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# Monochromatic Cycles in 2-Coloured Graphs

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Li, Nikiforov and Schelp [13] conjectured that any 2-edge coloured graph $G$ with order $n$ and minimum degree $\delta(G) > 3n/4$ contains a monochromatic cycle of length $\ell$, for all $\ell \in [4, \lceil n/2 \rceil]$. We prove this conjecture for sufficiently large $n$ and also find all 2-edge coloured graphs with $\delta(G) = 3n/4$ that do not contain all such cycles. Finally, we show that, for all $\delta > 0$ and $n > n_0(\delta)$, if $G$ is a 2-edge coloured graph of order $n$ with $\delta(G) \geqslant 3n/4$, then one colour class either contains a monochromatic cycle of length at least $(2/3 + \delta/2)n$, or contains monochromatic cycles of all lengths $\ell \in [3, (2/3 - \delta)n]$.

## 1. Introduction

A well-known theorem of Dirac [8] states that a graph with order $n \geqslant 3$ and minimum degree at least $n/2$ contains a cycle $C_n$ on $n$ vertices.

**Theorem 1.1 (Dirac [8]).** *Let $G$ be a graph of order $n \geqslant 3$. If $\delta(G) \geqslant n/2$, then $G$ is hamiltonian.* □

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In fact, as noted by Bondy [5], an immediate corollary of the following theorem is that such a graph will contain cycles of all lengths $\ell \in [3, n]$. We call such a graph *pancyclic*.

**Theorem 1.2 (Bondy [5]).** *If $G$ is a hamiltonian graph of order $n$ such that $|E(G)| \geqslant n^2/4$, then either $G$ is pancyclic or $n$ is even and $G$ is isomorphic to $K_{n/2,n/2}$.* □

**Corollary 1.3.** *Let $G$ be a graph of order $n \geqslant 3$. If $\delta(G) \geqslant n/2$, then either $G$ is pancyclic or $n$ is even and $G$ is isomorphic to $K_{n/2,n/2}$. In particular, if $\delta(G) > n/2$, then $G$ is pancyclic.* □

Given a graph $G$ with edge set $E(G)$, a *2-edge colouring* of $G$ is a partition $E(G) = E(R) \cup E(B)$, where $R$ and $B$ are spanning subgraphs of $G$. We define a *$k$-edge colouring* of $G$ similarly. In a recent paper [13], Li, Nikiforov and Schelp made the following conjecture, which would give an analogue of Corollary 1.3 for 2-edge coloured graphs.

**Conjecture 1.4 (Li, Nikiforov and Schelp [13]).** *Let $n \geqslant 4$ and let $G$ be a graph of order $n$ with $\delta(G) > 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of $G$, then for each $\ell \in [4, \lceil n/2 \rceil]$, either $C_\ell \subseteq R$ or $C_\ell \subseteq B$.*

Note that we may only ask for $\ell$ in this range. For example, take the 2-colouring of $K_5$ consisting of a red and a blue $C_5$ and blow it up, that is, replace each vertex of $K_5$ with an independent set of $p$ vertices and each edge with the complete monochromatic bipartite graph $K_{p,p}$ of the same colour. The resulting graph $G$ has minimum degree $\delta(G) = 4|G|/5$ but no monochromatic $C_3$. Similarly letting $R$ be the complete bipartite graph with vertex classes of order $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, and letting $B$ be the complement, we obtain a 2-colouring of the complete graph $K_n$ with no monochromatic odd cycle $C_\ell$ with $\ell > \lceil n/2 \rceil$. Li, Nikiforov and Schelp [13] proved the following partial result.

**Theorem 1.5 (Li, Nikiforov and Schelp [13]).** *Let $\epsilon > 0$, let $G$ be a graph of sufficiently large order $n$, with $\delta(G) > 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of $G$, then for all $\ell \in [4, \lfloor (1/8 - \epsilon)n \rfloor]$, either $C_\ell \subseteq R$ or $C_\ell \subseteq B$.* □

We will prove Conjecture 1.4 for sufficiently large $n$, but first we will define a set of 2-edge coloured graphs showing that the degree bound $3n/4$ is tight.

**Definition.** Let $n = 4p$ and let $G$ be a graph isomorphic to $K_{p,p,p,p}$. A *2-bipartite 2-edge colouring* of $G$ is a 2-edge colouring $E(G) = E(R) \cup E(B)$ such that both $R$ and $B$ are bipartite.

If $G \cong K_{p,p,p,p}$ and $E(G) = E(R) \cup E(B)$ is a 2-bipartite 2-edge colouring of $G$, let $V_1 \cup V_2$ be the bipartition of $R$ and $W_1 \cup W_2$ be the bipartition of $B$. Let $U_{i,j} = V_i \cap W_j$ for all $i, j \in \{1, 2\}$. Then the $U_{i,j}$ are four independent sets of $G$ covering all vertices, and so must be the four independent sets of order $p$. So, a 2-bipartite 2-edge colouring of $K_{p,p,p,p}$ forces a labelling of the independent sets $\{U_{i,j} : i, j \in \{1, 2\}\}$ such that:

- all edges between $U_{1,1}$ and $U_{1,2}$ and between $U_{2,1}$ and $U_{2,2}$ are blue;
- all edges between $U_{1,1}$ and $U_{2,1}$ and between $U_{1,2}$ and $U_{2,2}$ are red;
- edges between $U_{1,1}$ and $U_{2,2}$ and between $U_{2,1}$ and $U_{1,2}$ can be either colour.

If 4 divides $n$, then the graph $K_{n/4,n/4,n/4,n/4}$ with a 2-bipartite 2-edge colouring has minimal degree $3n/4$ and no monochromatic odd cycles. Note that for a fixed labelling of this graph, there are $2^{n^2/8}$ 2-bipartite 2-edge colourings of $K_{n/4,n/4,n/4,n/4}$. However, $K_{n/4,n/4,n/4,n/4}$ has $24((n/4)!)^4 = 2^{O(n \log n)}$ automorphisms and so there are $2^{n^2/8+O(n \log n)}$ distinct 2-bipartite 2-edge colourings of $K_{n/4,n/4,n/4,n/4}$. In fact we will prove that $K_{n/4,n/4,n/4,n/4}$ is the only extremal graph; although any 2-bipartite 2-edge colouring of $K_{n/4,n/4,n/4,n/4}$ is extremal. We state our main result now.

**Theorem 1.6.** *There exists a positive integer $n_0$ with the following property. Let $G$ be a graph of order $n > n_0$ with $\delta(G) \geqslant 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of $G$. Then either $C_\ell \subseteq R$ or $C_\ell \subseteq B$ for all $\ell \in [4, \lceil n/2 \rceil]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring.*

We define the *monochromatic circumference* of a $k$-edge coloured graph to be the length of its longest monochromatic cycle. Li, Nikiforov and Schelp [13] also posed the following question.

**Question 1.7.** *Let $0 < c < 1$ and $n$ be sufficiently large integer. If $G$ is a 2-coloured graph of order $n$ with $\delta(G) > cn$, what is the minimum possible monochromatic circumference of $G$?*

For graphs $G$ with $\delta(G) \geqslant 3n/4$ we show that the monochromatic circumference is at least $(2/3 + o(1))n$. In fact, we show the following result.

**Theorem 1.8.** *For every $0 < \delta \leqslant 1/180$, there exists an integer $n_0 = n_0(\delta)$ such that the following holds. Let $G$ be a graph of order $n > n_0$ with $\delta(G) \geqslant 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of $G$. Then either $G$ has monochromatic circumference at least $(2/3 + \delta/2)n$, or one of $R_G$ and $B_G$ contains cycles of all lengths $\ell \in [3, (2/3 - \delta)n]$.*

Note that the last statement requires monochromatic cycles of all lengths in some prescribed set of integers, as in Theorem 1.6. However, here these cycles are required to be of the *same* colour. Also, the upper bound on $\delta$ is of a technical nature, and we are only interested in small $\delta$. There are similar technical bounds throughout this paper.

For integers $s \leqslant t$, we define the following 2-edge coloured graph $F_{s,t}$, which with $t = 2s$ shows that Theorem 1.8 is asymptotically sharp.

**Definition.** For $s \leqslant t$, let $F_{s,t}$ be the 2-edge coloured complete graph on $s + t$ vertices in which the blue edges form the complete bipartite graph $K_{s,t}$ and all the other edges are red.

The longest cycle in the red subgraph of $F_{s,t}$ has length $t$. The blue subgraph is bipartite and has circumference $2s$. Thus the monochromatic circumference of $F_{s,t}$ is $\max\{t, 2s\}$.

Let $n = 3s$. Then $|F_{s,2s}| = n$, $\delta(F_{s,2s}) = n - 1$, and $F_{s,2s}$ has monochromatic circumference $2s = 2n/3$. Hence Theorem 1.8 is asymptotically sharp.

We shall show that the linear dependence between the two occurrences of $\delta$ in Theorem 1.8 is correct. Fix $\delta > 0$. Suppose that $n > 1/2\delta$, and let $G \cong F_{n-\lceil(2/3-\delta)n\rceil,\lceil(2/3-\delta)n\rceil}$. Then the monochromatic circumference of $G$ is $2\lfloor(1/3 + \delta)n\rfloor \geqslant (2/3 + \delta)n$. However, $G$ contains no monochromatic cycle of odd length $\ell \in \{\lceil(2/3 - \delta)n\rceil + 1, \lceil(2/3 - \delta)n\rceil + 2\}$.

In Section 2, we will introduce some theorems that will be used in our proofs. We will then prove Theorem 1.6 in two parts. Section 3 will deal with short (up to constant length) cycles and Section 4 will deal with long cycles. This will rely on a number of lemmas, whose proofs are postponed to Sections 5 and 6. In Section 7 we will look at the length of the longest monochromatic cycle, and in particular prove Theorem 1.8. We conclude in Section 8 with some open problems.

## 2. Results used in the proof

In the proof of Theorem 1.6, we shall use well-known extremal graph theory results and the regularity method to find long cycles. Before introducing these, we make some preliminary definitions and notation.

For a graph $G$, we denote by $e(G)$ its number of edges. Let $X$ and $Y$ be disjoint subsets of $V(G)$. We denote by $G[X]$ the subgraph of $G$ induced by the vertices of $X$. We also denote by $E(X, Y)$ the set of edges joining $X$ and $Y$, set $e(X, Y) := |E(X, Y)|$, and let $G[X, Y]$ be the bipartite subgraph of $G$ with partite sets $X$ and $Y$ and edge set $E(X, Y)$. For a set of vertices $S$, we denote by $\Gamma_G(S)$ the set of all vertices adjacent to some vertex in $S$. We drop the subscript when there is no danger of confusion. We also write $\Gamma_G(v)$ instead of $\Gamma_G(\{v\})$, and set $d_G(v) := |\Gamma_G(v)|$.

**Definition.** Let $G$ be a graph and $X$ and $Y$ be disjoint subsets of $V(G)$. The *density* of the pair $(X, Y)$ is the value

$$d(X, Y) := \frac{e(X, Y)}{|X||Y|}.$$

We define a regular pair to be one where the density between not-too-small subsets of $X$ and $Y$ is close to the density between $X$ and $Y$.

**Definition.** Let $\epsilon > 0$. Let $G$ be a graph and $X$ and $Y$ be disjoint subsets of $V(G)$. We call $(X, Y)$ an *$\epsilon$-regular pair for* $G$ if, for all $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geqslant \epsilon|X|$ and $|Y'| \geqslant \epsilon|Y|$, we have

$$|d(X, Y) - d(X', Y')| < \epsilon.$$

The next fact shows that almost all vertices in any regular pair have large degrees.

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**Fact 2.1.** *Let $G$ be a graph and let $(V_1, V_2)$ be an $\epsilon$-regular pair for $G$ with density $d := d(V_1, V_2)$. Then all but at most $\epsilon|V_1|$ vertices $v \in V_1$ satisfy $|\Gamma(v) \cap V_2| \geqslant (d - \epsilon)|V_2|$.* □

This is particularly useful when we want to find paths of prescribed length in a regular pair, as shown by the next lemma from [1, Lemma 10].

**Lemma 2.2.** *For every $0 < \beta < 1$ there is an $m_0(\beta)$ such that for every $m > m_0(\beta)$ the following holds. Let $G$ be a graph, and let $V_1, V_2$ be disjoint subsets of $V(G)$ such that $|V_1|, |V_2| \geqslant m$. Furthermore, suppose that, for some $\epsilon$ satisfying $0 < \epsilon < \beta/100$, the pair $(V_1, V_2)$ is $\epsilon$-regular for $G$, with density at least $\beta/4$.*

*Then, for every pair of vertices $v_1 \in V_1$, $v_2 \in V_2$ satisfying $|\Gamma(v_1) \cap V_2|$, $|\Gamma(v_2) \cap V_1| \geqslant \beta m/5$, and for every $i$, $1 \leqslant i \leqslant m - 5\epsilon m/\beta$, $G$ contains a $v_1$–$v_2$ path of length $2i + 1$.* □

Combining Lemma 2.2 with Fact 2.1 yields the following straightforward consequence.

**Corollary 2.3.** *For every $0 < \epsilon < 10^{-5}$ there is an $m_0(\epsilon)$ such that for every $m > m_0(\epsilon)$ the following holds. Let $G$ be a graph, and let $V_1, V_2, \ldots, V_\ell$ be disjoint subsets of $V(G)$ such that $|V_1|, |V_2|, \ldots, |V_\ell| \geqslant m$. Suppose that all pairs $(V_1, V_2), (V_2, V_3), \ldots, (V_{\ell-1}, V_\ell)$, $(V_\ell, V_1)$ are $\epsilon$-regular for $G$, with density at least $\sqrt{\epsilon}$.*

*Then, $C_i \subset G$ for every $i$, $\ell \leqslant i \leqslant (1 - 5\sqrt{\epsilon})(\ell - 1)m$, of the same parity as $\ell$.* □

We will use the following 2-coloured version of the Szemerédi Regularity Lemma (see, for example, the survey paper of Komlós and Simonovits [12]), which is not hard to deduce from the standard form of the regularity lemma [14].

