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SUBJECT-MATTER AND INTENSIONAL OPERATORS III: STATE-SENSITIVE SUBJECT-MATTER AND TOPIC SUFFICIENCY

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Abstract. Logical frameworks that are sensitive to features of sentences' subject-matter—like Berto's topic-sensitive intentional modals (TSIMs)—demand a maximally faithful model of the topics of sentences. This is an especially difficult task in the case in which topics are assigned to intensional formulae. In two previous papers, a framework was developed whose model of intensional subject-matter could accommodate a wider range of intuitions about particular intensional conditionals. Although resolving a number of counterintuitive features, the work made an implicit assumption that the subject-matter of an intensional conditional is a function of the subject-matters of its subformulae. This assumption—which I will call a principle of *topic sufficiency*—runs counter to some natural intuitions concerning topic. In this paper, we will investigate topic sufficiency and offer a semantic account that is *state-sensitive*, providing an implementation through the introduction of topic-sensitive logics related to William Parry's prototypical PAI.

§1. Introduction. This paper is the third in a sequence concerned with producing semantic frameworks including components that are adequate to the task of representing *subject-matter* or *topic* of intensional sentences. The work outlined in [13, 14] aimed to address perceived limitations in standard topic-sensitive frameworks on the market, *e.g.*, legacy systems of analytic implication studied by Parry [24], Angell [1], Deutsch [8] or more recent work on topic-sensitive intentional modals (TSIMs) studied by Berto and collaborators including Hawke and Özgün (in, *e.g.*, [5]).

The first piece [13] took issue with the tendency of standard frameworks to treat the assignment of topics to intensional sentences in the same way as the assignment of topics to extensional sentences, *i.e.*, by the coarse fusion of the topics of subformulae. The work investigated scenarios in which such an approach seemed implausible and identified a number of conditions characteristic of the topic-theoretic peculiarities of different types of intensional conditionals. These considerations drove the development of a flexible semantics and proof theory for a system of *conditional agnostic analytic implication* (CA/PAI) in which the subject-matter of intensional formulae is determined on the basis of more nuanced and fine-grained considerations than fusion alone.

Its sequel [14] was concerned with decoupling this framework from the Parry-style setting in order to apply the techniques and tools developed for CA/PAI to

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the setting of TSIMs.[1] The work applied these tools to introduce more granular and expressive variants of Berto's systems KRI (the logic of knowability relative to information) and PHB (the logic of plain hyperintensional belief). Notably, the novel emendations resolved several long-standing limitations of the frameworks. For example, by introducing adequate machinery for evaluating nested operators, the variant KRI of [14] improved on stock KRI by satisfying Fagin and Halpern's requirement of [11] that any "viable logic of knowledge" must capture an agent's meta-reasoning.

Despite the advances of the framework introduced in [12] and implemented across [13, 14] in freeing the standard topic-theoretic frameworks from several counterintuitive assumptions, there remain several limitations to the applicability of the underlying framework left to resolve.

Before identifying any one target in detail, it will help us to review some relevant notions to set the stage. Especially relevant are two conditions described as principles of *Negation Transparency* and *Junctive Transparency*. Where t is a function assigning sentences topics and $\oplus$ is a *fusion* operator that simply merges two topics into one, these can be described as:

- **Negation Transparency:** $t(\neg\varphi) = t(\varphi)$.
- **Junctive Transparency:** $t(\varphi \wedge \psi) = t(\varphi \vee \psi) = t(\varphi) \oplus t(\psi)$.

This is to say that the topic of a formula φ is the same as the topic of its negation $\neg\varphi$ while the topics of conjunctions and disjunctions are merely the fusions of the topics of their immediate subformulae.

Topic transparency is more-or-less unobjectionable in extensional cases. It is not implausible that extensional connectives are topic-theoretically inert and make no more contribution to the topic of a whole than punctuation marks. Indeed, in many sequent calculi, extensional conjunctions and disjunctions are interchangable with commas in a multiset.

In contrast to the topic-theoretic inertness of extensional connectives, many intensional operators—conditionals or modals—appear to play a *transformative* role in determining the topics of complexes in which they appear, *i.e.*, distinct types of intensional conditionals can have distinct topic-theoretic features. Some conditionals appear to be topic-theoretically additive, nontrivially contributing to a complex's subject-matter beyond that of its subformulae. For other conditionals, the topic of a complex may *omit* some elements of the topics of its antecedent and consequent. The potential topic-theoretic features of conditionals are as varied as intensional conditionals themselves.

The standard accounts of topic on offer entirely fail to acknowledge this diversity, effectively treating all operators as interchangeable. For an intensional conditional $\rightarrow$, this is reflected in a condition of *Intensional Transparency*—the coarse identification of $t(\varphi \rightarrow \psi)$ and $t(\varphi) \oplus t(\psi)$. We can briefly consider several of the consequences of *Intensional Transparency* that reveal distinct types of conditionals whose topic-theoretic features cannot be modeled on the standard approaches.

[1] TSIMs are binary operators X constructing formulae $X^{\varphi}\psi$. Particular applications are abundant and wide-ranging; X can be understood as *knowability relative to information* (see, *e.g.*, [6]), as *static belief revision* (see [3]), or *mental simulation* (see [2]).

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One consequence is a constraint requiring that intensional conditionals are topic-theoretically *non-ampliative* in the sense that a conditional $\varphi \to \psi$ may include no subject-matter not already present in those of φ and ψ. Formally:

- **Non-Ampliativity:** $t(\varphi \to \psi) \leq t(\varphi) \oplus t(\psi)$.

Reference [13] cites the intuitionistic conditional as a connective whose topic-theoretic properties are incompatible with this constraint. Following Brouwer–Heyting–Kolmogorov readings, it is natural to expect that a conditional $\varphi \to \psi$ is (in part) *about* a particular construction f transforming proofs of its antecedent into proofs of its consequent. Although this f is not counted among the topics of φ or ψ, it is reasonable to expect that f is a constituent of the overall conditional's subject-matter. Consequently, a framework in which *Non-Ampliativity* holds is incapable of faithfully capturing the topic-theoretic properties of the intuitionistic conditional.[2]

Complementing this constraint is a further consequence dictating that the subject-matter of an intensional conditional must preserve the subject-matters of its subformulae *in toto*, that is, imposing an expectation of *non-explicativity*. Formally:

- **Non-Explicativity:** $t(\varphi) \oplus t(\psi) \leq t(\varphi \to \psi)$.

For intensional conditionals that *shift contexts*, such topic-theoretic preservation is doubtful.

Reference [13] identifies counterfactuals as a species of conditional whose context-shifting leads to violations of *Non-Explicativity*. The incongruity between *Non-Explicativity* and counterfactual conditionals can be made clear by an example. Consider a sentence in which a necessary property is falsely attributed to an individual identified through a definite description, *e.g.*,

I: "The inventor of bifocals is a tiger."

Although false, on many theories—such as Hawke's *issue-based* theory of [18]—Benjamin Franklin constitutes part of the subject-matter of [I]. A counterfactual conditional φ in which [I] appears as antecedent, however, shifts contexts from the actual world to states in which [I] is true. In virtue of the impossibility of Franklin's being a tiger, Franklin will be excluded from the subject-matter in any context relevant to φ's evaluation and, plausibly, from the subject-matter of φ itself. But such phenomena cannot be captured in the presence of *Non-Explicativity*.

The system introduced in [13] of conditional agnostic analytic implication (CA/PAI) provided a concrete implementation of this intuition by a modification of Parry's PAI. The work served as a proof-of-concept witnessing that the framework was sufficiently general to accommodate a wide range of semantic conditions regulating the subject-matters of conditionals while remaining robust enough to continue to accommodate the standard theories' assumptions.

The formal framework at the heart of CA/PAI succeeds in dismantling many artificial barriers to the development of nuanced theories about intensional subject-matter. CA/PAI supports reasoning about topic-theoretic contexts in which the conditions

2 Heinrich Wansing has pointed out that this example has a further consequence that the topics of *disjunctions* must exceed the fusion of the topics of their disjuncts if the BHK interpretations are taken literally. This suggestion is not only natural, but the framework outlined in this series of papers is general enough to accommodate such an analysis of disjunction.

of *Non-Explicativity* and *Non-Ampliativity*—among others—are unsuitable. But for all the modularity and granularity made available by this more general framework, barriers remain hindering the expression of several exceedingly natural topic-theoretic intuitions.

§2. Topic sufficiency and intensional conditionals. Most importantly, the refined framework inherits one problematic feature from its predecessors, namely, its commitment to a constraint that *the topic of an intensional conditional must be a function of the subject-matters of its antecedent and consequent.*

In short, this tacit assumption stipulates that the topics of a conditional's antecedent and consequent suffice to determine the topic of the complex. Following this characterization, we formally introduce the constraint as *Topic Sufficiency*:

- **Topic Sufficiency:** For an intensional conditional $\rightarrow$, if $t(\varphi) = t(\xi)$ and $t(\psi) = t(\zeta)$, then $t(\varphi \rightarrow \psi) = t(\xi \rightarrow \zeta)$.

The condition is remarkably entrenched, reflected in virtually all topic-theoretic frameworks in which conditionals are assigned topics.[3]

There exists phenomenological evidence against this principle; however, this evidence motivates an examination of the thesis of *Topic Sufficiency* and discussion of generalizing semantic frameworks capable of more faithfully representing a broader range of cases. In this section, we will consider some informal examples, allowing the illustrations to direct the selection of criteria on the adequacy of formal accounts of topic.

2.1. State-sensitive subject-matter. Let us make a brief phenomenological observation concerning the following conditional:

> **II:** "If Dr. Xi had continued her research into autonomous vehicles, self-driving cars would have already reached a wide user-base."

Although the fictional Dr. Xi and her research appear to make up *part* of the subject-matter of [II],[4] an utterer of the above sentence seems to form an intention towards a still more complex subject-matter. The objects that the utterer speaks *about* are not limited to the constituents and roles mentioned in [II]. Rather, to consider [II] involves fixing and considering situations over which the conditional will be evaluated. If one takes seriously that to assert [II] is to speak *about* these situations, the situations themselves must constitute part of the conditional's overall subject-matter. In other words, in some intensional contexts, subject-matter appears to be *state-sensitive*.

The requirement of a state-sensitive account of subject-matters is reinforced by the way in which the relevant counterfactual situations are determined; [II] is apparently

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[3] Topic Sufficiency is, *e.g.*, assumed in nearly all work on Parry-style logics, including those of Dunn's [10], Urquhart's [28], Fine's [15], or Deutsch's [9]. Likewise for the work in the more recent TSIM tradition, including all the systems outlined in [4, 5], although Berto acknowledges in [5, p. 65] that such assumptions are compromises. The only exception of which I am aware is the framework offered by Hawke, Özgün, and Berto in [19], in which the model theory imparts no constraints on the topic assigned to a formula $K_i\varphi$ (with a TSIM-like knowledge operator), whence the topics of intensional sentences are entirely arbitrary.

[4] Although the earlier illustration in the sentence [I] signals that this need not hold, especially in case definite descriptions are in play.

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about a *restricted class* of situations—*e.g.*, those in which the antecedent is supposed to be *true*. But in general, the truth conditions of a sentence cannot be inferred or recovered from its subject-matter, in which case some non-topic-theoretic contribution must be made by the proposition to the overall subject-matter.

In short, if [II] is about a collection of counterfactual scenarios whose extent is not uniquely determined by the subject-matters of its antecedent and consequent, then [II] will act as a counterexample to *Topic Sufficiency*.

2.2. Counterexamples to topic sufficiency. In this section, we undertake a closer examination of the thesis of *Topic Sufficiency* reflected in the framework of [13, 14]. This discussion will help identify the requirements of a semantic framework capable of more faithfully representing state-sensitive subject-matter. We begin by surveying several scenarios in which our intuitions likely conflict with *Topic Sufficiency*.

For the first illustration, consider the following scenario: A team of coworkers is aware that a colleague, John, will likely soon resign in favor of a position at a different organization. Several of the team members hold a meeting to prepare for this contingency. A member of the team begins the discussion by asking:

Q1: "What steps should we take if John resigns from his position?"

Now, consider two potential responses to [Q1] that participants might offer:

R1: "Should John resign, we will have to find a replacement."
R2: "Should John *not* resign, we will have to find a replacement."

According to *Negation Transparency*, the subject-matters of the antecedents of [R1] and [R2] are identical. The subject-matters of the responses' consequents, too, are trivially identical. Thus, *Topic Sufficiency* amounts to a requirement that the subject-matters of [R1] and [R2] must coincide.

Consider, however, the responses to [R1] and [R2] that one would expect from the questioner. Clearly, the act of asking [Q1] fixes the scope of the discussion, limiting the range of acceptable topics to *contingencies in which John resigns*. Now, insofar as [R1] describes a recommendation that is *responsive to* or *conditioned on* these very contingencies, [R1] remains within the boundaries of the discussion. In other words, the questioner would likely believe [R1] to be *on-topic*. In contrast, the response [R2] *fails* to align with the contingencies at the heart of the discussion. One would therefore anticipate that the questioner would *reject* [R2] as *off-topic*. Simply put, [R1] is on-topic and [R2] is off-topic.

