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The multidisciplinary combinatorial approach (MCA) and its applications in engineering

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Abstract

The current paper describes the Multidisciplinary Combinatorial Approach (MCA), the idea of which is to develop discrete mathematical representations, called "Combinatorial Representations" (CR) and to represent with them various engineering systems. During the research, the properties and methods embedded in each representation and the connections between them were investigated thoroughly, after which they were associated with various engineering systems to solve related engineering problems. The CR developed up until now are based on graph theory, matroid theory, and discrete linear programming, whereas the current paper employs only the first two. The approach opens up new ways of working with representations, reasoning and design, some of which are reported in the paper, as follows: 1) Integrated multidisciplinary representation—systems which contain interrelating elements from different disciplines are represented by the same CR. Consequently, a uniform analysis process is performed on the representation, and thus on the whole system, irrespective of the specific disciplines, to which the elements belong. 2) Deriving known methods and theorems—new proofs to known methods and theorems are derived in a new way, this time on the basis of the combinatorial theorems embedded in the CR. This enables development of a meta-representation for engineering as a whole, through which the engineering reasoning becomes convenient. In the current paper, this issue is illustrated on structural analysis. 3) Deriving novel connections between remote fields—new connections are derived on the basis of the relations between the different combinatorial representations. An innovative connection between mechanisms and trusses, shown in the paper, has been derived on the basis of the mutual dualism between their corresponding CR. This new connection alone has opened several new avenues of research, since knowledge and algorithms from machine theory are now available for use in structural analysis and vice versa. Furthermore, it has opened opportunities for developing new design methods, in which, for instance, structures with special properties are developed on the basis of known mechanisms with special properties, as demonstrated in this paper. Conversely, one can use these techniques to develop special mechanisms from known trusses.

Keywords: Combinatorial Representations; Design; Graph Theory; Matroid Theory; Meta-Representation; Multidisciplinary Combinatorial Approach

1. INTRODUCTION

This paper presents an overview of a general approach called the Multidisciplinary Combinatorial Approach (MCA). During the research conducted with this approach, first the representations based on discrete mathematics, called Combinatorial Representations (CR) were developed. At this stage, the properties, theorems and methods embedded in each of the CR were investigated thoroughly and the connections between the individual CR were established. At the next stage, the CR were applied to represent different engineering systems from different fields and then to solve them.

From the results already achieved, it appears that the approach contributes to both practical and theoretical aspects of engineering. In the current paper, a few of these results are provided, while preserving the paper's main objective of giving a comprehensive perspective on MCA as a whole.

The use of graph theory in engineering and AI is widely accepted and many related works are reported in the literature, some of which are listed below. In structural analy-

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sis, the first work was done by Kron (1963), who used the analogy between electrical networks and elastic structures. Fenves was the first to develop a software program "STRESS" (Fenves, Logcher, & Mauch, 1965), which used a method based on graphs and networks for the formulation of the structural problems. Structure analysis and optimization using graph theory has been performed by Kaveh (1991, 1997). In machine theory, the first study of graph theory as a representation of mechanisms was conducted by Freudenstein and Dobrjanskyj (1964). In dynamics, Andrews (1971) associated vector algebra with graph theory, and called it the "vector-network model." Computer programs based on this formulation have been reported, among them, VECENT (Andrews & Kesavan, 1975). An approach that uses graph theory in a more general perspective was published by Bjorke (Wang & Bjorke, 1989). Bjorke found out that network theory is probably the best foundation for establishing a unified theory to represent a manufacturing system.

In a similar manner, many works published in Artificial Intelligence used graphs for knowledge representation. One of the first applications was to represent the state-space by graphs in which vertices corresponded to states and edges to the operators, causing the states to be changed (Nilsson, 1971).

The use of matroid theory to represent engineering systems is less known in the literature. In structural mechanics, the known works are by Kaveh, who used matroid theory to represent structures (Kaveh, 1997). A comprehensive report on the use of matroid theory in electrical networks and in statics can be found in Recski (1989). An extensive list of matroid theory applications can be found in Iri (1983).

The approach adopted in this paper is different from the works reported above. In this approach, the research at first was focused entirely on developing the CR and investigating their properties, their embedded methods, and the interrelations between them. Only then were the CR applied to represent engineering systems and to solve the related engineering problems. This conception has provided a general engineering perspective, which enabled obtaining the results to be reported in this paper through six sections, each presenting a different aspect of the approach, as follows.

Section 2 starts with providing the mathematical foundation for graph and matroid theories on which the CR presented in the current paper are based. Matroid theory is an advanced topic in discrete mathematics, which is not familiar to the engineering community and therefore graph theory terminology was employed in its explanation. In the current paper, the following four graph representations are introduced: Flow Graph Representation (FGR), which is applied to represent static systems; Potential Graph Representation (PGR), employed to represent mechanisms; Resistance Graph Representation (RGR), with its two embedded methods, employed to represent electrical, dynamical, hydraulic systems, and indeterminate trusses; and the Line Graph Representation (LGR), employed to repre-

Notations

$\mathbf{0}$	zero matrix
$\mathbf{A}$	incidence matrix
$\mathbf{B}$	scalar circuit matrix
$\vec{\mathbf{B}}$	vector circuit matrix
$\mathbf{B}(M)$	circuit matrix of a matroid
$\mathbf{C}$	set of matroid circuits
$\mathbf{D}$	vector of scalar displacements of truss elements
$\dim(\vec{F})$	dimension of the forces acting in the truss
E	set of graph edges
$e(G)$	number of edges in graph G
$\vec{\mathbf{F}}$	flow vector
$\vec{F}(e)$	flow in edge e
$F(e)$	magnitude of the flow in edge e
$\mathbf{F}^{\mathbf{I}}$	set independent subsets of a matroid
G	graph
G^*	the dual graph of graph G
G_F	flow graph
G_Δ	potential graph
G_R	resistance graph
$K(e)$	scalar conductance of edge e
$\mathbf{K}(e)$	matrix conductance of edge e
$\mathbf{K}_R$	square matrix containing the conductances of the resistance edges of a graph
$\mathbf{K}_{T'}$	conductance cutset matrix
$\mathbf{K}_\Delta$	conductance cutset matrix of the potential difference sources
M	matroid
M_Q	matroid defined by matrix **Q**
$\vec{\mathbf{P}}$	vector of flows in the flow sources
$\vec{\mathbf{Q}}$	vector cutset matrix
$\mathbf{Q}$	scalar cutset matrix
$\mathbf{Q}(M)$	cutset matrix of a matroid
$\hat{r}(e)$	unit vector in the direction of edge e
$R(e)$	scalar resistance of edge e
$\mathbf{R}(e)$	matrix resistance of edge e
$\mathbf{R}_{C'}$	resistance circuit matrix
$\mathbf{R}_P$	resistance circuit matrix of the flow sources
$\mathbf{R}_R$	square matrix containing the resistances of the resistance edges of a graph
$\mathbf{S}$	underlying set of a matroid
T	spanning tree
T'	spanning tree without sources
$\mathbf{T}$	set of matroid bases
$\vec{V}_i$	relative linear velocity of link *i*
V	set of the graph vertices
$v(G)$	number of vertices in graph G
$\vec{\Delta}$	potential difference vector
$\vec{\Delta}(e)$	potential difference in edge e
$\pi(i)$	potential of vertex *i*
$\emptyset$	empty set

Abbreviations

CCM	Conductance Cutset Method
CR	Combinatorial Representations
FGR	Flow Graph representation
PGR	Potential Graph Representation
LGR	Line Graph Representation
MCA	Multidisciplinary Combinatorial Approach
RCM	Resistance Circuit Method
RGR	Resistance Graph Representation
RMR	Resistance Matroid Representation

Matrices and sets in the current paper are designated by bold letters.

sent planetary gear systems. In addition, one matroid representation called Resistance Matroid Representation (RMR) is introduced and shown to be a generalization of the RGR.

Section 3 introduces one of the contributions of MCA—deriving new connections between remote engineering fields. In this section, based on the duality connection between FGR and PGR, a novel connection between trusses and mechanisms is derived. This innovation opens up new avenues in research and practical applications, some of which are reviewed in the following sections.

Section 4 gives a brief introduction to the contribution of MCA to the theoretical research. It postulates that the theorems embedded in the CR can be considered to be meta-theorems, from which known theorems and methods in engineering can be derived. This issue is demonstrated by proving that known theorems and methods in structural mechanics are derived from a theorem embedded in RGR, called Tellegen's theorem. This enables a new method of research where new theorems and methods will be developed on the basis of the knowledge embedded in the CR.

Section 5 highlights the contribution of MCA to dealing with integrated multidisciplinary systems. It is based on the fact that different engineering fields are represented by the same combinatorial representation, in this case RGR. This opens up the possibility of applying a unified method to deal with integrated systems consisting of elements from different fields. This section presents an example of a system composed of elements from dynamics, statics, and electricity interacting with one another. The graph representation of that system, on the other hand, does not distinguish between those different types of elements.

Section 6 introduces the further application of MCA that allows checking the validity of the engineering problem before applying the analysis methods to solve it. This issue is similar to the process done in the first representation used in AI—the logic representation, where the logic formulas should satisfy syntax rules, and if they do, they are called "well-formed formulas" (Genesereth & Nilsson, 1987). MCA deals differently with this issue: Its checking rules are based on the knowledge embedded in the CR. Demonstration of this ability is presented in subsection 6.5, where a problem of checking the rigidity of a truss, which was found to be difficult even for experts, is easily solved using MCA. This issue enhances Herbert Simon's postulate: "solving a problem simply means representing it so as to make the solution transparent. If the problem solving could actually be organized in these terms, the issue of representation would indeed become central" (p. 153) (Simon, 1981).

Section 7 introduces a possibility of developing new design techniques by using properties of MCA. Here, the idea for the design is derived from knowledge and ready designs from other fields. This idea is carried out by using the connection between mechanisms and trusses introduced in Section 3.

2. COMBINATORIAL REPRESENTATIONS

Combinatorial Representations (CR) are special representations based on discrete mathematics and used in MCA to represent various engineering systems. Combinatorial Representations are based on graph theory, matroid theory, and discrete linear programming. Table 1 lists combinatorial representations used in this paper and the engineering systems to which they are applied.

2.1. Mathematical foundation of the Combinatorial Representations

This section gives a brief introduction to the mathematical topics on which the combinatorial representations developed in this paper are based. These mathematical topics are network graphs and matroid theory. Network graphs are used in four graph representations: Flow Graph (FGR), Potential Graph (PGR), Resistance Graph (RGR) and Line-Graph (LGR) Representations. The matroid theory is used in the Resistance Matroid Representation (RMR).

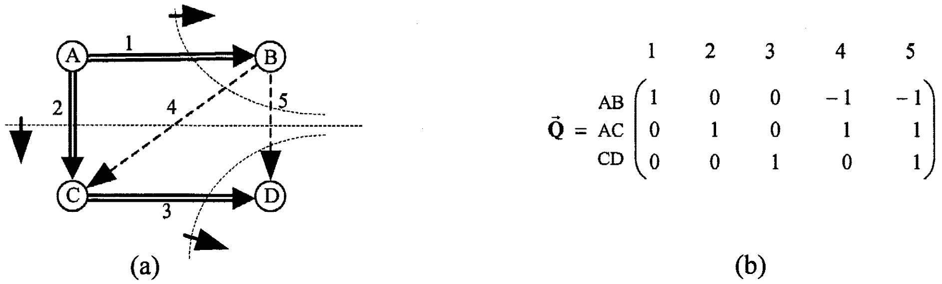
2.1.1. *Network graphs*

This section provides the reader with a brief survey on graph theory terminology. More details can be found in Shai (1997) and Shai and Preiss (1999b) or books on graph theory, such as Swamy and Thulasiraman (1981).