**Theorem 2.4 (Degree form of 2-coloured Regularity Lemma).** *For every $\epsilon > 0$ and positive integer $k_0$, there is an $M = M(\epsilon, k_0)$ such that if $G = (V, E)$ is any 2-edge coloured graph and $d \in [0, 1]$ is any real number, then there is $k_0 \leqslant k \leqslant M$, a partition $(V_i)_{i=0}^k$ of the vertex set $V$ and a subgraph $G' \subseteq G$ with the following properties:*

(R1) $|V_0| \leqslant \epsilon|V|$,
(R2) *all clusters* $V_i$, $i \in [k] := \{1, 2, \ldots, k\}$, *are of the same size* $m \leqslant \lceil \epsilon|V| \rceil$,
(R3) $d_{G'}(v) > d_G(v) - (2d + \epsilon)|V|$ *for all* $v \in V$,
(R4) $e(G'[V_i]) = 0$ *for all* $i \in [k]$,
(R5) *for all* $1 \leqslant i < j \leqslant k$, *the pair* $(V_i, V_j)$ *is* $\epsilon$*-regular for* $R_{G'}$ *with a density either* 0 *or greater than* $d$ *and* $\epsilon$*-regular for* $B_{G'}$ *with a density either* 0 *or greater than* $d$, *where* $E(G') = E(R_{G'}) \cup E(B_{G'})$ *is the induced 2-edge colouring of* $G'$. □

Having applied the above form of the Regularity Lemma to a 2-edge coloured graph $G$, we make the following definition, based on the clusters $\{V_i : 1 \leqslant i \leqslant k\}$. Note that this definition depends on the parameters $\epsilon$ and $d$.

**Definition.** Given a graph $G = (V, E)$ and a partition $(V_i)_{i=0}^k$ of $V$ satisfying conditions (R1)–(R5) above, we define the $(\epsilon, d)$*-reduced 2-edge coloured graph* $H$ on vertex set $\{v_i : 1 \leqslant i \leqslant k\}$ as follows. For $1 \leqslant i < j \leqslant k$,

- let $\{v_i, v_j\}$ be a blue edge of $H$ when $B_{G'}[V_i, V_j]$ has density at least $d$;
- let $\{v_i, v_j\}$ be a red edge of $H$ when it is not a blue edge and $R_{G'}[V_i, V_j]$ has density at least $d$.

**Remark.** Notice that the definition of the $(\epsilon, d)$-reduced 2-edge coloured graph $H$ is non-symmetric in the following sense. On one hand, if $\{v_i, v_j\}$ is a red edge of $H$, then we know that $R_{G'}[V_i, V_j]$ has density at least $d$ and $B_{G'}[V_i, V_j]$ has density less than $d$. On the other hand, if $\{v_i, v_j\}$ is a blue edge of $H$, then we know that $R_{B'}[V_i, V_j]$ has density at least $d$, but we have no information about the density of $R_{G'}[V_i, V_j]$.

This asymmetry will never cause a problem in our arguments because we only use the facts that a red edge $\{v_i, v_j\}$ in $H$ corresponds to $R_{G'}[V_i, V_j]$ with density at least $d$ and a blue edge $\{v_i, v_j\}$ in $H$ corresponds to $B_{G'}[V_i, V_j]$ with density at least $d$.

We aim to use matchings in the reduced graph $H$ to find long cycles in $G$. To do so we will use the following lemma.

**Lemma 2.5.** *Given $0 < c < 1$ and $0 < \delta < 1 - c$, let $\epsilon > 0$ and $d > 0$ be sufficiently small real numbers such that $d \geqslant \epsilon^{1/3}$. Let $G$ be a 2-edge coloured graph of sufficiently large order $n$ with a vertex partition $(V_i)_{i=0}^k$ satisfying conditions* (R1)–(R5), *and let $H$ be the corresponding $(\epsilon, d)$-reduced 2-edge coloured graph.*

*Suppose that $H$ has a monochromatic component $C$ that contains a matching on $(c+\delta)k$ vertices. Then*

(a) *$G$ contains monochromatic cycles of length $\ell$ for all even $4k \leqslant \ell \leqslant (c + \delta/2)n$,*
(b) *if $C$ also contains any odd cycle, then $G$ contains monochromatic cycles of length $\ell$ for all $4k \leqslant \ell \leqslant (c + \delta/2)n$.*

*Furthermore, all cycles in* (a) *and* (b) *have the same colour as $C$.*

Lemma 2.5 has by now a very standard proof using Lemma 2.2, which we omit here. The interested reader can easily modify the proof given in [1, Section 5, p. 696].

In the proof of Theorem 1.6, we shall frequently show that there is a subset $S$ of $V$ on which one of $R_G$ or $B_G$ is hamiltonian, and apply Theorem 1.2. To prove hamiltonicity, it will normally be sufficient to use Dirac's Theorem (Theorem 1.1). However, we will also need the following generalization and one of its consequences.

**Theorem 2.6 (Chvátal [7]).** *Let $G$ be a graph of order $n \geqslant 3$ with degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ such that*

$$d_k \leqslant k < \frac{n}{2} \Rightarrow d_{n-k} \geqslant n - k.$$

*Then $G$ contains a Hamilton cycle.* □

**Corollary 2.7.** *Let $G$ be a graph of order $n \geqslant 3$ with minimum degree $\delta(G) \geqslant n/2 + 2$. Then, for every two vertices $u, v$, there is a $u$–$v$ path containing all vertices of $G$.* □

In Section 5, we will need to use the following defect version of Tutte's 1-factor Theorem [15], due to Berge [2]. Here $q(G)$ denotes the number of components of a graph $G$ of odd order.

**Theorem 2.8 (Berge [2]).** *A graph $G$ contains a set of independent edges covering all but at most $d$ of the vertices if, and only if,*

$$q(G-S) \leqslant |S| + d$$

*for all $S \subseteq V$.* □

The next result of Bollobás [3, p. 150] will be used in Section 3, as will be the three following results. Note that by the length of a path or a cycle we mean the number of its edges.

**Theorem 2.9 (Bollobás [3]).** *If $G$ is a graph of order $n$, with $e(G) > n^2/4$, then $G$ contains $C_k$ for all $k \in [3, \lceil n/2 \rceil]$.* □

**Theorem 2.10 (Bondy and Simonovits [6]).** *Let $G$ be a graph of order $n$ and let $k$ be an integer. If $e(G) > 100kn^{1+1/k}$, then $G$ contains a cycle of length $2k$.* □

**Theorem 2.11 (Erdős and Gallai [9]).** *If $G$ is a graph with order $n$ and circumference at most $L$, then $e(G) \leqslant (n-1)L/2$. If $G$ is a graph with order $n$ and with no paths of length at least $L+1$, then $e(G) \leqslant nL/2$.* □

**Theorem 2.12 (Győri, Nikiforov and Schelp [11]).** *Let $k, m$ be positive integers. There exist $n_0 = n_0(k,m)$ and $c = c(k,m) > 0$ such that, for every nonbipartite graph $G$ on $n > n_0$ vertices with minimum degree*

$$\delta(G) > \frac{n}{2(2k+1)} + c,$$

*if $C_{2s+1} \subseteq G$, for some $k \leqslant s \leqslant 4k+1$, then $C_{2s+2j+1} \subseteq G$ for every $j \in [m]$.* □

## 3. Existence of short cycles

In this section we shall prove that unless we are in the extremal case, we have monochromatic cycles of all lengths $\ell \in [4, K]$ for a given integer $K$. To prove this we shall use the following claim.

**Claim 3.1.** *Let $L$ be an integer. Let $n$ be sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geqslant 3n/4$. Suppose that $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of $G$. If there is a monochromatic $C_3$ or $C_5$, then one of $R, B$ contains $C_\ell$ for all odd $\ell \in [5, 2L+1]$.*

The proof of Claim 3.1 follows exactly the method used in [13] to show the existence of short odd cycles. Note that we cannot appeal directly to Theorem 1.5, as the assumption there is that $\delta(G) > 3n/4$, whereas in Theorem 1.6 we assume only that $\delta(G) \geqslant 3n/4$.

**Proof.** Suppose first that $\Delta(B) > n/2 + 4L$. Let $v$ be a vertex with $d_B(v) = \Delta(B)$, and $U = \Gamma_B(v)$. If $B[U]$ contains a path of length $2L$, then using the vertex $v$, there is a blue $C_\ell$ for all $\ell \in [3, 2L+1]$. Hence $B[U]$ does not contain a path of length $2L$, and by Theorem 2.11, we have $e(B[U]) \leqslant L|U|$. However, any vertex $u \in U$ has at most $n/4 - 1$ non-neighbours in $U$ and so at least $|U| - n/4$ neighbours. Therefore,

$$\begin{aligned} e(G[U]) &= \frac{1}{2}\sum_{u\in U} d_{G[U]}(u) \\ &\geqslant \frac{1}{2}|U|\left(|U| - \frac{n}{4}\right) \\ &> \frac{1}{2}|U|\left(\frac{|U|}{2} + 2L\right). \end{aligned}$$

Hence $e(R[U]) = e(G[U]) - e(B[U]) > |U|^2/4$, and, consequently, by Theorem 2.9, $R[U]$ has cycles of all lengths from 3 to $|U|/2 \geqslant 2L+1$.

So we may assume that $\Delta(B) \leqslant n/2 + 4L$, and hence

$$\delta(R) \geqslant \frac{n}{4} - 4L > \frac{n}{6} + c(1, L),$$

where $c(1, L)$ is the constant from Theorem 2.12. Similarly we may assume that $\delta(B) > n/6 + c(1, L)$. Suppose that there is a monochromatic $C_3$ or $C_5$ and assume without loss of generality that it is red. Applying Theorem 2.12 to $R$ with $k = 1$ and $m = L$, there is a red $C_\ell$ for all odd $\ell \in [5, 2L+1]$ as required. □

Now we prove the main result of this section.

**Lemma 3.2.** *Let $K$ be an integer. Let $n$ be sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geqslant 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of $G$, then either $C_\ell \subseteq R$ or $C_\ell \subseteq B$ for all $\ell \in [4, K]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring of $G$ is a 2-bipartite 2-edge colouring.*

**Proof.** Note that the existence of monochromatic $C_\ell$ for all even $\ell \in [4, K]$ is immediate from Theorem 2.10 because one colour class has at least $e(G)/2 \geqslant 3n^2/16$ edges. Hence, by Claim 3.1, it is sufficient to prove that either there is a monochromatic $C_3$ or $C_5$, or $n = 4p$, $G \cong K_{p,p,p,p}$, and the colouring is a 2-bipartite 2-edge colouring.

Suppose that, in fact, none of these occur. Any 2-edge colouring of $K_5$ contains a monochromatic $C_3$ or $C_5$. Hence we may assume that $K_5 \nsubseteq G$. Since $e(G) \geqslant 3n^2/8 = t_4(n)$, by Turán's Theorem, we must therefore have that $G \cong T_4(n)$. However, $\delta(G) \geqslant 3n/4$ implies that $n = 4p$, and hence $G \cong K_{p,p,p,p}$. Let $U_i$, $1 \leqslant i \leqslant 4$, be the independent sets of $G$ of order $p$.

We may assume that $R$ is not bipartite. Let $C = v_1v_2\dots v_r$ be a shortest odd cycle of $R$; we may assume that $r \geqslant 7$. We may properly 4-vertex colour $C$ by setting $c(v_i) = j$ when $v_i \in U_j$. As $C$ is an odd cycle, there must be three consecutive vertices with different colours under $c$. Without loss of generality, assume that $c(v_3) = 1$, $c(v_4) = 2$ and $c(v_5) = 3$.

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We will aim to show that $G[V(C)]$ contains a triangle or 5-cycle which is edge-disjoint from $C$. Then we may assume that an edge of the triangle or 5-cycle is red, else we have a monochromatic $C_3$ or $C_5$. But this red edge, together with $C$, will create a shorter red odd cycle than $C$, contradicting our assumption that $C$ was minimal. We shall find such a triangle or 5-cycle by case analysis.

If $c(v_1)$ is 2 or 4, then $G$ contains the triangle $v_1v_3v_5$, as these vertices lie in different $U_j$. Hence $c(v_1) \in \{1,3\}$, and similarly $c(v_7) \in \{1,3\}$.

If $c(v_6) = 4$, then $G$ contains the triangle $v_1v_4v_6$. So we may assume that $c(v_6) \neq 4$ and similarly $c(v_2) \neq 4$. Hence $c(v_2) \in \{2,3\}$ and $c(v_6) \in \{1,2\}$. If $c(v_2) = 3$ and $c(v_6) = 1$, then $G$ contains the triangle $v_2v_4v_6$. Hence, by symmetry, we may assume that $c(v_2) = 2$ and $c(v_6) \in \{1,2\}$.

If $c(v_7) = 1$, then $G$ contains the triangle $v_2v_5v_7$. Hence $c(v_7) = 3$.

If $|C| = 7$, then $v_1v_7 \in E(C)$, and $c(v_1) = 1$ because $c$ is a proper colouring of $G$. But then $v_1v_5v_2v_7v_4$ is a 5-cycle in $G$, not containing any edges of $C$.

So we may assume that $|C| > 7$. If $c(v_1) = 1$, then $v_1v_4v_7$ is a triangle in $G$ which is edge-disjoint from $C$. Hence $c(v_1) = c(v_7) = 3$. But now, if $c(v_6) = 1$, then $G$ contains the triangle $v_1v_4v_6$, while if $c(v_6) = 2$, then $G$ contains the triangle $v_1v_3v_6$, giving a contradiction.

Hence, in fact, our assumption was false, and one of the cases of the lemma holds. □

## 4. Existence of long cycles

In order to find long monochromatic cycles, we will use the regularity method. Recall from Section 2 that having applied the Regularity Lemma to the graph $G$ on $n$ vertices, we define a reduced graph $H$ on $k$ vertices. Note that the Regularity Lemma implies that the minimum degree of the reduced graph $H$ is not too much smaller than $k/n$ times the minimum degree of $G$.