The prediction that the subject-matters of [R1] and [R2] are identical requires that [R1] is on-topic precisely when [R2] is on-topic; a single topic cannot be both on-topic and off-topic with respect to a single context. But we have just considered a context in which the responses' degrees of topicality are distinct. The condition of *Topic Sufficiency* therefore conflicts with very plausible assumptions concerning the topics of intensional conditionals.

I should acknowledge that—as a referee has pointed out—this is not the uniquely plausible explanation for a participant in a conversation initiated by [Q1] having different reactions to responses [R1] and [R2]. On another interpretation—an interpretation described clearly by Leahy in [21]—initiating a conversation with [Q1] serves to set the common ground of the discourse so that conversants tacitly agree to treat "John has resigned from his position" as true during the conversation. The utterance of a conditional, carrying with it the presupposition that its antecedent is true, acts to enter its antecedent into the common ground. Although the antecedent of [R1]—that John has resigned—is consistent with the common ground, the common ground cannot be consistently updated with the presupposition of [R2]. Consequently, to utter [R2] conflicts with conversational norms in a way not incurred by an utterance of [R1].

We need not adjudicate whether topic-theoretic or pragmatic explanations of this phenomenon are more correct. Indeed, it is not clear that the two are necessarily incompatible. But it is fair to note that the type of explanation laid out in [21] rests heavily on a conditional's *antecedent* when pinning down its presuppositions while a similar topic-theoretic phenomenon can be replicated without somehow privileging the antecedent, as we will now see.

Many accounts presume that the states a conditional is about are fixed—*e.g.*, hypothesized—in virtue of their *satisfying* or *making true* the *antecedent* of the conditional. For example, this presumption is a hallmark of the theory described by Kratzer in [20] in which the role of the antecedent (the "if-clause") is that of a *restrictor*, *i.e.*, the antecedent serves to restrict the set of worlds or states against which the consequent is to be evaluated. But as we've noted, determining the extent of the situations a conditional is about requires access to the antecedent's *truth conditions*. The subject-matter of a sentence, however, is in general insufficient to pin down its truth conditions. For example, on standard accounts, $\varphi \vee \neg\varphi$ and $\varphi \wedge \neg\varphi$ are predicted to have the same topic but have markedly different truth conditions. Consequently, possessing the topics of the conditional's subformulae is insufficient to decisively fix the selection of states.

Although in the foregoing examples the role of the antecedent plays an outsized role in determining the topics that are germane to a discussion, an adequate framework should acknowledge the possibility of the *consequent* playing a role.[5] Ed Mares' discussion of counterfactual conditionals in [22, pp. 144–146] recalls some remarks due to Gabbay's [16] that illustrate a consequent's role in the determination of situations against which a conditional is evaluated and—following our reasoning—that a conditional is about. One of Gabbay's examples centers on the following two conditionals:

III: "If I were the Pope, I would have allowed the use of the pill in India."
IV: "If I were the Pope, I would have dressed more humbly."

Gabbay detects an unmistakable influence of the consequents over constraining the acceptable states for the evaluation of [III] and [IV], noting "clearly, in the first statement, we must assume that India remains overpopulated and poor in resources, while in the second example nothing of the sort is required" [16, p. 98].

We can produce a more direct illustration of how topic may be influenced by the proposition expressed by a consequent. Consider a query:

Q2: "Why might Rebecca have left the party early?"

along with two potential responses to [Q2]:

R3: "If Rebecca were scheduled for an early shift at work, she would have left early."

5 I am indebted to Nicholas Ferenz for suggesting that this be considered.

> **R4:** "If Rebecca were scheduled for an early shift at work, she would *not* have left early."

Conventionally, the asking of [Q2] signals the initiation of collective *abductive activity*, serving as invitation to a collaborative investigation after explanations for a target proposition, *i.e.*, that Rebecca has departed. Intensional conditionals including this target as consequent are a prototypical class of candidate explanations; upon their introduction, the explanatory value and plausibility of the link between antecedent and the target proposition can be evaluated.

[R3] seems to express such a link between a potential explanation (the possibility of an early shift) and the target proposition made salient by [Q2]. *Prima facie*, its introduction to the discussion successfully poses the type of response necessary for the discussion to proceed. In contrast, [R4] describes no explanatory path to the target proposition. Informally, one is led to say that the utterer of [R4] has veered off topic. While the case remains analogous to the [Q1] case, the scenario of [Q2] illustrates that the *consequent* can contribute more to the topic of an intensional conditional than is predicted by Topic Sufficiency.

Collectively, these observations identify a collection of desiderata for topic-theoretic frameworks in which *Topic Sufficiency* is not assumed by default.

2.3. Compositionality and contents. The foregoing examples reinforce one insight—and uncover another—concerning *compositionality*. Together, these observations lead to the fundamental assumptions guiding the formal proposal to follow. We can review them in order.

First is that they provide a new perspective from which to recapitulate a particular lesson implicit in Parry's work as regards compositionality. One of Parry's insights is that truth conditions—or propositions in the sense of sets of worlds—are not "internally" compositional in the sense that truth conditions for complex sentences are not always determined by the truth conditions of their subsentences. Sentences including operators like Parry's analytic implication $\rightarrow$, Fine's subject-matter inclusion $\preccurlyeq$, or Berto's topic-sensitive intentional modals can only be evaluated after one has determined the *topics* of their parts. When a language is expanded to include such notions, truth conditions are no longer functions of further truth conditions and are compositional only for appeal to an *external* semantic category.

We draw an analogous conclusion here. Our considerations suggest that topic itself is not "internally" compositional, *i.e.*, the topic of a complex is not in general a function of the topics of its parts. Rather, cases exist in which truth conditions of subformulae play a hand in determining a sentence's topic. Thus, the task of assigning topic must in some cases reach beyond the *internal* resources of a class of topics to an *external* semantic category of truth conditions.

These two lessons require a *refinement* rather than a *rejection* of compositionality. As we have considered matters, the categories of proposition and topic joined by their *mutual* sufficiency. This harmonious relationship is reflected in a sort of *reciprocal* compositionality between the two.

The possibility of interdependence between propositions and topic suggests that we consider a class of semantic items that is internally compositional. This interdependence would be respected by the internal compositionality of the class of *contents*—pairs including both a topic and a proposition. This is in harmony with the account of topic offered by Berto in [5] as a development of the intuitions underlying

Yablo's [29]. Compare the implicit "two-component" model (2C-semantics) that falls out of considerations of interdependence with the model described by Berto and Hawke in [5], in which a meaning P is treated as a pair of truth conditions C_P and topic T_P.

The core features of two-component approaches follow closely to our needs—the possibility of interdependence and irreducibility—are described succinctly by Berto and Hawke:

> A 2C-semantics can allow all kinds of important connections between truth conditions and topics. But according to 2C, facts about either cannot be reduced to facts about the other in the sense described above. [5, p. 27]

In what follows, this notion of content will play an important role in our formal models and it will become clear that the logic to be introduced is authentically two-component in the sense described by Berto and Hawke.

§3. State-sensitive analytic implication. As in [13], we return to the setting of Parry's PAI in order to demonstrate the *robustness* of the framework to be proposed, *e.g.*, an intuitive axiomatization may substantiate the framework's constancy and coherence while introducing a range of natural extensions may underscore its versatility and breadth. Not only does Parry's logic have historical merit as the progenitor of topic-sensitive techniques in logic, but the elegance of the PAI model theory developed by Fine in [15] and the perspicuity and accessibility of its axiomatization make it an attractive setting.

Most importantly, the parallels between its topic-theoretic machinery and that of alternative frameworks guarantees the *portability* of insights won in the setting of analytic implication. This portability ensures that the results of the present investigation can immediately transfer to the topic-theoretic framework of topic-sensitive intentional modals developed by Berto and his collaborators. What we show here, *e.g.*, for the strict conditional of Parry applies equally well not only for the variably-strict conditionals but—with appropriate adjustments—to a whole host of topic-sensitive intensional operators. As seen in [14], a wide range of applications can be quickly drawn from modifications to Parry's system; their implementation in PAI can be viewed simply as a sort of stress test or proof of concept.

3.1. Conditional agnostic analytic implication. Now, let us take our first steps towards a positive formalization of the core intuition. In this section, we review the framework of [13]. This work modified Parry's PAI to yield the logic of CA/PAI offering not only the flexibility to drop conditions like *Non-Ampliativity* and *Non-Explicativity*, but the modularity to impose semantic constraints corresponding to a wide range of topic-theoretic assumptions as needed.

Although investigations into systems related to PAI were carried out by Dunn in [10] and Urquhart in [28], the first model theory for PAI was described by Kit Fine in [15].[6] Fine's semantics for Parry's system equips each world w of an S4 Kripke model

[6] Technically, [15], too, provides model theory for an *extension* of Parry's logic, carrying over an axiom introduced by Dunn in [10]. But the addition is very modest and natural and the resulting system is frequently treated as a sort of "completion" of the axioms in [24]; indeed, the Dunn–Fine axiom is ultimately endorsed by Parry in [25].

with join semilattices of *topics* $\langle \mathcal{T}, \oplus \rangle$ and forms the background for the following definitions.

Let $\mathcal{L}$ be a propositional language including negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and an intensional conditional $\rightarrow$. We informally refer as well to the material conditional $\supset$ defined from $\neg$ and $\vee$ in the standard way.

DEFINITION 1. *A* CA/PAI *Fine model is a tuple* $\langle W, R, \mathcal{T}, \oplus, \multimap, v, t, h \rangle$ *such that*:

- *R is a reflexive and transitive binary relation on W.*
- *For each $w \in W$, $\langle \mathcal{T}_w, \oplus_w \rangle$ is a join semilattice.*
- v *is a valuation from atomic formulae to W*
- *For each $w \in W$, t_w is a function mapping atomic formulae to $\mathcal{T}_w$.*
- *For each $w \in W$, $\multimap_w$ is a binary function from $\mathcal{T}_w \times \mathcal{T}_w \rightarrow \mathcal{T}_w$.*
- *For all w, w' such that wRw', $h_{w,w'} : \mathcal{T}_w \rightarrow \mathcal{T}_{w'}$ is a homomorphism such that*:
  - *for atoms p, $h_{w,w'}(t_w(p)) = t_{w'}(p)$,*
  - $h_{w,w'}(a \oplus_w b) = h_{w,w'}(a) \oplus_{w'} h_{w,w'}(b)$,
  - $h_{w,w'}(a \multimap_w b) = h_{w,w'}(a) \multimap_{w'} h_{w,w'}(b)$.

A couple of remarks are in order. First, as a join semilattice, each $\langle \mathcal{T}_w, \oplus_w \rangle$ defines a partial order $\leq_w$ so that $a \leq_w b$ if $a \oplus_w b = b$ which will also be preserved by a homomorphism $h_{w,w'}$. Moreover, the introduction of homomorphisms $h_{w,w'}$ acts as a generalization of Fine's stipulation that topic inclusion between atoms be preserved across accessible worlds. In other words, the requirement that $h_{w,w'}(t_w(p)) = t_{w'}(p)$ ensures that if wRw' then for atoms p and q, $t_w(p) \leq_w t_w(q)$ implies $t_{w'}(p) \leq_{w'} t_{w'}(q)$, whence topic inclusion between sentences will persist across accessible worlds. Finally, note that the assumption that R is reflexive and transitive is made only to preserve continuity with the system PAI; not only could Definition 1 be modified to include analytic strict implications for weaker systems of modal logic, but one could follow, *e.g.*, Sylvan's work in [27] to produce conditional-agnostic versions of relevant containment logics.

The novel element of Definition 1 is the introduction of a binary function $\multimap$. The features of our $\multimap$ function will become clear through its role in definitions.

DEFINITION 2. *The topic assignment function t_w is extended through the language*:

- $t_w(\neg\varphi) = t_w(\varphi)$.
- $t_w(\varphi \star \psi) = t_w(\varphi) \oplus_w t_w(\psi)$ *for $\star$ extensional.*
- $t_w(\varphi \rightarrow \psi) = t_w(\varphi) \multimap_w t_w(\psi)$.

Truth at a world is defined:

DEFINITION 3. *Truth conditions are defined recursively*:

- $w \Vdash p$ *if* $w \in v(p)$.
- $w \Vdash \neg\varphi$ *if* $w \nVdash \varphi$.
- $w \Vdash \varphi \wedge \psi$ *if* $w \Vdash \varphi$ *and* $w \Vdash \psi$.
- $w \Vdash \varphi \rightarrow \psi$ *if* $\begin{cases} \text{for all } w' \text{ such that } wRw', \text{ if } w' \Vdash \varphi \text{ then } w' \Vdash \psi, \\ t_w(\psi) \leq_w t_w(\varphi). \end{cases}$

Given truth conditions for negation and conjunction, those for disjunction ($\vee$) and material implication ($\supset$) can be inferred from the above.

