in \Omega, \\ -\min\limits_{y \in \partial\Omega} |x - y| & \text{if } x \notin \Omega. \end{cases} \tag{2.1}$$

We recall that $\mathcal{K}^n$, equipped with the *Hausdorff metric*

$$d_H(K, L) := \max\Big\{ \sup_{x \in K} \inf_{y \in L} |x - y|, \sup_{x \in L} \inf_{y \in K} |x - y| \Big\}, \quad K, L \in \mathcal{K}^n,$$

is a complete metric space (see [20, theorems 1.8.2, 1.8.5]). In the following, all metrical and topological notions occurring in connection with $\mathcal{K}^n$ are tacitly understood to refer to the Hausdorff metric and the topology induced by it.

A convex body $\Omega \in \mathcal{K}_0^n$ is said to be a *tangential body* of an n-dimensional ball B, if, through each boundary point of Ω, there exists a support plane to Ω that also supports B. (In other words, Ω is circumscribed to B.)

Recall that a *polytope* is the closed convex hull of a finite number of points. We denote by $\mathcal{P}_0^n \subset \mathcal{K}_0^n$ the class of all polytopes in $\mathbb{R}^n$ with non-empty interior.

The Lebesgue measure and the k-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ will be denoted, respectively, by $|A|$ and $\mathcal{H}^k(A)$, $0 \leqslant k \leqslant n$.

As is customary, $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, and $H_0^1(\Omega)$ will denote the Lebesgue and Sobolev spaces of functions defined in a set $\Omega \in \mathcal{K}_0^n$. The usual norms in these spaces will be denoted, respectively, by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H_0^1(\Omega)}$.

By B^n we denote the n-dimensional unit ball centred at the origin, and by $S^{n-1} = \partial B^n$ the unit sphere of $\mathbb{R}^n$. Moreover, we set $B_\rho^n := \rho B^n$ for every $\rho > 0$.

The n-dimensional measure of B^n is denoted by κ_n, and its surface area by ω_n. Thus

$$\kappa_n = |B^n| = \frac{\pi^{n/2}}{\Gamma(1+\frac{1}{2}n)}, \qquad \omega_n = \mathcal{H}^{n-1}(\partial B^n) = n\kappa_n = \frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)}.$$

With some abuse of notation, we shall denote by $|\partial\Omega|$ the surface area of a set $\Omega \in \mathcal{K}_0^n$, that is, $|\partial\Omega| \equiv \mathcal{H}^{n-1}(\partial\Omega)$.

3. The isoperimetric inequality

The aim of this section is to prove an isoperimetric inequality that will be used in the proof of proposition 4.2 in the next section. This inequality turns out to be an improvement of the classical isoperimetric inequality, which states that the volume $|\Omega|$ and the surface area $|\partial\Omega|$ of a set $\Omega \in \mathcal{K}^n$ are related in the following way:

$$\left(\frac{|\Omega|}{\kappa_n}\right)^{n-1} \leqslant \left(\frac{|\partial\Omega|}{\omega_n}\right)^{n}. \tag{3.1}$$

The equality sign holds if and only if Ω is a ball.

We recall that, if $\Omega \in \mathcal{K}_0^n$, then, by the Brunn–Minkowski theorem, the map

$$\gamma(t) := \sqrt[n-1]{|\partial\Omega_t|}, \quad t \in [0, r_\Omega], \tag{3.2}$$

is concave in $[0, r_\Omega]$ (see, for example, [4, §§ 24, 55] and [20, theorem 6.4.3]). This implies that, for every $t \in\,]0, r_\Omega[$, there exist the left and right derivatives $\gamma'_\pm(t)$ of γ at t (see [19, § 23]). Moreover, since γ is positive and strictly monotone decreasing in $[0, r_\Omega[$, there also exists the right derivative of γ at 0 (and it is negative). Throughout this section, for every $\Omega \in \mathcal{K}_0^n$, we shall use the notation

$$C_\Omega := -\gamma'_+(0) = \lim_{t\to 0+} \frac{\sqrt[n-1]{|\partial\Omega|} - \sqrt[n-1]{|\partial\Omega_t|}}{t}. \tag{3.3}$$

We now state the isoperimetric inequalities needed for the proof of theorem 1.1, which will be obtained as a consequence of the more general inequality proved in theorem 3.4 below.

THEOREM 3.1 (isoperimetric inequality). *For every $\Omega \in \mathcal{K}_0^n$, we have*

$$|\Omega| \leqslant \frac{1}{nC_\Omega}|\partial\Omega|^{n/(n-1)}, \tag{3.4}$$

where $C_\Omega \geqslant \sqrt[n-1]{\omega_n}$ is the constant defined in (3.3). Equality in (3.4) holds if and only if Ω is a tangential body of an n-dimensional ball.

REMARK 3.2. Since $C_\Omega \geqslant \sqrt[n-1]{\omega_n}$, it is clear that (3.1) is a particular case of (3.4).

COROLLARY 3.3. *Let $\Omega \in \mathcal{K}_0^n$ and assume that the map $t \mapsto |\partial\Omega_t|$ is differentiable at some point $t \in [0, r_\Omega[$. Then*

$$-\frac{\mathrm{d}}{\mathrm{d}t}|\partial\Omega_t| \leqslant \frac{n-1}{n}\frac{|\partial\Omega_t|^2}{|\Omega_t|}.$$

Proof. We apply the isoperimetric inequality (3.4) to the convex body Ω_t. Since

$$C_{\Omega_t} = -\gamma'(t) = -\frac{1}{n-1}|\partial\Omega_t|^{(2-n)/(n-1)}\frac{\mathrm{d}}{\mathrm{d}t}|\partial\Omega_t|,$$

from (3.4) we deduce that

$$-\frac{\mathrm{d}}{\mathrm{d}t}|\partial\Omega_t| \leqslant \frac{1}{n|\Omega_t|}|\partial\Omega_t|^{n/(n-1)}\frac{n-1}{|\partial\Omega_t|^{(2-n)/(n-1)}} = \frac{n-1}{n}\frac{|\partial\Omega_t|^2}{|\Omega_t|},$$

concluding the proof. □

We now prove the main theorem of this section; its consequences will be exploited in the subsequent corollaries.

THEOREM 3.4. *For every $\Omega \in \mathcal{K}_0^n$, we have*

$$|\partial\Omega|^{n/(n-1)} - nC_\Omega|\Omega| \geqslant [|\partial\Omega|^{1/(n-1)} - r_\Omega C_\Omega]^n, \tag{3.5}$$

where the constant C_Ω, defined in (3.3), satisfies the bound $C_\Omega \geqslant \sqrt[n-1]{\omega_n}$. Moreover, the right-hand side in (3.5) is non-negative.

Proof. Let $\Omega \in \mathcal{K}_0^n$. Since the map γ, defined in (3.2), is concave in $[0, r_\Omega]$, we have that $\gamma(t) \leqslant \gamma(0) - C_\Omega t$ for every $t \in [0, r_\Omega]$. Then

$$|\Omega| = \int_0^{r_\Omega} |\partial\Omega_t|\,\mathrm{d}t = \int_0^{r_\Omega} \gamma^{n-1}(t)\,\mathrm{d}t \leqslant \frac{\gamma(0)^n - [\gamma(0) - C_\Omega r_\Omega]^n}{nC_\Omega}, \tag{3.6}$$

and (3.5) follows by substituting $\gamma(0) = \sqrt[n-1]{|\partial\Omega|}$. We remark that

$$\gamma(0) - C_\Omega r_\Omega \geqslant \gamma(r_\Omega) \geqslant 0,$$

and hence the right-hand side in (3.5) is non-negative.