At this point we use the following lemma, proved in Section 5 using extremal arguments, which shows that either there is a monochromatic component of $H$ containing a large matching, or the reduced graph $H$ has one of two particular forms.

**Lemma 4.1.** *Let $0 < \delta < 1/36$ and let $G$ be a graph of sufficiently large order $n$ with $\delta(G) \geqslant (3/4 - \delta)n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$. Then one of the following holds.*

(i) *There is a component of $R$ or $B$ which contains a matching on at least $(2/3 + \delta)n$ vertices.*
(ii) *There is a set $S$ of order at least $(2/3 - \delta/2)n$ such that either $\Delta(R[S]) \leqslant 10\delta n$ or $\Delta(B[S]) \leqslant 10\delta n$.*
(iii) *There is a partition $V(G) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geqslant (1/4 - 3\delta)n$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.*

In the first case, we will need the following lemma, which is also proved in Section 5.

**Lemma 4.2.** *Let $0 < \delta < 1/36$ and let $G$ be a graph of sufficiently large order $n$ with $\delta(G) \geqslant (3/4 - \delta)n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.*

*Suppose that there is a monochromatic component containing a matching on at least* $(2/3+\delta)n$ *vertices. Then there is a monochromatic component* $C$ *containing a matching on at least* $(1/2+\delta)n$ *vertices such that either* $C$ *contains an odd cycle, or* $|V(C)| \geqslant (1-5\delta)n$.

To finish our argument in this case we shall use Lemma 2.5 and the next lemma.

**Lemma 4.3.** *Let* $n$ *be sufficiently large, and let* $G$ *be a graph of order* $n$ *with* $\delta(G) \geqslant 3n/4$. *Suppose that* $E(G) = E(R) \cup E(B)$ *is a 2-edge colouring of* $G$.

(a) *If* $B$ *has an independent set* $S$ *with* $|S| > n/2$, *then* $C_\ell \subseteq R$ *for all* $\ell \in [3, |S|]$.
(b) *If* $B$ *is bipartite, then either* $C_\ell \subseteq R$ *for all* $\ell \in [4, \lceil n/2 \rceil]$, *or* $n$ *is divisible by four,* $G \cong K_{n/4,n/4,n/4,n/4}$ *and the colouring is a 2-bipartite 2-edge colouring.*

*Both statements remain valid when we interchange* $B$ *and* $R$.

By analysing the original graph $G$, we will use the following two lemmas to show that, in cases (ii) and (iii) of Lemma 4.1, we will have the desired monochromatic cycles.

**Lemma 4.4.** *Given* $0 < \delta < 1/144$, *let* $\epsilon$ *and* $d$ *be real numbers such that* $0 < \epsilon \ll d \ll \delta$, *where, as usual,* $\ll$ *means 'sufficiently smaller than'. Let* $G$ *be a graph of sufficiently large order* $n$ *with* $\delta(G) \geqslant 3n/4$. *Suppose that* $E(G) = E(R_G) \cup E(B_G)$ *is a 2-edge colouring of* $G$, $(V_i)_{i=0}^k$ *is a partition of* $V(G)$ *satisfying conditions* (R1)–(R5), *and let* $H$ *be the corresponding* $(\epsilon, d)$*-reduced 2-edge coloured graph.*

(a) *If there is a set* $S \subseteq V(H)$ *of order at least* $(2/3-\delta/2)k$ *such that* $\Delta(R_H[S]) \leqslant 10\delta k$, *then* $G$ *contains a blue cycle of length* $\ell$ *for all* $\ell \in [3, (2/3-\delta)n]$.
(b) *If there is a set* $S \subseteq V(H)$ *of order at least* $(2/3-\delta/2)k$ *such that* $\Delta(B_H[S]) \leqslant 10\delta k$, *then* $G$ *contains a red cycle of length* $\ell$ *for all* $\ell \in [3, (2/3-\delta)n]$.

**Lemma 4.5.** *Given* $0 < \delta < 1/144$, *let* $\epsilon$ *and* $d$ *be real numbers such that* $0 < \epsilon \ll d \ll \delta$. *Let* $G$ *be a graph of sufficiently large order* $n$ *with* $\delta(G) \geqslant 3n/4$. *Suppose that* $E(G) = E(R_G) \cup E(B_G)$ *is a 2-edge colouring of* $G$, $(V_i)_{i=0}^k$ *is a partition of* $V(G)$ *satisfying conditions* (R1)–(R5), *and let* $H$ *be the corresponding* $(\epsilon, d)$*-reduced 2-edge coloured graph.*

*Suppose that there is a partition* $V(H) = U_1 \cup \cdots \cup U_4$ *with* $\min_i |U_i| \geqslant (1/4-3\delta)k$ *such that there are no red edges from* $U_1 \cup U_2$ *to* $U_3 \cup U_4$ *and no blue edges from* $U_1 \cup U_3$ *to* $U_2 \cup U_4$. *Then* $G$ *contains a monochromatic cycle of length at least* $(1-38\delta)n$ *and monochromatic cycles of length* $\ell$ *for all* $\ell \in [4, \lceil n/2 \rceil]$.

Lemmas 4.3–4.5 will be proved in Section 6. We now prove Theorem 1.6 as outlined above.

### 4.1. Proof of Theorem 1.6

Choose $0 < \delta < 1/144$ and $0 < \epsilon \ll d \ll \delta$. In particular, we may assume that $d \geqslant \epsilon^{1/3}$. Choose $k_0 > 1/\epsilon^2$ so that one can apply Lemmas 4.1 and 4.2 on any graph with at least $k_0$ vertices. From Theorem 2.4, we obtain $M = M(\epsilon, k_0)$ and set $K = 6M$. We take $n$ sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geqslant 3n/4$, with 2-edge colouring $E(G) = E(R_G) \cup E(B_G)$.

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By Lemma 3.2, we have that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring, or $G$ contains a monochromatic $C_\ell$ for all $\ell \in [4, K]$. Hence it is sufficient to prove that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring, or $G$ contains a monochromatic $C_\ell$ for all $\ell \in [K, \lceil n/2 \rceil]$.

We apply the degree form of the 2-colour Regularity Lemma to $G$, with parameters $d$ and $\epsilon$. Let $V_0, V_1, \ldots, V_k$ be the partition of $V(G)$ satisfying conditions (R1)–(R5), and $G'$ be the subgraph of $G$ defined by Theorem 2.4. Finally, let $H$ be the $(\epsilon, d)$-reduced graph defined from $G'$ earlier, with 2-edge colouring $E(H) = E(R_H) \cup E(B_H)$.

We first observe that

$$\delta(H) \geqslant \left(\frac{3}{4} - \delta\right)k. \tag{4.1}$$

Indeed, by (R3), we have $\delta(G') \geqslant (3/4 - 2d - \epsilon)n$. Suppose that $\delta(H) < (3/4 - \delta)k$. Then there exists some $i \geqslant 1$ with $d_H(v_i) < (3/4 - \delta)k$. For a vertex $v \in V_i$ , its neighbours in $G'$ are only in $V_0$, or in $V_j$ for those $j$ such that $v_i v_j$ is an edge of $H$. Hence

$$d_{G'}(v) < \left(\frac{3}{4} - \delta\right)km + |V_0| \leqslant \left(\frac{3}{4} - \delta + \epsilon\right)n,$$

which is a contradiction, as $\delta \gg 2d + 2\epsilon$.

Applying Lemma 4.1 to $H$, we have one of the following possibilities.

(i) There is a component of $R_H$ or $B_H$ which contains a matching on at least $(2/3 + \delta)k$ vertices.

(ii) There is a set $S$ of order at least $(2/3 - \delta/2)k$ such that either $\Delta(R_H[S]) \leqslant 10\delta k$ or $\Delta(B_H[S]) \leqslant 10\delta k$.

(iii) There is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geqslant (1/4 - 3\delta)k$ such that there are no blue edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no red edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

If (ii) or (iii) occurs, then we are done immediately by Lemma 4.4 and Lemma 4.5 respectively. Hence we assume that there is a component of $R_H$ or $B_H$ which contains a matching on at least $(2/3 + \delta)k$ vertices. By Lemma 4.2, we may assume that there is a monochromatic component $C$ which contains a matching on at least $(1/2 + \delta)k$ vertices, and that either $C$ contains an odd cycle or $|V(C)| \geqslant (1 - 5\delta)k$.

Assume first that

$$C = R'_H \text{ is a component of } R_H. \tag{4.2}$$

If the component $R'_H$ of $R_H$ contains an odd cycle, then we are done because, by Lemma 2.5(b), $G$ contains red cycles of any length between $4k$ and $(1/2 + \delta/2)n$ and $4k < K$.

Suppose now that $R'_H$ contains no odd cycles and hence $|V(R'_H)| \geqslant (1 - 5\delta)k$. Then $R'_H$ is bipartite, with classes $H_1$ and $H_2$. Applying Lemma 2.5(a), we deduce that $C_\ell \subseteq R_G$ for all even $\ell \in [4, (1/2 + \delta/2)n]$. Hence we are done if we can show that $G$ contains a monochromatic $C_\ell$ for all odd $\ell \in [K, \lceil n/2 \rceil]$. We start with the following claim.

**Claim 4.6.** *There are disjoint sets of vertices $X$ and $Y$ with $|X \cup Y| \geqslant (1 - 6\delta)n$ with the following properties.*

(a) *Between any vertices* $u, v \in X$ *(or* $u, v \in Y$*) there are, in* $R_G[X \cup Y]$*, paths of length* $\ell$ *for all even* $\ell \in [3k, (1/2 + \delta/2)n]$.

(b) *Between any* $u \in X$ *and any* $v \in Y$ *there are, in* $R_G[X \cup Y]$*, paths of length* $\ell$ *for all odd* $\ell \in [3k - 1, (1/2 + \delta/2)n]$.

**Proof.** Take a matching $M$ in $R'_H$ with a maximal number of vertices. Let $r \geqslant (1/2 + \delta)k/2$ be the number of edges in the matching. Since $R'_H$ is connected, there exists a spanning tree $T$ that contains all edges of $M$.

For every vertex $v_i \in V(R'_H)$, fix an edge $D_i = v_i v_{\ell_i}$ in $T$. Let $V'_i$ be the set of all vertices in $V_i$ with at least $(d - \epsilon)m$ red neighbours in $V_{\ell_i}$. By Fact 2.1, we have that $|V'_i| \geqslant (1 - \epsilon)|V_i|$.

Let $X$ be the union of all $V'_i$ such that $v_i \in H_1$, and let $Y$ be the union of all $V'_i$ such that $v_i \in H_2$, so that $X$ and $Y$ are subsets of $V(G)$. Notice that

$$|X \cup Y| \geqslant |H_1|(1 - \epsilon)m + |H_2|(1 - \epsilon)m \geqslant (1 - 5\delta)k(1 - \epsilon)^2 \frac{n}{k} > (1 - 6\delta)n.$$

Now we show that (a) holds. Let $u, v \in X$ and suppose that $u \in V'_i$ and $v \in V'_j$. We claim there is a walk in $R'_H$ from $v_i$ to $v_j$, starting with edge $D_i$, ending with edge $D_j$, containing all edges of $M$, and whose number of edges is even and at most $3k$.

Indeed, one starts at $v_i$, continues to $v_{\ell_i}$ (using $D_i$), then walks around $T$ using each edge of $T$ twice (including those in $M$) until coming back to $v_{\ell_i}$, then continues to $v_{\ell_j}$ via the unique $v_{\ell_i}$–$v_{\ell_j}$ path in $T$, and finally ends in $v_j$ (using $D_j$). This walk has at most $1 + 2(k - 1) + (k - 1) + 1 < 3k$ edges. Furthermore, the number of edges in this walk is even, because $R'_H$ is bipartite and both endpoints of the walk are from the same partition class.

For convenience, denote the vertices of this walk by $v_{t_1}, v_{t_2}, \ldots, v_{t_\ell}$, where $v_{t_1} = v_i$, $v_{t_2} = v_{\ell_i}$, $v_{t_{\ell-1}} = v_{\ell_j}$, and $v_{t_\ell} = v_j$. Using Fact 2.1 repeatedly, we construct a path $P = w_1, w_2 \ldots, w_\ell$ in $R_G$ such that $w_1 = u$, $w_\ell = v$, and, for $1 < j < \ell$, $w_j \in V_{t_j}$ has at least $(d - \epsilon)m$ red neighbours in both $V_{t_{j-1}}$ and $V_{t_{j+1}}$.

Consider the edge $w_{t_j} w_{t_{j+1}}$. The bipartite graph

$$R_j = R[(V_{t_j} \setminus P) \cup \{w_{t_j}\}, (V_{t_{j+1}} \setminus P) \cup \{w_{t_{j+1}}\}]$$

is $(2\epsilon)$-regular with density at least $d - \epsilon > \epsilon^{1/3}/2$, and both $w_{t_j}$ and $w_{t_{j+1}}$ have large degree in $R_j$. Using Lemma 2.2 with $\beta = \epsilon^{1/3}$, we obtain that $R_j$ contains a $w_{t_j}$–$w_{t_{j+1}}$ path of any odd length between 3 and $2(1 - 10\epsilon^{2/3})(m - 3k)$.

Hence, by replacing edge $w_{t_j} w_{t_{j+1}}$ in $P$ by a $w_{t_j}$–$w_{t_{j+1}}$ path of appropriate odd length in $B_j$ for one occurrence $v_{t_j} v_{t_{j+1}}$ of each edge in $M$, we obtain a red $u$–$v$ path of any even length between $3k$ and $2(1 - 10\epsilon^{2/3})(m - 3k)r$. Using $\epsilon \ll d \ll \delta$, a short calculation shows that $2(1 - 10\epsilon^{2/3})(m - 3k)r > (1/2 + \delta/2)n$.