The truth conditions for analytic implication are "double-barrelled" (as pejoratively described by Sylvan in [26]) by dividing into parallel truth-theoretic and topic-theoretic components, framing analytic implication as an early case of a "two-component" approach to semantic content in the sense of [7]. We will return to this double-barrelled representation in Section 4.2, where it will play an important role. Having reviewed the features of CA/PAI models acting as a point of departure for our investigation, we introduce the structure of the state-sensitive model theory.

3.2. Models for state-sensitive analytic implication. We have considered reasons to reject *Topic Sufficiency* in the previous section, discussing natural language examples of conditionals whose subject-matter is *not* determined functionally by the subject-matters of their parts. The intuition common to these cases rests in an observation that often an intensional conditional $\varphi \rightarrow \psi$ is *about* classes of states (or situations) whose extent cannot be inferred by the topics of φ and ψ alone, *i.e.*, $t(\varphi \rightarrow \psi)$ is not determined merely by $t(\varphi)$ and $t(\psi)$. Recast as a formal critique of the $\multimap_w$ function of Definition 1, this intuition diagnoses a defect the failure to admit *sets of situations or worlds* as additional arguments.

To accommodate such arguments, we need not add new argument places to the function $\multimap_w$. Rather, we preserve the arity of $\multimap_w$ by modifying its *domain* so that its arguments are pairs of *contents* rather than *topics*.

This is reflected in an even more refined model—a *state-sensitive model*—upon minimal modifications to Definition 1. Let $w\uparrow$ denote the R-cone of w. Let $(\!|\varphi|\!)_w$ denote the set $\{w' \in w\uparrow \mid w' \Vdash \varphi\}$, *i.e.*, the collection of worlds accessible from w at which φ is true. Then:

DEFINITION 4. *An* S/PAI *Fine model is a tuple* $\langle W, R, \mathcal{T}, \mathcal{C}, \oplus, \multimap, v, t\rangle$ *revising Definition 1 by the clauses*:

- *For each* $w \in W$, $\mathcal{C}_w$ *is a set of* contents *defined recursively below.*
- *For any* $\langle a, X\rangle \in \mathcal{C}_w$, $h_{w,w'}(\langle a, X\rangle) = \langle h_{w,w'}(a), X \cap (w\uparrow)\rangle$.
- *For each* $w \in W$, $\multimap_w$ *is a binary function from* $\mathcal{C}_w \times \mathcal{C}_w \rightarrow \mathcal{T}_w$.
- *Whenever* wRw', $h_{w,w'}(c \multimap_w d) = h_{w,w'}(c) \multimap_{w'} h_{w,w'}(d)$.

Note that the functions $h_{w,w'}$ are now *polymorphic*, taking both topics from $\mathcal{T}_w$ and contents from $\mathcal{C}_w$ as arguments. Having made this remark, it is an appropriate time to be more explicit in how such contents are being represented in our models. We construct each set $\mathcal{C}_w$ by means of an intermediary—a function c_w assigning contents to sentences:

DEFINITION 5. *We define a function* c_w. *Where* π_0 *and* π_1 *are projection functions onto first and second coordinates, respectively, let* $c^0_w = \pi_0 \circ c_w$ *and* $c^1_w = \pi_1 \circ c_w$. *Then*:

- $c_w(p) = \langle t_w(p), v(p) \cap w\uparrow\rangle$.
- $c_w(\neg\varphi) = \langle c^0_w(\varphi), w\uparrow \setminus c^1_w(\varphi)\rangle$.
- $c_w(\varphi \wedge \psi) = \langle c^0_w(\varphi) \oplus_w c^0_w(\psi), c^1_w(\varphi) \cap c^1_w(\psi)\rangle$.
- $c_w(\varphi \vee \psi) = \langle c^0_w(\varphi) \oplus_w c^0_w(\psi), c^1_w(\varphi) \cup c^1_w(\psi)\rangle$.
- $c_w(\varphi \rightarrow \psi) = \langle c_w(\varphi) \multimap_w c_w(\psi), (\!|\varphi \rightarrow \psi|\!)_w\rangle$.

c_w assigns an element $c_w(\varphi)$ to each sentence in $\mathcal{L}$, which induces a succinct definition for each $\mathcal{C}_w$:

DEFINITION 6. *The set of contents* $\mathcal{C}_w$ *at a world* w *is the set* $c_w[\mathcal{L}]$*—the image of the language under* c_w

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One can note that an extension of each t_w over the language follows by setting $t_w(\varphi) = c^0_w(\varphi)$. This allows us to retain the stock CA/PAI truth conditions from Definition 3 and aids in the illumination of an important feature of the content assigned to a conditional $\varphi \rightarrow \psi$.

We had used the notation $(\!|\varphi \rightarrow \psi|\!)_w$ for purposes of economy in Definition 5, which could be fully expressed in the following terms:

$$(\!|\varphi \rightarrow \psi|\!)_w = \{w' \in w\!\uparrow \mid c^1_{w'}(\varphi) \subseteq c^1_{w'}(\psi), \text{ and } c^0_{w'}(\psi) \leq_{w'} c^0_{w'}(\varphi)\}.$$

The interdependence of topic and proposition is clearly recognizable between this presentation and the assignment of $t_w(\varphi \rightarrow \psi)$ in Definition 5. The influence of truth conditions over topic is reflected in the sensitivity of the value $c^0_w(\varphi \rightarrow \psi)$ to $c^1_w(\varphi)$ and $c^1_w(\psi)$ while the converse case is seen for the topic-theoretic clause $c^0_{w'}(\psi) \leq_{w'} c^0_{w'}(\varphi)$ that shapes the proposition $c^1_w(\varphi \rightarrow \psi)$.

This interdependence also brings corresponding risks of *interdefinition*, *e.g.*, hidden circularity in apparently recursive definitions. For this reason it is fitting to show that the models described above are in fact well-defined.

OBSERVATION 1. S/PAI *models are well-defined.*

Proof. The content of any formula requires only that contents have been assigned to simpler formulae, *e.g.*, $\varphi \rightarrow \psi$ requires that the contents $c_w(\varphi)$ and $c_w(\psi)$ have been determined only up to the complexity of φ. Thus, all the machinery is determined iteratively from the functions t_w and v in tandem with the determination of $\Vdash$. □

We now turn our attention to matters of topic preservation.

LEMMA 1. *If* wRw' *then* $h_{w,w'}(t_w(\varphi)) = t_{w'}(\varphi)$ *and* $h_{w,w'}(c_w(\varphi)) = c_{w'}(\varphi)$.

Proof. The basis steps are established by definition, so suppose that the property holds of subformulae of φ. In case φ is a conjunction, we have

$$\begin{array}{rcl} h_{w,w'}(c_w(\psi \wedge \xi)) & = & h_{w,w'}(\langle t_w(\psi) \oplus_w t_w(\xi), c^1_w(\psi) \cap c^1_w(\xi)\rangle) \\ & = & \langle h_{w,w'}(t_w(\psi)) \oplus_{w'} h_{w,w'}(t_w(\xi)), (c^1_w(\psi) \cap c^1_w(\xi)) \cap w'\!\uparrow\rangle \\ & = & \langle t_{w'}(\psi) \oplus_{w'} t_{w'}(\xi), (c^1_w(\psi) \cap w'\!\uparrow) \cap (c^1_w(\xi) \cap w'\!\uparrow)\rangle \\ & = & \langle t_{w'}(\psi) \oplus_{w'} t_{w'}(\xi), c^1_{w'}(\psi) \cap c^1_{w'}(\xi)\rangle \\ & = & c_{w'}(\psi \wedge \xi). \end{array}$$

This gives us identity of $h_{w,w'}(t_w(\varphi))$ and $t_{w'}(\varphi)$ for free. Simple modifications yield arguments to establish the cases of disjunction or material conditional.

The proof of a conditional $\psi \rightarrow \xi$ is slightly more interesting. Note briefly that $(\!|\varphi|\!)_{w'} = (\!|\varphi|\!)_w \cap w'\!\uparrow$ when wRw'. Then:

$$\begin{array}{rcl} h_{w,w'}(c_w(\psi \rightarrow \xi)) & = & \langle h_{w,w'}(c_w(\psi) \multimap_w c_w(\xi)), (\!|\psi \rightarrow \xi|\!)_w \cap w'\!\uparrow\rangle \\ & = & \langle c_{w'}(\psi) \multimap_{w'} c_{w'}(\xi), (\!|\psi \rightarrow \xi|\!)_{w'}\rangle \\ & = & c_{w'}(\psi \rightarrow \xi). \end{array}$$

Again, as $t_w(\psi \rightarrow \xi) = c^0_w(\psi \rightarrow \xi)$, the case of t_w follows trivially from the case of c_w. □

Lemma 1 straightforwardly leads to the persistence of topic inclusion for all formulae.

LEMMA 2. *If* $t_w(\varphi) \leq_w t_w(\psi)$ *and* wRw' *then* $t_{w'}(\varphi) \leq_{w'} t_{w'}(\psi)$.

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Proof. Expand the identity $t_w(\varphi) \leq_w t_w(\psi)$ to $t_w(\varphi) \oplus_w t_w(\psi) = t_w(\psi)$ and apply $h_{w,w'}$ to yield $h_{w,w'}(t_w(\varphi) \oplus_w t_w(\psi)) = h_{w,w'}(t_w(\psi))$. As $h_{w,w'}$ preserves structure, we infer $h_{w,w'}(t_w(\varphi)) \oplus_{w'} h_{w,w'}(t_w(\psi)) = h_{w,w'}(t_w(\psi))$, which by appeal to Lemma 1 yields $t_{w'}(\varphi) \oplus_{w'} t_{w'}(\psi) = t_{w'}(\psi)$. Of course, this is just to say that $t_{w'}(\varphi) \leq_{w'} t_{w'}(\psi)$ as needed. □

Now, we offer a definition for validity in S/PAI in a standard fashion:

Definition 7. $\Gamma \vDash_{\mathsf{S/PAI}} \varphi$ *if for all points* w *in all* S/PAI *models, if* $w \Vdash \gamma$ *for each* $\gamma \in \Gamma$, *also* $w \Vdash \varphi$.

Having defined validity, we can now proceed to examine some logical features of S/PAI. As a consequence of Lemma 2, we get the following easily:

Lemma 3. *In an* S/PAI *model, if* wRw', *then if* $w \Vdash \varphi \to \psi$ *then* $w' \Vdash \varphi \to \psi$.

In order to produce a maximally modular and minimally dogmatic tool, no properties beyond its functionality are imposed on each $\multimap_w$. As it turns out, functionality alone is enough to secure a number of intuitively correct validities:

Proposition 1. *The following are valid in state-sensitive* S/PAI *models*:

- $((\varphi \wedge \psi) \to \xi) \to ((\psi \wedge \varphi) \to \xi)$.
- $(\varphi \to (\psi \vee \xi)) \to (\varphi \to (\xi \vee \psi))$.

In other words, applying commutation—or distribution or DeMorgan's laws—to the antecedent or consequent of an intensional conditional remains topic-preserving in this setting. In contrast, there are some surprising cases in which entailments valid in CA/PAI turn out invalid in the weaker S/PAI. Consider the axiom [*R*]:

$$[R] \quad ((\varphi \vee \psi) \to \xi) \to ((\varphi \wedge \psi) \to \xi).$$

Despite some intuitive appeal—which we will soon review—the framework is not adequate to ensure its validity.

Proposition 2. *The axiom* [*R*] *is not valid in state-sensitive* S/PAI *models.*

Proof. Consider a simple countermodel with two worlds, w and w' with wRw', both of which make p and q true, while r is true only at w'. Also, let $\mathcal{T}_w$ include distinct elements a and b with $a \leq_w b$, taking special note that $t_w(p \vee q) = t_w(p \wedge q)$. Finally, assume of $\multimap_w$ merely that for contents $d, e \in \mathcal{C}_w$, $\pi_0(d \multimap_w e) = a$ if $\pi_0(d) = \{w, w'\}$ and $\pi_0(d \multimap_w e) = b$ if $\pi_0(d) = \{w'\}$. Then because $(\!|p \vee q|\!)_w = \{w, w'\}$ and $(\!|p \wedge q|\!)_w = \{w'\}$,

$$t_w((p \wedge q) \to r) = b \not\leq_w a = t_w((p \vee q) \to r).$$

Thus, $w \nVdash ((p \vee q) \to r) \to ((p \wedge q) \to r)$ as the requisite subject-matter inclusion condition is not satisfied. □

The state-sensitive S/PAI models can be shown to allow the same modularity and degree of semantic control as the CA/PAI models of Definition 1. This can receive a tentative illustration through describing a semantic condition on $\multimap_w$ that guarantees the validity of $((\varphi \vee \psi) \to \xi) \to ((\varphi \wedge \psi) \to \xi)$. It is desirable that such semantic conditions are elegant and admit natural interpretations. To this end, we pause to reconsider the proof of Proposition 2 (an activity serving as a preparatory exercise in anticipation of investigating extensions of S/PAI in the sequel).