A graph is defined by the ordered pair G = ⟨V,E⟩, where V is the vertex set and E is the edge set, and every edge is defined by its two end vertices. If each edge in the graph is assigned a direction, then the graph is known as a directed graph. The directed graph is a network graph, if each edge

Table 1. *Combinatorial representations, their applications, and the corresponding sections in the current paper*

The Combinatorial Representation	Represented engineering systems	Section
Flow Graph Representation (FGR)	determinate trusses, geometric constraint systems	2.2
Potential Graph Representation (PGR)	mechanisms	2.3
Resistance Graph Representation (RGR)	mass-spring-damper dynamic systems, electric circuits, hydraulic systems, multidimensional indeterminate trusses, integrated systems	2.4
Resistance Matroid Representation (RMR)	indeterminate trusses	2.5
Line Graph Representation (LGR)	planetary gear systems	2.6



$$\vec{\mathbf{Q}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ AB & 1 & 0 & 0 & -1 & -1 \\ AC & 0 & 1 & 0 & 1 & 1 \\ CD & 0 & 0 & 1 & 0 & 1 \end{array}$$

Fig. 1. Example of a vector cutset matrix. (a) The cutsets of the graph. (b) The vector cutset matrix.

and vertex is assigned properties of flow and potential, respectively.

For convenience, this paper uses linetype attributes, which are:

- *A solid line*—represents an edge with unknown value of flow or potential difference.
- *A bold line*—represents an edge for which the flow or potential difference is known.
- *A dashed line*—represents a chord, which is an edge not included in the spanning tree. If the value of the flow in the chord is known, then it is both dashed and bold.
- *A double line*—represents a branch of a spanning tree.

In some of the graphs, one of the vertices will be chosen to be a special vertex called a "reference vertex," highlighted with a gray color.

To deal with the graph representations used in this paper, one should first define cutset and circuit matrices in their vector and scalar forms. Given a connected network graph, choosing a spanning tree within it defines its branches and chords.

A *cutset* in a connected graph is a minimal set of edges whose removal results in a disconnected graph. It can be proved (Swamy & Thulasiraman, 1981) that a cutset separates the graph into two components (maximal connected subgraphs). When a cutset includes only one branch of the spanning tree, it is called a "fundamental cutset," since any cutset in the graph can be obtained as a linear combination of these cutsets. Thus, this paper deals only with fundamental cutsets; hence, for brevity, they will be called cutsets. Each cutset is defined by the corresponding branch and is labeled with its index. The direction of the cutset is defined by the direction of its branch, as shown in Figure 1a.

The *vector cutset matrix* $\vec{\mathbf{Q}}$ is a matrix that describes all the graph cutsets, but contains only topological information. The matrix has e(G) columns (corresponding to the edges of the graph) and v(G) − 1 rows (corresponding to the cutsets or the branches that define them). The value of the matrix element $[\vec{\mathbf{Q}}]_{ij}$ may be +1, 0, or −1. It is +1 if edge j is included in the cutset that is defined by branch i and has the same orientation as the cutset, −1 if it has the opposite orientation, and 0 if it is not included in the cutset. The vector cutset matrix of the graph of Figure 1a is shown in Figure 1b.

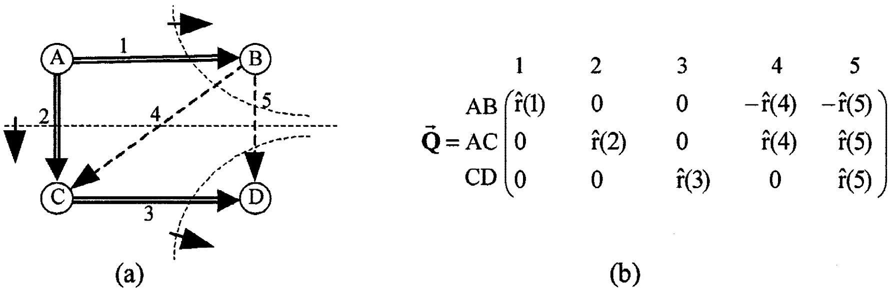
The *scalar cutset matrix* **Q** is obtained from the vector cutset matrix $\vec{\mathbf{Q}}$ by multiplying each column with a unit vector in the direction of the edge to which it corresponds. For example, the scalar cutset matrix of the graph of Figure 2a is given in Figure 2b.

It is well known from vector algebra that the unit vector with angle α can also be written as

$$\hat{\mathrm{r}} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}.$$

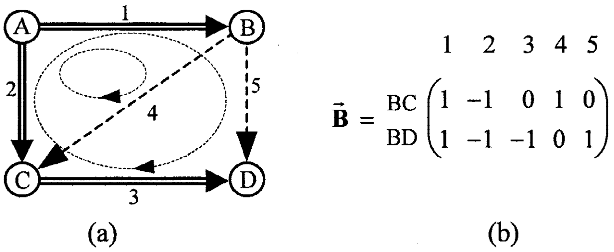
This notation is extensively used in the current paper.

A *circuit* is a set of edges that form a closed path. A circuit is called a fundamental circuit if it includes exactly



$$\vec{\mathbf{Q}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ AB & \hat{r}(1) & 0 & 0 & -\hat{r}(4) & -\hat{r}(5) \\ AC & 0 & \hat{r}(2) & 0 & \hat{r}(4) & \hat{r}(5) \\ CD & 0 & 0 & \hat{r}(3) & 0 & \hat{r}(5) \end{array}$$

Fig. 2. Example of scalar cutset matrix. (a) The cutsets of the graph. (b) The scalar cutset matrix.



$$\vec{\mathbf{B}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline BC & 1 & -1 & 0 & 1 & 0 \\ BD & 1 & -1 & -1 & 0 & 1 \end{array}$$

Fig. 3. Example of a vector circuit matrix. (a) The circuits of a graph. (b) The corresponding vector circuit matrix.

one chord. This paper deals only with fundamental circuits, and for brevity they will be called circuits. Each circuit will be labeled with the index of the chord that defines it. The direction of the circuit is defined by the direction of its chord, as shown in Figure 3a.

The *vector circuit matrix* $\vec{\mathbf{B}}$, demonstrated in Figure 3b, has e(G) columns as for the vector cutset matrix and $e(G) - v(G) + 1$ rows corresponding to the circuits. Each circuit is defined by a chord; therefore the number of rows is equal to the number of chords defined by the spanning tree. The element $[\vec{\mathbf{B}}]_{ij}$ is $+1$ if edge j is included in the circuit defined by chord i, and has the same orientation as the circuit, -1 if it has opposite orientation, 0 otherwise.

Every edge e is assigned a vector called the *flow* and designated by $\vec{F}(e)$ that can correspond to a force, flow of liquid, money, goods, or the like.[1]

Every vertex v is assigned a vector called the *potential*[2] and designated by $\vec{\pi}(v)$. The potential may represent a physical quantity such as displacement, pressure, or voltage, but it can also be used for other attributes. For instance, in the shortest path algorithm, it represents the lower bound of the distance (or the combined weights of the edges) from the current vertex to the target vertex (Shai, 1997). The potentials of the vertices of edge $e = \langle v_1, v_2 \rangle$ define the potential difference of that edge, designated $\vec{\Delta}(e)$, as follows:

$$\vec{\Delta}(e) = \vec{\pi}(v_2) - \vec{\pi}(v_1) \tag{1}$$

2.1.2. Basics of matroid theory

Matroid theory is a branch of discrete mathematics that possesses various important features, among them, the feature of generality that allows consideration of matroid theory as a generalization of graph theory. To simplify the explanation, matroid theory is introduced in this paper using terminology of graph theory. Matroid representation is used in MCA to represent various engineering systems, whereas in this paper, it is used to represent and analyze indeterminate trusses.

DEFINITION. If we denote **S** to be a finite set and $\mathbf{F^I}$ to be a collection of certain subsets of **S**, then the pair $M = \langle \mathbf{S}, \mathbf{F^I} \rangle$ is called a matroid if the following properties are satisfied:

1. $\varnothing \in \mathbf{F^I}$
2. If $\mathbf{X} \in \mathbf{F^I}$ and $\mathbf{Y} \subseteq \mathbf{X}$ then $\mathbf{Y} \in \mathbf{F^I}$ must also hold.
3. If $\mathbf{X} \in \mathbf{F^I}$ and $\mathbf{Y} \in \mathbf{F^I}$ and $|\mathbf{X}| > |\mathbf{Y}|$, then there exists an element $x \in \mathbf{X} - \mathbf{Y}$, so that $\mathbf{Y} \cup \{x\} \in \mathbf{F^I}$.

S is said to be *the underlying set of matroid* M. The subsets of **S** which belong to $\mathbf{F^I}$ are said to be *independent subsets*; otherwise they are called *dependent* subsets.

Maximal independent sets of M, that is, independent sets that are not contained in any other independent set of M, are called *bases of* M. For every base of M there is a corresponding *cobase* which is the complement of the base to **S**. It can be proved (Recski, 1989) that the sizes of all the bases of a matroid are equal. In graph theory terminology, a base is a spanning tree of the matroid. Thus, every matroid can be described by the collection of all its bases **T**, instead of the collection of all its independent sets $\mathbf{F^I}$. Minimal dependent sets of M, that is, dependent sets which do not contain other dependent sets, are called *circuits of M*. The collection of all the circuits of M is denoted by **C**. It also can be used instead of $\mathbf{F^I}$ to describe the matroid.

DEFINITION OF A MATROID CUTSET. The subset $\mathbf{X} \subseteq \mathbf{S}$ is called a *cutset of M* if and only if it satisfies the following conditions:

1. $\mathbf{X} \neq \varnothing$;
2. $|\mathbf{X} \cap \mathbf{Y}| \neq 1$ for every $\mathbf{Y} \in \mathbf{C}$;
3. **X** is minimal with respect to these properties.

Since a base is a maximal possible set of independent elements, adding an additional element to the base turns it into a dependent set, that is, a set that contains a circuit. Therefore every cobase element defines exactly one circuit which contains the element itself and all the other elements are from the base only. Such a circuit is called a fundamental circuit.

It can also be shown that every base element defines a unique cutset that contains the element itself and all the other elements are cobase elements. Such a cutset is called a *fundamental cutset*.

2.1.2.1. Representing graph as a matroid. Consider the graph of Figure 4a and its corresponding vector cutset matrix in Figure 4b. One can define a matroid associated with this matrix as follows: $M_Q = \langle \mathbf{S}_Q, F_Q^{\mathbf{I}} \rangle$, where the underlying set S_Q is equal to the set of columns of $\vec{\mathbf{Q}}$, and the family of independent sets $F_Q^{\mathbf{I}}$ is the collection of all sets of columns which are linearly independent. According to lin-

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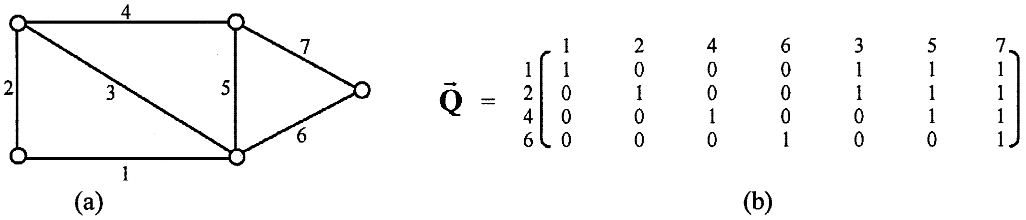
[1] In control theory, this is called the "through variable," but the word "flow" is more suitable for the work reported here.