The proof of (b) is straightforward: if $u \in X$ and $v \in Y$, take any neighbour $w \in X$ of $v$ and apply part (a) to $u, w$. □

If either $X$ or $Y$ contains an internal red edge $uv$, then, by Claim 4.6(a), we have red cycles of all odd lengths between $3k + 1$ and $(1/2 + \delta/2)n$. Hence we may assume that $X$ and $Y$ are independent sets in $R_G$. However, if $|X| > n/2$ or $|Y| > n/2$, then we

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are done by Lemma 4.3. Hence we may assume that $|X|, |Y| \leqslant n/2$ and, consequently, $\min\{|X|, |Y|\} \geqslant (1/2 - 6\delta)n$.

If any vertex $w$ of $V(G) \setminus (X \cup Y)$ has at least one red neighbour in both $X$ and $Y$, then using the paths between a red neighbour $u$ of $w$ in $X$ and a red neighbour $v$ of $w$ in $Y$ (see Claim 4.6(b)), we have cycles of all odd lengths between $3k+1$ and $(1/2+\delta/2)n$. Hence all vertices of $V(G) \setminus (X \cup Y)$ have no red neighbours in at least one of $X$ or $Y$.

Define disjoint sets $X'$ and $Y'$ by letting $X'$ be the set of vertices of $V(G) \setminus (X \cup Y)$ with at least two red neighbours in $Y$, and $Y'$ be the set of vertices of $V(G) \setminus (X \cup Y)$ with at least two red neighbours in $X$. Then there are no red edges between $X'$ and $X$ or between $Y'$ and $Y$. If there is a red edge $u'v'$ within $X'$, let $u$ and $v$ be distinct vertices of $Y$ with $uu'$ and $vv'$ both red edges. Then $uu'v'v$ is a red path of length three between vertices of $Y$, with internal vertices in $V \setminus (X \cup Y)$. Using the $u$–$v$ paths obtained from Claim 4.6(a), we have red cycles of all odd lengths between $3k+3$ and $(1/2+\delta/2)n$.

So we may assume that $R_G[X \cup X' \cup Y \cup Y']$ is bipartite with partite sets $X \cup X'$ and $Y \cup Y'$. We may assume that both $X \cup X'$ and $Y \cup Y'$ have order at most $n/2$; otherwise, we are done by Lemma 4.3.

A vertex not in $X \cup X' \cup Y \cup Y'$ has at least $(3/4 - 6\delta)n$ neighbours in $X \cup Y$. Let $X''$ be the set of vertices not in $X \cup X' \cup Y \cup Y'$ with at least $(3/8 - 3\delta)n$ neighbours in $X$, and let $Y''$ be the set of vertices not in $X \cup X' \cup Y \cup Y'$ with at least $(3/8 - 3\delta)n$ neighbours in $Y$. Letting $X_0 = X \cup X' \cup X''$ and $Y_0 = Y \cup Y' \cup Y''$, we see that $V(G)$ is the (not necessarily disjoint) union of $X_0$ and $Y_0$.

Without loss of generality, we may assume that $|X_0| \geqslant n/2$. By definition, all vertices in $X''$ have at least $(3/8 - 3\delta)n$ neighbours in $X$. However, vertices in $X''$ have at most one red neighbour in $X$, else they would have been in $Y'$. All vertices in $X \cup X'$ have at most $n/4$ non-neighbours in $G$ and so at least $|X_0| - n/4$ neighbours in $X_0$. Since there are at most $|X''|$ red edges between $X''$ and $X$, the set $X'''$ of vertices in $X$ with a red neighbour in $X''$ has order at most $|X''|$. Vertices in $X \setminus X'''$ have no red neighbours in $X_0$, while vertices in $X' \cup X'''$ have no red neighbours in $X_0 \setminus X''$. Hence

$$d_{B[X_0]}(v) \geqslant \begin{cases} |X_0| - \frac{n}{4} & v \in X \setminus X''', \\ |X_0| - \frac{n}{4} - |X''| & v \in X''' \cup X', \\ \left(\frac{3}{8} - 3\delta\right)n - 1 & v \in X''. \end{cases} \tag{4.3}$$

Since $|X' \cup X'' \cup X'''| \leqslant |X' \cup X''| + |X''| \leqslant 12\delta n$, the conditions of Theorem 2.6 are satisfied on the graph $B_G[X_0]$, and so $B_G[X_0]$ is hamiltonian.

Furthermore, using (4.3), we have

$$\begin{aligned} e(B_G[X_0]) \geqslant & \frac{1}{2}\left(|X_0| - \frac{n}{4}\right)|X \setminus X'''| + \left(\left(\frac{3}{16} - \frac{3\delta}{2}\right)n - \frac{1}{2}\right)|X''| \\ & + \frac{1}{2}\left(|X_0| - \frac{n}{4} - |X''|\right)(|X'''| + |X'|) \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}|X_0|\left(|X_0| - \frac{n}{4}\right) \\
&\quad + \left(\left(\frac{3}{16} - \frac{3\delta}{2}\right)n - \frac{1}{2} - \frac{1}{2}\left(|X_0| - \frac{n}{4}\right) - \frac{1}{2}(|X'''| + |X'|)\right)|X''| \\
&\geqslant \frac{1}{4}|X_0|^2 + \left(\left(\frac{1}{16} - \frac{15\delta}{2}\right)n - \frac{1}{2}\right)|X''|.
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
\frac{1}{2}\left(|X_0| - \frac{n}{4}\right) + \frac{1}{2}(|X'''| + |X'|) &\leqslant \frac{1}{2}\left(|X| - \frac{n}{4}\right) + |X'| + |X''| \\
&\leqslant \frac{n}{8} + |V(G) \setminus (X \cup Y)| \\
&\leqslant \left(\frac{1}{8} + 6\delta\right)n.
\end{aligned}
$$

From Theorem 1.2 we deduce that either $B_G[X_0]$ is pancyclic, in which case $C_\ell \subseteq B_G$ for all $\ell \in [3, |X_0|]$, or $B_G[X_0] \cong K_{|X_0|/2,|X_0|/2}$ and $e(B_G[X_0]) = |X_0|^2/4$. In the latter case, this means that $X'' = \emptyset$. Similarly, if $|Y_0| \geqslant n/2$, then either $B_G[Y_0]$ is pancyclic, or $Y'' = \emptyset$. Hence we may assume that $X'' = Y'' = \emptyset$ and, therefore, $B_G$ is bipartite. Thus, by Lemma 4.3, we are done.

Now we only need to discuss what changes are needed if, instead of (4.2), we assume that the component $C$ is a blue component $B'_H$ of $B_H$.

If the component $B'_H$ contains an odd cycle, then we are again done by Lemma 2.5(b). Otherwise, $B'_H$ is bipartite and $|V(B'_H)| \geqslant (1 - 5\delta)k$. Applying Lemma 2.5(a), we again deduce that $C_\ell \subseteq B_G$ for all even $\ell \in [4, (1/2 + \delta/2)n]$. By following the proof of Claim 4.6 with colours red and blue swapped, we realize that the following holds.

**Claim 4.7.** *There are disjoint sets of vertices $X$ and $Y$ with $|X \cup Y| \geqslant (1 - 6\delta)n$ with the following properties.*

(a) *Between any vertices $u, v \in X$ (or $u, v \in Y$) there are, in $B_G[X \cup Y]$, paths of length $\ell$ for all even $\ell \in [3k, (1/2 + \delta/2)n]$.*
(b) *Between any $u \in X$ and any $v \in Y$ there are, in $B_G[X \cup Y]$, paths of length $\ell$ for all odd $\ell \in [3k - 1, (1/2 + \delta/2)n]$.* □

Notice that the asymmetry in the definition of the colouring of $H$ is not an issue here. In the proof of Claim 4.6, we only use the fact that the red edges in $H$ correspond to $\epsilon$-regular pairs in $G'$ whose density of red edges is at least $d$. Hence, in the proof of Claim 4.7, we only need the fact that the blue edges in $H$ correspond to $\epsilon$-regular pairs in $G'$ whose density of blue edges is at least $d$.

Having Claim 4.7, we follow the lines of the proof after Claim 4.6 with colours red and blue interchanged. This completes the proof of Theorem 1.6.

## 5. Proof of Lemmas 4.1 and 4.2

In this section and in Section 6 we shall prove the lemmas used in the proof of Theorem 1.6. More precisely, we give the proofs of Lemma 4.1 and Lemma 4.2 here and leave the proofs of Lemmas 4.3–4.5 to the next section.

Throughout both proofs we shall assume that for a given 2-edge colouring $E(G) = E(R) \cup E(B)$ of a graph $G$, $R'$ is a largest component of $R$, and that $B'$ is a largest component of $B$.

We will need the following claim about the component structure.

**Lemma 5.1.** *Let $0 < \delta < 1/36$ and let $G$ be a graph of sufficiently large order $n$ with $\delta(G) \geqslant (3/4 - \delta)n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.*

*Then one of the following holds.*

(a) *One of $R$ or $B$ is connected.*

(b) *For a largest component $R'$ of $R$, and a largest component $B'$ of $B$, we have that $V(G) = V(R') \cup V(B')$ and both $R'$ and $B'$ have order at least $(3/4 - \delta)n$.*

(c) *There is a partition $V(G) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geqslant (1/4 - 3\delta)n$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.*

**Proof.** Assume that (a) does not hold, that is, both $R$ and $B$ are disconnected. Suppose first that $|V(R')| \leqslant (5/12 - \delta)n$. Then $\Delta(R) < (5/12 - \delta)n$ and hence $\delta(B) \geqslant \delta(G) - \Delta(R) > n/3$. Since $B$ is disconnected, we see that $B$ has exactly two components $B_1 = B'$ and $B_2$ with $n/3 < |V(B_2)| \leqslant |V(B_1)| < 2n/3$. Set $W_i = V(B_i) \cap V(R')$, for $i \in \{1,2\}$.

Suppose that $W_i \neq \emptyset$, for some $i \in \{1,2\}$. Let $v \in W_i$. Then $v$ has no neighbours outside $R' \cup B_i$, and so $\Gamma_G(v) \subseteq R' \cup B_i$. Hence, as $W_{3-i} = V(R') \setminus V(B_i)$,

$$|W_{3-i}| \geqslant |\Gamma_G(v)| - |V(B_i)| > \left(\frac{1}{12} - \delta\right)n.$$

In particular $W_{3-i} \neq \emptyset$, and so $W_1$ is non-empty if and only if $W_2$ is non-empty. As $V(R') = W_1 \cup W_2$, we see that both $W_1$ and $W_2$ are therefore non-empty. But then

$$\begin{aligned} |V(R')| = |W_1| + |W_2| &\geqslant \left(\frac{3}{4} - \delta\right)n - |V(B_1)| + \left(\frac{3}{4} - \delta\right)n - |V(B_2)| \\ &= \left(\frac{1}{2} - 2\delta\right)n. \end{aligned}$$

This contradicts our assumption that $|V(R')| \leqslant (5/12 - \delta)n$.

We may thus assume that $|V(R')| > (5/12 - \delta)n$, and similarly $|V(B')| > (5/12 - \delta)n$. Let $W_1 = V(B') \cap V(R')$, $W_2 = V(R') \setminus V(B')$, $W_3 = V(B') \setminus V(R')$ and $W_4 = V(G) - (W_1 \cup W_2 \cup W_3)$. Note that there are no edges (of either colour) between $W_1$ and $W_4$ or between $W_2$ and $W_3$.

If $W_4 = \emptyset$, then $V(G) = V(R') \cup V(B')$. As neither $R$ nor $B$ is connected, we must have that both $W_2$ and $W_3$ are non-empty. Let $v \in W_2$. Since $\Gamma_G(v) \cap W_3 = \emptyset$, we see that $|W_3| \leqslant (1/4 + \delta)n$ and so $|V(R')| = |W_1| + |W_2| = n - |W_3| \geqslant (3/4 - \delta)n$. We may similarly show that $|V(B')| \geqslant (3/4 - \delta)n$. Hence, (b) holds.

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If, however, $W_4 \neq \emptyset$, choose any $x \in W_4$. As $\Gamma_G(x) \cap W_1 = \emptyset$, we have $|W_1| \leqslant (1/4+\delta)n$. However, both $R'$ and $B'$ have order at least $(5/12-\delta)n$ and hence both $W_2$ and $W_3$ are non-empty. Thus, arguing as for $W_4$, we see that both $W_2$ and $W_3$ have order at most $(1/4+\delta)n$ and so $W_1$ is also non-empty. This, in turn, implies that $W_4$ has order at most $(1/4+\delta)n$. Hence each $W_i$ has order at least $(1/4-3\delta)n$. Furthermore, there is no red edge between $W_1 \cup W_2 = V(R')$ and $W_3 \cup W_4 = V(G) \setminus V(R')$ and no blue edge between $W_1 \cup W_3 = V(B')$ and $W_2 \cup W_4 = V(G) \setminus V(B')$. Hence, (c) holds. □

### 5.1. Proof of Lemma 4.1

Let $0 < \delta < 1/36$. We assume throughout that $n$ is sufficiently large. Suppose that $G$ is a graph of order $n$ with $\delta(G) \geqslant (3/4-\delta)n$ and that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$ such that the conclusions (i) and (iii) do not hold. Since (i) is not true, we assume that

$$\text{neither } R' \text{ nor } B' \text{ contains a matching on at least } \left(\tfrac{2}{3}+\delta\right)n \text{ vertices.} \tag{5.1}$$

We are aiming to show that (ii) holds, that is, that there is a large set on which one of the colours has a very low density. We first show that the orders of $B'$ and $R'$ cannot take certain values.

**Claim 5.2.** *Either $|V(B')| < (1/3+\delta/2)n$ or $|V(B')| > (2/3-\delta/2)n$.*

**Proof.** Suppose that $(1/3+\delta/2)n \leqslant |V(B')| \leqslant (2/3-\delta/2)n$. We apply Lemma 5.1: (c) cannot be true because (iii) does not hold, (b) fails because $|V(B')| < (3/4-\delta)n$, and so (a) must be true. Since $B$ is disconnected, we conclude that $R$ is connected.