The countermodel of Proposition 2 was constructed to interdict the content-inclusion requirement that $c_w(p \wedge q) \multimap_w c_w(r) \leq_w c_w(p \vee q) \multimap_w c_w(r)$, thereby falsifying the target formula. Intuitively, such a move appears to conflict with our earlier discussion of this section. To make the potential conflict more clear, note that $(\!|p \wedge q|\!)_w \subseteq (\!|p \vee q|\!)_w$ guarantees that the states that $(p \wedge q) \rightarrow r$ is *about* are *included among* the states that $(p \vee q) \rightarrow r$ is *about*, *i.e.*, $(p \vee q) \rightarrow r$ is about at least as much as $(p \wedge q) \rightarrow r$ (and possibly more). Moreover, the subject-matters of the sentences' antecedents and consequents, respectively, are identical to one another. Consequently, every *constituent* determining the subject-matter of $(p \wedge q) \rightarrow r$ is *included* in a constituent of the subject-matter of $(p \wedge q) \rightarrow r$. In a very real sense, the topic of $(p \vee q) \rightarrow r$ ought to include the topic of $(p \wedge q) \rightarrow r$.

It might be seen as counterintuitive that this inclusion would not be respected by semantic conditions governing the situation-theoretic subject-matters of the complexes. The foregoing reflections on the determination of the subject-matter of intensional conditionals in practice directly translate to a semantic condition that enforces the validity of [R]:

Definition 8. *An* S/PAI *model is propositionally monotonic if for all* w *and contents* $d, e, f \in \mathcal{C}_w$,

$$d \multimap_w f \leq_w e \multimap_w f \text{ if } \pi_0(d) = \pi_0(e) \text{ and } \pi_1(d) \subseteq \pi_1(e).$$

This property of propositional monotonicity is *at least* strong enough to ensure the validity of [R]:

Observation 2. *The axiom* [R] *is valid in propositionally monotonic models.*

Proof. To prove that the satisfaction of the condition entails validity of the sentence, assume that the condition holds. First, we cover the alethic component by observing that for arbitrary w, whenever $w \Vdash (\varphi \vee \psi) \rightarrow \xi$, it follows that $w \Vdash (\varphi \wedge \psi) \rightarrow \xi$. Assume that $w \Vdash (\varphi \vee \psi) \rightarrow \xi$. Then at any w' such that wRw', should $w' \Vdash \varphi \wedge \psi$, also $w' \Vdash \varphi \vee \psi$, which, by hypothesis, entails that $w' \Vdash \xi$. Also, because $t_w(\varphi \vee \psi) = t_w(\varphi \wedge \psi)$, whenever $t_w(\xi) \leq_w t_w(\varphi \vee \psi)$, it follows that $t(\xi)_w \leq_w t_w(\varphi \wedge \psi)$. Thus $w \Vdash (\varphi \wedge \psi) \rightarrow \xi$. To establish content inclusion, note that because $t_w(\varphi \vee \psi) = t_w(\varphi \wedge \psi)$ and $(\!|\varphi \wedge \psi|\!)_w \subseteq (\!|\varphi \vee \psi|\!)_w$, the condition guarantees that $c_w(\varphi \wedge \psi) \multimap_w c_w(\xi) \leq_w c_w(\varphi \vee \psi) \multimap_w c_w(\xi)$, *i.e.*, $t_w((\varphi \wedge \psi) \rightarrow \xi) \leq_w t_w((\varphi \vee \psi) \rightarrow \xi)$. Between these two observations, we conclude that for every w, $w \Vdash ((\varphi \vee \psi) \rightarrow \xi) \rightarrow ((\varphi \wedge \psi) \rightarrow \xi)$. ☐

We have not yet introduced sufficient machinery to go further and investigate whether the condition is *characteristic* for the extension of S/PAI including $((\varphi \vee \psi) \rightarrow \xi) \rightarrow ((\varphi \wedge \psi) \rightarrow \xi)$. We will return to this question in Section 4.4.

3.3. Axioms for state sensitive analytic implication. Having introduced its model theory, we now turn to a modification of Parry's axioms for PAI to introduce a Hilbert-style calculus for S/PAI.

Our formulation borrows the following idiom from the presentation of CA/PAI in [13]: Interpreting axioms of the form $\varphi \rightarrow \psi$ in which φ or ψ have $\rightarrow$ as the primary connective encapsulate principles concerning the topics of conditionals, the adoption of any such axiom restricts the generality of the framework. In such cases, we follow a pattern of replacing the main $\rightarrow$ operator of an axiom [An] by a material $\supset$ to yield an axiom labeled as $[An^{\dagger}]$. In all cases, we preserve as much of PAI and CA/PAI as possible without running afoul of our guiding principles.

Two notational comments must precede the introduction of the Hilbert calculus. First, we inherit Parry's notation so that when $f(\varphi)$ is a formula in which φ appears, $f(\psi)$ is a formula resulting from the replacement of one or more instances of φ with ψ in $f(\varphi)$. Second, we use the notation $\mathbf{t}_\varphi$ as shorthand for the formula $\varphi\supset\varphi$.[7] With these notational issues covered, we are equipped to describe an axiomatization of S/PAI:

Definition 9. *The logic of* state-sensitive analytic implication S/PAI *is determined by the following axioms*:

$$
\begin{array}{rl}
[A1] & (\varphi \wedge \psi) \to (\psi \wedge \varphi), \\
[A2] & \varphi \to (\varphi \wedge \varphi), \\
[A3] & \varphi \to \neg\neg\varphi, \\
[A4] & \neg\neg\varphi \to \varphi, \\
[A5] & (\varphi \wedge (\psi \vee \xi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \xi)), \\
[A6] & (\varphi \vee (\psi \wedge \neg\psi)) \to \varphi, \\
[A7^{\dagger}] & ((\varphi \to \psi) \wedge (\psi \to \xi))\supset(\varphi \to \xi), \\
[A8^{\dagger}] & (\varphi \to (\psi \wedge \xi))\supset(\varphi \to \psi), \\
[A9^{\dagger}] & ((\varphi \to \xi) \wedge (\psi \to \zeta))\supset((\varphi \wedge \psi) \to (\xi \wedge \zeta)), \\
[A10^{\dagger}] & ((\varphi \to \xi) \wedge (\psi \to \zeta))\supset((\varphi \vee \psi) \to (\xi \vee \zeta)), \\
[A11^{\dagger}] & (\varphi \to \psi)\supset(\varphi\supset\psi), \\
[A12^{\dagger}] & ((\varphi \leftrightarrow \psi) \wedge f(\varphi))\supset f(\psi), \\
[A14^{\dagger}] & ((\neg\varphi \to \varphi) \wedge (\varphi \to \psi))\supset(\neg\psi \to \psi), \\
[D2] & (\varphi \to \psi)\supset(\neg(\varphi \to \psi) \to (\varphi \to \psi)), \\
[S1] & \varphi \to \mathbf{t}_\varphi, \\
[S3] & (\mathbf{t}_\varphi \vee \psi \vee \xi) \to (\mathbf{t}_\varphi \vee \xi),
\end{array}
$$

and rules:

$$
\begin{array}{rl}
[MP1] & \varphi, \varphi\supset\psi \Rightarrow \psi, \\
[MP2] & \varphi, \varphi \to \psi \Rightarrow \psi, \\
[ADJ] & \varphi, \psi \Rightarrow \varphi \wedge \psi, \\
[MOD] & \varnothing \Rightarrow \neg\varphi \to \varphi \textit{ for } \varphi \textit{ an axiom.}
\end{array}
$$

Consequence can be defined in a familiar way:

Definition 10. $\Gamma \vdash_{\mathsf{S/PAI}} \varphi$ *if there is a Hilbert-style proof from assumptions* Γ *terminating with the formula* φ.

[7] The use of such devices has a long history in topic-sensitive contexts, especially in their utility in proving completeness. Deutsch uses such a device in [8] in proving completeness of his S, while more recent cases include Giordani in axiomatizing the logic of imagination in [17], Hawke, Özgün, and Berto in [19], and Özgün and Berto in axiomatizations for hyperintensional belief in [23].

Let us conclude with further proof-theoretic observations. It was shown in [13] that the simple PAI theorems described by Dunn as "normal form theorems" in [10] continue to hold in CA/PAI. Happily, these "normal form theorems" remain provable in S/PAI as well. We single out several useful theorems for their utility in what follows:

LEMMA 4. *The following are provable*:

$$[T9]\quad \varphi \vee \varphi \leftrightarrow \varphi, \quad [T16]\quad \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi),$$
$$[ID]\quad \varphi \to \varphi, \quad [S2]\quad (\varphi \vee \neg\varphi \vee \psi) \to (\varphi \vee \neg\varphi).$$

§4. Metatheoretical features of S/PAI.

The foregoing section laid out a number of definitions whose utility critically depends on our establishing certain metatheoretical results. This section includes the metatheoretical investigations necessary to justify our definitions concerning S/PAI.

4.1. Soundness of the axioms. Soundness requires establishing that the axioms of Definition 9 are valid, that is, true at every point in any S/PAI model and that the rules preserve truth. In this section, we will consider the axioms in order, prioritizing proofs of the less obvious cases.

OBSERVATION 3. *Axioms* $[A1]$–$[A5]$ *are valid.*

Proof. Each of axioms $[A1]$–$[A5]$ is of the form $\varphi \to \varphi'$ where both $t_w(\varphi) = t_w(\varphi')$ and $(\!|\varphi|\!)_w = (\!|\varphi'|\!)_w$. These facts suffice to ensure that $w \Vdash \varphi \to \varphi'$ holds. □

OBSERVATION 4. *Axiom* $[A6]$ *and axioms* $[A7^\dagger]$–$[A11^\dagger]$ *are valid.*

Proof. Because $t_w(\varphi) \leq_w t_w(\varphi) \oplus_w t_w(\psi \wedge \neg\psi) = t_w(\varphi \vee (\psi \wedge \neg\psi))$, an appropriate topic inclusion relationship is guaranteed to hold for $[A6]$ at any point w. Further, consistency dictates that for any w at which $\varphi \vee (\psi \wedge \neg\psi)$ is true, the formula φ is the only satisfiable disjunct, whence $w \Vdash \varphi$. These observations jointly ensure the validity of $[A6]$. The validity of the axioms $[A7^\dagger]$–$[A11^\dagger]$ can be easily established along similar lines. □

To prove validity of axiom $[A12^\dagger]$, we provide an introductory lemma.

LEMMA 5. *Say that formulae* φ *and* ψ *are* w-*indiscernible if* $c_w(\varphi) = c_w(\psi)$. *Then for* w-*indiscernible* φ *and* ψ, $f(\varphi)$ *and* $f(\psi)$ *will also be* w-*indiscernible.*

Proof. We prove this by induction on complexity of the depth in which φ and ψ are nested in $f(\varphi)$ and $f(\psi)$, respectively. For basis step, note that when the depth is zero, $f(\varphi) = \varphi$, in which case $f(\psi) = \psi$ and w-indiscernibility follows trivially. As induction hypothesis, assume w-indiscernibility of φ and ψ and let ψ be an immediate subformula of $f(\psi)$ in each of the following cases:

- When $f(\psi) = \neg\psi$, Negation Transparency entails $t_w(\neg\varphi) = t_w(\neg\psi)$ and because $(\!|\neg\xi|\!)_w = w{\uparrow} \setminus (\!|\xi|\!)_w$, it also follows that $(\!|\neg\varphi|\!)_w = (\!|\neg\psi|\!)_w$. So $c_w(f(\varphi)) = c_w(f(\psi))$.
- For conjunction, suppose without loss of generality that $f(\psi) = \psi \wedge \xi$. The definition of $\oplus_w$ ensures that $t_w(\varphi \wedge \xi) = t_w(\psi \wedge \xi)$ and set-theoretic operations ensure that $(\!|\varphi \wedge \xi|\!)_w = (\!|\psi \wedge \xi|\!)_w$, whence $c_w(f(\varphi)) = c_w(f(\psi))$. The argument holds *mutatis mutandis* for disjunction and the material conditional.
- For the case in which $f(\psi) = \psi \to \xi$, the functionality of $\multimap_w$ and w-indiscernibility of φ and ψ jointly ensure that $c_w(\varphi) \multimap_w c_w(\xi) = c_w(\psi) \multimap_w c_w(\xi)$,

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whence $c_w^0(\varphi \to \xi) = c_w^0(\psi \to \xi)$. Likewise, the truth-theoretic and topic-theoretic identity between the antecedents establishes that $(\!|\varphi \to \xi|\!)_w = (\!|\psi \to \xi|\!)_w$, which ensures that $c_w^1(\varphi \to \xi) = c_w^1(\psi \to \xi)$. Together, $c_w(\varphi \to \xi) = c_w(\psi \to \xi)$ as well. The case in which $f(\psi) = \xi \to \psi$ follows along similar lines.