Let us prove the inequality $C_\Omega \geqslant \sqrt[n-1]{\omega_n}$. Let $\Omega^\rho = \Omega + \rho B^n$, $\rho > 0$, be the family of outer parallel bodies of Ω. Let us fix $\rho > 0$. From the concavity of the map $t \mapsto \sqrt[n-1]{|\partial\Omega_t^\rho|}$, $t \in [0, \rho + r_\Omega]$, and the fact that its right derivative at $t = \rho$ is $-C_\Omega$, we deduce that

$$|\partial\Omega^\rho| \leqslant [\sqrt[n-1]{|\partial\Omega|} + \rho C_\Omega]^{n-1}.$$

On the other hand, $|\Omega^\rho| \geqslant |\rho B^n| = \kappa_n \rho^n$, and hence, using (3.1), we conclude that

$$\frac{\kappa_n^{n-1}}{\omega_n^n} \geqslant \limsup_{\rho\to+\infty} \frac{|\Omega^\rho|^{n-1}}{|\partial\Omega^\rho|^n} \geqslant \frac{\kappa_n^{n-1}}{C_\Omega^{n(n-1)}},$$

which gives the required inequality. □

REMARK 3.5. The right-hand side in (3.5) gives an estimate of the so-called *isoperimetric deficit*.

REMARK 3.6. A sharper estimate on the constant C_Ω will be proved in lemma 3.8.

The following lemma is needed in order to discuss the equality case in theorem 3.1.

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LEMMA 3.7. *The condition*

$$|\partial\Omega_t| = |\partial\Omega|\left(1 - \frac{t}{r_\Omega}\right)^{n-1} \quad \forall t \in [0, r_\Omega] \tag{3.7}$$

holds if and only if, for every $t \in [0, r_\Omega]$*,* Ω_t *is homothetic to* Ω *and it is a tangential body of an* n*-dimensional ball. In particular, up to a translation,* $\Omega_t = (1 - t/r_\Omega)\Omega$ *for every* $t \in [0, r_\Omega]$.

Proof. From the Brunn–Minkowski theorem, condition (3.7) holds if and only if every Ω_t is homothetic to Ω. By lemma 3.1.10 in [20], this last condition is satisfied if and only if every Ω_t is a tangential body of an n-dimensional ball. □

Proof of theorem 3.1. Inequality (3.4) follows directly from the fact that the right-hand side in (3.5) is non-negative. From (3.6), it is also clear that equality in (3.4) holds if and only if $\gamma(t) = \gamma(0)(1 - t/r_\Omega)$, that is, if and only if (3.7) holds. By lemma 3.7, this last condition is satisfied if and only if Ω is a tangential body of an n-dimensional ball. □

In order to state the consequences of theorem 3.4, a further piece of notation is needed. For every $\Omega \in \mathcal{K}^n$ and every $u \in S^{n-1}$, denote by $H^-(K, u)$ the supporting half-space of Ω with exterior normal u. Let $\Omega \in \mathcal{K}^n_0$, and let S be the smallest closed set contained in S^{n-1} such that Ω is determined by S, that is,

$$\Omega = \bigcap_{u \in S} H^-(\Omega, u). \tag{3.8}$$

Then the *form body* of Ω is defined by

$$\Omega_* := \bigcap_{u \in S} H^-(B^n, u). \tag{3.9}$$

We are now in a position to prove a sharper estimate on the constant C_Ω. We remark that this estimate can be quickly proved using the tools of convex geometry (mixed volumes and mixed areas (see, for example, [4, 20])). However, we prefer to give a simple proof, which can be followed without knowing this machinery, and which uses only a formula for computing the measure of a polytope (see (3.11) below).

LEMMA 3.8. *Let* $\Omega \in \mathcal{K}^n_0$ *and let* Ω_* *be its form body. Then* $C_\Omega \geqslant C_{\Omega_*} \geqslant \sqrt[n-1]{\omega_n}$.

Proof. The inequality $C_{\Omega_*} \geqslant \sqrt[n-1]{\omega_n}$ follows from theorem 3.4. Since C_Ω is invariant under homotheties, without loss of generality, we can assume $r_\Omega = 1$. As a second reduction, by an approximation argument, it is enough to consider $\Omega \in \mathcal{P}^n_0$. In this case, the set S appearing in (3.8) and (3.9) is finite, say, $S = \{u_1, \dots, u_m\}$. The sets Ω_t and $(\Omega_*)_t$, for t small enough, say, $t \leqslant t_0 \in \,]0, 1]$, can be represented in the following way:

$$\Omega_t = \bigcap_{i=1}^m H^-_{u_i, h_i - t}, \quad (\Omega_*)_t = \bigcap_{i=1}^m H^-_{u_i, 1-t}, \quad t \in [0, t_0]$$

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(see [20, corollary 2.4.4]). Here,

$$H^-_{u,\alpha} := \{x \in \mathbb{R}^n;\ x \cdot u \leqslant \alpha\} \tag{3.10}$$

and $h_1, \dots, h_m \in \mathbb{R}$ are defined in such a way that $H^-_{u_i,h_i} = H^-(\Omega, u_i)$ for every $i = 1, \dots, m$. Since $r_\Omega = 1$, it is clear that $\min_j h_j = 1$. From lemma 5.1.2 in [20], we have that there exist symmetric coefficients $(a_{j_1,\dots,j_n})$, $j_1, \dots, j_n \in \{1, \dots, m\}$, such that, for every $t \in [0, t_0]$,

$$|\Omega_t| = \sum a_{j_1,\dots,j_n}(h_{j_1} - t)\dots(h_{j_n} - t), \qquad |(\Omega_*)_t| = \sum a_{j_1,\dots,j_n}(1-t)^n. \tag{3.11}$$

Computing the second derivative of the maps $t \mapsto |\Omega_t|$ and $t \mapsto |(\Omega_*)_t|$ in $t = 0$, we get

$$C_\Omega = \sum a_{j_1,\dots,j_n} \sum_{\substack{h,k \\ h\neq k}} \prod_{r\neq h,k} h_{j_r}, \qquad C_{\Omega_*} = n(n-1)\sum a_{j_1,\dots,j_n}.$$

Since $\min h_i = 1$, we have that

$$\sum_{\substack{h,k \\ h\neq k}} \prod_{r\neq h,k} h_{j_r} \geqslant n(n-1),$$

and hence the proof is complete. □

The following corollary of theorem 3.4 is a result already known in the literature, due to Aleksandrov [1] and extended by Diskant [9] (also see [20, p. 321]).