Let $V_1$ be the smaller of $V(B')$ and $V \setminus V(B')$. Let $V_2 = V \setminus V_1$ and $F = G[V_1, V_2]$ be the bipartite graph between $V_1$ and $V_2$. There are no blue edges between $V_1$ and $V_2$ and so all edges of $F$ are red. For a subset $S$ of $V_1$ we shall find a lower bound on $|\Gamma_F(S)|$ by splitting into the cases that $|S| > (1/4+\delta)n$ and $|S| \leqslant (1/4+\delta)n$.

For $S \subseteq V_1$ with $|S| > (1/4+\delta)n$, consider a vertex $v \in V_2$. Then, as $d_G(v) \geqslant (3/4-\delta)n$, $v$ must have a neighbour in $S$. Hence $\Gamma_F(S) = V_2$, and so $|\Gamma_F(S)| = |V_2| \geqslant |V_1| \geqslant |S|$.

If $S \subseteq V_1$ and $|S| \leqslant (1/4+\delta)n$, then every vertex in $S$ has at least $|V_2| - (1/4+\delta)n$ neighbours in $V_2$. Hence

$$\begin{aligned}
|\Gamma_F(S)| &\geqslant |V_2| - \left(\frac{1}{4}+\delta\right)n \\
&= |S| - \left(\left(\frac{1}{4}+\delta\right)n + |S| - |V_2|\right) \\
&\geqslant |S| - \left(\left(\frac{1}{2}+2\delta\right)n - |V_2|\right).
\end{aligned}$$

Thus, by the defect form of Hall's Theorem, $F$ contains a matching with at least

$$|V_1| - \max\left\{0, \left(\frac{1}{2}+2\delta\right)n - |V_2|\right\} = \min\left\{|V_1|, |V_1|+|V_2| - \left(\frac{1}{2}+2\delta\right)n\right\}$$

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edges. As $|V_1| + |V_2| = n$ and $|V_1| \geqslant (1/3 + \delta/2)n$, this matching contains at least $(2/3 + \delta)n$ vertices. As $R$ is connected, this contradicts (5.1). Hence Claim 5.2 holds. □

Let $X_R = V(G) \setminus V(R')$ and $X_B = V(G) \setminus V(B')$. We define the following sets when $R'$ or $B'$ is large.

**Definition.** Suppose that $|V(R')| \geqslant (2/3 + \delta)n$.

- Let $S_R \subseteq V(R')$ be a set such that

$$q(R[V(R') - S_R]) > |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n.$$

  Note that, in view of (5.1), such a set exists by Theorem 2.8 applied with $d = |V(R')| - (2/3 + \delta)n$.
- For $1 \leqslant i \leqslant n$, let $T_{R,i}$ be the set of vertices which lie in components of $R[V(R') - S_R]$ of order $i$.
- Let $T_R = \bigcup_{1 \leqslant i \leqslant t} T_{R,i}$, where $t = \lceil \delta^{-1} \rceil$ .

If $|V(B')| \geqslant (2/3 + \delta)n$, we define $S_B$, $T_{B,i}$ and $T_B$ similarly.

We shall use the following result throughout. Note that, as with Claim 5.2, we may exchange the roles of $R$ and $B$ to obtain a symmetrical version of this result.

**Claim 5.3.** *Suppose that $|V(R')| \geqslant (2/3 + \delta)n$. Then $|S_R| < (1/3 + \delta/2)n$ and*

$$|X_R \cup T_R| > |S_R| + \left(\frac{1}{3} - 2\delta\right)n.$$

*Further, if $C_B$ is a component of $B$ with $|V(C_B)| \leqslant (5/12 - 2\delta)n$, then $V(C_B) \subseteq S_R$.*

**Proof.** All vertices of $V(R')$ lie in $S_R$ or some component of $R[V(R') - S_R]$. Hence

$$\begin{aligned} |V(R')| &\geqslant |S_R| + q(R[V(R') - S_R]) \\ &> 2|S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n. \end{aligned}$$

This implies that $|S_R| < (1/3 + \delta/2)n$.

There are at most $|T_R|$ components of $R[V(R') - S_R]$ of order at most $t$. However, there are at most $n/t \leqslant \delta n$ components of $R[V(R') - S_R]$ of order at least $t$. Hence $|T_R| \geqslant q(R[V(R') - S_R]) - \delta n$. As $X_R$ and $T_R$ are disjoint, we have

$$\begin{aligned} |X_R \cup T_R| &\geqslant n - |V(R')| + q(R[V(R') - S_R]) - \delta n \\ &> (1 - \delta)n - |V(R')| + |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n \\ &= |S_R| + \left(\frac{1}{3} - 2\delta\right)n. \end{aligned}$$

Finally, suppose that $C_B$ is a component of $B$ with $|V(C_B)| \leqslant (5/12 - 2\delta)n$. A vertex in $C_B$ has blue degree at most $|V(C_B)| - 1$. Hence any vertex in $C_B$ must have red degree at

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least

$$\begin{aligned}\delta(G) - |V(C_B)| + 1 &\geqslant \left(\frac{3}{4} - \delta\right)n - \left(\frac{5}{12} - 2\delta\right)n + 1 \\ &= \left(\frac{1}{3} + \delta\right)n + 1. \end{aligned} \tag{5.2}$$

A vertex in $X_R$ has red degree at most

$$|X_R| - 1 = n - |V(R')| - 1 \leqslant \left(\frac{1}{3} - \delta\right)n - 1.$$

However, a vertex in $T_R$ is in a component of $R[V(R') - S_R]$ of order at most $t$. Hence, for all $v \in T_R$,

$$\begin{aligned} d_R(v) &\leqslant t + |S_R| \\ &< \left(\frac{1}{3} + \frac{1}{2}\delta\right)n + t. \end{aligned}$$

Hence (5.2) and $n \gg 1/\delta$ imply that $V(C_B) \cap (X_R \cup T_R) = \emptyset$.

Suppose that there exists $v \in V(C_B) \setminus S_R$. All blue neighbours of $v$ lie in $C_B$, and so $v$ has no blue neighbours in $X_R \cup T_R$. However, $V(C_B) \subseteq V(R')$ because $V(C_B) \cap X_R = \emptyset$, and so $v$ has no red neighbours in $X_R$. The only vertices with red neighbours in $T_R$ are those in $S_R \cup T_R$, and so we see that $v$ also has no red neighbours in $T_R$. Hence $v$ has no neighbours in $X_R \cup T_R$, and so

$$d_G(v) \leqslant n - |X_R \cup T_R| < \left(\frac{2}{3} + 2\delta\right)n.$$

This contradicts $\delta(G) \geqslant (3/4 - \delta)n$, and so $V(C_B) \subseteq S_R$. □

We may thus assume that $S_R$ is not much bigger than $n/3$. The following result shows that if $R'$ is very large and $S_R$ has order approaching $n/3$, then $S = V(G) \setminus S_R$ is the set we are looking for in (ii).

**Claim 5.4.** *Suppose that $|V(R')| \geqslant (1 - 5\delta/2)n$ and that $|S_R| \geqslant (1/3 - 2\delta)n$. Then $R[V(G) \setminus S_R]$ is a graph on at least $(2/3 - \delta/2)n$ vertices with maximum degree at most $10\delta n$.*

**Proof.** That $R[V(G) \setminus S_R]$ is a graph of order at least $(2/3 - \delta/2)n$ follows immediately from Claim 5.3.

For all $1 \leqslant i \leqslant n$, there are exactly $|T_{R,i}|/i$ components of order $i$ in $R[V(R') \setminus S_R]$. Hence

$$\begin{aligned} \sum_{i \geqslant 1} \frac{1}{2i-1} |T_{R,2i-1}| &= q(R[V(R') \setminus S_R]) \\ &> |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n. \end{aligned}$$

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However,

$$\sum_{i\geqslant 1}\frac{1}{2i-1}|T_{R,2i-1}| \leqslant |T_{R,1}| + \frac{1}{3}\sum_{i\geqslant 2}|T_{R,2i-1}|$$
$$\leqslant |T_{R,1}| + \frac{1}{3}\big(|V(R')| - |S_R| - |T_{R,1}|\big).$$

Combining these inequalities, and using the bounds on $|V(R')|$ and $|S_R|$, we have

$$\begin{aligned}\frac{2}{3}|T_{R,1}| &> \frac{4}{3}|S_R| + \frac{2}{3}|V(R')| - \left(\frac{2}{3}+\delta\right)n\\ &\geqslant \frac{4}{3}\left(\frac{1}{3}-2\delta\right)n + \frac{2}{3}\left(1-\frac{5\delta}{2}\right)n - \left(\frac{2}{3}+\delta\right)n\\ &= \left(\frac{4}{9}-\frac{16\delta}{3}\right)n.\end{aligned}$$

Hence $|T_{R,1}| > (2/3-8\delta)n$.

However, $T_{R,1}$ is a set of isolated vertices in $R[V(G)\setminus S_R]$. As $|V(G)\setminus S_R| \leqslant (2/3+2\delta)n$, we see that $R[V(G)\setminus S_R]$ has maximum degree at most $10\delta n$. □

We may now complete the proof of Lemma 4.1, using the preceding claims. Since (iii) does not hold, by Lemma 5.1, we may assume that

$$\text{either one of } R \text{ or } B \text{ is connected, or } V(G) = V(R')\cup V(B') \text{ and } \min\{|V(R')|,|V(B')|\} \geqslant (3/4-\delta)n. \tag{5.3}$$

In either case, there will be a monochromatic component of order at least $(3/4-\delta)n > (2/3+\delta)n$. We may without loss of generality assume that this component is $R'$. We consider several cases depending on the order of $B'$.

If $|V(B')| < (1/3+\delta/2)n$, then every component of $B$ has order at most $(1/3+\delta/2)n$. By Claim 5.3, $|S_R| < (1/3+\delta/2)n$ and $S_R$ contains all blue components of order at most $(5/12-2\delta)n$. Since $(5/12-2\delta)n > (1/3+\delta/2)n$, $S_R$ contains all components of $B$, and hence has order $n$, a contradiction.

We cannot have $(1/3+\delta/2)n \leqslant |V(B')| \leqslant (2/3-\delta/2)n$ by Claim 5.2.

If $(2/3-\delta/2)n < |V(B')| < (2/3+\delta)n$, then, by (5.3), $R$ is connected. Also, all components of $B$ other than $B'$ have order at most $(1/3+\delta/2)n$. Hence, by Claim 5.3, $S_R$ contains $X_B = V(G)\setminus V(B')$ and so $|S_R| > (1/3-\delta)n$. Thus we are done by Claim 5.4.

Finally, suppose that $|V(B')| \geqslant (2/3+\delta)n$. In this case, sets $S_B \subseteq V(B')$ and $T_B$ are defined, and, in particular, we have that

$$q(B[V(B')-S_B]) > |S_B| + |V(B')| - \left(\frac{2}{3}+\delta\right)n.$$

By Claim 5.3, we see that $X_R \subseteq S_B$ and $X_B \subseteq S_R$.

Suppose that there is a vertex $v \in T_R \cap T_B$. Then $v$ has at most $|S_R| + t$ red neighbours and at most $|S_B| + t$ blue neighbours. As $|S_R|$ and $|S_B|$ both have order at most $(1/3+\delta/2)n$ and $n \gg 1/\delta$, this contradicts $d_G(v) \geqslant \delta(G) \geqslant (3/4-\delta)n$. Hence $T_R \cap T_B = \emptyset$.

Suppose that $T_B \setminus S_R$ is non-empty, and let $v \in T_B \setminus S_R$. As $X_R \subseteq S_B$, we have $T_B \subseteq V(R')$. Hence $v$ has no red neighbours in $X_R$. Vertices in $T_R$ only have red neighbours in

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$T_R \cup S_R$. However, $T_B \cap T_R = \emptyset$, and so $v \notin T_R \cup S_R$. In particular, $v$ has no red neighbours in $X_R \cup T_R$.

Hence, $v$ has at least $|X_R \cup T_R| - (1/4+\delta)n$ blue neighbours in $X_R \cup T_R$, as $v$ has at most $(1/4+\delta)n$ non-neighbours. However, $v \in T_B$, and so all but $t$ of its blue neighbours are in $S_B$. Hence

$$|S_B| \geqslant |X_R \cup T_R| - \left(\frac{1}{4}+\delta\right)n - t > |S_R| + \left(\frac{1}{12} - 3\delta\right)n - t,$$

where the second inequality uses Claim 5.3.

Similarly, if $T_R \setminus S_B$ is non-empty, then

$$|S_R| > |S_B| + \left(\frac{1}{12} - 3\delta\right)n - t.$$

As these cannot both occur, one of $T_R \setminus S_B$ or $T_B \setminus S_R$ is empty.

We assume without loss of generality that $T_B \subseteq S_R$. Then $S_R$ contains the disjoint sets $T_B$ and $X_B$. Hence, using Claim 5.3, again

$$|S_R| \geqslant |T_B \cup X_B| > |S_B| + \left(\frac{1}{3} - 2\delta\right)n.$$

Thus $|S_R| \geqslant (1/3-2\delta)n$. As $|S_R| < (1/3+\delta/2)n$, we must have $|S_B| \leqslant 5\delta n/2$. Since $X_R \subseteq S_B$, we see that $|V(R')| \geqslant (1-5\delta/2)n$. Hence, by Claim 5.4, we are done.

This concludes the proof Lemma 4.1. We now prove Lemma 4.2, using similar methods to those used in the proof of Lemma 4.1.

### 5.2. Proof of Lemma 4.2

Let $0 < \delta < 1/36$. We assume throughout that $n$ is sufficiently large. Suppose that $G$ is a graph of order $n$ with $\delta(G) \geqslant (3/4-\delta)n$ and that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.