Induction easily establishes the property that w-indiscernibility of φ and ψ entails w-indiscernibility of any $f(\varphi)$ and $f(\psi)$ irrespective of the depth in which φ and ψ are nested. □

OBSERVATION 5. *Axiom* $[A12^{\dagger}]$ *is valid.*

Proof. To prove validity, assume that $w \Vdash (\varphi \leftrightarrow \psi) \wedge f(\varphi)$. That $w \Vdash \varphi \leftrightarrow \psi$ guarantees that φ and ψ are w-indiscernible. By Lemma 5, we infer w-indiscernibility of $f(\varphi)$ and $f(\psi)$, whence $(\!|f(\varphi)|\!)_w = (\!|f(\psi)|\!)_w$. Thus, because $w \Vdash f(\varphi)$, it follows that $w \Vdash f(\psi)$. □

OBSERVATION 6. *Axiom* $[A14^{\dagger}]$ *is valid.*

Proof. If $\neg\varphi \to \varphi$ holds at w, φ must hold at every point in $w\uparrow$. In case $\varphi \to \psi$ holds at w as well, ψ must also hold at every point in $w\uparrow$, whence it holds vacuously that for every $w' \in w\uparrow$ at which $\neg\psi$ is true, ψ is true. By Negation Transparency, $t_w(\neg\psi) = t_w(\psi)$, ensuring that the axiom is valid. □

OBSERVATION 7. *Axiom* $[D2]$ *is valid.*

Proof. Suppose $w \Vdash \varphi \to \psi$. Then by Lemma 2, $\varphi \to \psi$ will continue to hold at all $w' \in w\uparrow$, whence it vacuously holds that all accessible points making true $\neg(\varphi \to \psi)$ also make true $\varphi \to \psi$. By Negation Transparency, $t_w(\neg(\varphi \to \psi)) = t_w(\varphi \to \psi)$. As the truth-theoretic and topic-theoretic conditions are met, it follows that $w \Vdash \neg(\varphi \to \psi) \to (\varphi \to \psi)$. □

OBSERVATION 8. *Axioms* $[S1]$ *and* $[S3]$ *are valid.*

Proof. As $\mathbf{t}_\varphi$—and thus $\mathbf{t}_\varphi \vee \xi$—is true at every point, every point w trivially satisfies the truth-theoretic conditions for both $[S1]$ and $[S3]$. It is easily confirmed that both $t_w(\mathbf{t}_\varphi) \leq_w t_w(\varphi)$ and $t_w(\mathbf{t}_\varphi \vee \xi) \leq_w t_w(\mathbf{t}_\varphi \vee \psi \vee \xi)$ hold at any w, whence the axioms' respective topic-theoretic conditions are also met. □

We will skip rules, although showing them to be validity preserving is straightforward. These considerations together provide the necessary components for a standard proof of soundness:

THEOREM 1. *If* $\Gamma \vdash_{\mathsf{S/PAI}} \varphi$ *then* $\Gamma \vDash_{\mathsf{S/PAI}} \varphi$.

4.2. Some important facts. Having established soundness, we now turn to the more difficult task of proving completeness. This preliminary section will lay the necessary groundwork to prepare for a completeness proof through the technique of canonical models.

The ultimate target of this section is a demonstration of a syntactic representation of Sylvan's "double-barrelled" analysis of analytic implication as a joint condition including distinct truth-theoretic and topic-theoretic components. To adequately describe this representation, we briefly revisit our modest notational device $\mathbf{t}_\varphi$, which will now be actively conscripted. Interpreted as a tautology with an identifiable topic, the presence of $\mathbf{t}_\varphi$ aids in the encoding of a number of important properties within the language itself. For example:

• $\mathbf{t}_\varphi$ can be leveraged as a mark of topic-theoretic equivalence by interpreting the formula $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi$ as a syntactic proxy for the identity of the subject-matters of formulae φ and ψ. Because the truth-theoretic conditions for $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi$ are trivially satisfied in virtue of the tautological character of each $\mathbf{t}_\xi$, the truth of $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi$ stands or falls precisely with the identity of the topics of φ and ψ.

• $\mathbf{t}_\varphi$ is critical in adequately representing S4 necessity by interpreting the sentence $\mathbf{t}_\varphi \to \varphi$ as a proxy for the necessity of φ. Insofar as the topics of φ and $\mathbf{t}_\varphi$ are identical, the topic-theoretic conditions on the truth of $\mathbf{t}_\varphi \to \varphi$ are trivially satisfied. As $\mathbf{t}_\varphi$ is a tautology, the satisfaction of the formula's truth-theoretic conditions requires that φ is true at every accessible world.

Returning to the syntactic surrogate of the double-barrelled analysis, the adequacy of the double-barrelled analysis will appear through the following equivalence:

$$\Gamma \vdash \varphi \to \psi \text{ iff } \begin{cases} \Gamma \vdash \mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi), \text{ and} \\ \Gamma \vdash \mathbf{t}_\varphi \to \mathbf{t}_\psi. \end{cases}$$

We set off to establish the fidelity of this representation by observing some important facts and lemmas. We do this in stages determined by individual components of the double-barrelled analysis. The goal for the first stage is to establish that an analytic implication $\varphi \to \psi$ encodes appropriate topic-theoretic conditions, namely, that the topic of φ includes the topic of ψ. The topic-theoretic condition itself receives a representation in the formula $\mathbf{t}_\varphi \to \mathbf{t}_\psi$, whence the task requires showing that provability of $\varphi \to \psi$ entails provability of $\mathbf{t}_\varphi \to \mathbf{t}_\psi$.

LEMMA 6. *If* $\Gamma \vdash \varphi \to \psi$ *then* $\Gamma \vdash \mathbf{t}_{\varphi \to \psi} \to (\varphi \to \psi)$.

Proof. Suppose that $\Gamma \vdash \varphi \to \psi$. By $[D2]$ and $[MP]$, $\Gamma \vdash \neg(\varphi \to \psi) \to (\varphi \to \psi)$. By Lemma 4, also $\Gamma \vdash (\varphi \to \psi) \to (\varphi \to \psi)$. So by $[A10^\dagger]$ and $[T9]$, conclude that $\Gamma \vdash (\neg(\varphi \to \psi) \vee (\varphi \to \psi)) \to (\varphi \to \psi)$, *i.e.*, $\Gamma \vdash \mathbf{t}_{\varphi \to \psi} \to (\varphi \to \psi)$. □

LEMMA 7. $\vdash (\mathbf{t}_\varphi \wedge \mathbf{t}_\psi) \leftrightarrow \mathbf{t}_{\varphi \wedge \psi}$.

Proof. For left-to-right, note that by some permutations and definitions, we can establish that $\vdash (\mathbf{t}_\varphi \wedge \mathbf{t}_\psi) \leftrightarrow [[(\varphi \wedge \psi)] \vee [(\neg\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)]]$. First, by Lemma 4, $\vdash (\varphi \wedge \psi) \to (\varphi \wedge \psi)$. Second, straightforward appeals to $[A10^\dagger]$, $[T9]$, and $[T16]$ show that $\vdash [(\neg\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)] \to \neg(\varphi \wedge \psi)$. Thus, $\vdash [[(\varphi \wedge \psi)] \vee [(\neg\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)]] \to [\neg(\varphi \wedge \psi) \vee (\varphi \wedge \psi)]$. Recognizing the consequent of this formula as $\mathbf{t}_{\varphi \wedge \psi}$, appeal to $[A7^\dagger]$ establishes that $\vdash (\mathbf{t}_\varphi \wedge \mathbf{t}_\psi) \to \mathbf{t}_{\varphi \wedge \psi}$.

For right-to-left, it is straightforward to show that $\vdash \mathbf{t}_{\varphi \wedge \psi} \to [(\varphi \vee \neg\varphi \vee \neg\psi) \wedge (\psi \vee \neg\psi \vee \neg\varphi)]$. By Lemma 4, both $\vdash (\varphi \vee \neg\varphi \vee \neg\psi) \to \mathbf{t}_\varphi$ and $\vdash (\psi \vee \neg\psi \vee \neg\varphi) \to \mathbf{t}_\psi$, whence appeals to $[A7^\dagger]$ and $[A9^\dagger]$ establish that $\vdash \mathbf{t}_{\varphi \wedge \psi} \to (\mathbf{t}_\varphi \wedge \mathbf{t}_\psi)$. □

LEMMA 8. *If* $\Gamma \vdash \varphi \to \psi$ *then* $\Gamma \vdash \varphi \leftrightarrow (\varphi \wedge \psi)$.

Proof. Applying $[A9^\dagger]$ to the identity $(\varphi \wedge \psi) \to (\varphi \wedge \psi)$ yields $(\varphi \wedge \psi) \to \varphi$. For the other direction, an application of $[A9^\dagger]$ secures $((\varphi \to \varphi) \wedge (\varphi \to \psi)) \supset ((\varphi \wedge \varphi) \to (\varphi \wedge \psi))$, so by $[ADJ]$ and hypothetical syllogism, we easily infer $\varphi \to (\varphi \wedge \psi)$. □

LEMMA 9. *If* $\Gamma \vdash \varphi \to \psi$ *then* $\Gamma \vdash \mathbf{t}_\varphi \to \mathbf{t}_\psi$.

Proof. Suppose that $\Gamma \vdash \varphi \to \psi$. Then by Lemma 8, we have $\Gamma \vdash \varphi \leftrightarrow (\varphi \wedge \psi)$ and by Lemma 4 we have $\Gamma \vdash (\varphi \supset \varphi) \to (\varphi \supset \varphi)$. By $[ADJ]$, we can conjoin

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these and use an instance of $[A12^{\dagger}]$ to yield $\Gamma \vdash (\varphi \supset \varphi) \to ((\varphi \wedge \psi) \supset (\varphi \wedge \psi))$. By definitions, instances of distribution, and applications of $[A7^{\dagger}]$, we can infer that $\Gamma \vdash (\varphi \supset \varphi) \to [(\varphi \vee \neg\varphi \vee \neg\psi) \wedge (\psi \vee \neg\psi \vee \neg\varphi)]$; by $[A8^{\dagger}]$, $\Gamma \vdash (\varphi \supset \varphi) \to (\psi \vee \neg\psi \vee \neg\varphi)$. But $(\psi \vee \neg\psi \vee \neg\varphi) \to (\psi \vee \neg\psi)$ is a theorem by Lemma 4, and by applying $[A7^{\dagger}]$, we infer that $(\varphi \supset \varphi) \to (\psi \vee \neg\psi)$, *i.e.*, $\mathbf{t}_{\varphi} \to \mathbf{t}_{\psi}$. □

The second stage of the double-barrelled analysis—that analytic implication $\to$ exhibits the properties of an S4 strict conditional—can now be approached. Because a strict conditional is emulated by a formula of the form, $\mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$, the task is then to show that $\varphi \to \psi$ entails $\mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$. We require a lemma:

LEMMA 10. $\vdash \mathbf{t}_{\varphi \supset \psi} \to \mathbf{t}_{\varphi}$.

Proof. By definitions and applications of distribution, we infer theoremhood of $((\varphi \supset \psi) \supset (\varphi \supset \psi)) \to ((\varphi \vee \neg\varphi \vee \psi) \wedge (\psi \vee \neg\psi \vee \varphi))$ and, by $[A8^{\dagger}]$, $((\varphi \supset \psi) \supset (\varphi \supset \psi)) \to (\varphi \vee \neg\varphi \vee \psi)$. By Lemma 4, we infer that $(\varphi \vee \neg\varphi \vee \psi) \to (\varphi \vee \neg\varphi)$, and by $[A7^{\dagger}]$, $((\varphi \supset \psi) \supset (\varphi \supset \psi)) \to (\varphi \supset \varphi)$, *i.e.*, $\mathbf{t}_{\varphi \supset \psi} \to \mathbf{t}_{\varphi}$. □

LEMMA 11. *If* $\Gamma \vdash \varphi \to \psi$ *then* $\Gamma \vdash \mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$.

Proof. By Lemma 4 and hypothesis, $[ADJ]$ yields $\Gamma \vdash (\varphi \to \psi) \wedge (\neg\varphi \to \neg\varphi)$; by an instance of $[A10^{\dagger}]$, we thereby may infer that $\Gamma \vdash (\varphi \vee \neg\varphi) \to (\neg\varphi \vee \psi)$, *i.e.*, $\mathbf{t}_{\varphi} \to (\varphi \supset \psi)$. By an appropriate instance of $[A9^{\dagger}]$, we appeal to Lemma 10 to infer that $\Gamma \vdash \mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$. □

Jointly, Lemmas 9 and 11 amount to the left-to-right direction of our desired equivalence. Our third stage involves establishing the right-to-left direction.

LEMMA 12. *If* $\Gamma \vdash (\varphi \to (\varphi \supset \psi))$ *then* $\Gamma \vdash \varphi \to \psi$.