[2] The potential difference between the vertices defining an edge is known in control theory as the "across variable."



$$\vec{\mathbf{Q}} = \begin{array}{c|ccccccc} & 1 & 2 & 4 & 6 & 3 & 5 & 7 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Fig. 4. Graph to be represented by a matroid.

ear algebra, every set of graph edges that corresponds to a minimal dependent set of columns of $\vec{\mathbf{Q}}$ forms a circuit in the graph. By matroid properties, such a set is also a circuit in the matroid. Thus, the circuits in M_Q completely correspond to the circuits in the graph. Moreover, a similar claim is true for the bases of M_Q and spanning trees of the graph, cutsets in M_Q and cutsets in the graph, etc.

One can see, for example, that columns 1, 2, and 3 of $\vec{\mathbf{Q}}$ of Figure 4b are linearly dependent, whereas edges 1, 2, and 3 form a circuit in the graph (Fig. 4a). On the other hand, columns 1, 2, 4, and 6 of the matrix form the maximum possible independent set of columns, that is, the base of M_Q, whereas edges 1, 2, 4, and 6 in the graph form a spanning tree.

2.1.2.2. Representing matrix as a matroid. Previously, it was explained that every vector cutset matrix of a graph can be considered as a matroid and that such a matroid actually represents the graph. In the current section, this issue is expanded and it is shown that every matrix corresponds to a matroid. This time it is possible that this matroid does not have a corresponding graph.

Let $\mathbf{Q}$ be a m × n matrix. The matroid $M_Q = \{\mathbf{S}_Q, F_Q^{\mathbf{I}}\}$ can be defined as follows.

1. The underlying set S_Q is the set of *n* column vectors of $\mathbf{Q}$.
2. Every subset of linearly independent columns of $\mathbf{Q}$ belongs to $F_Q^{\mathbf{I}}$.

Consider, for example, the matrix in Fig. 5.

The underlying set $\mathbf{S}_Q$ of the matroid representing the matrix of Figure 5 is:

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

Fig. 5. Matrix to be represented by a matroid.

$$\mathbf{S}_Q = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Some of the independent subsets of $F_Q^{\mathbf{I}}$ are:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\},$$

the first two of which are also the bases of M_Q, since any additional column from $\mathbf{S}_Q$ will cause a linear dependence. And some of the circuits (elements of $\mathbf{C}_Q$) are:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

2.2. The Flow Graph Representation (FGR)

DEFINITION OF THE FLOW GRAPH REPRESENTATION (FGR). A network graph G_F is a flow graph, designated by G_F, if the flows in the edges are independent of the potential differences and satisfy the Flow Law, stated as follows: The vector sum of the flows in every cutset of G is equal to zero.

The flow law may be recognized as a generalization of the well-known Kirchhoff's Current Law (KCL). Note that KCL is restricted only to one dimension, which is appropriate for electrical circuits, while the flow law can be multidimensional; thus it can also be used for structures and other engineering systems that require two or three dimensions.

The matrix form of the Flow Law is:

$$\vec{\mathbf{Q}} \cdot \vec{\mathbf{F}} = \mathbf{0}, \qquad (2)$$

where $\vec{\mathbf{F}}$ is the vector of the flows, or *Flow Vector*.

The FGR can be used to represent various engineering systems, such as simple electrical circuits, mass-cable systems in force equilibrium, and so forth.

The important property of the flow graph is that it should not contain cutsets consisting entirely of the flow sources, namely, edges whose flows are given. For if such cutsets of sources existed, then, by the flow law, there would be a linear dependence between the flows in these sources, violating the definition of the flow sources. Therefore, the spanning tree of the flow graph should be chosen in such a way that it does not include the flow sources.

2.2.1. *Representing mass-cable systems by the flow graph*

Figure 6a shows an example of a system of masses connected by cables which is known to be in static equilibrium. The objective is to find the gravitation force acting on the mass B. Since the system is in static equilibrium, it is obvious that it should be solved using statical equations relating the forces acting in the system. The most convenient representation for this purpose is FGR. Each edge in FGR corresponds to an acting force, no matter whether it is tension in the cable or an external force. Each vertex corresponds to a point or a body, on which a number of forces is acting, like a mass or a pulley. The flow graph representing the system of Figure 6a is shown in Figure 6b.

The analysis equations for this graph are as follows.

$$\vec{\mathbf{Q}}\cdot\vec{\mathbf{F}} = \mathbf{0} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_5 \\ T_4 \\ M_A g \\ M_B g \end{pmatrix}$$

$$= \mathbf{0} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ T_5 \\ T_4 \\ M_B g \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \times M_A g.$$

Note, that since the engineering system of Figure 6a is one-dimensional, its vector and scalar cutset matrices are identical.

2.2.2. *Employing the Flow Graph Representation (FGR) in representing determinate trusses*

The conventional procedure used to analyze determinate trusses is based on building the force equilibrium equation for each joint of the truss and for each coordinate axis. This strongly correlates with the flow law, according to which, the sum of the flows at each cutset (and thus vertex) of the graph is equal to zero. Accordingly, one can represent a determinate truss by d(G) one-dimensional FGRs, each corresponding to a different coordinate axis. For example, a plane determinate truss of Figure 7a is represented by two flow graphs—one corresponding to the X coordinate (Fig. 7b) and the other to the Y coordinate (Fig. 7c).

The graphs presented in Figure 7b and Figure 7c are of the same topology as the represented truss. One can think of FGR as if the flow comes out from vertex O, flows through the flow sources (external forces), then through the edges (rods) and returns back to the vertex O through the reaction edges. Vertex O is called the "*reference vertex*" (gray vertex) since it assures that the sum of all the external forces acting on the truss is also equal to zero. The reference vertex can be considered to be the equivalent of the ground in electrical circuits.

One can see that if the flows in the edges of the corresponding graph are equated to the correct forces in the truss, they would be valid, that is, would satisfy the flow law.

To perform the statical analysis of the truss, there is a need to use the angles of the rods and the external forces. The latter knowledge affects the ratio between the flows in

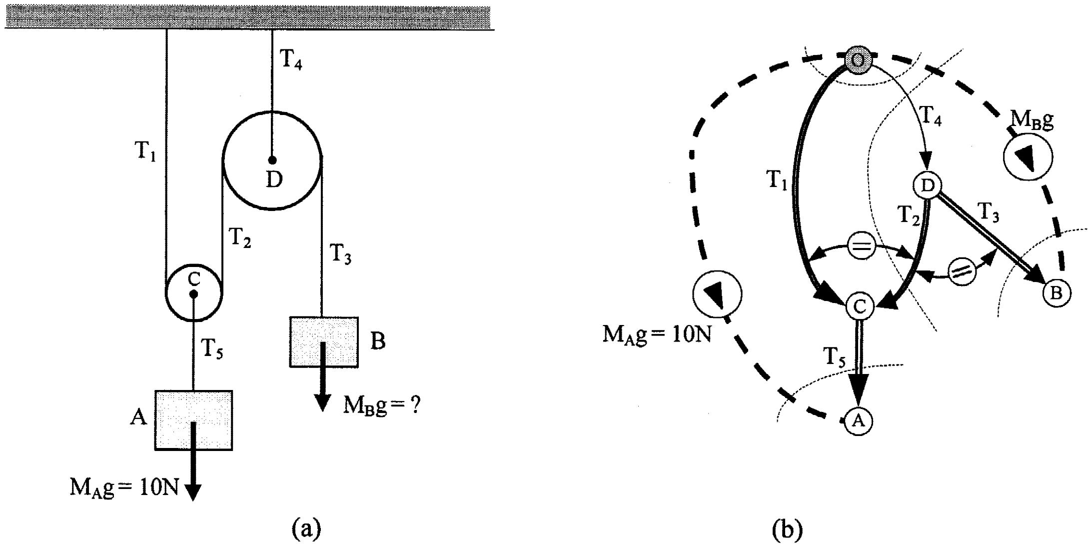
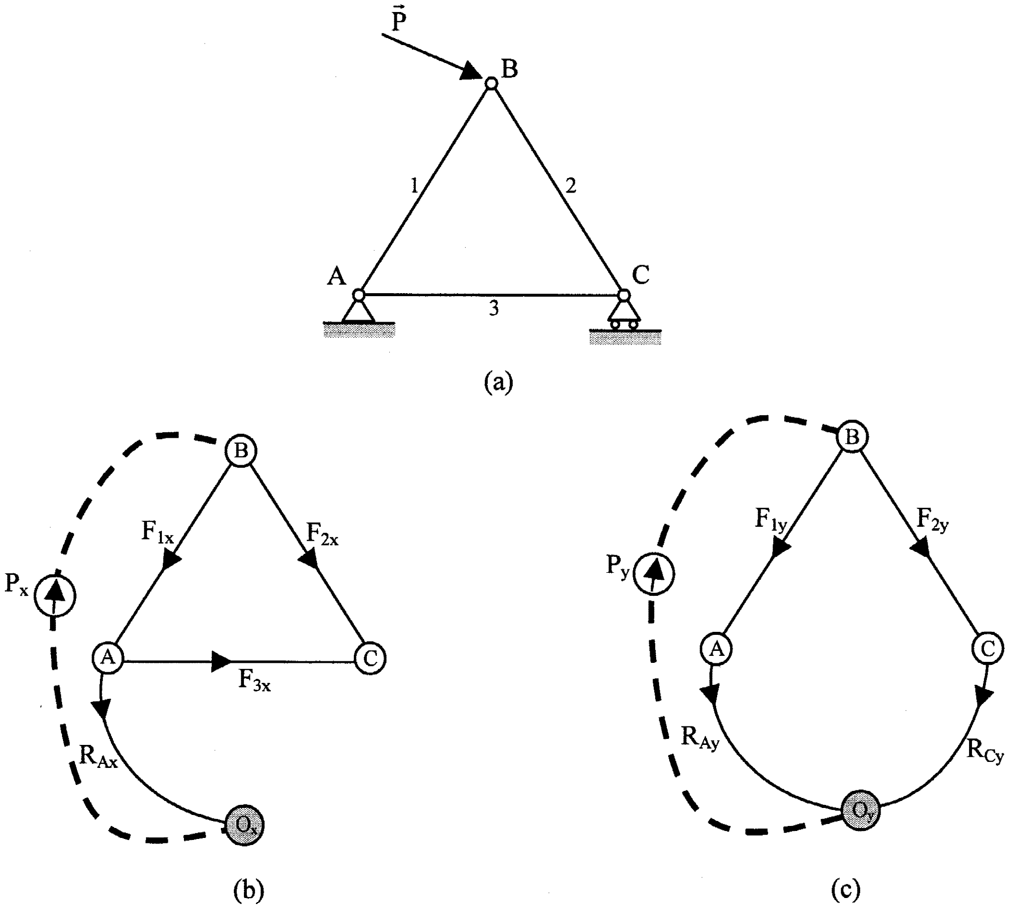


Fig. 6. Example of a mass–cable system and its corresponding graph. (a) System of cables and masses in static equilibrium. (b) FGR representing the system.



(a)

(b)

(c)

Fig. 7. Plane determinate truss and its two one-dimensional Flow Graph Representations. (a) Plane determinate truss. (b),(c) FGRs corresponding to the X and Y coordinate axes.

the two FGRs corresponding to the X and Y coordinates. The most efficient way to store such information is by merging the two one-dimensional graphs into one two-dimensional graph, setting the angles of the flows to be constant. In other words, instead of representing the truss by two FGRs with one dimensional flow in each, one FGR with multidimensional flow is used, as explained below.

The steps for representing the truss by a multidimensional flow graph are:

STEP 1. Create a vertex in the graph for every pinned joint in the truss.

STEP 2. For every rod create an edge in the graph, called a "truss edge"; its end vertices correspond to the joints that connect the corresponding rod to the truss. Assign an arbitrary orientation to each truss edge and a unit vector $\hat{r}(e)$ indicating the direction from the tail joint to the head joint. The engineering meaning of the flow in the edge is the force applied on the head joint by the rod in the direction of the unit vector $\hat{r}(e)$. As is explained in detail in Shai (2001a), if the flow in the edge is positive, then the rod is in a compression state, otherwise it is in a tension state.