Suppose that $R'$ contains a matching on at least $(2/3+\delta)n$ vertices. We may assume that $|V(R')| < (1-5\delta)n$ and $R'$ is bipartite with classes $Y_R$ and $Z_R$, otherwise we are done. Without loss of generality, we assume that $|Z_R| \geqslant |Y_R|$ and so $|Y_R| \leqslant |V(R')|/2 < (1/2-5\delta/2)n$. As each edge of the matching contains one vertex from $Z_R$ and one from $Y_R$, we have

$$\left(\frac{1}{3}+\frac{\delta}{2}\right)n \leqslant |Y_R| \leqslant |Z_R| < \left(\frac{2}{3} - \frac{11\delta}{2}\right)n. \tag{5.4}$$

As in the proof of Lemma 4.1, we let $X_R = V(G) \setminus V(R')$ and $X_B = V(G) \setminus V(B')$. Note that $5\delta n < |X_R| \leqslant (1/3-\delta)n$. We apply Lemma 5.1. Since $|V(R')| \geqslant (2/3+\delta)n$, we are not in case (c) of this lemma because, in that case, each monochromatic component has at most $(1/2+6\delta)n$ vertices. Hence, either $B$ is connected by (a), or, by (b), both $R'$ and $B'$ have order at least $(3/4-\delta)n$. Consequently, $|X_B| \leqslant (1/4+\delta)n$ and $|V(R') \cap V(B')| \geqslant (1/2-2\delta)n$. Thus, every vertex in $X_B \cap X_R$ must have a neighbour in $V(R') \cap V(B')$, which is a contradiction. So, we must have $X_B \cap X_R = \emptyset$.

**Claim 5.5.** *$B'$ contains a matching on at least $(1/2+\delta)n$ vertices.*

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**Proof.** Suppose not; then by Theorem 2.8 there is a set $S \subseteq V(B')$ such that

$$q(B[V(B') - S]) > |S| + |V(B')| - \left(\frac{1}{2} + \delta\right)n.$$

We will apply the same arguments as used in Claim 5.3 to the set $S$. All vertices of $V(B')$ lie in $S$ or some component of $B[V(B') - S]$. Hence

$$\begin{aligned}|V(B')| &\geqslant |S| + q(B[V(B') - S]) \\ &> 2|S| + |V(B')| - \left(\frac{1}{2} + \delta\right)n.\end{aligned}$$

This implies that $|S| < (1/4 + \delta/2)n$.

We let $T_B$ be the set of vertices in components of $B[V(B') - S]$ with order at most $t = \lceil \delta^{-1} \rceil$. Then, as in the proof of Claim 5.3, $|T_B| \geqslant q(B[V(B') - S]) - \delta n$.

Any vertex in $T$ has blue degree at most

$$|S| + t - 1 \leqslant \left(\frac{1}{4} + \frac{\delta}{2}\right)n + t - 1,$$

and any vertex in $X_B$ has blue degree at most

$$|X_B| - 1 \leqslant \left(\frac{1}{4} + \delta\right)n - 1.$$

Also any vertex in $X_R$ has red degree at most

$$|X_R| - 1 = n - |V(R')| - 1 \leqslant \left(\frac{1}{3} - \delta\right)n - 1,$$

and any vertex in $Z_R$ has red degree at most

$$|Y_R| < \left(\frac{1}{2} - \frac{5\delta}{2}\right)n.$$

Hence, any vertex in the intersection of $T_B \cup X_B$ and $Z_R \cup X_R$ has degree at most $(3/4 - 3\delta/2)n - 1$. Since $\delta(G) \geqslant (3/4 - \delta)n$, we deduce that $T_B \cup X_B$ does not intersect $Z_R \cup X_R$.

Hence $T_B \cup X_B \subseteq Y_R$. However, $T_B$ and $X_B$ are disjoint sets, and so

$$\begin{aligned}|Y_R| &\geqslant |T_B \cup X_B| \\ &\geqslant q(B[V(B') - S]) - \delta n + n - |V(B')| \\ &> |S| + |V(B')| - \left(\frac{1}{2} + \delta\right)n + (1 - \delta)n - |V(B')| \\ &\geqslant \left(\frac{1}{2} - 2\delta\right)n,\end{aligned}$$

a contradiction with $|Y_R| \leqslant (1/2 - 5\delta/2)$. So $B'$ contains a matching on at least $(1/2 + \delta)n$ vertices. □

We will show that $B'$ contains all vertices in $X_R \cup Z_R$. All vertices of $G$ have at most $(1/4 + \delta)n$ non-neighbours, and so any two vertices have at least $(1/2 - 2\delta)n$

common neighbours. As $|Y_R| \leqslant (1/2 - 5\delta/2)n$, every pair of vertices in $Z_R$ have a common neighbour in $V(G) \setminus Y_R$. Since all vertices in $Z_R$ have no red neighbours in $V(G) \setminus Y_R$, any two vertices in $Z_R$ have a common blue neighbour. Hence all vertices of $Z_R$ lie in the same blue component. Similarly, if $|Z_R| < (1/2 - 2\delta)n$ all vertices of $Y_R$ lie in a single blue component.

Every vertex in $X_R$ has at most $(1/4 + \delta)n$ non-neighbours in both $Y_R$ and $Z_R$. Thus, by (5.4), every vertex in $X_R$ has at least one neighbour in both $Z_R$ and $Y_R$, which is necessarily blue. Hence $X_R \cup Z_R$ lies within one component of $B$. If $|Z_R| < (1/2 - 2\delta)n$, then $B$ is connected and $B' = B$. If $B$ is not connected, then the component of $B$ containing $X_R \cup Z_R$ has order at least $n - |Y_R| \geqslant (1/2 + 5\delta/2)n$, and hence this component is $B'$.

If $|V(B')| \geqslant (1 - 5\delta)n$ or if $B'$ is not bipartite, then, in view of Claim 5.5, we are done. So, suppose now that $|V(B')| < (1 - 5\delta)n$ and $B'$ is bipartite, with classes $Z_B$ and $Y_B$.

Both $Z_B \cap Z_R$ and $Y_B \cap Z_R$ are independent sets of $G$, and hence have order at most $(1/4 + \delta)n$. If $|Z_R| < (1/2 - 2\delta)n$, then, by the above argument, $B$ is connected – a contradiction. So we may assume that $|Z_R| \geqslant (1/2 - 2\delta)n$. Hence, as $Z_R \subseteq V(B') = Y_B \cup Z_B$, both $Z_B \cap Z_R$ and $Y_B \cap Z_R$ have order at least $(1/4 - 3\delta)n$.

Let $v \in X_R$. Since $X_R \cap X_B = \emptyset$, we see that $v \in Z_B \cup Y_B$. We may assume without loss of generality that $v \in Z_B$. Then $v$ has no blue neighbours in $Z_B$, and no red neighbours in $Z_R$. In particular, $v$ has no neighbours of either colour in $Z_B \cap Z_R$, which is a set of order at least $(1/4 - 3\delta)n$. As $v$ has at most $(1/4 + \delta)n$ non-neighbours, it thus has at most $4\delta n$ non-neighbours in $Y_R \subseteq V(G) \setminus (Z_R \cap Z_B)$. However, all edges from $v$ to $Y_R$ are blue. Thus all but at most $4\delta n$ vertices in $Y_R$ lie in the same blue component as $v$. However, $v \in V(B')$, and $X_R \cup Z_R \subseteq V(B')$. Hence $B'$ contains all but $4\delta n$ vertices, contradicting our assumption that $|V(B')| < (1 - 5\delta)n$, and concluding the proof of Lemma 4.2.

## 6. Proof of Lemmas 4.3–4.5

We shall now prove Lemmas 4.3–4.5, which deal with particular cases arising from the reduced graph. In the proof of Lemma 4.5, we shall be using the graph $G' \subseteq G$ defined by the Regularity Lemma.

**Proof of Lemma 4.3.** Suppose that $B$ has an independent set $S$ with $|S| \geqslant n/2$. All vertices in $S$ have at most $n/4 - 1$ non-neighbours in $G$, and so

$$\delta(R[S]) \geqslant (|S| - 1) - \left(\frac{n}{4} - 1\right) \geqslant \frac{|S|}{2}.$$

By Corollary 1.3, either $R[S]$ is pancyclic, or $R[S] \cong K_{|S|/2,|S|/2}$. In the latter case, $\delta(R[S]) = |S|/2$, and so $|S| = n/2$. Hence, if $|S| > n/2$, then $C_\ell \subseteq R_G$ for all $\ell \in [3, |S|]$. This completes the proof of part (a).

In order to see (b), suppose that $B$ is bipartite with classes $S_1$ and $S_2$, chosen so that $|S_1| \geqslant |S_2|$. If $|S_1| > n/2$, then $C_\ell \subseteq R$ for all $\ell \in [3, |S_1|]$ by part (a), and we are done. Hence we may assume that $n$ is even and $|S_1| = |S_2| = n/2$. But by the proof of (a) above, we must have either that $C_\ell \subseteq R$ for all $\ell \in [3, n/2]$, or both $R[S_1]$ and $R[S_2]$ are isomorphic to $K_{n/4,n/4}$. This implies that $n$ is divisible by four. Also, both $B[S_1]$ and $B[S_2]$ are isomorphic to the empty graph and so $G \cong K_{n/4,n/4,n/4,n/4}$.

For $i \in \{1,2\}$, let $S_{i,1}$ and $S_{i,2}$ be the independent sets of $G$ partitioning $S_i$. Then if $R$ is not bipartite, without loss of generality, there are red edges between $S_{1,1}$ and both $S_{2,1}$ and $S_{2,2}$. Hence there is a red path of length either two or four between a vertex of $S_{2,1}$ and a vertex of $S_{2,2}$, with all internal vertices in $S_1$. As $R$ is complete between $S_{2,1}$ and $S_{2,2}$, $R$ contains $C_\ell$ for all $\ell \in [4, \lceil n/2 \rceil]$. If, however, $R$ is bipartite then the colouring is a 2-bipartite 2-edge colouring. □

**Proof of Lemma 4.4.** We shall prove part (a) first. Suppose that $S \subset V(H)$ is such that $|S| \geqslant (2/3 - \delta/2)k$ and $\Delta(R_H[S]) \leqslant 10\delta k$. We know that $\delta(H) \geqslant (3/4 - (2d + \epsilon))k$. Hence,

$$\delta(B_H[S]) > \left(\frac{3}{4} - (2d+\epsilon)\right)k - 10\delta k > \frac{k}{2} \geqslant \frac{|S|}{2}.$$

By Corollary 1.3, the graph $B_H[S]$ is pancyclic. We repeatedly apply Corollary 2.3 with $\ell = 3, \ldots, |S|$ and conclude that $B_G$ contains a monochromatic cycle of every length between 3 and $(1 - 5\sqrt{\epsilon})(|S| - 2)m$. As $m \geqslant (1-\epsilon)n/k$, $|S| \geqslant (2/3 - \delta/2)k$ and $\delta \gg \epsilon \gg 1/k$, we see that

$$(1 - 5\sqrt{\epsilon})(|S| - 2)m > (1 - 5\sqrt{\epsilon})(1-\epsilon)^2\left(\frac{2}{3} - \frac{\delta}{2}\right)n > \left(\frac{2}{3} - \delta\right)n.$$

The proof of (b) follows the same lines, with colours red and blue interchanged.

Notice that the asymmetry of $H$ is not a problem in both proofs because we only use the fact that a red (blue, respectively) edge of $H$ corresponds to an $\epsilon$-regular pair of density at least $d$ in $R_{G'}$ (in $B_{G'}$, respectively). □

### 6.1. Proof of Lemma 4.5

Our main tools to prove the lemma will be the following two claims. The first excludes a particular structure in a given graph.

**Claim 6.1.** *Let $G$ be a graph on $n$ vertices such that $\delta(G) \geqslant 3n/4$. Then there is no set $S$ of order at most three such that $V(G) \setminus S$ can be partitioned into non-empty sets $X_1, \ldots, X_4$ such that, for $i = 1, \ldots, 4$ $G$ has no edges between $X_i$ and $X_{5-i}$.*

**Proof.** Suppose that there is such a set $S$. Then $\sum_{i=1}^{4} |X_i| \geqslant n - 3$, and so, for some $1 \leqslant i \leqslant 4$, we have $|X_i| \geqslant (n-3)/4$. As $X_{5-i} \neq \emptyset$, we may consider a vertex $v \in X_{5-i}$. Then $v$ has no neighbours in $X_{5-i}$ and is also not adjacent to itself. Hence $d_G(v) \leqslant n - (|X_i| + 1) < 3n/4$, contradicting the minimal degree of $G$. □

The second claim gives us long monochromatic paths in bipartite subgraphs with large minimum degree. We first need one definition.

**Definition.** Let $G$ be a graph and $U$ and $W$ be two disjoint subsets of vertices. We say that the bipartite graph $G[U, W]$ is *t-complete* if every vertex in $U$ has at least $|W| - t$ neighbours in $W$ and every vertex in $W$ has at least $|U| - t$ neighbours in $U$.