Corollary 3.9. *Let $\Omega \in \mathcal{K}_0^n$ and let Ω_* be its form body. Then*

$$\left(\frac{|\partial\Omega|}{n}\right)^{n/(n-1)} - |\Omega||\Omega_*|^{1/(n-1)} \geqslant \left[\left(\frac{|\partial\Omega|}{n}\right)^{1/(n-1)} - r_\Omega|\Omega_*|^{1/(n-1)}\right]^n, \tag{3.12}$$

where the right-hand side is non-negative. In particular,

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geqslant \frac{|\partial\Omega_*|^n}{|\Omega_*|^{n-1}} = n^n|\Omega_*|, \tag{3.13}$$

with equality if and only if Ω is homothetic to Ω_*.

Proof. Let us consider the function

$$g(s) = |\partial\Omega|^{n/(n-1)} - ns|\Omega| - [|\partial\Omega|^{1/(n-1)} - sr_\Omega]^n, \quad s \geqslant 0.$$

We have that $g(0) = 0$ and $g(C_\Omega) \geqslant 0$, by (3.5). Since, by lemma 3.8, $C_{\Omega_*} \leqslant C_\Omega$, in order to prove (3.12) it is enough to show that $g(s) \geqslant 0$ for every $s \in [0, C_\Omega]$. Let

$$s_0 := \frac{\sqrt[n-1]{|\partial\Omega|}}{r_\Omega} - \frac{\sqrt[n-1]{|\Omega|}}{r_\Omega^{n/(n-1)}}, \qquad s_1 := \frac{\sqrt[n-1]{|\partial\Omega|}}{r_\Omega} + \frac{\sqrt[n-1]{|\Omega|}}{r_\Omega^{n/(n-1)}}.$$

Computing the first derivative of g, it is easily seen that g is monotone increasing for $s \leqslant s_0$. If n is even, then g is monotone decreasing for $s \geqslant s_0$, whereas if n is odd, then g is monotone decreasing for $s_0 \leqslant s \leqslant s_1$ and again monotone increasing for $s \geqslant s_1$. Since $C_\Omega \leqslant \sqrt[n-1]{|\partial\Omega|}/r_\Omega$, we have that $s_1 > C_\Omega$, and hence, in both cases

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(n even or n odd), we conclude that either g is monotone increasing in $[0, C_\Omega]$ or it has one maximum point s_0 in that interval. Thus $g(s) \geqslant 0$ for every $s \in [0, C_\Omega]$, and (3.12) is proved.

Inequality (3.13) now follows from (3.12). However, in order to discuss the equality case, we provide a self-contained proof of it. Since the left-hand side in (3.13) is invariant under homotheties, it is not restrictive to assume $r_\Omega = 1$. Let

$$\gamma_*(t) := \sqrt[n-1]{|\partial(\Omega_*)_t|}, \quad t \in [0, 1].$$

Since Ω_* is a tangential body of an n-dimensional ball, we have $\gamma_*(t) = C_{\Omega_*}(1 - t)$, $t \in [0, 1]$, so that

$$|\Omega_*| = \frac{C_{\Omega_*}^{n-1}}{n}, \qquad |\partial\Omega_*| = C_{\Omega_*}^{n-1}. \tag{3.14}$$

Finally, from (3.4), (3.14) and lemma 3.8, we deduce that

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geqslant (nC_\Omega)^{n-1} \geqslant (nC_{\Omega_*})^{n-1} = \frac{|\partial\Omega_*|^n}{|\Omega_*|^{n-1}} = n^n|\Omega_*|, \tag{3.15}$$

with equality if and only if Ω is a tangential body of an n-dimensional ball, that is, if and only if Ω is homothetic to Ω_*. We remark that, since Ω_* contains a unit ball, from (3.15), we deduce that $C_{\Omega_*} \geqslant \sqrt[n-1]{\omega_n}$, with equality if and only if Ω_* is a ball. □

Remark 3.10. The case of polytopes is described in detail in [10]. In particular, inequality (3.13) was proved in [10, theorem 34] when $\Omega \in \mathcal{P}_0^n$. We recall that the form polytope of Ω is the convex polytope circumscribed about a unit ball which has the same number of $(n-1)$-dimensional faces and the same set of outer normals to these faces.

Example 3.11. If $\Omega \subset \mathbb{R}^2$ is a convex polygon, then inequality (3.4), which is equivalent to (3.13), becomes

$$|\Omega| \leqslant \frac{|\partial\Omega|^2}{2C_\Omega}, \tag{3.16}$$

where $C_\Omega = 2\sum_{i=1}^N \operatorname{cotan}\frac{1}{2}\theta_i$, and $\theta_1, \dots, \theta_N$ denote the inner angles of the polygon. We have that

$$C_\Omega \geqslant 2N \operatorname{cotan}\frac{\pi(N-2)}{2N} = 2N\tan\frac{\pi}{N} > 2\pi,$$

and hence inequality (3.16) is stronger than the classical isoperimetric inequality $|\Omega| \leqslant |\partial\Omega|^2/4\pi$ (see also [2, § 1.1]). Inequality (3.16) was proved by Lhuilier in 1782 (see, for example, [3, 16]).

4. Proof of theorem 1.1

We first prove a result that will be useful in the sequel (see [8, proposition 1]).

Proposition 4.1. *Let $\Omega \in \mathcal{K}_0^n$. Then $\mathcal{E}(\Omega) = 1$ if and only if Ω is a ball.*

Proof. Let $u_\Omega \in H_0^1(\Omega)$ be the solution to (1.3). If Ω is a ball, then u_Ω is radially symmetric, and hence $\mathcal{E}(\Omega) = 1$. Conversely, let us assume that $\mathcal{E}(\Omega) = 1$. This means that $u_\Omega \in \mathcal{W}(\Omega)$, that is, $u_\Omega(x) = \phi(d(x, \partial\Omega))$, with

$$\phi(t) := \int_0^t \frac{|\Omega_s|}{|\partial\Omega_s|}\,\mathrm{d}s, \quad t \in [0, r_\Omega] \tag{4.1}$$

(see [5, theorem 3.1]). Then, on $\partial\Omega$, the normal derivative of u_Ω is given by

$$\frac{\partial u_\Omega}{\partial n} = \phi'(d)\nabla d \cdot n = -\phi'(0).$$

Now, from a result of Serrin [21] (which is valid only if $\partial\Omega$ is of class C^2, but can be extended to convex domains thanks to the simplified proof given in [23]), it follows that Ω must be a ball. □

In the following, a major role will be played by the functional

$$\Lambda(\Omega) := \int_0^{r_\Omega} \left\{ \int_t^{r_\Omega} |\Omega_s|\,\mathrm{d}s \right\} \mathrm{d}t, \quad \Omega \in \mathcal{K}^n. \tag{4.2}$$

By the continuity of the maps $\Omega \mapsto r_\Omega$ and $\Omega \mapsto |\Omega|$ with respect to the Hausdorff metric, we also deduce that $\Omega \mapsto \Lambda(\Omega)$ is a continuous functional on $\mathcal{K}^n$.

In order to deal with positive quantities and make the inequalities more readable, we shall use the notation $\mathcal{E}(\Omega) = \mathcal{N}(\Omega)/\mathcal{D}(\Omega)$, where

$$\mathcal{N}(\Omega) := -2 \min_{u \in \mathcal{W}(\Omega)} J(u; \Omega), \tag{4.3}$$

$$\mathcal{D}(\Omega) := -2 \min_{u \in H_0^1(\Omega)} J(u; \Omega) = -2J(u_\Omega; \Omega), \tag{4.4}$$

in such a way that $0 < \mathcal{N}(\Omega) \leqslant \mathcal{D}(\Omega)$. We remark that, from theorem 6.1 in [5] and proposition A.1 in the appendix, the functionals $\Omega \mapsto \mathcal{N}(\Omega)$ and $\Omega \mapsto \mathcal{D}(\Omega)$ are both continuous in $\mathcal{K}^n$.