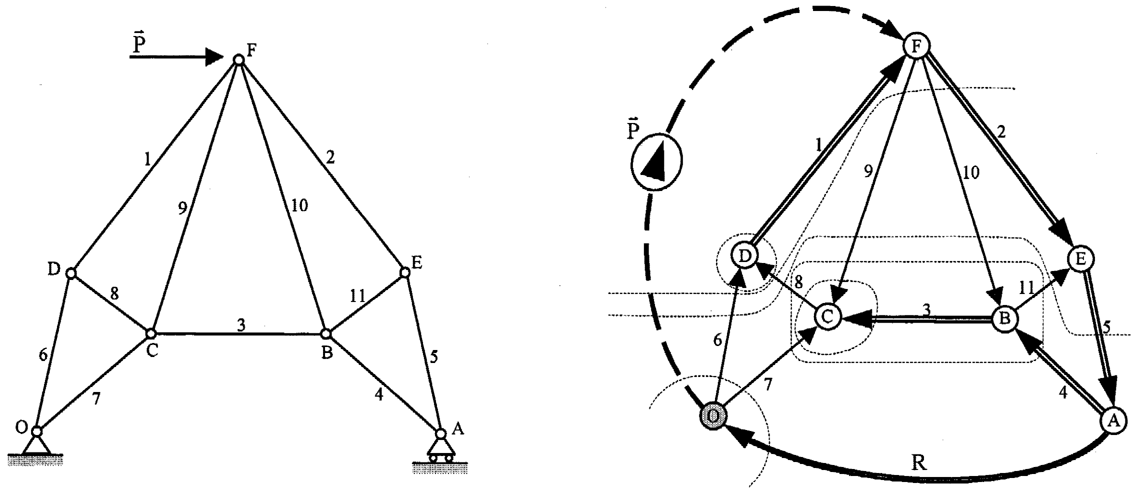
STEP 3. Create a reference vertex (gray vertex) or choose one of the vertices corresponding to a joint connected to a hinged support to be the reference vertex of the graph.

STEP 4. For each externally applied force and reaction, add an edge to the graph as follows. For each external applied force, a "*flow source edge*" is added. Its tail vertex is the reference vertex and the head vertex is the vertex corresponding to the joint upon which the external force acts. As was explained earlier, these edges should always be chosen to be chords. Since flow source edges are chords and their values are known, they appear in the graph as bold dashed lines. For each roller support reaction a "*reaction edge*" is added. Its tail vertex is the vertex corresponding to the joint upon which the reaction acts and the head vertex is the reference vertex. The reaction edge is assigned a unit vector directed along the reaction. For each hinged support (except for the one corresponding to the reference vertex), two "reaction edges" are added, the first having the corresponding unit vector directed along the X axis and the second along the Y axis.

In statically determinate trusses, the sum of forces at every joint is equal to zero. In the terminology of the FGR, this means that in the graph corresponding to the truss, the flow law is satisfied. Thus, the force analysis process of the truss is transformed into a search for flows that satisfy the flow law in the corresponding flow graph, while the flows in the flow sources are given. An example of a truss, its corresponding graph, and the equations written according to the flow law [Eq. (2)] are given in Figure 8.

2.3. The Potential Graph Representation (PGR)

Definition of a Potential Graph Representation (PGR). A network graph G is a Potential Graph, designated G_Δ, if the potential differences in its edges are independent of the flows in these edges and satisfy the Potential Law, which states: For every circuit in the graph, the sum of



(a) (b)

$$\begin{array}{c|cccccccccccc}
 & R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
R & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1
\end{array} \cdot \begin{pmatrix} \vec{F}_R \\ \vec{F}_1 \\ \vec{F}_2 \\ \vec{F}_3 \\ \vec{F}_4 \\ \vec{F}_5 \\ \vec{F}_6 \\ \vec{F}_7 \\ \vec{F}_8 \\ \vec{F}_9 \\ \vec{F}_{10} \\ \vec{F}_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \vec{P}$$

(c)

$$\begin{array}{c|cccccccccccc}
 & R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
R_x & \mathrm{Cos}(270) & 0 & 0 & 0 & 0 & 0 & -\mathrm{Cos}(77) & -\mathrm{Cos}(45) & 0 & 0 & 0 & 0 \\
R_y & \mathrm{Sin}(270) & 0 & 0 & 0 & 0 & 0 & -\mathrm{Sin}(77) & -\mathrm{Sin}(45) & 0 & 0 & 0 & 0 \\
1_x & 0 & \mathrm{Cos}(53) & 0 & 0 & 0 & 0 & -\mathrm{Cos}(77) & 0 & -\mathrm{Cos}(150) & 0 & 0 & 0 \\
1_y & 0 & \mathrm{Sin}(53) & 0 & 0 & 0 & 0 & -\mathrm{Sin}(77) & 0 & -\mathrm{Sin}(150) & 0 & 0 & 0 \\
2_x & 0 & 0 & \mathrm{Cos}(233) & 0 & 0 & 0 & -\mathrm{Cos}(77) & 0 & -\mathrm{Cos}(150) & \mathrm{Cos}(250) & \mathrm{Cos}(340) & 0 \\
2_y & 0 & 0 & \mathrm{Sin}(233) & 0 & 0 & 0 & -\mathrm{Sin}(77) & 0 & -\mathrm{Sin}(150) & \mathrm{Sin}(250) & \mathrm{Sin}(340) & 0 \\
3_x & 0 & 0 & 0 & \mathrm{Cos}(180) & 0 & 0 & 0 & \mathrm{Cos}(45) & -\mathrm{Cos}(150) & \mathrm{Cos}(250) & 0 & 0 \\
3_y & 0 & 0 & 0 & \mathrm{Sin}(180) & 0 & 0 & 0 & \mathrm{Sin}(45) & -\mathrm{Sin}(150) & \mathrm{Sin}(250) & 0 & 0 \\
4_x & 0 & 0 & 0 & 0 & \mathrm{Cos}(135) & 0 & 0 & \mathrm{Cos}(45) & -\mathrm{Cos}(150) & \mathrm{Cos}(250) & \mathrm{Cos}(340) & -\mathrm{Cos}(40) \\
4_y & 0 & 0 & 0 & 0 & \mathrm{Sin}(135) & 0 & 0 & \mathrm{Sin}(45) & -\mathrm{Sin}(150) & \mathrm{Sin}(250) & \mathrm{Sin}(340) & -\mathrm{Sin}(40) \\
5_x & 0 & 0 & 0 & 0 & 0 & \mathrm{Cos}(257) & -\mathrm{Cos}(77) & 0 & -\mathrm{Cos}(150) & \mathrm{Cos}(250) & \mathrm{Cos}(340) & -\mathrm{Cos}(40) \\
5_y & 0 & 0 & 0 & 0 & 0 & \mathrm{Sin}(257) & -\mathrm{Sin}(77) & 0 & -\mathrm{Sin}(150) & \mathrm{Sin}(250) & \mathrm{Sin}(340) & -\mathrm{Sin}(40)
\end{array} \cdot \begin{bmatrix} F_R \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \end{bmatrix} = \begin{bmatrix} \mathrm{Cos}(0) \\ \mathrm{Sin}(0) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot P$$

(d)

Fig. 8. Example for analysis of a determinate truss using the FGR (Flow Graph Representation). (a) Statically determinate truss. (b) Corresponding flow graph. (c),(d) Force analysis equations in the vector and scalar cutset matrix forms.

the potential differences of the circuit edges is equal to zero. In matrix form this is written:

$$\vec{\mathbf{B}}\cdot\vec{\Delta} = \mathbf{0} \tag{3}$$

where $\vec{\Delta}$ is the vector of potential differences, or *Potential Difference Vector*.

Proof of the potential law: In the summation of potential differences of all circuit edges, the potential of each vertex appears twice with opposite signs—either because it is once a head vertex and once a tail vertex, or since it appears once in the direction of the circuit and once in the opposite direction. Therefore this summation is equal to zero. ■

This law is a vectorial generalization to several dimensions of KVL (Kirchhoff's Voltage Law) which is stated for a one-dimensional or scalar system.

The important property of PGR is that there can be no circuits consisting of only potential difference sources (the potential differences of the edges are given). If such circuits of sources existed, then there would be a linear dependence between the potential differences in these sources, thus violating the definition of potential difference sources. Therefore, the spanning tree of PGR should be chosen so that it includes all the potential difference sources.

2.3.1. *The potential graph representation (PGR) of a mechanism*

The main property of a mechanism is that the vector sum of the link relative velocities is equal to zero in every circuit formed by its links. This property suffices for the analysis, so it is reasonable to represent it by a PGR. In this representation, the potential difference of the edges will correspond to the relative velocities of the links in the mechanism. Note that this is different from the graph representation that is commonly used for mechanisms (Erdman, 1993).

The steps for representing a mechanism by PGR are as follows.

STEP 1. For every joint of the mechanism having individual velocity, create a corresponding vertex in the graph. The potential of vertex i, designated $\vec{\pi}(i)$, is equal to the linear velocity of the corresponding joint. The velocities of all the fixed joints in the mechanism are zero, thus all these joints are represented by the same vertex—the reference (gray) vertex.

STEP 2. For every link of the mechanism, create a corresponding edge in the graph; its end vertices correspond to the joints that connect the link to the mechanism. The potential difference of this edge, designated $\vec{\Delta}(\mathrm{e})$ is equal to the relative velocity of the corresponding link, and can be written $\vec{\Delta}(\mathrm{e}) = \vec{\mathrm{V}}(\mathrm{e}) = \mathrm{V}(\mathrm{e})\cdot\hat{\mathrm{v}}(\mathrm{e})$, where $\mathrm{V}(\mathrm{e})$ is the magnitude of the relative linear velocity and $\hat{\mathrm{v}}(\mathrm{e})$ is a unit vector in the direction of the relative linear velocity of the link.

STEP 3. Label the edge corresponding to the driving link with a bold line since its potential difference (corresponding to the relative velocity between its end vertices) is known. This edge is the potential difference source edge.

Step 4. The relative velocity of a link is the velocity of the head joint minus the velocity of the tail joint. As was mentioned above, the property needed for analyzing the velocities in a mechanism is that in each circuit formed by links, the sum of their relative velocities is equal to zero. Since the relative velocity of a link is represented by the potential difference of the corresponding edge, the Potential Law [Eq. (3)] is actually the implementation of this property.

Figure 9 shows an example of a mechanism and its corresponding PGR.

2.3.2. *The analysis algorithm*

The algorithm is based on the principle that every fundamental circuit must satisfy the Potential Law, as follows.

STEP 1. Find a spanning tree, and label each branch with a double line. Every chord defines a circuit.

STEP 2. For this spanning tree, write the vector circuit matrix $\vec{\mathbf{B}}$, as defined in Section 2.1.1.

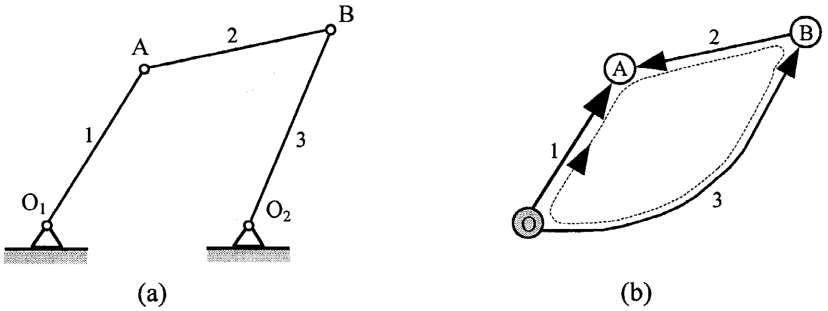


Fig. 9. Example of representing a mechanism by PGR (Potential Graph Representation). (a) The mechanism. (b) The corresponding PGR.