Proof. By Lemma 4, the hypothesis, and $[ADJ]$, $\Gamma \vdash (\varphi \to \varphi) \wedge (\varphi \to (\neg\varphi \vee \psi))$. By appeal to appropriate instances of $[A2]$ and $[A9^{\dagger}]$, we infer that $\Gamma \vdash \varphi \to (\varphi \wedge (\neg\varphi \vee \psi))$ and, by $[A5]$, $\Gamma \vdash \varphi \to ((\varphi \wedge \neg\varphi) \vee (\varphi \wedge \psi))$. By an instance of $[A6]$, we continue to infer that $\Gamma \vdash \varphi \to (\varphi \wedge \psi)$ and by theoremhood of $(\varphi \wedge \psi) \to \psi$, $[ADJ]$ and $[A7^{\dagger}]$ give us $\Gamma \vdash \varphi \to \psi$. □

LEMMA 13. *If* $\Gamma \vdash \mathbf{t}_{\varphi} \to \mathbf{t}_{\psi}$ *then* $\Gamma \vdash \mathbf{t}_{\varphi} \to \mathbf{t}_{\varphi \supset \psi}$.

Proof. Suppose that $\Gamma \vdash \mathbf{t}_{\varphi} \to \mathbf{t}_{\psi}$, *i.e.*, $\Gamma \vdash (\varphi \vee \neg\varphi) \to (\psi \vee \neg\psi)$. Then by Lemma 4, $(\varphi \vee \neg\varphi) \to (\varphi \vee \neg\varphi)$ is a theorem, and an appeal to $[A10^{\dagger}]$ ensures that $\Gamma \vdash (\varphi \vee \neg\varphi) \to (\varphi \vee \neg\varphi \vee \psi \vee \neg\psi)$.

Now, by Lemma 4, $(\varphi \vee \neg\varphi \vee \psi \vee \neg\psi) \to (\varphi \vee \neg\varphi \vee \psi)$ and $(\varphi \vee \psi \vee \neg\psi) \to (\varphi \vee \neg\varphi \vee \psi \vee \neg\psi)$ are also theorems. Between $[A2]$ and $[A9^{\dagger}]$, then, we can infer that $(\varphi \vee \neg\varphi) \to ((\varphi \vee \neg\varphi \vee \psi) \wedge (\varphi \vee \neg\varphi \vee \psi \vee \neg\psi))$. But because, as we saw in earlier lemmas, distributions and definitions ensure theoremhood of $((\varphi \vee \neg\varphi \vee \psi) \wedge (\varphi \vee \neg\varphi \vee \psi \vee \neg\psi)) \to ((\varphi \supset \psi) \supset (\varphi \supset \psi))$, $[A9^{\dagger}]$ ensures that $\Gamma \vdash (\varphi \vee \neg\varphi) \to ((\varphi \supset \psi) \supset (\varphi \supset \psi))$, *i.e.*, $\Gamma \vdash \mathbf{t}_{\varphi} \to \mathbf{t}_{\varphi \supset \psi}$. □

We now show the right-to-left direction, whereby the joint provability of topic inclusion and strict conditional behavior suffice to prove the corresponding conditional $\varphi \to \psi$:

LEMMA 14. *If* $\Gamma \vdash \mathbf{t}_{\varphi} \to \mathbf{t}_{\psi}$ *and* $\Gamma \vdash \mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$ *then* $\Gamma \vdash \varphi \to \psi$.

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Proof. Suppose that $\Gamma \vdash \mathbf{t}_\varphi \to \mathbf{t}_\psi$. Then by Lemma 13, $\Gamma \vdash \mathbf{t}_\varphi \to \mathbf{t}_{\varphi\supset\psi}$. By Lemma 4, $\varphi \to \mathbf{t}_\varphi$, whence $[A7^\dagger]$ guarantees that $\varphi \to \mathbf{t}_{\varphi\supset\psi}$. Applying $[A7^\dagger]$ again to the hypothesis that $\Gamma \vdash \mathbf{t}_{\varphi\supset\psi} \to (\varphi\supset\psi)$ in turn yields that $\Gamma \vdash \varphi \to (\varphi\supset\psi)$. But by Lemma 12, this entails that $\Gamma \vdash \varphi \to \psi$. □

With Lemmas 9, 11, and 14 established, we conclude this section with a justification of the proposed syntactic characterization of the double-barrelled analysis:

THEOREM 2. $\Gamma \vdash \varphi \to \psi$ *iff* $\begin{cases}\Gamma \vdash \mathbf{t}_{\varphi\supset\psi} \to (\varphi\supset\psi), \textit{ and}\\ \Gamma \vdash \mathbf{t}_\varphi \to \mathbf{t}_\psi.\end{cases}$

With Theorem 2 in hand, we are able to proceed to proving completeness.

4.3. Completeness of the axioms. We prove completeness through the method of canonical models. Before defining the canonical model itself, we introduce some definitions. A satisfactory account of the canonical model's accessibility relation will make use of the following definition:

DEFINITION 11. *For a theory* Γ, *the set* $\Gamma^\Box = \{\varphi \mid \mathbf{t}_\varphi \to \varphi \in \Gamma\}$.

We anticipate the definition of the canonical model by introducing the notation:

DEFINITION 12. *For maximally consistent prime theories* Γ *and* Ξ, $\Gamma R^\star \Xi$ *if* $\Gamma^\Box \subseteq \Xi$. *We give expedient syntactic interpretations to familiar notions*:

- $\Gamma\uparrow^\star = \{\Xi \mid \Gamma R^\star \Xi\}$.
- $(\!|\varphi|\!)^\star_\Gamma = \{\Xi \in \Gamma\uparrow \mid \varphi \in \Xi\}$.

Although the decorations $\_^\star$ indicate that the above notions are defined syntactically and prevent confusion with earlier semantic definitions, eventually, it will become clear that the syntactic abuses of notation are justified.

The importance of the relationship between a set $\Gamma^\Box$ and other maximally consistent prime theories is immeasurable and it is worth pausing to get a better understanding of the properties of any such $\Gamma^\Box$. Thus, we make a detour and prove some important *closure properties* holding of $\Gamma^\Box$. The initial lemma shows that any such set $\Gamma^\Box$ at least includes all instances of axioms:

LEMMA 15 (Closure Under Axioms). *For all axioms* φ, $\varphi \in \Gamma^\Box$.

Proof. Let φ be an instance of an axiom. By Lemma 4, $\varphi \to \varphi \in \Gamma$ and by rule $[MOD]$ $\neg\varphi \to \varphi \in \Gamma$. Between $[A10^\dagger]$ and $[T9]$, we can infer that $\mathbf{t}_\varphi \to \varphi \in \Gamma$, whence $\varphi \in \Gamma^\Box$. □

Having established that each $\Gamma^\Box$ is populated, it now merits investigating its closure under the four rules of our calculus. First is adjunction ($[ADJ]$):

LEMMA 16 (Closure Under $[ADJ]$). *If* $\varphi \in \Gamma^\Box$ *and* $\varphi \in \Gamma^\Box$ *then* $\varphi \wedge \psi \in \Gamma^\Box$.

Proof. By hypothesis, Γ includes $\mathbf{t}_\varphi \to \varphi$ and $\mathbf{t}_\psi \to \psi$. Through the use of $[A9^\dagger]$, we can infer that Γ includes $(\mathbf{t}_\varphi \wedge \mathbf{t}_\psi) \to \varphi \wedge \psi$ and by Lemma 7, ultimately infer that $\mathbf{t}_{\varphi\wedge\psi} \to \varphi \wedge \psi \in \Gamma$, whence $\varphi \wedge \psi \in \Gamma^\Box$. □

Closure can be established for modus ponens with respect to $\supset$ ($[MP1]$):

LEMMA 17 (Closure Under $[MP1]$). *If* $\varphi\supset\psi \in \Gamma^\Box$ *and* $\varphi \in \Gamma^\Box$ *then* $\psi \in \Gamma^\Box$.

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Proof. Suppose that $\varphi \supset \psi \in \Gamma^{\Box}$ and $\varphi \in \Gamma^{\Box}$. Then $\mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi) \in \Gamma$ and $\mathbf{t}_{\varphi} \to \varphi \in \Gamma$. By an appeal to [$A9^{\dagger}$], we can infer that Γ includes $(\mathbf{t}_{\varphi \supset \psi} \wedge \mathbf{t}_{\varphi}) \to ((\varphi \supset \psi) \wedge \varphi)$. By appeal to Lemma 7, we infer the inclusion of $\mathbf{t}_{(\varphi \supset \psi) \wedge \varphi} \to (\mathbf{t}_{\varphi \supset \psi} \wedge \mathbf{t}_{\varphi})$ and—with a bit of permutation—an appeal to [$A6$] will get $((\varphi \supset \psi) \wedge \varphi) \to \psi$. Putting these together, we get

$$\mathbf{t}_{(\varphi \supset \psi) \wedge \varphi} \to \psi.$$

Of course, $\neg \mathbf{t}_{(\varphi \supset \psi) \wedge \varphi} \to \mathbf{t}_{(\varphi \supset \psi) \wedge \varphi}$ is provable, whence by an appeal to an appropriate instance of [$A14^{\dagger}$], we are able to conclude that $\neg \psi \to \psi \in \Gamma$. Because $\psi \to \psi \in \Gamma$, it nearly immediately follows that $\mathbf{t}_{\psi} \to \psi$, *i.e.*, that $\psi \in \Gamma^{\Box}$. □

Closure under [$MP1$] very quickly establishes closure under modus ponens with respect to $\to$ ([$MP2$]):

LEMMA 18 (Closure Under [$MP2$]). *If* $\varphi \to \psi \in \Gamma^{\Box}$ *and* $\varphi \in \Gamma^{\Box}$ *then* $\psi \in \Gamma^{\Box}$.

Proof. By appeal to Lemma 15, we infer that $\Gamma^{\Box}$ includes an appropriate copy of [$A11^{\dagger}$] from which an appeal to Lemma 17 establishes that $\varphi \supset \psi \in \Gamma^{\Box}$. That $\psi \in \Gamma^{\Box}$ then follows from a second appeal to Lemma 17. □

Finally, we establish closure of $\Gamma^{\Box}$ under the last of our rules:

LEMMA 19 (Closure Under [MOD]). $\neg \varphi \to \varphi \in \Gamma^{\Box}$ *for any axiom* φ.

Proof. An application of [MOD] entails that $\neg \varphi \to \varphi \in \Gamma$. An appeal to our earlier Lemma 6 ensures that Γ also includes $\mathbf{t}_{\neg \varphi \to \varphi} \to (\neg \varphi \to \varphi)$, *i.e.*, that $\neg \varphi \to \varphi \in \Gamma^{\Box}$. □

The merits of the foregoing Lemmas 15–19 can be recognized through the utility of the following type of closure they establish, namely, the closure of $\Gamma^{\Box}$ under theoremhood:

LEMMA 20. *For* φ *a theorem of* S/PAI, $\varphi \in \Gamma^{\Box}$.

Proof. This follows from induction on complexity of proofs. If φ is an axiom, then the property follows from Lemma 15. Otherwise, the theoremhood follows from the application of one of the four rules. But Lemmas 16–19 show that $\Gamma^{\Box}$ is closed under each of these rules. □

The foregoing lemmas permit us to infer a critical lemma and corollary that illuminate the workings of elements of $\Gamma^{\Box}$ as regards the theories in the family $\Gamma{\uparrow}^{\star}$:

LEMMA 21. $\Gamma^{\Box} \vdash \varphi$ *iff for all* $\Xi \in \Gamma{\uparrow}^{\star}$, $\varphi \in \Xi$.

Proof. Left-to-right holds by definition. Suppose that φ is a member of every theory $\Xi \in \Gamma{\uparrow}^{\star}$. Then $\Gamma^{\Box} \vdash \varphi$ (for otherwise, simple reasoning would show the existence of an extension in which $\neg \varphi$ holds). By the compactness of S/PAI, there exists a sequence of formulae $\xi_0, ..., \xi_{n-1} \in \Gamma^{\Box}$ such that $\bigwedge_{i<n} \xi_i \vdash \varphi$. Importantly, by Lemma 16, $\bigwedge_{i<n} \xi_i \in \Gamma^{\Box}$. By the classical deduction theorem, $\bigwedge_{i<n} \xi_i \supset \varphi$ is an S/PAI theorem, whence by Lemma 20, $\bigwedge_{i<n} \xi_i \supset \varphi \in \Gamma^{\Box}$. As $\bigwedge_{i<n} \xi_i \in \Gamma^{\Box}$, Lemma 17 can be applied to establish that $\varphi \in \Gamma^{\Box}$, whence $\Gamma^{\Box} \vdash \varphi$. □

COROLLARY 1. $\Gamma^{\Box} \vdash \varphi \supset \psi$ *iff for all* $\Xi \in \Gamma{\uparrow}^{\star}$, $\varphi \in \Xi$ *only if* $\psi \in \Xi$.

The importance of Corollary 1 lies in the image it depicts in which the syntactic behavior of the inclusion of $\varphi \supset \psi \in \Gamma^{\Box}$ mimics the semantics of a strict conditional.