The proof of theorem 1.1 will be achieved in four steps.

(1) Give a lower bound for $\mathcal{N}(\Omega)$ in terms of $\Lambda(\Omega)$ (see proposition 4.2).

(2) Give an upper bound for $\mathcal{D}(\Omega)$ in terms of $\Lambda(\Omega)$ (see proposition 4.7). Here we need a strict inequality, in order to prove that the infimum of $\mathcal{E}$ is not attained.

(3) Combine the previous estimates in order to find a lower bound for $\mathcal{E}(\Omega)$ (see (4.16)).

(4) Show that this lower bound is sharp by exhibiting a minimizing sequence.

Step 1 (bound from below for $\mathcal{N}(\Omega)$).

Proposition 4.2. *Let $\Omega \in \mathcal{K}_0^n$ and let $\Lambda(\Omega)$, $\mathcal{N}(\Omega)$ be the quantities defined, respectively, in (4.2) and (4.3). Then*

$$\mathcal{N}(\Omega) \geqslant \frac{n+1}{n}\Lambda(\Omega), \tag{4.5}$$

with equality if and only if Ω is a tangential body of a ball.

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Proof. Using the explicit form (4.1) of the solution to (1.5), we have that

$$\mathcal{N}(\Omega) = \int_0^{r_\Omega} \frac{|\Omega_t|^2}{|\partial\Omega_t|}\,\mathrm{d}t$$

(also see [7, § 5] and [17, eqn (3.30)]). Let us define the function

$$\psi(t) = \frac{|\Omega_t|^2}{|\partial\Omega_t|}, \quad t \in [0, r_\Omega[.$$

From the isoperimetric inequality (3.1), we have that

$$0 < \psi(t) \leqslant c_n |\Omega_t| \sqrt[n-1]{|\partial\Omega_t|}$$

for every $t \in [0, r_\Omega[$, where $c_n = 1/(n \sqrt[n-1]{\omega_n})$. Hence ψ can be continuously extended to $[0, r_\Omega]$ by setting $\psi(r_\Omega) = 0$. Furthermore, the map $t \mapsto |\Omega_t|$ is continuously differentiable on $[0, r_\Omega]$, whereas the map $t \mapsto |\partial\Omega_t|$ is locally Lipschitz continuous on $]0, r_\Omega[$. We can then deduce that also ψ is locally Lipschitz continuous on $]0, r_\Omega[$.

Let $\mathcal{T} \subset [0, r_\Omega]$ denote the set (with vanishing Lebesgue measure) of points of non-differentiability of ψ. For every $t \in [0, r_\Omega] \setminus \mathcal{T}$, we have

$$\psi'(t) = -2|\Omega_t| - \frac{|\Omega_t|^2}{|\partial\Omega_t|^2} \frac{\mathrm{d}}{\mathrm{d}t} |\partial\Omega_t|.$$

From corollary 3.3 and the fact that $\mathrm{d}/\mathrm{d}t|\partial\Omega_t| < 0$, we obtain the estimates

$$-2|\Omega_t| \leqslant \psi'(t) \leqslant -\frac{n+1}{n}|\Omega_t|. \tag{4.6}$$

It follows that $|\psi'(t)| \leqslant 2|\Omega|$ for every $t \in [0, r_\Omega] \setminus \mathcal{T}$, and hence ψ is a Lipschitz continuous function on $[0, r_\Omega]$. Moreover,

$$\psi(t) = -\int_t^{r_\Omega} \psi'(s)\,\mathrm{d}s \geqslant \frac{n+1}{n} \int_t^{r_\Omega} |\Omega_s|\,\mathrm{d}s,$$

and hence

$$\mathcal{N}(\Omega) = \int_0^{r_\Omega} \psi(t)\,\mathrm{d}t \geqslant \frac{n+1}{n} \Lambda(\Omega),$$

concluding the proof. □

STEP 2 (bound from above for $\mathcal{D}(\Omega)$). In order to give an estimate from above of the denominator $\mathcal{D}(\Omega)$ in (4.4), some preparation is needed. We recall that, for every $\Omega \in \mathcal{K}_0^n$, u_Ω denotes the minimizer of the functional $J(\cdot; \Omega)$ in $H_0^1(\Omega)$. It is known that $u_\Omega \in C^\infty(\operatorname{int} \Omega)$ (see [12, corollary 8.11]). If $\Omega \in \mathcal{K}^n$ but $|\Omega| = 0$ (that is, $\Omega \notin \mathcal{K}_0^n$), we set $u_\Omega = 0$ and $J(u_\Omega; \Omega) = 0$.

Let $\Omega \in \mathcal{K}_0^n$. Since the distance function $d(\cdot, \partial\Omega)$ is concave in Ω, the vector $n(x) := -\nabla d(x, \partial\Omega)$ is defined a.e. on Ω, and coincides with the exterior normal on $\partial\Omega$ (in the regular points of $\partial\Omega$). For every $u \in H_0^1(\Omega)$, let us define the following vectors:

$$D^{\mathrm{N}} u(x) := (\nabla u(x) \cdot n(x)) n(x), \quad D^{\mathrm{T}} u(x) := \nabla u(x) - D^{\mathrm{N}} u(x), \quad x \in \Omega.$$

Clearly, we have $J(u;\Omega) = J^{\mathrm{N}}(u;\Omega) + J^{\mathrm{T}}(u;\Omega)$, where

$$J^{\mathrm{N}}(u;\Omega) := \int_\Omega (\tfrac{1}{2}|D^{\mathrm{N}}u|^2 - u)\,\mathrm{d}x, \qquad J^{\mathrm{T}}(u;\Omega) := \int_\Omega \tfrac{1}{2}|D^{\mathrm{T}}u|^2\,\mathrm{d}x.$$

Our first task will be to give a lower bound for $J^{\mathrm{N}}(\cdot;\Omega)$ in terms of the functional $\Lambda(\Omega)$ defined in (4.2), which is valid when $\Omega \in \mathcal{P}_0^n$ is a polytope with non-empty interior (see lemma 4.4 below).

We recall that, if $\Omega \in \mathcal{P}_0^n$ is a polytope and $F_1,\dots,F_m$ are its $(n-1)$-dimensional faces, then it admits the following representation,

$$\Omega = \bigcap_{i=1}^m H^-_{u_i,\alpha_i}, \tag{4.7}$$

where $H^-_{u,\alpha}$ is defined in (3.10) and $u_i \in S^{n-1}$ is the exterior normal vector to F_i, $i = 1,\dots,m$ (see [20, § 2.4]). On $\partial\Omega$, let us define the *piercing function*

$$\lambda_0(y) := \sup\{\mu \geqslant 0;\ y - \mu u_i \in \Omega,\ \Pi(y-\mu u_i) = y\} \quad \text{if } y \in F_i, \tag{4.8}$$

where $\Pi(x)$ denotes the projection of the point $x \in \Omega$ on $\partial\Omega$. It is convenient to extend λ_0 to all Ω by

$$\lambda(x) := \lambda_0(\Pi(x)) - |x - \Pi(x)|, \quad x \in \Omega. \tag{4.9}$$

The main properties of λ_0 are collected in the following lemma.