Step 3. On the basis of equation (3), write the equations:

$$(\vec{\mathbf{B}}\,\vec{\mathbf{B}}_\Delta)\cdot\begin{pmatrix}\vec{\Delta}\\ \vec{\Delta}_\Delta\end{pmatrix} = \mathbf{0} \rightarrow \vec{\mathbf{B}}\cdot\vec{\Delta} = -\vec{\mathbf{B}}_\Delta\cdot\vec{\Delta}_\Delta, \tag{4}$$

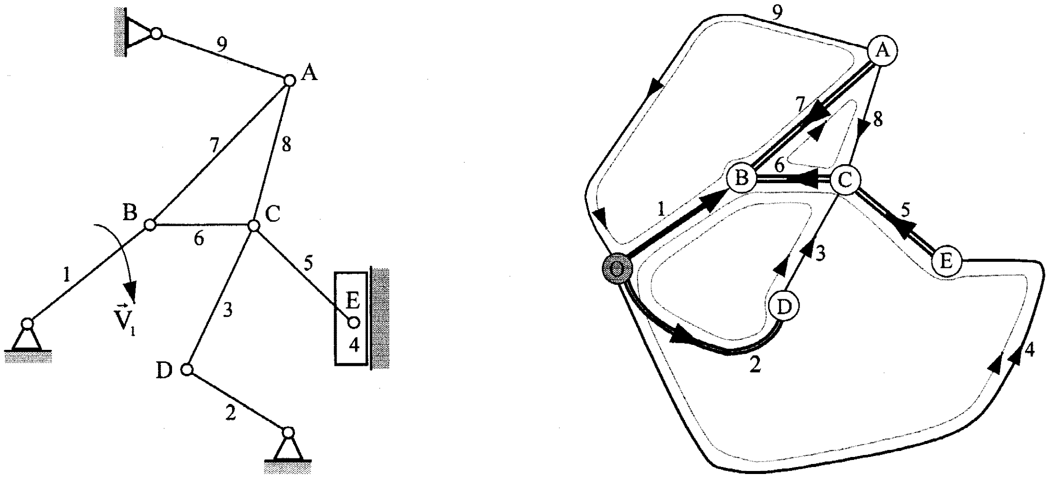
where $\vec{\mathbf{B}}_\Delta$ is the part of the vector circuit matrix corresponding to the potential difference sources and $\vec{\Delta}_\Delta$ is the vector of the potential difference sources (velocities of the driving links).

Step 4. Solve the $2\cdot(\mathrm{e(G)} - (\mathrm{v(G)} - \mathrm{dr(G)} - 1))$ equations obtained in step 3.

An example of a mechanism analysis using the PGR is shown in Figure 10.

2.4. Resistance Graph Representation (RGR)

In the representations introduced until now, there was no relation between the flows of the edges and potentials of



(a) (b)

$$\begin{array}{cc} & \begin{array}{cccccccc}3 & 4 & 8 & 9 & 2 & 5 & 6 & 7\end{array} \\ \begin{array}{c}3\\4\\8\\9\end{array} & \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\end{pmatrix}\end{array} \cdot \begin{pmatrix}\vec{V}_3\\ \vec{V}_4\\ \vec{V}_8\\ \vec{V}_9\\ \vec{V}_2\\ \vec{V}_5\\ \vec{V}_6\\ \vec{V}_7\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 0\\ -1\end{pmatrix}\cdot\vec{V}_1$$

(c)

$$\begin{array}{cc} & \begin{array}{cccccccc}3 & 4 & 8 & 9 & 2 & 5 & 6 & 7\end{array} \\ \begin{array}{c}3_x\\3_y\\4_x\\4_y\\8_x\\8_y\\9_x\\9_y\end{array} & \begin{pmatrix}\cos\alpha_3 & 0 & 0 & 0 & \cos\alpha_2 & 0 & \cos\alpha_6 & 0\\ \sin\alpha_3 & 0 & 0 & 0 & \sin\alpha_2 & 0 & \sin\alpha_6 & 0\\ 0 & \cos\alpha_4 & 0 & 0 & 0 & \cos\alpha_5 & \cos\alpha_6 & 0\\ 0 & \sin\alpha_4 & 0 & 0 & 0 & \sin\alpha_5 & \sin\alpha_6 & 0\\ 0 & 0 & \cos\alpha_8 & 0 & 0 & 0 & \cos\alpha_6 & -\cos\alpha_7\\ 0 & 0 & \sin\alpha_8 & 0 & 0 & 0 & \sin\alpha_6 & -\sin\alpha_7\\ 0 & 0 & 0 & \cos\alpha_9 & 0 & 0 & 0 & -\cos\alpha_7\\ 0 & 0 & 0 & \sin\alpha_9 & 0 & 0 & 0 & -\sin\alpha_7\end{pmatrix}\end{array} \cdot \begin{pmatrix}V_3\\ V_4\\ V_8\\ V_9\\ V_2\\ V_5\\ V_6\\ V_7\end{pmatrix} = \begin{pmatrix}\cos\alpha_1\\ \sin\alpha_1\\ \cos\alpha_1\\ \sin\alpha_1\\ 0\\ 0\\ -\cos\alpha_1\\ -\sin\alpha_1\end{pmatrix}\cdot V_1$$

(d)

Fig. 10. Kinematic analysis of a mechanism using PGR. (a) A mechanism. (b) The corresponding potential graph. (c),(d) Sets of vector and scalar equations for its analysis.

the vertices. This section introduces a graph representation that possesses such a relation. To clarify the explanation, the representation is first applied to electrical circuits. Doing so enables us to demonstrate the transition from one-dimensional problems, which are well known in the literature, to the multidimensional ones, thus obtaining an insight into the generality of the approach. Moreover, since the representation is applied to different engineering fields consisting of elements with different dimensions, it is suitable to be the representation of multidimensional integrated systems, as shown in the current section. In the next section, this representation is extended on the basis of matroid theory.

2.4.1. *Description of the representation*

The Resistance Graph Representation (RGR) is a generalization of FGR and PGR. RGR is a network graph, where there are edges with a dependence between the flow and the potential difference. Such a dependence is characterized by either a scalar or a matrix. The scalar is used if there is an explicit dependence between the vector magnitudes of the flow and potential difference; otherwise the matrix is used. For both scalar and matrix possibilities there are two presentations—resistance presentation (designated by R(e) and **R**(e) respectively) and conductance presentation (designated by K(e) and **K(e)** respectively), as follows:

$$|\vec{\Delta}(\mathrm{e})| = \mathrm{R}(\mathrm{e})\cdot|\vec{\mathrm{F}}(\mathrm{e})|; \qquad |\vec{\mathrm{F}}(\mathrm{e})| = \mathrm{K}(\mathrm{e})\cdot|\vec{\Delta}(\mathrm{e})| \tag{5}$$

$$\vec{\Delta}(\mathrm{e}) = \mathbf{R}(\mathrm{e})\cdot\vec{F}(\mathrm{e}); \qquad \vec{\mathrm{F}}(\mathrm{e}) = \mathbf{K}(e)\cdot\vec{\Delta}(\mathrm{e}), \tag{6}$$

where $\vec{\Delta}(\mathrm{e})$ is the potential difference in edge e and $\vec{\mathrm{F}}(\mathrm{e})$ is the flow. In addition, flows and potential differences of the resistance graph must satisfy the flow and potential laws, respectively.

When dealing with resistance graph representation, an important theorem from the graph theory, called the orthogonality principle, becomes essential.

THEOREM 1. *The orthogonality principle: Vectorial cutset and circuit matrices of a graph are orthogonal:*

$$\vec{\mathbf{B}}\cdot\vec{\mathbf{Q}}^{\mathrm{t}} = \mathbf{0}. \tag{7}$$

■

As is shown in Swamy and Thulasiraman (1981), from this principle the following equations are derived:

$$\vec{\Delta} = \vec{\mathbf{Q}}^{\mathrm{t}}\cdot\vec{\Delta}_{\mathrm{T}} \tag{8}$$

$$\vec{\mathbf{F}} = \vec{\mathbf{B}}^{\mathrm{t}}\cdot\vec{\mathbf{F}}_{\mathrm{C}}, \tag{9}$$

where $\vec{\Delta}_{\mathrm{T}}$ is the vector of potential differences in the branches of the spanning tree and $\vec{\mathbf{F}}_{\mathrm{C}}$ is the vector of flows in the chords of the graph.

The edges in the resistance graph are divided into three principal groups: flow sources, potential difference sources, and resistance edges. Flow sources, denoted by bold dashed lines, are edges for which the value of the flow is known and is independent of the potential difference. Potential difference sources, denoted by bold solid lines, are the edges in which the potential difference is known and is independent of the flow in that edge. Resistance edges, denoted by black solid lines, are the edges at which there is a dependence between the flow and the potential difference.

2.4.2. *Conductance Cutset Method (CCM) for analyzing the RGR*

The analysis problem for the resistance graph is as follows: Given the flows in the flow sources, the potential differences in the potential difference sources, and the resistances (or conductances) of the resistance edges, find the flows and potential differences in all the e(G) edges of the graph. The obvious method for solving the resistance graphs is to write all the equations based on Eq. (2), (3), (5), and (6) and then to solve them simultaneously. This method has a high computational complexity; thus in the current and next sections, efficient methods based on graph theory theorems will be shown, with the method called the "Conductance Cutset Method" (CCM) explained first.

The first step in solving the resistance graph is to find a suitable spanning tree, which, for the reasons explained in Sections 2.2 and 2.3, contains all the potential difference sources and does not contain any flow sources. Then, using Eq. (10), derived from Eqs. (2), (3), (5), and (6), as shown in Shai (1999), a set of linear equations is obtained:

$$(\vec{\mathbf{Q}}_{\mathrm{T'R}}\cdot\mathbf{K}_{\mathrm{R}}\cdot\vec{\mathbf{Q}}^{\mathrm{t}}_{\mathrm{T'R}})\cdot\vec{\Delta}_{\mathrm{T'}} = -(\vec{\mathbf{Q}}_{\mathrm{T'R}}\cdot\mathbf{K}_{\mathrm{R}}\cdot\vec{\mathbf{Q}}^{\mathrm{t}}_{\Delta\mathrm{R}})\cdot\vec{\Delta}_{\Delta} - \vec{\mathbf{Q}}_{\mathrm{T'P}}\cdot\vec{\mathbf{F}}_{\mathrm{P}}, \tag{10}$$

where Δ and P are the edges corresponding to the potential difference and flow sources, respectively, and R are all the other edges of the graph—the edges with resistance. T′ are those branches of the spanning tree which are not sources.

For convenience, the matrix $(\vec{\mathbf{Q}}_{\mathrm{T'R}}\cdot\mathbf{K}_{\mathrm{R}}\cdot\vec{\mathbf{Q}}^{\mathrm{t}}_{\mathrm{T'R}})$ is designated as $\mathbf{K}_{\mathrm{T'}}$ and is termed the "conductance matrix of the spanning tree T′." Matrix $(\vec{\mathbf{Q}}_{\mathrm{T'R}}\cdot\mathbf{K}_{\mathrm{R}}\cdot\vec{\mathbf{Q}}^{\mathrm{t}}_{\Delta\mathrm{R}})$ is designated as **K** and is called "the conductance matrix of the potential sources." These are shown in Eq. (11).

$$(\mathbf{K}_{\mathrm{T'}})\cdot\vec{\Delta}_{\mathrm{T'}} = -(\mathbf{K}_{\Delta})\cdot\vec{\Delta}_{\Delta} - \vec{\mathbf{Q}}_{\mathrm{T'P}}\cdot\vec{\mathbf{F}}_{\mathrm{P}}. \tag{11}$$

The values of the elements in these conductance matrices can be derived on the basis of linear algebra considerations as follows. $[\mathbf{K}_{\mathrm{T'}}]_{ij}$ equals the sum of conductances of the edges which belong to both cutsets *i* and *j* defined by branches with resistance; the sign of the conductance is taken positive if it is directed similarly relative to both cutsets and negative otherwise. $[\mathbf{K}_{\Delta}]_{ij}$ also equals the sum of conductances of the edges that belong to both cutsets *i* and *j*, although this time *j* is a branch which is a potential difference source, while *i* is a branch with resistance.