**Claim 6.2.** *Let $G$ be a graph and $U$ and $W$ be two disjoint subsets of vertices. If $G[U,W]$ is $t$-complete, then the following holds.*

(a) *For any two vertices $u,w \in U$, the graph $G[U,W]$ contains a $u$–$w$ path of length $\ell$ for all even $2 \leqslant \ell \leqslant 2\min\{|U|,|W|-2t\}$. If $G[U]$ or $G[W]$ contains an edge other than $uw$, then $G$ also contains a $u$–$w$ path of length $\ell$ for every odd $7 \leqslant \ell \leqslant 2\min\{|U|,|W|-2t\}-1$.*

(b) *For any two vertices $u \in U$ and $w \in W$, the graph $G[U,W]$ contains a $u$–$w$ path of length $\ell$ for all odd $3 \leqslant \ell \leqslant 2\min\{|U|,|W|-2t\}-1$. If $G[U]$ or $G[W]$ contains an edge, then $G$ also contains a $u$–$w$ path of length $\ell$ for every even $6 \leqslant \ell \leqslant 2\min\{|U|,|W| - 2t\}-2$.*

(c) *The graph $G$ contains cycles of all even lengths between 4 and $2\min\{|U|,|W|-2t\}$. If $G[U]$ is non-empty, then $G$ also contains cycles of all odd lengths between 3 and $2\min\{|U|,|W|-2t\}-1$.*

**Proof.** Suppose that $u,w$ are two vertices in $U$, and let $1 \leqslant r \leqslant \min\{|U|,|W|-2t\}$ be given. Consider any sequence $v_1,v_2,\dots,v_r,v_{r+1}$ of distinct vertices in $U$ such that $v_1 = u$ and $v_{r+1} = w$. Clearly, any two vertices in $U$ have at least $|W|-2t$ common neighbours. Hence, there are distinct vertices $w_1,\dots,w_r \in W$ such that, for all $1 \leqslant i \leqslant r$, $w_i$ is a common neighbour of $v_i$ and $v_{i+1}$. Hence,

$$v_1w_1v_2w_2\dots v_rw_rv_{r+1}$$

is a $u$–$w$ path of length $2r$.

Let $xy \neq uw$ be an edge in $G[U]$ such that $\{x,y\} \cap \{u,w\} = \emptyset$. Then in the proof above for $r \geqslant 3$ we take $v_1 = u$, $v_2 = x$, $v_3 = y$, and $v_{r+1} = w$, and find distinct common neighbours $w_i$ of $v_i$ and $v_{i+1}$ for all $1 \leqslant i \leqslant r$, $i \neq 2$. Hence,

$$v_1w_1v_2v_3w_3\dots v_rw_rv_{r+1}$$

is a $u$–$w$ path of length $2r-1$.

Let $xy \neq uw$ be an edge in $G[U]$ such that $x = u$. In this case, we take $v_1 = u$, $v_2 = y$, and $v_{r+1} = w$, and find distinct common neighbours $w_i$ of $v_i$ and $v_{i+1}$ for all $2 \leqslant i \leqslant r$. Hence,

$$v_1v_2w_2v_3w_3\dots v_rw_rv_{r+1}$$

is a $u$–$w$ path of length $2r-1$, $r \geqslant 3$.

Finally, let $xy$ be an edge in $G[W]$. Then in the proof above we take $v_1 = u$, $v_2$ to be any neighbour of $w_2 = x$, $v_4$ to be any neighbour of $w_3 = y$, and $v_{r+1} = w$. We again find distinct common neighbours $w_i$ of $v_i$ and $v_{i+1}$ for all $1 \leqslant i \leqslant r$, $i \neq 2,3$. Hence,

$$v_1w_1v_2w_2w_3v_4\dots v_rw_rv_{r+1}$$

is a $u$–$w$ path of length $2r-1$, $r \geqslant 4$.

To see (b), take any $u \in U$ and $w \in W$, and let $2 \leqslant r \leqslant \min\{|U|,|W|-2t\}$. We again consider any sequence $v_1,v_2,\dots,v_r$ of distinct vertices in $U$ such that $v_1 = u$ and $v_r \neq u$ is any neighbour of $w$. For all $1 \leqslant i \leqslant r$, $v_i$ and $v_{i+1}$ have $|W|-2t-1$ common neighbours other than $w$. Hence, there are distinct vertices $w_1,\dots,w_{r-1} \in W$ such that, for all $1 \leqslant i \leqslant r$,

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$w_i \neq w$ is a common neighbour of $v_i$ and $v_{i+1}$. Hence,

$$v_1 w_1 v_2 w_2 \ldots v_{r-1} w_{r-1} v_r w$$

is a $u$–$w$ path of length $2r-1$. The proof of the second part of (b) is similar to (a).

The first part of (c) follows from (b) by taking an edge $uw$ in $G[U, W]$. For the second part, take an edge $uw \in G[U]$ and apply part (a). □

**Proof of Lemma 4.5.** We shall first prove that there exists a partition $V(G) = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$ such that

(i) $|W_0| \leqslant \delta n$ and $\min_i |W_i| \geqslant (1/4 - 4\delta)n$,
(ii) the graphs $G'[W_1, W_4]$ and $G'[W_2, W_3]$ are empty,
(iii) there are no blue edges from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G'$.

For $1 \leqslant i \leqslant 4$, let $W_i \subseteq V(G)$ be the union of the clusters in $U_i$, so that $|W_i| \geqslant (1/4 - 3\delta)(1-\epsilon)n$. Note that in $G'$ there are no blue edges from $W_1 \cup W_3$ to $W_2 \cup W_4$. Since there are no edges between $U_1$ and $U_4$ and between $U_2$ and $U_3$ in $H$, it follows that there are no edges in $G'[W_1, W_4]$ and $G'[W_2, W_3]$. Furthermore, as $\epsilon \ll \delta$, we have that

$$|W_i| \geqslant \left(\frac{1}{4} - 3\delta\right)(1-\epsilon)n \geqslant \left(\frac{1}{4} - 4\delta\right).$$

The set $W_0$ contains the vertices from $V_0$, so $|W_0| \leqslant \epsilon n < \delta n$. Notice that if we later move a constant number of vertices to $W_0$, then (i)–(iii) will still hold.

Recall that $\delta(G') \geqslant (3/4 - \delta)n$ and hence vertices in $W_1 \cup \cdots \cup W_4$ have at most $(1/4 + \delta)n$ non-neighbours in $G'$. For a vertex in $W_1$, at least $(1/4 - 4\delta)n$ of these non-neighbours are in $W_4$. Hence vertices in $W_1$ are adjacent in $G'$ (and hence in $G$) to all but at most $5\delta n$ vertices in $W_1 \cup W_2 \cup W_3$. Similar results hold for $W_2$, $W_3$ and $W_4$. Hence,

(iv) $\delta(G[W_i]) \geqslant |W_i| - 5\delta n - 1$ for all $1 \leqslant i \leqslant 4$.
(v) The graphs $R_G[W_1, W_2]$ and $R_G[W_3, W_4]$ are $5\delta n$-complete.

If there is a red edge $uv$ in $W_1$ or $W_3$, then, by Claim 6.2, $G$ contains red cycles of length $\ell$ for all $\ell \in [3, (1/2 - 19\delta)n]$. Otherwise, all edges in $G[W_1]$ and $G[W_3]$ are blue. Since $|W_1| > 10\delta n + 2$, we have that $\delta(G[W_1]) > |W_1|/2$. By Corollary 1.3, $G$ contains blue cycles of length $\ell$ for all $\ell \in [3, |W_1|]$. But there are also two vertex-disjoint blue edges between $W_1$ and $W_3$ because there was a blue edge between $U_1$ and $U_3$ in $H$. By (iv), we can join greedily their endpoints in $W_i$, $i \in \{1, 3\}$, by a blue path of any length between 2 and $|W_i| - 10\delta n - 2$. By concatenating these two paths and two edges, we get a cycle of any length between $|W_1|$ and $|W_1| + |W_3| - 20\delta - 4 > (1/2 - 29\delta)n$.

To complete the proof, we need to show that $G$ contains a monochromatic cycle of length $\ell$ for all $\ell \in [(1/2 - 29\delta)n, \lceil n/2 \rceil]$, and a monochromatic cycle of length at least $(1 - 38\delta)n$. We distinguish two cases.

**Case 1:** there are two red edges in $G[W_i]$ for some $i \in \{1, 2, 3, 4\}$.

Without loss of generality, assume that $G[W_1]$ contains two red edges. Suppose that there are two disjoint paths $P_1$ and $P_2$ from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $R_G$. Let $P_1$ have endpoints $u$ in $W_1 \cup W_2$ and $u'$ in $W_3 \cup W_4$ and let $P_2$ have endpoints $w$ in $W_1 \cup W_2$

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and $w'$ in $W_3 \cup W_4$. By restricting to a smaller path if necessary, we may assume that all internal vertices of $P_1$ and $P_2$ are in $W_0$. Note that this also includes the case when there are two vertex-disjoint red edges between $W_1 \cup W_2$ and $W_3 \cup W_4$.

We now use Claim 6.2 to find $u$–$w$ paths of length $\ell$ for all $\ell \in [6, (1/2 - 19\delta)n]$ in $R[W_1, W_2]$. However, Claim 6.2 also implies that $R_G[W_3, W_4]$ contains $u'$–$w'$ paths of length $\ell$ for all $\ell \in [6, (1/2 - 19\delta)n]$ of a given parity. By concatenating these paths with $P_1$ and $P_2$, we see that in this case we have monochromatic cycles of length $\ell$ for all $\ell \in [14, (1 - 38\delta)n]$.

So, we may assume that there are no two vertex-disjoint red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $R_G$. By a corollary of Menger's Theorem, there is a vertex $v_R$ such that there are no red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $G - \{v_R\}$. Hence, the set $W_0$ splits into sets $W_{12}, W_{34}$ such that there are no red edges between $W_1 \cup W_2 \cup W_{12}$ and $W_3 \cup W_4 \cup W_{34}$ in $G - \{v_R\}$. If there is also a vertex $v_B$ such that there are no blue paths from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v_B\}$, then we can split $W_{12}$ into $W_1', W_2'$ and $W_{34}$ into $W_3', W_4'$ so that there are no edges between $X_i := W_i \cup W_i'$ and $X_{5-i} = W_{5-i} \cup W_{5-i}'$. Taking $S = \{v_R, v_B\}$, we have a contradiction to Claim 6.1. Hence we may assume that there are two disjoint blue paths between $W_1 \cup W_3$ and $W_2 \cup W_4$. Similarly to the above, applying Claim 6.2 to the ends of these paths in $B_G[W_1, W_3]$ and $B_G[W_2, W_4]$, there is a blue cycle of length at least $(1 - 38\delta)n$. Moreover, if there are two blue edges in $G[W_i]$ for some $i$, then we get blue cycles of length $\ell$ for all $\ell \in [14, (1 - 38\delta)n]$.

So, by moving at most eight vertices to $W_0$, we may assume that all edges of each $G[W_i]$ are red. To complete the proof in Case 1, we need to show that $G$ contains a monochromatic cycle of length $\ell$ for all $\ell \in [(1/2 - 29\delta)n, \lceil n/2 \rceil]$.

Suppose that some vertex $v \in W_0$ has at least $(1/2 + 8\delta)n + 3$ blue neighbours. Then it must have at least two blue neighbours in at least three of the sets $W_i$. If there is no blue path $P$ from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v\}$, then, as above, we would have a contradiction with Claim 6.1 applied with $S = \{v, v_R\}$. Hence, let $P$ be a blue path from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v\}$. Without loss of generality, we may assume that $P$ has endpoints $u_1 \in W_1$ and $u_2 \in W_2$ and all internal vertices of $P$ are in $W_0$. Again, this includes the case of a single edge between $W_1 \cup W_3$ an $W_2 \cup W_4$. Suppose that $v$ has at least two blue neighbours in each of $W_1$, $W_2$ and $W_3$, the other cases being similar. We may find $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ with $\{u_1, u_2\} \cap \{w_1, w_2, w_3\} = \emptyset$ such that each of $w_1$, $w_2$ and $w_3$ are blue neighbours of $v$.

By Claim 6.2 we have the following paths:

- for all even $\ell \in [6, (1/2 - 19\delta)n]$, $B_G[W_2, W_4]$ contains a $u_2$–$w_2$ path $P_\ell$ of length $\ell$;
- for all even $\ell' \in [6, (1/2 - 19\delta)n]$, $B_G[W_1, W_3]$ contains a $u_1$–$w_1$ path $P'_{\ell'}$ of length $\ell'$;
- for all odd $\ell'' \in [7, (1/2 - 19\delta)n]$, $B_G[W_1, W_3]$ contains a $u_1$–$w_3$ path $P''_{\ell''}$ of length $\ell''$.

Then, for all even $\ell, \ell' \in [6, (1/2 - 19\delta)n]$, the path

$$u_2 P_\ell w_2 v w_1 P'_{\ell'} u_1$$

is a blue $u_1$–$u_2$ path of length $2 + \ell + \ell'$ which is internally disjoint from $P$. Similarly, for all even $\ell \in [6, (1/2 - 19\delta)n]$ and odd $\ell'' \in [6, (1/2 - 19\delta)n]$, the path

$$u_2 P_\ell w_2 v w_3 P''_{\ell''} u_1$$

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is a blue $u_1$–$u_2$ path of length $2+\ell+\ell''$ which is internally disjoint from $P$.

Hence, for all $L \in [14,(1-38\delta)n]$, there is a blue $u_1$–$u_2$ path of length $L$ which is internally disjoint from $P$. Since $|P| \leqslant |W_0|+2 \leqslant 2\delta n$, this gives blue cycles of length $L$ for all $L \in [2\delta n+14,(1-38\delta)n]$. Since $2\delta n+14 < (1/2-29\delta)n$, we are done.

Thus, we may assume that each vertex in $W_0$ has blue degree at most $(1/2+8\delta)n+3$, and so red degree at least $(1/4-8\delta)n-3$. Let $C_1$ be the red component of $G-\{v_R\}$ containing $W_1 \cup W_2$ and $C_2$ be the red component of $G-\{v_R\}$ containing $W_3 \cup W_4$. We know that $R_G[W_1 \cup W_2]$ and $R_G[W_3 \cup W_4]$ are connected, and the minimal red degree condition on $W_0$ ensures that there are at most two components in $R_G[V-\{v_R\}]$. As $v_R$ has red degree at least $(1/4-8\delta)n-3$, it has at least $(1/8-5\delta)n$ red neighbours in at least one of $C_1$ or $C_2$. Let $C_i'$ be the set $C_i$, with $v_R$ added if it has at least $(1/8-5\delta)n$ red neighbours in $C_i$.