LEMMA 4.3. *Let $\Omega \in \mathcal{P}_0^n$ and let $F_1,\dots,F_m$ be its $(n-1)$-dimensional faces. Then the piercing function λ_0 satisfies the following properties.*

(i) $0 \leqslant \lambda_0(y) \leqslant r_\Omega$ *for every* $y \in \partial\Omega$.

(ii) λ_0 *vanishes on the relative boundaries of* $F_1,\dots,F_m$, *and is strictly positive on their relative interior.*

(iii) λ_0 *is a Lipschitz continuous function on* $\partial\Omega$.

(iv) *For every* $i \in \{1,\dots,m\}$, *the restriction of* λ_0 *to* F_i *is a concave function on* F_i.

Proof. Properties (i) and (ii) follow directly from the definition (4.3) of λ_0. Let (4.7) be the representation of Ω. In order to prove (iii) and (iv), it is convenient to rewrite λ_0 in terms of (u_i) and (α_i).

We claim that

$$\lambda_0(y) = \min_{j\neq i} \frac{\alpha_j - y\cdot u_j}{1 - u_i\cdot u_j} \quad \forall y \in F_i. \tag{4.10}$$

Remark that $-1 \leqslant u_i \cdot u_j < 1$ for every $j \neq i$, and hence the right-hand side in (4.10) is well defined. Let $y \in F_i$. The set of all $\mu \geqslant 0$ such that $y - \mu u_i \in \Omega$ and $\Pi(y-\mu u_i) = y$ is characterized by

$$\alpha_i - (y-\mu u_i)\cdot u_i \leqslant \alpha_j - (y - \mu u_i)\cdot u_j \quad \forall j \neq i.$$

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Since $y \in F_i$, we have that $y \cdot u_i = \alpha_i$, and hence the inequality above becomes

$$\mu \leqslant \frac{\alpha_j - y \cdot u_j}{1 - u_i \cdot u_j} \quad \forall j \neq i.$$

Therefore, equation (4.10) easily follows from the definition of λ_0.

Now let $i \in \{1, \dots, m\}$ and let $y_1, y_2 \in F_i$. From (4.10), we deduce that there exist $j_1, j_2 \in \{1, \dots, m\} \setminus \{i\}$ such that, for $h = 1, 2$,

$$\lambda_0(y_h) = \frac{\alpha_{j_h} - y \cdot u_{j_h}}{1 - u_i \cdot u_{j_h}} \leqslant \frac{\alpha_j - y \cdot u_j}{1 - u_i \cdot u_j} \quad \forall j \neq i.$$

Hence

$$\begin{aligned} \lambda_0(y_1) - \lambda_0(y_2) &= \frac{\alpha_{j_1} - y_1 \cdot u_{j_1}}{1 - u_i \cdot u_{j_1}} - \frac{\alpha_{j_2} - y_2 \cdot u_{j_2}}{1 - u_i \cdot u_{j_2}} \\ &\leqslant \frac{\alpha_{j_2} - y_1 \cdot u_{j_2}}{1 - u_i \cdot u_{j_2}} - \frac{\alpha_{j_2} - y_2 \cdot u_{j_2}}{1 - u_i \cdot u_{j_2}} \\ &= -\frac{u_{j_2}}{1 - u_i \cdot u_{j_2}} \cdot (y_1 - y_2). \end{aligned} \tag{4.11}$$

In particular, if y_2 is fixed, then

$$\lambda_0(y_1) - \lambda_0(y_2) \leqslant p \cdot (y_1 - y_2) \quad \forall y_1 \in F_i,$$

where

$$p := -\frac{u_{j_2}}{1 - u_i \cdot u_{j_2}},$$

so that (iv) follows. The Lipschitz continuity of λ_0 also follows from (4.11). Namely, if we define

$$L := \max_{1=i<j\leqslant m} \frac{1}{1 - u_i \cdot u_j}, \tag{4.12}$$

then, from (4.11), we obtain $\lambda_0(y_1) - \lambda_0(y_2) \leqslant L|y_1 - y_2|$. Exchanging the role of y_1 and y_2 in (4.11), we also have $\lambda_0(y_2) - \lambda_0(y_1) \leqslant L|y_1 - y_2|$, and therefore λ_0 is Lipschitz continuous on F_i, with Lipschitz constant L. Clearly, this property holds for every $i = 1, \dots, m$. Since the sets $F_1, \dots, F_m$ are closed, property (iii) follows. ☐

LEMMA 4.4. *Let $\Omega \in \mathcal{P}_0^n$. Then $J^{\mathrm{N}}(u; \Omega) \geqslant -\Lambda(\Omega)$ for every $u \in H_0^1(\Omega)$.*

Proof. Let $F_1, \dots, F_m$ be the $(n-1)$-dimensional faces of the polytope Ω and let (4.7) be its representation. From lemma 4.3, the sets

$$\{y - tu_i;\ y \in F_i,\ 0 \leqslant t \leqslant \lambda_0(y)\}, \quad i = 1, \dots, m,$$

belong to $\mathcal{K}_0^n$ and form a covering of Ω, with pairwise disjoint interior.

Let $u \in H_0^1(\Omega)$. From Fubini's theorem, we have that

$$J^{\mathrm{N}}(u; \Omega) = \sum_{i=1}^{m} \int_{F_i} \left\{ \int_0^{\lambda_0(y)} [\tfrac{1}{2}|\nabla u(y - tu_i)|^2 - u(y - tu_i)]\, \mathrm{d}t \right\} \mathrm{d}\mathcal{H}^{n-1}(y).$$

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Let us fix $y \in \partial\Omega$. The quantity in braces can be estimated by

$$\begin{aligned}\{\cdots\} &\geqslant \min\biggl\{\int_0^{\lambda_0(y)} [\tfrac{1}{2}|v'(t)|^2 - v(t)]\,\mathrm{d}t;\ v \in H^1(0,\lambda_0(y)),\ v(0)=0\biggr\} \\ &= -\tfrac{1}{6}\lambda_0(y)^3,\end{aligned}$$

and hence

$$J^{\mathrm{N}}(u;\Omega) \geqslant -\frac{1}{6}\sum_{i=1}^{m}\int_{F_i}\lambda_0(y)^3\,\mathrm{d}\mathcal{H}^{n-1}(y) = -\frac{1}{6}\int_{\partial\Omega}\lambda_0(y)^3\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

The proof will be concluded if we show that

$$\Lambda(\Omega) = \frac{1}{6}\int_{\partial\Omega}\lambda_0(y)^3\,\mathrm{d}\mathcal{H}^{n-1}(y). \tag{4.13}$$

By Fubini's theorem, we have that

$$\int_{\partial\Omega_s}\lambda(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) = |\Omega_s|,$$

where λ is the extension of λ_0 defined in (4.9). In general, if $g \in C[0,r_\Omega]$, $g(0)=0$ and $G(s) = \int_0^s g(\sigma)\,\mathrm{d}\sigma$, $s \in [0,r_\Omega]$, we have

$$\begin{aligned}\int_{\Omega_t} g(\lambda(x))\,\mathrm{d}x &= \int_{\partial\Omega_t}\biggl(\int_0^{\lambda(y)} g(\lambda(y)-s)\,\mathrm{d}s\biggr)\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_{\partial\Omega_t} G(\lambda(y))\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned}$$