After solving Eq. (10) or (11), all the potential differences in the branches are known. All the potential differences in the graph are obtained by using Eq. (8) and after

Table 2. *Correspondence between the RGR and its dual graph*

RGR	Dual graph
e—edge	e′—edge
T—spanning tree	C—set of chords
T′—branches which are not sources	C′—chords which are not sources
R(e), **R(e)**—resistance	K(e′), **K(e′)**—conductance
$\vec{\Delta}$(e)—potential difference	$\vec{F}$(e′)—flow
$\vec{\Delta}_{\Delta}$—potential difference source	$\vec{F}_P$—flow source
$\vec{\mathbf{Q}}$—vector cutset matrix	$\vec{\mathbf{B}}$—vector circuit matrix
CCM—Conductance Cutset Method	RCM—Resistance Circuit Method
The dualism is also applied to the sub-matrices of B and Q, for instance	
$\vec{Q}_{T'R}$	$\vec{B}_{C'R}$
$\vec{Q}^t_{\Delta R}$	$\vec{B}^t_{PR}$

that, all the flows in the graph are obtained by applying Eq. (5) or (6).

2.4.3. *Resistance Circuit Method (RCM) for solving the RGR*

It is well known in graph theory (Deo, 1974), that for each planar graph, there exists a dual graph. This section shows that on the basis of this dualism, Resistance Circuit Method (RCM) for RGR analysis can be derived from CCM. The relations between RGR and its dual graph are given in Table 2. Thus, one can develop a new method for analysis of resistance graphs by just rewriting the CCM method in the terminology of the dual resistance graph. This method is called Resistance Circuit Method (RCM) and is formulated by Eq. (12) (Shai, 1999).

$$(\vec{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\vec{\mathbf{B}}^t_{C'R})\cdot\vec{\mathbf{F}}_{C'} = -(\vec{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\vec{\mathbf{B}}^t_{PR})\cdot\vec{\mathbf{F}}_P - \vec{\mathbf{B}}_{C'P}\cdot\vec{\Delta}_{\Delta}, \tag{12}$$

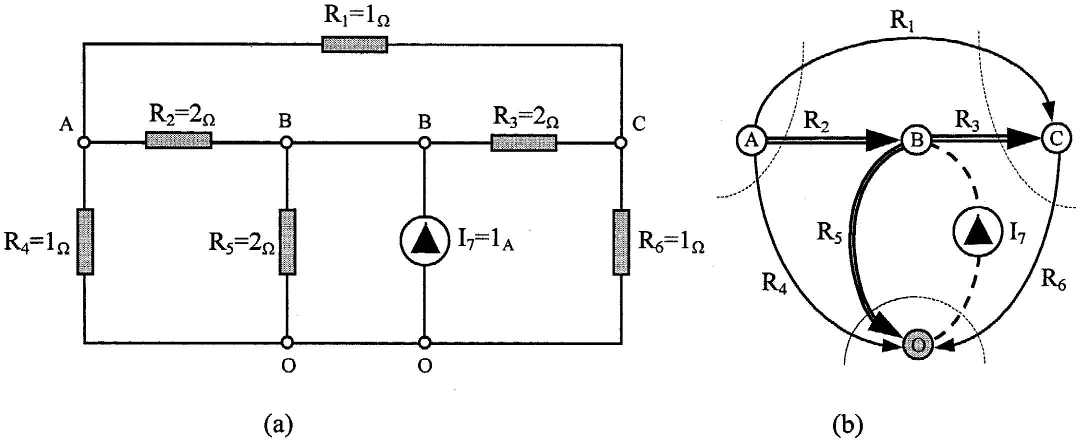
where $\mathbf{R}_R$ is a square diagonal matrix, the components of which correspond to the resistances in the resistance edges. For convenience, the matrix $(\vec{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\vec{\mathbf{B}}^t_{C'R})$ is designated by $\mathbf{R}_{C'}$ and is termed the resistance matrix of the chord set C′. Matrix $(\vec{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\vec{\mathbf{B}}^t_{PR})$ is designated by $\mathbf{R}_P$ and is called the resistance matrix of the flow sources. Using those notations we rewrite Eq. (12) in the following way:

$$(\mathbf{R}_{C'})\cdot\vec{\mathbf{F}}_{C'} = -(\mathbf{R}_P)\cdot\vec{\mathbf{F}}_P - \vec{\mathbf{B}}_{C'\Delta}\cdot\vec{\Delta}_{\Delta}. \tag{13}$$

The values of the elements of the resistance matrices can be derived on the basis of linear algebra considerations of the above dualism relation, and they are as follows. $[\mathbf{R}_{C'}]_{ij}$ is the sum of the resistances of the edges which belong to both circuits *i* and *j*, defined by the chords which are not sources. The sign of the resistance is positive if the corresponding edge is directed similarly relative to both circuits and negative otherwise. $[\mathbf{R}_P]_{ij}$ is calculated in the same way, except that this time the circuit *j* is defined by the chord which is a flow source. Equation 13 is actually a set of linear equations, the unknowns of which are the flows in the resistance chords of the graph. After solving it, all the flows in the graph are obtained by using Eq. (9) and after that, all the potential differences in the graph are obtained using Eq. (5) or (6).

Table 3. *Representing engineering systems with RGR*

	Engineering system interpretation			
Engineering system	Edge and flow	Vertex and potential	Name of the element	Flow–potential difference relation
Electrical circuit	Edge corresponds to electrical elements: resistor, condenser, coil, current and voltage sources. Flow corresponds to electrical current through the element	Vertex corresponds to the junction in the circuit. Potential corresponds to the electric potential (voltage) of the junction.	Resistor	$\Delta_i = F_iR_i$
			Capacitor	$\mathbf{F}_i = C_i\frac{d\Delta_i}{dt} = C_is\Delta_i$
			Coil	$\Delta_i = L_i\frac{dF_i}{dt} = sL_iF_i$
Dynamical system	Edge corresponds to dynamical elements: mass, damper, spring external force, initial tension or velocity. Flow corresponds to the internal force in the element.	Vertex corresponds to junction having independent velocity. Potential corresponds to the velocity of the junction.	Mass	$\Delta_i = \frac{F_i}{m_is}$
			Spring	$\Delta_i = \frac{sF_i}{k_i}$
			Damper	$\Delta_i = b_iF_i$
Static system	Edge corresponds to a system element with an internal force and the flow to the force in that element.	Vertex is a joint connecting system elements. The potential corresponds to the displacement of the joint.	Rod	$F_i = K\cdot\Delta_i$
			Reaction	No relation



(a) (b)

Fig. 11. Example of an electrical system and its corresponding graph representation. (a) The electrical circuit. (b) The resistance graph representation.

2.4.4. *Representing one-dimensional engineering systems with Resistance Graph Representation*

This section shows how to represent various engineering systems with the resistance graph. The fact that the same representation has been applied to represent systems that belong to remote engineering fields opens two far ranging possibilities: first, to use methods developed in one field for the other field, and second, to solve integrated multidisciplinary engineering systems. Both issues are described later in the paper.

The systems that are described in this section are electrical systems, dynamic systems, and multidimensional statical systems.

Table 3 gives all relevant information on how to represent these engineering systems with RGR, including the information that can be found in the literature (e.g., Shearer et al., 1971) about representation of one-dimensional systems.

2.4.5. *Representing one-dimensional engineering systems with Resistance Graph Representation*

In this section, RGR is applied to electrical systems, since applications of graph theory to these systems are well known in the literature (Balabanian & Bickart, 1969). The two methods embedded in this representation, CCM and RCM (Sections 2.4.2 and 2.4.3), are applied to analyze both electrical circuits and multidimensional indeterminate trusses. Doing so emphasizes one of the main advances of MCA, that is, obtaining a unified perspective on systems consisting of elements with different dimensions and from different engineering fields.

The representation of an electrical circuit is quite simple: each junction corresponds to a vertex and an element to an edge as shown in Figure 11. The solution equations derived by applying the CCM to the graph of Figure 11 are as follows:

$$Q_{T'T'} = \begin{array}{c|ccc} & 2 & 3 & 5 \\ \hline 2 & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_4} & \frac{1}{R_1} & \frac{1}{R_4} \\ 3 & \frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_6} & -\frac{1}{R_6} \\ 5 & \frac{1}{R_4} & -\frac{1}{R_6} & \frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_6} \end{array}$$

$$= \begin{array}{c|ccc} & 2 & 3 & 5 \\ \hline 2 & 2.5 & 1 & 1 \\ 3 & 1 & 2.5 & -1 \\ 5 & 1 & -1 & 2.5 \end{array}$$

$$Q_F = \begin{array}{c|c} & 7 \\ \hline 2 & 0 \\ 3 & 0 \\ 5 & -1 \end{array} \begin{pmatrix} 2.5 & 1 & 1 \\ 1 & 2.5 & -1 \\ 1 & -1 & 2.5 \end{pmatrix} \cdot \begin{pmatrix} V_2 \\ V_3 \\ V_5 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} I_7$$

2.4.6. *Using Resistance Graph Representation for analysis of multidimensional trusses*

Let d(G) be the dimension of the engineering system, that is, the dimension of potential or flow vectors. The explanation provided in this section is for two dimensions, but the approach is valid for three dimensions as well. Equation 6 can be rewritten:

$$\vec{\mathbf{F}} = \mathbf{K}\cdot\vec{\Delta},$$

where **K** is built from the conductivity matrices of the resistance edges, each being a square matrix of size d(G) × d(G). Figure 12 shows the initial and deformed states of a rod.

Let $\Delta_i(e)$ be the potential difference between the two end vertices of edge e in coordinate direction i. As one can see from Figure 12, under the small deflection assumption (West, 1993), the following equation describes the scalar magni-

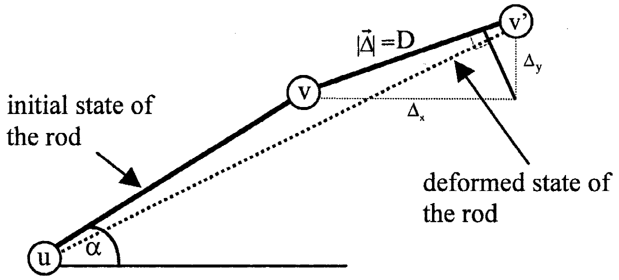


Fig. 12. Rod deformation.

tude of the potential difference as a combination of its coordinate components:

$$|\vec{\Delta}(\mathrm{e})| = \Delta_x(\mathrm{e})\cdot\cos\alpha + \Delta_y(\mathrm{e})\cdot\sin\alpha, \tag{14}$$

where α is the angle of the element.

Combining Eqs. (6) and (14) we obtain:

$$\vec{\mathrm{F}}(\mathrm{e}) = \begin{pmatrix} \mathrm{F}_x \\ \mathrm{F}_y \end{pmatrix} = \mathrm{K}(\mathrm{e})\cdot\begin{pmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{pmatrix}\cdot\begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix}$$

$$= \mathbf{K(e)}\begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix}, \tag{15}$$

where the square matrix **K(e)** is the "conductance matrix" of the element.

This two dimensional conductance matrix of the graph edges (designated by **K(e)**) is the product of the constant conductivity K(e) and the transformation matrix. For edges corresponding to hinged support reactions, the constant should be taken as 0, since there is no dependence between the displacement of the support and the reaction force.