Having thoroughly examined the device of $\Gamma^{\Box}$, we now aim to provide a satisfactory account of the canonical model's concept semilattices. This task requires the introduction of some preliminary definitions:

DEFINITION 13. *For a theory* Γ, $\sim_{\Gamma}$ *is an equivalence relation such that* $\varphi \sim_{\Gamma} \psi$ *if* $\mathbf{t}_{\varphi} \leftrightarrow \mathbf{t}_{\psi} \in \Gamma$ *and* $[\![\varphi]\!]_{\Gamma}$ *is the equivalence class of formulae induced by* $\sim_{\Gamma}$.

DEFINITION 14. *For a maximally consistent prime theory* Γ, $[\![\![\varphi]\!]\!]_{\Gamma} = \langle [\![\varphi]\!]_{\Gamma}, (\!|\varphi|\!)^{\star}_{\Gamma} \rangle$.

It should be clear that the value of these definitions for $[\![\varphi]\!]_{\Gamma}$ and $[\![\![\varphi]\!]\!]_{\Gamma}$ rests on their serviceability as syntactic proxies for topics and contents of φ, respectively. This observation motivates the next definition:

DEFINITION 15. *The functions* $\oplus_{\Gamma}$ *and* $\multimap_{\Gamma}$ *are defined so that*:

- $[\![\varphi]\!]_{\Gamma} \oplus_{\Gamma} [\![\psi]\!]_{\Gamma} = [\![\varphi \wedge \psi]\!]_{\Gamma}$,
- $[\![\![\varphi]\!]\!]_{\Gamma} \multimap_{\Gamma} [\![\![\psi]\!]\!]_{\Gamma} = [\![\varphi \to \psi]\!]_{\Gamma}$.

With these functions defined we have introduced all of the definitions necessary to describe canonical models.

DEFINITION 16. *The* S/PAI *canonical model is* $\mathfrak{M} = \langle W, R, \mathcal{T}, \mathcal{C}, \oplus, \multimap, v, t, h \rangle$ *where*:

- $W = \{\Delta \mid \Delta$ *a maximally consistent prime theory*$\}$,
- $R = \{\langle \Gamma, \Xi \rangle \mid \Gamma^{\Box} \subseteq \Xi\}$,
- $\Gamma \in v(p)$ *iff* $p \in \Gamma$,
- *For all* $\Gamma \in W$:

$$\begin{array}{lll} \mathcal{T}_{\Gamma} = t_{\Gamma}[\mathcal{L}] & \textit{and} & \mathcal{C}_{\Gamma} = c_{\Gamma}[\mathcal{L}], \\ t_{\Gamma}(\varphi) = [\![\varphi]\!]_{\Gamma} & \textit{and} & c_{\Gamma}(\varphi) = [\![\![\varphi]\!]\!]_{\Gamma}, \\ h_{\Gamma,\Xi}([\![\varphi]\!]_{\Gamma}) = [\![\varphi]\!]_{\Xi} & \textit{and} & h_{\Gamma,\Xi}([\![\![\varphi]\!]\!]_{\Gamma}) = [\![\![\varphi]\!]\!]_{\Xi}. \end{array}$$

Definition 16 induces an ancillary definition associating any maximally consistent prime theory with a particular canonical model:

DEFINITION 17. *For* Γ *a maximally consistent prime* S/PAI *theory, its canonical model* $\mathfrak{M}_{\Gamma}$ *is the submodel of* $\mathfrak{M}$ *where* $W_{\Gamma} = \{\Delta \mid \Gamma R \Delta\}$.

Having offered the definition, we must now establish its utility by showing that the objects so defined are models appropriate to our requirements.

First, we show that topic assignments persist along R as required. This is made easier by the following observation:

LEMMA 22. $[\![\varphi]\!]_{\Gamma} \leq_{\Gamma} [\![\psi]\!]_{\Gamma}$ *iff* $\mathbf{t}_{\psi} \to \mathbf{t}_{\varphi} \in \Gamma$.

Proof. Because $a \leq_{\Gamma} b$ is shorthand for $b = a \oplus_{\Gamma} b$, the requirement is to prove that $\Gamma \vdash \mathbf{t}_{\psi \wedge \varphi} \leftrightarrow \mathbf{t}_{\psi}$ iff $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\varphi}$. But because $\vdash \mathbf{t}_{\psi \wedge \varphi} \to \mathbf{t}_{\psi}$ can be easily established, the requirement is simply to show that $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\psi \wedge \varphi}$ iff $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\varphi}$.

For left-to-right, suppose that $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\psi \wedge \varphi}$. Then appeal to Lemma 7 and [$A8^{\dagger}$] assures us that $\mathbf{t}_{\psi \wedge \varphi} \to \mathbf{t}_{\varphi}$; by [$A7^{\dagger}$], then $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\varphi}$.

For right-to-left, suppose that $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\varphi}$. By, *e.g.*, Lemma 4 and [$A9^{\dagger}$], $\Gamma \vdash \mathbf{t}_{\psi} \to (\mathbf{t}_{\psi} \wedge \mathbf{t}_{\varphi})$; but Lemma 7 again establishes equivalence between the consequent and $\mathbf{t}_{\psi \wedge \varphi}$ itself, whence $\Gamma \vdash \mathbf{t}_{\psi} \to \mathbf{t}_{\psi \wedge \varphi}$. □

OBSERVATION 9. *If* $\Gamma R \Delta$ *and* $t_{\Gamma}(\varphi) \leq_{\Gamma} t_{\Gamma}(\psi)$ *then* $t_{\Delta}(\varphi) \leq_{\Delta} t_{\Delta}(\psi)$.

Proof. The hypothesis that $t_\Gamma(\varphi) \leq_\Gamma t_\Gamma(\psi)$ means that $\mathbf{t}_\psi \to \mathbf{t}_\varphi \in \Gamma$. By Lemma 6, $\Gamma \vdash \mathbf{t}_{\mathbf{t}_\psi \to \mathbf{t}_\varphi} \to (\mathbf{t}_\psi \to \mathbf{t}_\varphi)$, whence $\mathbf{t}_\psi \to \mathbf{t}_\varphi \in \Gamma^\Box$. The hypothesis that $\Gamma R \Delta$ means that $\Gamma^\Box \subseteq \Delta$, whence $\mathbf{t}_\psi \to \mathbf{t}_\varphi \in \Delta$ and $t_\Delta(\varphi) \leq_\Delta t_\Delta(\psi)$. □

We show that R has the necessary properties so that $\langle W, R\rangle$ is an S4 Kripke frame:

OBSERVATION 10. *R is reflexive and transitive.*

Proof. For reflexivity, note that for any $\varphi \in \Gamma^\Box$, $\mathbf{t}_\varphi \to \varphi \in \Gamma$. But as a tautology, $\mathbf{t}_\varphi \in \Gamma$ and by [*MP*], $\varphi \in \Gamma$. Thus, $\Gamma^\Box \subseteq \Gamma$, *i.e.*, $\Gamma R \Gamma$. For transitivity, suppose that $\Gamma R \Xi R \Theta$. We show $\Gamma R \Theta$. Pick a $\varphi \in \Gamma^\Box$. Then $\mathbf{t}_\varphi \to \varphi \in \Gamma$. By Lemma 6, also $\mathbf{t}_{\mathbf{t}_\varphi \to \varphi} \to (\mathbf{t}_\varphi \to \varphi) \in \Gamma$. Thus, $\mathbf{t}_\varphi \to \varphi$ is an element of $\Gamma^\Box$, and thus an element of Ξ. But if $\mathbf{t}_\varphi \to \varphi \in \Xi$, $\varphi \in \Xi^\Box$, whence $\varphi \in \Theta$. □

We furthermore need to demonstrate that our definitions for $\oplus$ and $\multimap$ are well-defined, that is, that these are in fact functions.

LEMMA 23. *Each $\oplus_\Gamma$ is a well-defined and total function.*

Proof. As $\mathcal{T}_\Gamma$ is defined in terms of the language, Definition 16 assigns a value to $\oplus_\Gamma$ for all pairs of arguments. It remains to show that such values are unique. That $\oplus_\Gamma$ is a function means that if $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi \in \Gamma$ then $\mathbf{t}_{\varphi \wedge \xi} \leftrightarrow \mathbf{t}_{\psi \wedge \xi} \in \Gamma$. Assume that the former condition holds; then by selecting instances of [$A9^\dagger$] and the theorem $\mathbf{t}_\xi \to \mathbf{t}_\xi$, we may straightforwardly infer that $(\mathbf{t}_\varphi \wedge \mathbf{t}_\xi) \leftrightarrow (\mathbf{t}_\psi \wedge \mathbf{t}_\xi) \in \Gamma$. By Lemma 7, we have as theorems $\mathbf{t}_{\varphi \wedge \xi} \leftrightarrow (\mathbf{t}_\varphi \wedge \mathbf{t}_\xi)$ and $\mathbf{t}_{\psi \wedge \xi} \leftrightarrow (\mathbf{t}_\psi \wedge \mathbf{t}_\xi)$. Thus, by several applications of rules to appropriate instances of [$A7^\dagger$], we can infer that $\mathbf{t}_{\varphi \wedge \xi} \leftrightarrow \mathbf{t}_{\psi \wedge \xi} \in \Gamma$. So in case $[\![\varphi]\!]_\Gamma = [\![\varphi']\!]_\Gamma$ and $[\![\psi]\!]_\Gamma = [\![\psi']\!]_\Gamma$, we are guaranteed that $[\![\varphi]\!]_\Gamma \oplus_\Gamma [\![\psi]\!]_\Gamma = [\![\varphi']\!]_\Gamma \oplus_\Gamma [\![\psi']\!]_\Gamma$. □

We turn next to the task of showing $\multimap_\Gamma$ to be well-defined, starting with a lemma:

LEMMA 24. *$c_\Gamma(\varphi) = c_\Gamma(\psi)$ iff $\varphi \leftrightarrow \psi \in \Gamma$.*

Proof. • For right-to-left, suppose that $\varphi \leftrightarrow \psi$. Then note two things: First, by Lemma 9, Γ includes $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi$, whence $[\![\varphi]\!]_\Gamma = [\![\psi]\!]_\Gamma$. Second, by Lemma 6, Γ includes $\mathbf{t}_{\varphi \to \psi} \to (\varphi \to \psi)$ and $\mathbf{t}_{\psi \to \varphi} \to (\psi \to \varphi)$, whence $\Gamma^\Box$ includes $\varphi \leftrightarrow \psi$. Thus, for any Ξ such that $\Gamma R \Xi$, $\varphi \in \Xi$ iff $\psi \in \Xi$, *i.e.*, $(\!|\varphi|\!)^\star_\Gamma = (\!|\psi|\!)^\star_\Gamma$. Together, this establishes that $[\![\![\varphi]\!]\!]_\Gamma = [\![\![\psi]\!]\!]_\Gamma$, *i.e.*, $c_\Gamma(\varphi) = c_\Gamma(\psi)$.

• For left-to-right, suppose that $c_\Gamma(\varphi) = c_\Gamma(\psi)$. As $[\![\varphi]\!]_\Gamma = [\![\psi]\!]_\Gamma$, Γ includes $\mathbf{t}_\varphi \leftrightarrow \mathbf{t}_\psi$ by definition. Likewise, because $(\!|\varphi|\!)^\star_\Gamma = (\!|\psi|\!)^\star_\Gamma$, Lemma 21 ensures that $\mathbf{t}_{\varphi \supset \psi} \to (\varphi \supset \psi)$ and $\mathbf{t}_{\psi \supset \varphi} \to (\psi \supset \varphi)$ are elements of Γ. These facts are sufficient for an appeal to Lemma 14 to establish that $\Gamma \vdash \varphi \leftrightarrow \psi$. □

LEMMA 25. *Each $\multimap_\Gamma$ is a well-defined and total function.*

Proof. By Definition 16, *some* value for $\multimap_\Gamma$ is defined for all pairs of arguments from $\mathcal{C}_\Gamma$. To show that this value is unique, assume that $c_\Gamma(\varphi) = c_\Gamma(\varphi')$ and $c_\Gamma(\psi) = c_\Gamma(\psi')$. By Lemma 24, Γ includes $\varphi \leftrightarrow \varphi'$ and $\psi \leftrightarrow \psi'$. By Lemma 4 we also know that Γ includes $\mathbf{t}_{\varphi \to \psi} \to \mathbf{t}_{\varphi \to \psi}$, providing the grounds to use modus ponens on instance of [$A12^\dagger$]:

$$((\varphi \leftrightarrow \varphi') \wedge (\mathbf{t}_{\varphi \to \psi} \to \mathbf{t}_{\varphi \to \psi})) \supset (\mathbf{t}_{\varphi \to \psi} \to \mathbf{t}_{\varphi' \to \psi})$$

to ensure that $\mathbf{t}_{\varphi \to \psi} \to \mathbf{t}_{\varphi' \to \psi} \in \Gamma$. A similar procedure establishes that $\mathbf{t}_{\varphi' \to \psi} \to \mathbf{t}_{\varphi' \to \psi'} \in \Gamma$. By small modifications to this process, one eventually can infer that Γ

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includes $\mathbf{t}_{\varphi\to\psi} \leftrightarrow \mathbf{t}_{\varphi'\to\psi'}$, *i.e.*, that $[\![\varphi \to \psi]\!]_\Gamma = [\![\varphi' \to \psi']\!]_\Gamma$. Consequently, $c_\Gamma(\varphi) \multimap_\Gamma c_\Gamma(\psi) = c_\Gamma(\varphi') \multimap_\Gamma c_\Gamma(\psi')$. □

The final step is to establish that the functions $h_{\Gamma,\Xi}$ exhibit the properties required by Definition 4:

LEMMA 26. *If* $\Gamma R \Xi$ *the function* $h_{\Gamma,\Xi}$ *enjoys the following properties*:

- $h_{\Gamma,\Xi}(t_\Gamma(p)) = t_\Xi(p)$ *and* $h_{\Gamma,\Xi}(c_\Gamma(p)) = c_\Xi(p)$ *for atoms* p.
- $h_{\Gamma,\Xi}([\![\varphi]\!]_\Gamma \oplus_\Gamma [\![\psi]\!]_\Gamma) = h_{\Gamma,\Xi}([\![\varphi]\!]_\Gamma) \oplus_\Xi h_{\Gamma,\Xi}([\![\psi]\!]_\Gamma)$.
- $h_{\Gamma,\Xi}(\llbracket\varphi\rrbracket_\Gamma \multimap_\Gamma \llbracket\psi\rrbracket_\Gamma) = h_{\Gamma,\Xi}(\llbracket\varphi\rrbracket_\Gamma) \multimap_\Xi h_{\Gamma,\Xi}(\llbracket\psi\rrbracket_\Gamma)$.