Hence we have

$$\begin{aligned}\Lambda(\Omega) &= \int_0^{r_\Omega}\biggl(\int_t^{r_\Omega}|\Omega_s|\,\mathrm{d}s\biggr)\mathrm{d}t \\ &= \int_0^{r_\Omega}\biggl[\int_t^{r_\Omega}\biggl(\int_{\partial\Omega_s}\lambda(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\biggr)\mathrm{d}s\biggr]\mathrm{d}t \\ &= \int_0^{r_\Omega}\biggl(\int_{\Omega_t}\lambda(x)\,\mathrm{d}x\biggr)\mathrm{d}t \\ &= \frac{1}{2}\int_0^{r_\Omega}\biggl[\int_{\partial\Omega_t}\lambda(y)^2\,\mathrm{d}\mathcal{H}^{n-1}(y)\biggr]\mathrm{d}t = \frac{1}{2}\int_\Omega\lambda(x)^2\,\mathrm{d}x \\ &= \frac{1}{6}\int_{\partial\Omega}\lambda(y)^3\,\mathrm{d}\mathcal{H}^{n-1}(y),\end{aligned}$$

and the proof is complete. □

LEMMA 4.5. *The functional $\Omega \mapsto J^{\mathrm{T}}(u_\Omega;\Omega)$ is continuous in $\mathcal{K}^n$.*

Proof. Let $\Omega \in \mathcal{K}^n$ and let $(\Omega_k)_k \subset \mathcal{K}^n$ be a sequence converging to Ω. If $|\Omega| = 0$, then $|\Omega_k| \to 0$ and $J^{\mathrm{T}}(u_{\Omega_k};\Omega_k) \to 0$. Assume now that $|\Omega| > 0$, that is, $\Omega \in \mathcal{K}^n_0$. It is not restrictive to assume that $\Omega_k \in \mathcal{K}^n_0$ for every $k \in \mathbb{N}$. Let $R > 0$ be such that Ω and every Ω_k are contained in B^n_R. Let us extend the function u_Ω to B^n_R by

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setting $u_\Omega = 0$ on $B^n_R \setminus \Omega$, so that $u_\Omega \in H^1_0(B^n_R)$, and let us extend the functions u_{Ω_k} in the same way on $B^n_R \setminus \Omega_k$. Let us define

$$n(x) := -\nabla d(x, \partial\Omega), \quad n_k(x) := -\nabla d(x, \partial\Omega_k), \quad x \in B^n_R$$

(recall that the distance function is defined on all $\mathbb{R}^n$ by (2.1)). Let $N \subset B^n_R$, $|N| = 0$, denote the union of the sets of points of non-differentiability of the functions $d(\cdot, \partial\Omega_k)$ and $d(\cdot, \partial\Omega)$. Since the sequence of concave functions $(d(\cdot, \partial\Omega_k))_k$ converges to $d(\cdot, \partial\Omega)$ pointwise on B^n_R, we have that $n_k(x) \to n(x)$ for every $x \in B^n_R \setminus N$ (see [19, theorem 24.5]). Moreover, from proposition A.1 in the appendix, $u_{\Omega_k} \to u_\Omega$ strongly in $H^1_0(B^n_R)$, and hence $\nabla u_{\Omega_k} \to \nabla u_\Omega$ a.e. in B^n_R. Then, for a.e. $x \in B^n_R$,

$$D^{\mathrm{T}} u_{\Omega_k}(x) \to D^{\mathrm{T}} u_\Omega(x), \qquad |D^{\mathrm{T}} u_{\Omega_k}(x)|^2 \leqslant |\nabla u_{\Omega_k}(x)|^2 \to |\nabla u_\Omega(x)|^2.$$

Therefore, $(|D^{\mathrm{T}} u_{\Omega_k}|^2)_k$ is an equi-integrable sequence of functions converging a.e. to the integrable function $|D^{\mathrm{T}} u_\Omega|^2$. From Vitali's theorem, we have $D^{\mathrm{T}} u_{\Omega_k} \to D^{\mathrm{T}} u_\Omega$ in $L^2(B^n_R)$, concluding the proof. □

Lemma 4.6. *Let $\Omega \in \mathcal{K}^n_0$. Then $J^{\mathrm{T}}(u_\Omega; \Omega) = 0$ if and only if $u_\Omega \in \mathcal{W}(\Omega)$.*

Proof. If $u_\Omega \in \mathcal{W}(\Omega)$, then $D^{\mathrm{T}} u_\Omega = 0$ a.e. on Ω, and hence $J^{\mathrm{T}}(u_\Omega; \Omega) = 0$. Conversely, assume that $J^{\mathrm{T}}(u_\Omega; \Omega) = 0$. In order to prove that $u_\Omega \in \mathcal{W}(\Omega)$, it is enough to show that, if $t \in \,]0, r_\Omega[$, then u_Ω is constant on $\partial\Omega_t$. Let $x_0 \in \partial\Omega_t$. We are going to show that, for every $x \in \partial\Omega_t$, we have $u_\Omega(x) = u_\Omega(x_0)$. Let $S \subset \partial\Omega_t$ be the set of singular points of $\partial\Omega_t$, that is, the subset of $\partial\Omega_t$ where the exterior normal is not defined. Since $\mathcal{H}^{n-1}(S) = 0$ (see [20, theorem 2.2.4]), there exists a Lipschitz continuous curve $\gamma : [0, 1] \to \mathbb{R}^n$ with the following properties: $\gamma(0) = x_0$; $\gamma(1) = x_1$; $\gamma(s) \in \partial\Omega_t$ for every $s \in [0, 1]$; $\gamma(s) \notin S$ for a.e. $s \in [0, 1]$. Moreover, $u_\Omega \in C^\infty(\operatorname{int} \Omega)$, and hence the composed map $s \mapsto u_\Omega(\gamma(s))$ is Lipschitz continuous in $[0, 1]$, and

$$\frac{\mathrm{d}}{\mathrm{d}s} u_\Omega(\gamma(s)) = \nabla u_\Omega(\gamma(s)) \cdot \gamma'(s)$$

for a.e. $s \in [0, 1]$. Now, $J^{\mathrm{T}}(u_\Omega; \Omega) = 0$ entails $D^{\mathrm{T}} u_\Omega = 0$ a.e. on Ω, that is, $\nabla u_\Omega(x)$ is parallel to $n(x)$ for a.e. $x \in \Omega$. On the other hand, $\gamma'(s) \cdot n(\gamma(s)) = 0$ for a.e. $s \in [0, 1]$, and hence $(\mathrm{d}/\mathrm{d}s) u_\Omega(\gamma(s)) = 0$ for a.e. $s \in [0, 1]$. Thus $u_\Omega(x) = u_\Omega(x_0)$, and the proof is complete. □

Proposition 4.7. *For every $\Omega \in \mathcal{K}^n$, we have*

$$\mathcal{D}(\Omega) = 2\Lambda(\Omega) - \delta(\Omega), \tag{4.14}$$

where $\Lambda(\Omega)$, $\mathcal{D}(\Omega)$ are the quantities defined, respectively, in (4.2) and (4.4), and δ is a non-negative continuous functional in $\mathcal{K}^n$. Furthermore, $\delta(\Omega) > 0$ for every $\Omega \in \mathcal{K}^n_0$.