In indeterminate trusses, the forces in the rods cannot be determined by the laws of statics alone, and one must also consider the compatibility conditions. In the terminology of graph representation, this means that the corresponding graph of the indeterminate truss should be analyzed by using the flow and potential laws simultaneously.

The components of the potential vector correspond to the displacements of the joints in the directions of the coordinate axes. The flow in an edge corresponds to the internal force in the corresponding rod.

The building process of the RGR corresponding to an indeterminate truss can be summarized as follows: Build a graph following the same steps as were explained in Section 2.2.2 for building the FGR of a determinate truss. The FGR becomes RGR when all of its edges that are not sources are assigned conductances (or resistances), as shown in Table 4.

The analysis process is based on applying the CCM to the RGR of the indeterminate truss. The first step is choosing a suitable spanning tree. Since specific components of the potential differences in the reaction edges are known to be equal to zero, these edges are somewhat similar to the potential difference sources. Thus, the spanning tree must include the reaction edges and it should not include the flow source edges.

An example of an indeterminate truss, its corresponding graph, the spanning tree, and the equations derived from CCM is given in Figure 13.

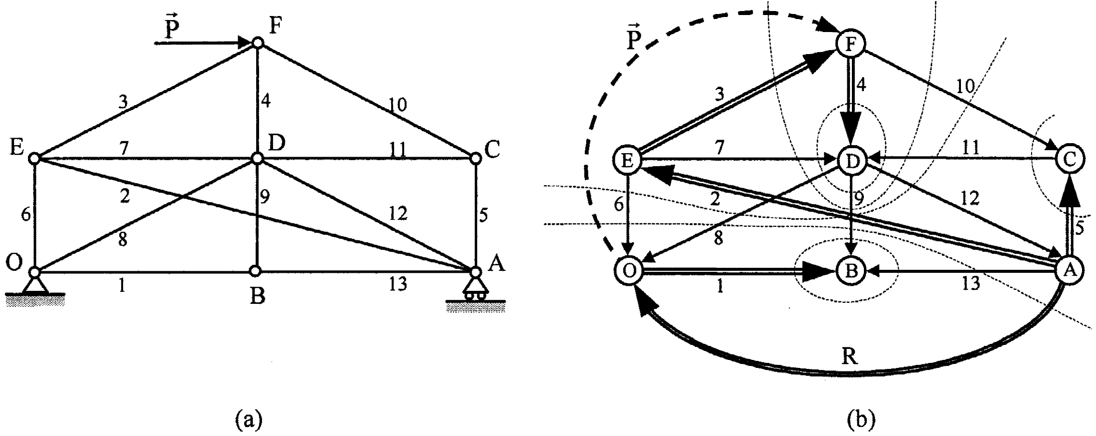
2.5. Resistance Matroid Representation (RMR)

The previous section showed the application of the CCM method embedded in the graph representation to analysis of indeterminate trusses. However, this approach has been shown to have its limitations, one of which is the fact that the dual of CCM–RCM is not applicable to trusses (Shai, 1999). This is due to the fact that the conductance matrix of a truss edge is singular. Thus, it does not have an inverse matrix; hence the rod edge in RGR cannot be assigned a resistance matrix.

According to the idea underlying MCA, such a limitation can be overcome by changing the representation or, as is done here, extending the representation. In this case, such an extended representation is the matroid representation, whose definitions and properties are given in Section 2.1.2. During the research it was found that representing engineering systems by matroid theory enables one to obtain a more general perspective. Such a generalization is demonstrated in this section by representing indeterminate trusses by Resistance Matroid Representation (RMR). The direct consequence of such a generalization is the fact that RCM in

Table 4. *Types of edges in the graph representation of an indeterminate truss and their conductances*

	Type of edge	The conductance of the edge
a.	Truss rod—Resistance edge with finite conductance	$\frac{\mathrm{A}(\mathrm{e})\cdot\mathrm{E}(\mathrm{e})}{\mathrm{L}(\mathrm{e})}\cdot\begin{pmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{pmatrix}$
b.	Fixed and roller supports	Zero, since there is no dependence between the reaction force and the displacement.
c.	Force applied to the truss—Flow source edge	Flow sources are not assigned conductance, since the flows in them are given, while the potential differences are not.



$$
\left(\begin{array}{c|cccccc}
 & R & 1 & 2 & 3 & 4 & 5 \\
R & K_R+K_6+K_8+K_9+K_{13} & K_9+K_{13} & -K_6-K_8-K_9 & -K_8-K_9 & -K_8-K_9 & 0 \\
1 & K_9+K_{13} & K_1+K_9+K_{13} & -K_9 & -K_9 & -K_9 & 0 \\
2 & -K_6-K_8-K_9 & -K_9 & K_2+K_6+K_8+K_9+K_{10}+K_{11}+K_{12} & K_8+K_9+K_{10}+K_{11}+K_{12} & K_8+K_9+K_{11}+K_{12} & -K_{10}-K_{11} \\
3 & -K_8-K_9 & -K_9 & K_8+K_9+K_{10}+K_{11}+K_{12} & K_3+K_7+K_8+K_9+K_{10}+K_{11}+K_{12} & K_7+K_8+K_9+K_{11}+K_{12} & -K_{10}-K_{11} \\
4 & -K_8-K_9 & -K_9 & K_8+K_9+K_{11}+K_{12} & K_7+K_8+K_9+K_{11}+K_{12} & K_4+K_7+K_8+K_9+K_{11}+K_{12} & -K_{11} \\
5 & 0 & 0 & -K_{10}-K_{11} & -K_{10}-K_{11} & -K_{11} & K_5+K_{10}+K_1
\end{array}\right)
\cdot
\begin{bmatrix} \Delta_{Rx} \\ \Delta_{Ry} \\ \Delta_{1x} \\ \Delta_{1y} \\ \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \\ \Delta_{4x} \\ \Delta_{4y} \\ \Delta_{5x} \\ \Delta_{5y} \end{bmatrix}
= -
\begin{bmatrix} -\mathrm{Cos}(0) \\ -\mathrm{Sin}(0) \\ 0 \\ 0 \\ \mathrm{Cos}(0) \\ \mathrm{Sin}(0) \\ \mathrm{Cos}(0) \\ \mathrm{Sin}(0) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\cdot P
$$

(c)

Fig. 13. Statically indeterminate truss and its resistance graph. (a) The truss. (b) Corresponding RGR. (c) CCM analysis equations.

RMR is shown to be applicable to indeterminate trusses in contrast to RCM in RGR.

2.5.1. Matroid representation for indeterminate trusses

The first step of representing an indeterminate truss by a matroid is representing it by a resistance graph (Section 2.4.6). Let G_R be a RGR of an indeterminate truss, and $\mathbf{Q}(G)$ be its scalar cutset matrix. The scalar cutset matrix defines the matroid $M_Q = \langle \mathbf{S}, \mathbf{F^I} \rangle$ where $\mathbf{S}$ is the set of columns of $\mathbf{Q}(G)$ and $\mathbf{F^I}$ is a family of all linearly independent subsets of $\mathbf{S}$. The subscript Q in M_Q is used to emphasize that the matroid corresponds to the scalar cutset matrix $\mathbf{Q}$. Each element of M_Q is a scalar cutset matrix column that, in its turn, corresponds to a truss element, which can be one of the following: rod, external reaction, or external force. An example of a truss with its corresponding matroid is given in Figure 14a and Figure 14d, respectively.

2.5.2. Structural interpretation of matroid components

2.5.2.1. Dependent sets of M_Q. The flow law for RGR is given by

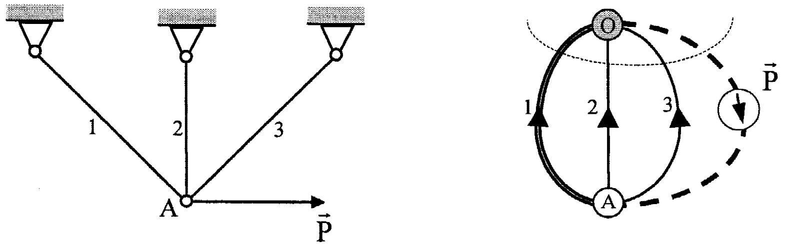
$$\mathbf{Q}(G)\cdot\mathbf{F} = \mathbf{0}, \tag{16}$$

where $\mathbf{F}$ is a vector of force scalar values acting in the truss elements. Therefore the nonzero entries of the vector $\mathbf{F}$ define a set of linearly dependent columns of the scalar cutset matrix. By definition, such a set is also the set of dependent elements in the matroid M_Q. Thus, a dependent set in M_Q corresponds to a set of truss elements in which internal forces can act simultaneously, that is, the truss elements that have nonzero internal forces during some state of self-stress. Such a set forms an indeterminate subset of truss rods (a *subtruss*).

2.5.2.2. Circuits of M_Q. A circuit of the matroid is a minimal dependent set, that is, removing even one of its elements results in an independent set. Therefore, in the terminology of structures, a circuit in M_Q corresponds to a minimal indeterminate subtruss, which is a rigid subtruss indeterminate to the first degree. Such a subtruss has the properties of a circuit, since removing any of the rods from such a truss will create a determinate truss or even a mechanism.

2.5.2.3. Base of M_Q. The base of a matroid is the maximal independent subset of $\mathbf{S}$, that is, adding any element to the base results in a dependent set. Thus, the base in M_Q corresponds to a determinate subtruss that contains all the pinned joints of the truss. It is well known that adding a rod to a determinate truss, without adding a pinned joint, makes the truss indeterminate. For the sake of consistency with the graph theory, the base of the matroid representing the truss is chosen so that it does not contain any external forces (flow sources) acting on the truss.

2.5.2.4. Cobase of M_Q. The cobase of M, that is, the set of elements which are not in the base is the set of external forces and redundant rods of the truss. The notation that is used in this paper for graphs (Section 2.1.1) is also applied to matroids. For this reason, the base elements (the deter-



(a) (b)

$$\mathbf{Q}(G)=\begin{array}{c} \\ 1x \\ 1y \end{array}\!\!\begin{array}{cccc} 1 & 2 & 3 & P \\ \cos(135°) & \cos(90°) & \cos(45°) & -\cos(0°) \\ \sin(135°) & \sin(90°) & \sin(45°) & -\sin(0°) \end{array}$$

$$= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & -1 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \end{pmatrix}$$

(c)

$\mathrm{M}{=}\langle \mathbf{S},\mathbf{F}^{\mathbf{I}}\rangle$
$\mathbf{S}{=}\{1,2,3,\mathrm{P}\}$
$\mathbf{F}^{\mathbf{I}} = \{\{1\}, \{2\},\{3\},\{\mathrm{P}\},\{1,2\}, \{1,3\},\{1,\mathrm{P}\},$
$\{2,3\},\{2,\mathrm{P}\},\{3,\mathrm{P}\}\}.$

(d)

Fig. 14. Example of a truss and its corresponding matroid. (a) The truss. (b) The graph. (c) The scalar cutset matrix. (d) The matroid.

minate subtruss elements) are represented by double lines, the cobase elements (redundant truss elements and external forces) by dashed lines, whereas the cobase elements which correspond to the external forces are both dashed and bold.

Figure 15 shows the truss from Figure 14, with highlighted base and cobase elements (a) and the two fundamental circuits (circuits containing only one redundant rod or external force) (b) and (c).

2.5.2.5. Circuit matrix of M_Q. By definition of circuit in matroid, each fundamental circuit in $\mathrm{M_Q}$ corresponds to a minimal set of linearly dependent columns in $\mathbf{Q}$. In other words, for each fundamental circuit C_i it can be written:

$$\sum_{j\in c_i} \lambda_{ij}\mathbf{Q}_{\downarrow_j} = \mathbf{0}, \tag{17}$$

where $\mathbf{Q}_{\downarrow_j}$ is the jth column of matrix $\mathbf{Q}$. In the terminology of trusses, λ_{ij} is the force acting in the truss element j while the state of self-stress produced by a force in the redundant truss element i.