Proof. The definition of $h_{\Gamma,\Xi}$ establishes the initial atomic cases. For induction, we first examine the case of $\oplus_\Gamma$:

$$\begin{aligned} h_{\Gamma,\Xi}([\![\varphi]\!]_\Gamma \oplus_\Gamma [\![\psi]\!]_\Gamma) &= h_{\Gamma,\Xi}([\![\varphi \wedge \psi]\!]_\Gamma) \\ &= [\![\varphi \wedge \psi]\!]_\Xi \\ &= [\![\varphi]\!]_\Xi \oplus_\Xi [\![\psi]\!]_\Xi \\ &= h_{\Gamma,\Xi}([\![\varphi]\!]_\Gamma) \oplus_\Xi h_{\Gamma,\Xi}([\![\psi]\!]_\Gamma). \end{aligned}$$

We examine the case of $\multimap_\Gamma$:

$$\begin{aligned} h_{\Gamma,\Xi}(\llbracket\varphi\rrbracket_\Gamma \multimap_\Gamma \llbracket\psi\rrbracket_\Gamma) &= h_{\Gamma,\Xi}(\llbracket\varphi \to \psi\rrbracket_\Gamma) \\ &= \llbracket\varphi \to \psi\rrbracket_\Xi \\ &= \llbracket\varphi\rrbracket_\Xi \multimap_\Xi \llbracket\psi\rrbracket_\Xi \\ &= h_{\Gamma,\Xi}(\llbracket\varphi\rrbracket_\Gamma) \multimap_\Xi h_{\Gamma,\Xi}(\llbracket\psi\rrbracket_\Gamma). \end{aligned}$$ □

Collectively, the foregoing lemmas tender an assurance that the relation R, concept semi-lattices, and other model-theoretic components suffice to satisfy Definition 4. We may thereby conclude that $\mathfrak{M}$ (and each $\mathfrak{M}_\Gamma$) is in fact an S/PAI model.

LEMMA 27. $\varphi \in \Gamma$ *iff* $\Gamma \Vdash \varphi$.

Proof. In the atomic case, this is handled by the valuation function v. Thus, as induction hypothesis, suppose that the result has been established for all formulae of lesser complexity than φ.

• In case $\varphi = \neg\psi$, $\neg\psi \in \Gamma$ holds if (by maximality of Γ) and only if (by consistency of Γ) $\psi \notin \Gamma$. But by induction hypothesis, this is equivalent to $\Gamma \nVdash \psi$, *i.e.*, $\Gamma \Vdash \neg\psi$.

• In case $\varphi = \psi \wedge \xi$, $\psi \wedge \xi \in \Gamma$ if (by $[ADJ]$) and only if (by theoremhood of $(\psi \wedge \xi) \to \psi$ and $(\psi \wedge \xi) \to \xi$) both $\psi \in \Gamma$ and $\xi \in \Gamma$. By induction hypothesis, this is equivalent to the conjunction of $\Gamma \Vdash \psi$ and $\Gamma \Vdash \xi$, a condition which is equivalent to $\Gamma \Vdash \psi \wedge \xi$.

• Finally, consider the case in which $\varphi = \psi \to \xi$. By Theorem 2, $\psi \to \xi \in \Gamma$ holds precisely when $\mathbf{t}_{\psi\supset\xi} \to (\psi\supset\xi) \in \Gamma$ and $\mathbf{t}_\psi \to \mathbf{t}_\xi \in \Gamma$.

- $\mathbf{t}_{\psi\supset\xi} \to (\psi\supset\xi) \in \Gamma$ holds exactly when $\psi\supset\xi \in \Gamma^\Box$. By Corollary 1, this is equivalent to the case when at every Ξ such that $\Gamma R \Xi$, if $\psi \in \Xi$ then $\xi \in \Xi$. By induction hypothesis, this is equivalent to saying that at all accessible Ξ, if $\Xi \Vdash \psi$ then $\Xi \Vdash \xi$
- $\mathbf{t}_\psi \to \mathbf{t}_\xi \in \Gamma$ holds if and only if $\psi \wedge \xi \sim_\Gamma \psi$, *i.e.*, if $[\![\psi]\!]_\Gamma = [\![\psi]\!]_\Gamma \oplus_\Gamma [\![\xi]\!]_\Gamma$. But by construction, this is equivalent to $t_\Gamma(\xi) \leq_\Gamma t_\Gamma(\psi)$ by Lemma 22.

Jointly, the two above semantic conditions are equivalent to $\Gamma \Vdash \psi \to \xi$. □

Lemma 27, of course, immediately captures completeness of the axioms.

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THEOREM 3. *If* $\Gamma \vDash_{\mathsf{S/PAI}} \varphi$ *then* $\Gamma \vdash_{\mathsf{S/PAI}} \varphi$.

4.4. The restrictor axiom, revisited. To conclude, we return to make good on a promissory note issued in Section 3.2, in which we had considered an axiom *en passant*:

$$[R] \quad ((\varphi \vee \psi) \to \xi) \to ((\varphi \wedge \psi) \to \xi).$$

This axiom $[R]$ might be thought of as a *restrictor axiom* as it encapsulates an intuition that *restrictions* in the states that a conditional is about should be tracked by corresponding restrictions in the conditional's overall subject-matter. This principle seems extremely natural—if not necessitated—if one takes states to make up a part of a conditional's topic.

To illustrate, suppose that $(\varphi \wedge \psi) \to \xi$ is evaluated over the states at which $\varphi \wedge \psi$ is true while $(\varphi \vee \psi) \to \xi$ is evaluated over the states at which $\varphi \vee \psi$ is true. Intuitively, the states that $(\varphi \wedge \psi) \to \xi$ is *about* are a subset of the states that $(\varphi \vee \psi) \to \xi$ is *about*. One might expect to see this reflected as a principle of topic inclusion. In response, we can define a system resS/PAI—*restrictor monotone* S/PAI—as follows:

DEFINITION 18. $\mathsf{resS/PAI} = \mathsf{S/PAI} \oplus ((\varphi \vee \psi) \to \xi) \to ((\varphi \wedge \psi) \to \xi)$.

Recall from Section 3.2 the description of a particular feature of propositional monotonicity—that for all $d, e, f \in \mathcal{C}_w$,

$$d \multimap_w f \leq_w e \multimap_w f$$

in case $\pi_0(d) = \pi_0(e)$ and $\pi_1(d) \subseteq \pi_1(e)$—whose imposition sufficed to determine models for which this resS/PAI is sound. We had ended prematurely, having not yet defined the necessary tools for a completeness proof.

Serendipitously—and somewhat surprisingly—it turns out that this property does serve to characterize a model theory for resS/PAI:

OBSERVATION 11. resS/PAI *is characterized by propositional monotonicity.*

Proof. Soundness follows from Observation 2, leaving only completeness. Fix contents $[\![\varphi]\!]_\Gamma, [\![\psi]\!]_\Gamma, [\![\xi]\!]_\Gamma \in \mathcal{C}_\Gamma$ and make two assumptions: A first, topic-theoretic assumption that $\pi_0([\![\varphi]\!]_\Gamma) = \pi_0([\![\psi]\!]_\Gamma)$ and a second, propositional assumption that $\pi_1([\![\varphi]\!]_\Gamma) \subseteq \pi_1([\![\psi]\!]_\Gamma)$. Now, take note of two facts. From the first assumption on topic, we note that $\pi_0([\![\varphi]\!]_\Gamma) \oplus_\Gamma \pi_0([\![\psi]\!]_\Gamma)$ is identical to both $\pi_0([\![\varphi]\!]_\Gamma)$ and $\pi_0([\![\psi]\!]_\Gamma)$. From the assumption, basic set theory ensures that $\pi_1([\![\varphi]\!]_\Gamma) \cup \pi_1([\![\psi]\!]_\Gamma) = \pi_1([\![\psi]\!]_\Gamma)$ and $\pi_1([\![\varphi]\!]_\Gamma) \cap \pi_1([\![\psi]\!]_\Gamma) = \pi_1([\![\varphi]\!]_\Gamma)$.

Jointly, these entail that $c_\Gamma(\varphi \wedge \psi) = \langle \pi_0([\![\varphi]\!]_\Gamma), \pi_1([\![\varphi]\!]_\Gamma) \rangle = c_\Gamma(\varphi)$ (on the one hand) and that $c_\Gamma(\varphi \vee \psi) = \langle \pi_0([\![\psi]\!]_\Gamma), \pi_1([\![\psi]\!]_\Gamma) \rangle = c_\Gamma(\psi)$ (on the other). These identities ensure two further equivalences:

$$c_\Gamma(\varphi) \multimap_\Gamma c_\Gamma(\xi) = c_\Gamma(\varphi \wedge \psi) \multimap_\Gamma c_\Gamma(\xi) = [\![(\varphi \wedge \psi) \to \xi]\!]_\Gamma,$$
$$c_\Gamma(\psi) \multimap_\Gamma c_\Gamma(\xi) = c_\Gamma(\varphi \vee \psi) \multimap_\Gamma c_\Gamma(\xi) = [\![(\varphi \vee \psi) \to \xi]\!]_\Gamma.$$

Now, in virtue of the restrictor axiom $[R]$, by Lemmas 9 and 22, $[\![(\varphi \wedge \psi) \to \xi]\!]_\Gamma \leq_\Gamma [\![(\varphi \vee \psi) \to \xi]\!]_\Gamma$. Putting this all together, we get the following picture:

$$c_\Gamma(\varphi) \multimap_\Gamma c_\Gamma(\xi) = [\![(\varphi \wedge \psi) \to \xi]\!]_\Gamma \leq_\Gamma [\![(\varphi \vee \psi) \to \xi]\!]_\Gamma = c_\Gamma(\psi) \multimap_\Gamma c_\Gamma(\xi).$$

But, given the arbitrary choice of elements from $\mathcal{C}_\Gamma$, this is just the condition of propositional monotonicity. □

Clearly, there remains an abundance of axiomatic extensions that could be investigated. Insofar as each serves as an axiomatic representation of a distinct topic-theoretic condition, this setting invites the articulation and modeling of a remarkable range of detailed theories of intensional subject-matter as extensions of S/PAI.

§5. Concluding remarks. As we conclude, we might pause to summarize what we have accomplished. We began by considering several settings displaying plausible topic-theoretic features for which standard topic-theoretic frameworks appear inadequate. Reflecting on features common to these scenarios led us to identify among the standard frameworks potential causes for these inadequacies and permitted us to rough out the types of constraints that might replace them. As this sketch took form, we continued to sharpen the image by providing a concrete implementation in which the topic-theoretic machinery of Parry's PAI was remolded until our intuitions could be accommodated. We concluded by demonstrating the modularity and versatility of the new framework by examining several of its extensions and their corresponding semantic assumptions.

It is worth underscoring that the above systems have played the role of a *vehicle* through which a more fundamental contribution could be conveyed, a *laboratory* in which to formally assess the contribution on its philosophical merits and its mathematical tractability. Behind this particular implementation is a more general framework whose underlying topic-theoretic representation is subtle enough to account for situations in which *Topic Sufficiency* breaks down. (And given some reflection on the examples we have reviewed, I suspect that such situations are more common than not.) This continues the earlier work of [13], further refining the core Parry intuitions to produce a model of topic suitable for a wider range of settings.

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