Proof. From lemma 4.4, we have that, for every $\Omega \in \mathcal{P}^n_0$,

$$J(u; \Omega) = J^{\mathrm{N}}(u; \Omega) + J^{\mathrm{T}}(u; \Omega) \geqslant -\Lambda(\Omega) + J^{\mathrm{T}}(u; \Omega) \quad \forall u \in H^1_0(\Omega),$$

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and hence

$$\min_{u \in H_0^1(\Omega)} J(u;\Omega) = J(u_\Omega;\Omega) \geqslant -\Lambda(\Omega) + J^{\mathrm{T}}(u_\Omega;\Omega). \tag{4.15}$$

From proposition A.1, the continuity of the volume and lemma 4.5, the functionals $\Omega \mapsto J(u_\Omega;\Omega)$, $\Omega \mapsto \Lambda(\Omega)$ and $\Omega \mapsto J^{\mathrm{T}}(u_\Omega;\Omega)$ are continuous in $\mathcal{K}^n$, so we can conclude that inequality (4.15) holds for every $\Omega \in \mathcal{K}^n$. Since $\mathcal{D}(\Omega) = -2J(u_\Omega;\Omega)$, we have that $\mathcal{D}(\Omega) = 2\Lambda(\Omega) - \delta(\Omega)$, with $\delta(\Omega) \geqslant 2J^{\mathrm{T}}(u_\Omega;\Omega) \geqslant 0$.

Finally, let $\Omega \in \mathcal{K}_0^n$ and prove that $\delta(\Omega) > 0$. If $\mathcal{E}(\Omega) = 1$, that is, if $\mathcal{D}(\Omega) = \mathcal{N}(\Omega)$, then, by proposition 4.1, we have that Ω is a ball. Then, from proposition 4.2, we conclude that

$$\mathcal{D}(\Omega) = \mathcal{N}(\Omega) = \frac{n+1}{n}\Lambda(\Omega) < 2\Lambda(\Omega),$$

so that $\delta(\Omega) > 0$, by the definition of δ in (4.14). On the other hand, if $\mathcal{E}(\Omega) < 1$, then $u_\Omega \notin \mathcal{W}(\Omega)$, and hence, from lemma 4.6, we conclude again that $\delta(\Omega) \geqslant 2J^{\mathrm{T}}(u_\Omega;\Omega) > 0$. □

Step 3 (bound from below for $\mathcal{E}(\Omega)$). From propositions 4.2 and 4.7, it is straightforward to conclude that

$$\mathcal{E}(\Omega) > \frac{n+1}{2n} \quad \forall \Omega \in \mathcal{K}_0^n. \tag{4.16}$$

Step 4 (the bound from below for $\mathcal{E}$ is the infimum). It remains to show that

$$\inf_{\Omega \in \mathcal{K}_0^n} \mathcal{E}(\Omega) = \frac{n+1}{2n}.$$

It is enough to construct a sequence $(\Omega^k)_k \subset \mathcal{K}_0^n$ such that

$$\limsup_k \mathcal{E}(\Omega^k) \leqslant \frac{n+1}{2n}.$$

We start by providing an upper bound to $\mathcal{E}$ for domains of the following type,

$$\Omega = \{(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R};\ y \in D,\ \alpha(y) \leqslant t \leqslant \beta(y)\}, \tag{4.17}$$

where $D \subset \mathbb{R}^{n-1}$ belongs to $\mathcal{K}_0^{n-1}$ and α, β are, respectively, a convex non-positive function and a concave non-negative function on D, vanishing on ∂D.

We consider the functionals

$$J_0(u) \doteq \int_\Omega [\tfrac{1}{2}u_t^2 - u]\,\mathrm{d}x, \qquad J_1(u) \doteq \int_\Omega \tfrac{1}{2}|\nabla_y u|^2\,\mathrm{d}x,$$

where $\nabla_y u$ denotes the vector of the first $n-1$ partial derivatives of u, while u_t denotes the nth partial derivative of u. We remark that $J(u;\Omega) = J_0(u) + J_1(u)$.

It is easy to show that, for the function $u^*(y,t) = -\frac{1}{2}[t-\alpha(y)][t-\beta(y)]$, we have

$$J_0(u^*) = -\frac{1}{24}\int_D [\beta(y) - \alpha(y)]^3\,\mathrm{d}y,$$

so that

$$\mathcal{D}(\Omega) = -2\min J \geqslant -2J_0(u^*) - 2J_1(u^*) = \frac{1}{12}\int_D [\beta(y)-\alpha(y)]^3\,\mathrm{d}y - 2J_1(u^*).$$

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For $R > 0$, let Ω_R be the convex domain defined by (4.17) with $D = B_R^{n-1}$, $\alpha(y) \equiv 0$ and $\beta(y) = 1 - |y|/R$. The previous estimate gives

$$\mathcal{D}(\Omega_R) \geqslant \frac{\kappa_{n-1}}{2n(n+1)(n+2)} R^{n-1}\left(1 - \frac{1}{R^2}\right).$$

The term $\mathcal{N}(\Omega)$ can be computed explicitly. Namely, if we denote by r_R the inradius of Ω_R (whose precise expression is not needed), we have that

$$\mathcal{N}(\Omega) = \frac{2\kappa_{n-1}}{n^2(n+2)} R^{n-1} r_R^2,$$

and hence

$$\mathcal{E}(\Omega_R) \leqslant \frac{2(n+1)}{n} r_R^2 \left(1 - \frac{1}{R^2}\right)^{-1}.$$

Since $r_R \to \frac{1}{2}$ as $R \to +\infty$, we conclude that

$$\limsup_{R\to+\infty} \mathcal{E}(\Omega_R) \leqslant \frac{n+1}{2n}.$$

We conclude this section with some remarks about the minimizing sequences of $\mathcal{E}$. Since $\mathcal{E}$ is invariant under dilations, it is enough to restrict our analysis to a subset of convex domains with prescribed measure, for example $\mathcal{K}_1^n := \{\Omega \in \mathcal{K}_0^n;\ |\Omega| = 1\}$. Let $(\Omega^k)_k \subset \mathcal{K}_1^n$ be a minimizing sequence for $\mathcal{E}$.