Then $|C_1'|+|C_2'| \geqslant n$ and so we may assume without loss of generality that $|C_1'| \geqslant \lceil n/2 \rceil$. All vertices in $C_1'$ have degree in $R_G[C_1']$ at least $(1/8-5\delta)n$. Further, all vertices in $C_1' \setminus |W_0|$ have degree in $R_G[C_1']$ at least $|C_1'|-6\delta n$. As $|C_1'| \leqslant (1/2+8\delta)n$ and $|W_0| \leqslant \delta n$, the condition of Theorem 2.6 holds on $R_G[C_1']$ and so $R_G[C_1']$ is hamiltonian. But we also have

$$\begin{aligned} e(R_G[C_1']) &\geqslant \frac{1}{2}(|C_1'|-6\delta n)(|C_1'|-|V_0|) \\ &> \frac{1}{4}|C_1'|^2. \end{aligned}$$

Hence, by Theorem 1.2, $R_G[C_1']$ is pancyclic and we are done with Case 1.

**Case 2:** for every $i=1,2,3,4$, there is at most one red edge in $G[W_i]$.

By moving at most four vertices attached to a red edge in $G[W_i]$ to $W_0$, we may assume that all edges of $G[W_i]$ are blue for all $i=1,2,3,4$.

By (iv) and Corollary 1.3, $G$ contains blue cycles of length $\ell$ for all $\ell \in [3,|W_1|]$. But there are also two vertex-disjoint blue edges between $W_1$ and $W_3$ because there was a blue edge between $U_1$ and $U_3$ in $H$. Using (iv), we can join greedily their endpoints in $W_i$, $i \in \{1,3\}$, by a blue path of any length between 2 and $|W_i|-10\delta n-2$. By concatenating these two paths and two edges, we get a cycle of any length between $|W_1|$ and $|W_1|+|W_3|-20\delta-4 > (1/2-29\delta)n$. Hence, $G$ contains blue cycles of all lengths between 3 and $|W_1|+|W_3|-20\delta n$. Moreover, the same argument gives that for any two vertices $u,w$ in $W_1 \cup W_3$, there are blue $u$–$w$ paths of every length between 8 and $|W_1|+|W_3|-20\delta n$. The same is true in $B_G[W_2 \cup W_4]$.

Consequently, there are no two internally disjoint paths between $W_1 \cup W_3$ and $W_2 \cup W_4$ in $B_G$. Hence, there exists a vertex $v_B$ such that, in $G-v_B$, there are no blue paths from $W_1 \cup W_3$ to $W_2 \cup W_4$.

Now we essentially follow the proof in Case 1, with colours red and blue interchanged. There must exist two internally disjoint red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$, otherwise we would get a contradiction with Claim 6.1. Consequently, we join their endpoints by red paths in $R_G[W_1 \cup W_2]$ and $R_G[W_3 \cup W_4]$ to get a red cycle of length at least $(1-38\delta)n$.

If there is a vertex in $W_0$ with at least $(1/2+8\delta)n+3$ red neighbours, then we are done as in Case 1. Hence, we may assume that every vertex of $W_0$ has at least

$(1/4 - 8\delta)n - 3$ blue neighbours. Hence, we may partition $W_0$ into sets $W_1', \dots, W_4'$, so that each vertex in $W_i'$ has at least $(1/16 - 3\delta)n$ blue neighbours in $W_i$. It follows that either $|W_1 \cup W_1' \cup W_3 \cup W_3'| \geqslant \lceil n/2 \rceil$ or $|W_2 \cup W_2' \cup W_4 \cup W_4'| \geqslant \lceil n/2 \rceil$.

Without loss of generality, suppose that $|W_1 \cup W_1' \cup W_3 \cup W_3'| \geqslant \lceil n/2 \rceil$. By removing vertices, if necessary, we may assume that $|W_1 \cup W_1' \cup W_3 \cup W_3'| = \lceil n/2 \rceil$. We construct a blue cycle on $\lceil n/2 \rceil$ vertices as follows. Take two vertex-disjoint blue edges $u_1u_3$, $v_1v_3$ such that $u_1, v_1 \in W_1$ and $u_3, v_3 \in W_3$. Take any two vertices $w_1 \in W_1$ and $w_3 \in W_3$ distinct from $u_1, v_1, u_3, v_3$. By (iv) and by the definition of $W_i'$, one can greedily construct blue $u_i$–$w_i$ path $P_i$ containing all the vertices of $W_i'$, avoiding $v_i$, and not having more than $3\delta n$ vertices. Then, by (iv), the induced sub-graph $B_G[(W_i \setminus V(P_i)) \cup \{w_i\}]$ satisfies the assumptions of Corollary 2.7, and so it must contain a blue $v_i$–$w_i$ path $P_i'$. By concatenating paths $P_1, P_1', P_3, P_3'$ and edges $u_1u_3$, $v_1v_3$, we obtain a blue cycle on $\lceil n/2 \rceil$ vertices. Clearly, by omitting some vertices from $W_1', W_3'$ and $W_1$, we can obtain a blue cycle of any length between $(1/2 - 29\delta)n$ and $\lceil n/2 \rceil$. □

## 7. Monochromatic circumference

In this section we shall look at the monochromatic circumference of a graph. We begin by proving Theorem 1.8.

**Proof of Theorem 1.8.** As in the proof of Theorem 1.6, we consider the reduced graph $H$, which has order $k$ and minimal degree at least $(3/4 - \delta)k$. Applying Lemma 4.1 to $H$, we have one of the following.

(i) There is a component of $R_H$ or $B_H$ which contains a matching on at least $(2/3 + \delta)k$ vertices.

(ii) There is a set $S$ of order at least $(2/3 - \delta/2)k$ such that either $\Delta(R_H[S]) \leqslant 10\delta k$ or $\Delta(B_H[S]) \leqslant 10\delta k$.

(iii) There is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geqslant (1/4 - 3\delta)k$ such that there are no blue edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no red edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

In the first case, we use Lemma 2.5(a) to find a monochromatic cycle of length at least $(2/3 + \delta/2)n$. In the second case, assume without loss of generality that $\Delta(R_H[S]) \leqslant 10\delta k$. Then, by Lemma 4.4, $G$ contains a blue cycle of length $\ell$ for all $\ell \in [3, (2/3 - \delta)n]$. In the third case, Lemma 4.5 implies that $G$ contains a monochromatic cycle of length at least $(1 - 38\delta)n \geqslant (2/3 + \delta)n$. □

We will make the following definition.

**Definition.** For $0 < c < 1$, let $\Phi = \Phi_c$ be the supremum of values $\phi$ such that any graph $G$ of sufficiently large order $n$ with $\delta(G) > cn$ and a 2-colouring $E(G) = E(R) \cup E(B)$ has monochromatic circumference at least $\phi n$.

For $c \geqslant 3/4$, Theorem 1.8 implies that $\Phi_c \geqslant 2/3$. However, the example given after Theorem 1.8 shows that $\Phi_c \leqslant 2/3$ for all $c$. We can also find upper and lower bounds for $\Phi_c$ when $c < 3/4$, and we collect them into the following theorem.

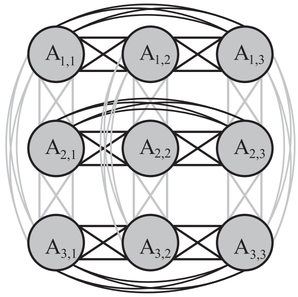


*Figure 1.* The graph $G_t^{(3)}$.

**Theorem 7.1.** *For all $c \geqslant 3/4$, we have $\Phi_c = 2/3$. For all $c \in (0,1)$, we have $\Phi_c \geqslant c/2$. Also, there are the following upper bounds on $\Phi_c$:*

$$\Phi_c \leqslant \begin{cases} \frac{1}{2} & c \in \left[\frac{3}{5}, \frac{3}{4}\right), \\ \frac{2}{5} & c \in \left[\frac{5}{9}, \frac{3}{5}\right), \\ \frac{1}{r} & c < \frac{2r-1}{r^2} \text{ for all } r \geqslant 3. \end{cases}$$

Note that, as $c \to 0$, we may use the last upper bound to show that $\Phi_c/(c/2) \to 1$. Hence, asymptotically, as $c \to 0$, the upper and lower bounds on $\Phi_c$ agree.

**Proof of Theorem 7.1.** For $c \in (0,1)$, every 2-edge coloured graph with $\delta(G) > cn$ has at least $cn^2/2$ edges. Hence, there are at least $cn^2/4$ edges of one colour. We may deduce from Theorem 2.11 that, in that colour, there is a cycle of length at least $cn/2$. Hence $\Phi_c \geqslant c/2$ for all $c \in (0,1)$. Next, we prove the upper bounds on $\Phi_c$.

For $c \in [5/9, 3/5)$, let $t$ be an integer such that $t > 1/(3-5c)$ and let $n = 5t$. We define a graph $G_t'$ as follows. Let $S_1$ and $S_2$ be sets of order $2t$ and let $T$ be a set of order $t$. Let $R$ be the union of the complete graph on $S_1$ and the complete graph on $S_2$. Then $R$ has circumference $2t$. Let $B$ be the union of the complete graph on $T$ and the complete bipartite graph between $T$ and $S_1 \cup S_2$. Then, any two consecutive vertices of a cycle in $B$ must contain a vertex of $T$ and hence $B$ has circumference at most $2t$. Let $G_t'$ be the union of $R$ and $B$. Therefore $\delta(G) = 3t - 1 > 5tc = c|G_t'|$ and so $\Phi_c \leqslant 2/5$.

Assume now that $c \in \left(0, \frac{2r-1}{r^2}\right)$ for a given $r \geqslant 2$. We aim to show that $\Phi_c \leqslant 1/r$. Note that when $r = 2$ this gives the bound $\Phi_c \leqslant 1/2$ for $c < 3/4$. Let $t$ be an integer such that $t > 1/(2r-1-r^2c)$ and let $n = tr^2$. Consider a family $\{A_{i,j} : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r\}$ of sets of order $t$. We define the following graphs on vertex set $\bigcup_{i,j} A_{i,j}$:

$$E(B) = \{uv : u \in A_{i,j}, v \in A_{i,j'} \text{ for some } 1 \leqslant i \leqslant r \text{ and } j \neq j'\},$$
$$E(R) = \{uv : u \in A_{i,j}, v \in A_{i',j} \text{ for some } 1 \leqslant j \leqslant r\}.$$

Let $G_t^{(r)}$ be the union of the graphs $R$ and $B$, as illustrated in Figure 1, for the case $r = 3$. Then $\delta(G_{r,t}'') = (2r-1)t - 1 > ctr^2 = c|G_t^{(r)}|$. However, as all monochromatic components

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have order $rt$, there are no monochromatic cycles of length greater than $n/r$. Hence $\Phi_c \leqslant 1/r$. □

## 8. Conclusion

Theorem 1.6 is a 2-colour version of the uncoloured (or 1-coloured) result of Bondy that all graphs with order $n \geqslant 3$ and minimum degree at least $n/2$ are either pancyclic or isomorphic to $K_{n/2,n/2}$. We may hope to generalize it to $k$ colours. In this case, we let $E(G) = \bigcup_{i=1}^{k} E(G_i)$ be an edge colouring, where each $G_i$ is a spanning subgraph of $G$, representing the edges coloured $i$. Our extremal graph was found by letting both $R$ and $B$ be subgraphs of the extremal graph in the uncoloured case, and we again use this method to find $k$-coloured graphs with high minimum degree but no odd cycles.

**Definition.** Let $n = 2^k p$ and let $G$ be isomorphic to the $2^k$-partite graph with classes all of order $p$. A *$k$-bipartite $k$-edge colouring* of $G$ is a $k$-edge colouring $E(G) = \bigcup_{i=1}^{k} E(G_i)$ such that each $G_i$ is bipartite.

As in the 2-coloured case, we can deduce that a $k$-bipartite $k$-edge colouring of the $2^k$-partite graph with classes all of order $p$ induces a labelling $U_\alpha$, $\alpha \in \{1,2\}^k$, of the classes such that, for all $i$, the graph $G_i$ is bipartite with classes

$$\bigcup_{\alpha:\alpha_i=1} U_\alpha$$

and

$$\bigcup_{\alpha:\alpha_i=2} U_\alpha.$$

Note that this implies that, if $\alpha$ and $\beta$ in $\{1,2\}^k$ differ only in the $i$th place, then all edges between $U_\alpha$ and $U_\beta$ are coloured with $i$. As this graph has minimum degree $(1-2^{-k})n$, we make the following conjecture.

**Conjecture 8.1.** *Let $n \geqslant 3$, and $k$ be an integer. Let $G$ be a graph of order $n$ with $\delta(G) \geqslant (1-2^{-k})n$. If $E(G) = \bigcup_{i=1}^{k} E(G_i)$ is a $k$-edge colouring, then either:*

- *for all $\ell \in [\min\{2^k, 3\}, \lceil n/2^{k-1} \rceil]$ there is some $1 \leqslant i \leqslant k$ such that $C_\ell \subseteq G_i$, or*
- *$n = 2^k p$, $G$ is the complete $2^k$-partite graph with classes of order $p$, and the colouring is a $k$-bipartite $k$-edge colouring.*

Note that the case when $k = 1$ is Bondy's Theorem, and Theorem 1.6 is the case $k = 2$ for large $n$. We pose the following problem about the monochromatic circumference.

**Problem 8.2.** *What is the value of $\Phi_c$ for $c < 3/4$?*

Note that Theorem 7.1 shows that $\Phi_c = 2/3$ for all $c \geqslant 3/4$. In this case, we make the following conjecture with an exact bound on the monochromatic circumference.

**Conjecture 8.3.** *Let $G$ be a graph of order $n$ with $\delta(G) \geqslant 3n/4$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. If $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring, then $G$ has monochromatic circumference at least $2t + r$.*

Note that Theorem 1.8 is an asymptotic version of this conjecture. The same result was proved independently by Gyárfás and Sárközy in [10]. By considering the graph $F_{t,2t+r}$ as defined in Section 1, we see that this conjecture is best possible. For the latest progress on Conjecture 8.3, the reader should consult [4].

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