We claim that

$$\lim_{k\to+\infty} r_{\Omega^k} = 0. \tag{4.18}$$

Namely, assume by contradiction that $r := \limsup_k r_{\Omega^k} > 0$. Up to a subsequence, which we do not relabel, we may assume that $\lim_k r_{\Omega^k} = r$ and $r_{\Omega^k} \geqslant \frac{1}{2} r$ for every $k \in \mathbb{N}$. Let $\Omega \in \mathcal{K}_1^n$. From the concavity of the map $t \mapsto \sqrt[n-1]{|\partial\Omega_t|}$, we obtain the inequality

$$|\partial\Omega_t| \geqslant |\partial\Omega| \left(1 - \frac{t}{r_\Omega}\right)^{n-1}, \quad t \in [0, r_\Omega],$$

which entails

$$1 = |\Omega| = \int_0^{r_\Omega} |\partial\Omega_t| \, \mathrm{d}t \geqslant \frac{r_\Omega}{n} |\partial\Omega| \quad \forall \Omega \in \mathcal{K}_1^n.$$

Therefore, $|\partial\Omega^k| \leqslant n/r_{\Omega^k} \leqslant 2n/r$ for every $k \in \mathbb{N}$. It follows that, up to a translation, the sets (Ω^k) are equibounded. Hence, from the Blaschke selection theorem (see [20, theorem 1.8.6]), there exists a subsequence of (Ω^k) converging to a set $\Omega \in \mathcal{K}^n$. By the continuity of the volume (see [20, theorem 1.8.16]), we have that $\Omega \in \mathcal{K}_1^n$. From the continuity of $\mathcal{E}$ on $\mathcal{K}_0^n$, we conclude that Ω is a minimizer of $\mathcal{E}$, in contradiction with the fact that $\mathcal{E}$ does not attain its infimum.

We remark that condition (4.18) is only necessary in order to have a minimizing sequence. Indeed, let us define, for every $k \in \mathbb{N}^+$,

$$\Omega^k := \{(y, t);\ y \in B_k^{n-1},\ 0 \leqslant t \leqslant 2r_k\}, \qquad r_k \doteq \frac{1}{2\kappa_{n-1} k^{n-1}}. \tag{4.19}$$

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Then $(\Omega^k) \subset \mathcal{K}_1^n$ and $\lim_k r_{\Omega^k} = \lim_k r_k = 0$. On the other hand, a straightforward computation shows that $\mathcal{N}(\Omega^k), \mathcal{D}(\Omega^k) \sim \frac{1}{3} r_k^2$ for $k \to +\infty$, and hence $\lim_k \mathcal{E}(\Omega^k) = 1$.

This different behaviour can be explained by the following argument, for which we are not exhibiting a formal proof. Let $(\Omega^k) \subset \mathcal{K}_1^n$ be a minimizing sequence of $\mathcal{E}$. From propositions 4.2 and 4.7, it is clear that

$$\mathcal{N}(\Omega^k) = \frac{n+1}{n} \Lambda(\Omega^k)(1 + \Delta_k), \qquad \mathcal{D}(\Omega^k) = 2\Lambda(\Omega^k)(1 - \delta_k),$$

with $\Delta_k, \delta_k \geqslant 0$ and $\lim_k \Delta_k = \lim_k \delta_k = 0$. If the first condition ($\lim_k \Delta_k = 0$) holds, we say that the sequence (Ω^k) is *asymptotically tangential to a ball*. Indeed, from proposition 4.2, for k large, Ω^k tends to be a tangential body of a ball. This requirement is fulfilled by the minimizing sequence chosen in step 4 above, but it is not satisfied by the sequence of cylinders defined in (4.19). The second condition, that is, $\lim_k \delta_k = 0$, is automatically satisfied when $\lim_k r_{\Omega^k} = 0$. (For sets of the form (4.17), this assertion can be checked using the argument given in step 4 above.)

Therefore, we can conclude that, if the sequence $(\Omega^k) \subset \mathcal{K}_1^n$ is asymptotically tangential to a ball and $\lim_k r_{\Omega^k} = 0$, then (Ω^k) is a minimizing sequence of $\mathcal{E}$.

Appendix A.

The next well-known proposition (see [15]) concerns the convergence of solutions to the Dirichlet problem (1.1) on variable domains. In what follows, it is understood that, if $A \subset B$ for some sets $A, B \in \mathcal{K}_0^n$, then $H_0^1(A)$ is embedded in $H_0^1(B)$ by setting $v = 0$ on $B \setminus A$ for every $v \in H_0^1(A)$.

PROPOSITION A.1. *Let $(\Omega^k)_{\mathcal{K}} \subset \mathcal{K}_0^n$ be a sequence converging to $\Omega \in \mathcal{K}_0^n$ in the Hausdorff metric. Let $u \in H_0^1(\Omega)$ be the solution to (1.1), and, for every $k \in \mathbb{N}$, let $u_k \in H_0^1(\Omega^k)$ be the solution of the same Dirichlet problem on Ω^k. Let $K \in \mathcal{K}_0^n$ be a set containing Ω and every Ω^k, $k \in \mathbb{N}$. Then $u_k \to u$ in the strong topology of $H_0^1(K)$.*

Proof. Since

$$\int_K |\nabla u_k|^2 \,\mathrm{d}x = \int_K u_k \,\mathrm{d}x$$

(see (1.4)), from the Poincaré inequality (see [12, § 7.8]), we get

$$\|u_k\|^2_{H_0^1(K)} \leqslant C \|u_k\|_{H_0^1(K)},$$

so that $\|u_k\|_{H_0^1(K)} \leqslant C$ for every $k \in \mathbb{N}$. Thus there exists $v \in H_0^1(K)$ such that $u_k \rightharpoonup v$ in the weak topology of $H_0^1(K)$.

We claim that $v \in H_0^1(\Omega)$. Namely, it is enough to prove that $v = 0$ a.e. on $K \setminus \Omega$. Indeed, if $\rho > 0$ is fixed, then there exists $k_\rho \in \mathbb{N}$ such that $\Omega^k \subset \Omega + B_\rho^n$ for every $k \geqslant k_\rho$. Hence $u_k = 0$ a.e. in $K \setminus (\Omega + B_\rho^n)$ for every $k \geqslant k_\rho$, which entails $v = 0$ a.e. in $K \setminus (\Omega + B_\rho^n)$. Now, this condition if fulfilled for every $\rho > 0$, and hence $v = 0$ a.e. in $K \setminus \Omega$.

Let us show that v is a solution to (1.1). Let $\varphi \in C_0^\infty(\Omega)$ be a test function. Since $\operatorname{supp} \varphi$ (the support of φ) is compactly contained in Ω, there exists $k_\varphi \in \mathbb{N}$ such

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that $\operatorname{supp}\varphi \subset \Omega^k$ for every $k \geqslant k_\varphi$. Therefore, from the fact that u_k satisfies (1.2) in Ω^k, we obtain

$$\int_{\Omega^k} \nabla u_k \cdot \nabla\varphi \,\mathrm{d}x = \int_{\Omega^k} \varphi \,\mathrm{d}x \quad \forall k \geqslant k_\varphi.$$

Then the convergence of $(u_k)_k$ to v in the weak topology of $H_0^1(K)$ entails

$$\int_{\Omega} \nabla v \cdot \nabla\varphi \,\mathrm{d}x = \int_{\Omega} \varphi \,\mathrm{d}x. \tag{A 1}$$

Since (A 1) holds for every test function $\varphi \in C_0^\infty(\Omega)$, we have that v is a solution to (1.1). From the uniqueness of the solution, we conclude that $v = u$.

Finally, from the convergence of (u_k) to u in the weak topology of $H_0^1(K)$, we obtain

$$\int_K |\nabla u_k|^2 = \int_K u_k \to \int_K u = \int_K |\nabla u|^2,$$

which allows us to conclude that $u_k \to u$ in the strong topology of $H_0^1(K)$. □

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