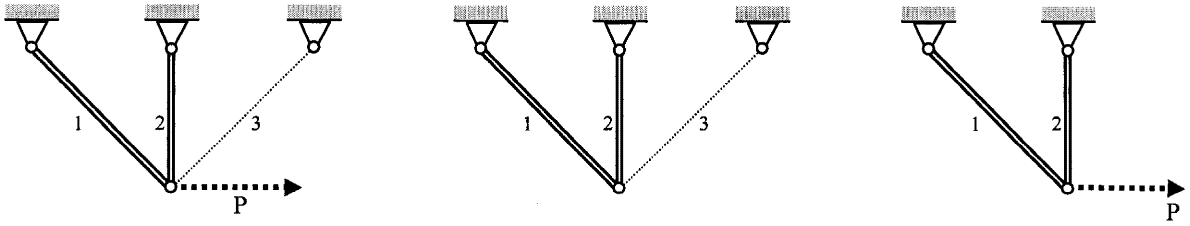
The set of fundamental circuits is represented by a special matrix $\mathbf{B}(\mathrm{M_Q})$, called *a circuit matrix of M_Q*. The rows of $\mathbf{B}(\mathrm{M})$ correspond to the cobase elements of M and the columns to all the elements of M. An entry ij of the matroid circuit matrix is defined:

$$[\mathbf{B}(\mathrm{M})]_{ij} = \lambda_{ij}. \tag{18}$$

Obviously, Eq. (17) still holds, when for some i, all λ_{ij} are multiplied by the same arbitrary scalar. Therefore, it is legitimate to “normalize” the circuit matrix, that is, to multiply the rows of $\mathbf{B}(\mathrm{M})$ so that the matrix is written as follows:

$$\mathbf{B}(\mathrm{M}) = (\mathbf{B}(\mathrm{M})_{\mathrm{T}}|\mathbf{I}), \tag{19}$$

where $\mathbf{I}$ is a unit matrix whose size is equal to the number of cobase elements, and $\mathbf{B}(\mathrm{M})_{\mathrm{T}}$ is a matrix with rows and columns corresponding to the cobase and base elements, respectively. In structural mechanics terminology the value of $[\mathbf{B}(\mathrm{M})]_{ij}$ becomes equal to the force in the truss rod or



(a) (b) (c)

Fig. 15. Example of fundamental circuits in the matroid of a truss. (a) The base and cobase of M. (b) A fundamental circuit defined by the redundant rod 3. (c) A fundamental circuit defined by the external force P.

reaction i when a unit force is applied in a redundant element j and the forces in all the other redundant elements are set to zero.

For example, the circuit matrix of matroid M_Q that represents the truss of Figure 14 is developed as follows. For cobase elements 3 and P, equations based on Eq. (17) are written, respectively,

$$\lambda_{3,1}\cdot\mathbf{Q}_{\downarrow 1} + \lambda_{3,2}\cdot\mathbf{Q}_{\downarrow 2} + \lambda_{3,3}\cdot\mathbf{Q}_{\downarrow 3}$$

$$= 1\cdot\begin{pmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{pmatrix} - 1.414\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

$$\lambda_{P,1}\cdot\mathbf{Q}_{\downarrow 1} + \lambda_{P,2}\cdot\mathbf{Q}_{\downarrow 2} + \lambda_{P,P}\cdot\mathbf{Q}_{\downarrow P}$$

$$= -1.414\cdot\begin{pmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Hence the circuit matrix of M_Q is

$$\mathbf{B}(\mathrm{M}) = \begin{array}{c} \\ 3 \\ \mathrm{P} \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & \mathrm{P} \end{array} \\ \begin{pmatrix} 1 & -1.414 & 1 & 0 \\ -1.414 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Proposition 1. *Every admissible force vector* $\mathbf{F}$ *is a linear combination of rows of* $\mathbf{B}(\mathrm{M})$. ■

Proof: Forces in the determinate subtruss (base) are uniquely defined by the forces in the redundant elements. Moreover, each row of $\mathbf{B}(\mathrm{M})$ corresponds to the forces in the determinate subtruss yielded by a unit force in the corresponding redundant element. Therefore, by the superposition principle, every admissible force vector is derived by summing over all the rows of $\mathbf{B}(\mathrm{M})$ each multiplied by the force in the corresponding redundant element. ■

Proposition 2. *The matroid potential law:*

$$\mathbf{B}(\mathrm{M})\cdot\mathbf{D} = \mathbf{0}, \tag{20}$$

where $\mathbf{D}$ *is a vector of scalar displacements in truss elements.* ■

Proof: According to the definition of matroid M_Q, each row of $\mathbf{B}(\mathrm{M})$ corresponds to a state of self-stress, which is a vector of admissible flows in G_R. On the other hand, vector $\mathbf{D}$ corresponds to a vector of admissible scalar potential differences in G_R. Thus, according to the equilibrium between the internal strain energy of the truss and the work done by the external forces (West, 1993), multiplication of every row in $\mathbf{B}(\mathrm{M})$ by vector $\mathbf{D}$ is equal to zero. ■

2.5.2.6. The cutset matrix of matroid. The cutsets of a matroid are represented by a cutset matrix as explained below.

Proposition 3. *The matrix* $\mathbf{Q}(\mathrm{M}) = (\mathbf{I}|-\mathbf{B}_{\mathrm{T}}^{\mathrm{t}})$ *is the cutset matrix of matroid* M_Q*, that is, each row of* $\mathbf{Q}(\mathrm{M})$ *defines a fundamental cutset in the matroid.*

Proof: To prove this property, one has to prove that every row of $\mathbf{Q}(\mathrm{M})$ satisfies the conditions of a cutset (given in Section 2.1.2).

Conditions (a) and (c) of the Section 2.1.2 are satisfied since $\mathbf{Q}(\mathrm{M})$ contains a unit matrix, whose rows are nonempty and do not contain other rows of the matrix. Condition (b) requires that for any circuit i and any cutset j the number of common elements is not equal to one. This can be proved by considering the forms of circuit and cutset matrices (Fig. 16). The number of common elements in circuit i and cutset j is the number of elements corresponding to the nonzero entries in rows i and j in the circuit and the cutset matrices, respectively. From Figure 16 one can see that this number can be either 0 or 2 depending on whether element B_{ij} is equal to zero or not. Thus, the number of common elements in circuit and cutset can never be equal to one. ■

Proposition 4. *The orthogonality principle:*

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{B}^{\mathrm{t}}(\mathrm{M}) = \mathbf{0} \tag{21}$$

■

Proof: By substituting Eq. (19) into Eq. (21), we obtain

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{B}^{\mathrm{t}}(\mathrm{M}) = (\mathbf{I}|-\mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}})\begin{pmatrix} \mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}} \\ \mathbf{I} \end{pmatrix}. \tag{22}$$

After the multiplication we get $\mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}} - \mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}} = \mathbf{0}$. ■

Proposition 5. *Matroid flow law:*

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{F} = \mathbf{0} \tag{23}$$

■

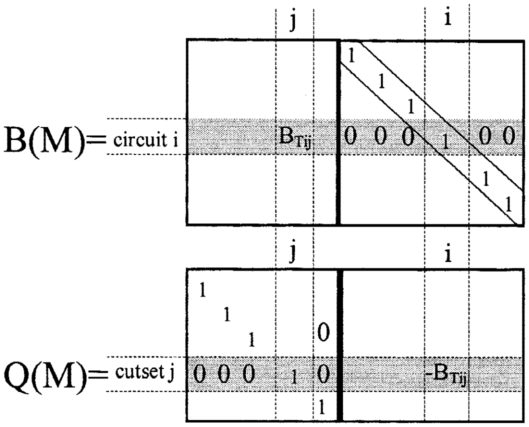


Fig. 16. The form of circuit and cutset matrices.

Proof: By proposition 1, every admissible force vector **F** is a linear combination of $\mathbf{B}(\mathrm{M})$ rows. According to proposition 4, $\mathbf{Q}(\mathrm{M})$ is orthogonal to $\mathbf{B}(\mathrm{M})$; hence it is orthogonal to every linear combination of its rows, that is, **F**. ■

Because of the validity of propositions 1 to 5, the flow, potential, and orthogonality laws are all valid for matroid $\mathrm{M_Q}$. Such a matroid is called a Resistance Matroid Representation (RMR). Since Eq. (13) was derived using only these properties of the resistance graph, it is also valid for the matroid $\mathrm{M_Q}$. Substituting to Eq. (13) the matrices corresponding to $\mathrm{M_Q}$ instead of those corresponding to $\mathrm{G_R}$, we obtain:

$$(\mathbf{B}(\mathrm{M})_{\mathrm{C'R}}\cdot\mathbf{R}_{\mathrm{R}}\cdot\mathbf{B}(\mathrm{M})^{\mathrm{t}}_{\mathrm{C'R}})\cdot\mathbf{F}_{\mathrm{C'}} = -(\mathbf{B}(\mathrm{M})_{\mathrm{C'R}}\cdot\mathbf{R}_{\mathrm{R}}\cdot\mathbf{B}(\mathrm{M})^{\mathrm{t}}_{\mathrm{PR}})\cdot\mathbf{F}_{\mathrm{P}} - \mathbf{B}_{\mathrm{C'P}}\cdot\mathbf{D}_{\mathrm{D}}. \tag{24}$$

2.5.3. *Example of application of the RCM in RMR to an indeterminate truss*

The method derived above is demonstrated in the following example. First, RGR representing the truss of Figure 17a is built (Fig. 17b). At the next stage, the cutset matrix of the RGR is found, Figure 17c. Finally the circuit matrix of the RMR is built from the cutset matrix of the RGR, Figure 17d. The elements of the circuit matrix (Fig. 17e and f) were substituted into Eq. (24) and the analysis equations were obtained (Fig. 17g).

A base (statically determinate subtruss) is obtained by removing from the truss the redundant rods 7 and 10. Hence the cobase elements of the resistance matroid representing the truss are 7, 10, and P, where the latter is the flow source. The circuit matrix in Figure 17d is now built by calculating three self-stresses, each having a unit force in one of the cobase elements. Then the parts of the circuit matrix are substituted into Eq. (24) and the analysis equations are obtained (Fig. 17g). After solving the equations of Figure 17g, the flows in all the cobase elements are known, and by applying Eq. (9), the flows in all the rest of the elements of the matroid are obtained. Then, using the resistance relations, the potential differences are obtained as well.

2.6. Line Graph Representation (LGR)

Line Graph Representation (LGR) is the only graph representation dealt with in this paper, which has no knowledge embedded in it. LGR is a regular graph, the main property of which is the way it is used to represent engineering systems. In contrast to FGR, PGR, and RGR, the elements of the engineering systems are represented in LGR not by edges, but by vertices. This enables us to use the edges of LGR to describe the connections between the elements. LGR was used to represent planetary gear systems, a traffic control problem (Shai, 1997), and various network optimization problems (Shai, 1997). The current paper uses LGR to represent planetary gear systems.

2.6.1. *LGR for planetary gear systems*

All the links of planetary gear systems are represented by vertices in LGR, whereas the connections between them are represented by the edges connecting the corresponding vertices. There are two types of connections, so there are two types of edges, marked as bold and double as explained below.

a. Bold edge—knowing the gear ratio between two engaged gear wheels one can calculate the ratio between the angular velocities (potentials) of the gear wheels. In the terminology of this paper, the edge representing the engagement between the wheels is a dependent potential source and for this reason it appears in the graph as a bold line.

b. Double edge—an edge which represents a turning connection. It will be shown below that the turning edges form a spanning tree.

Additional information about the labeled edges and the vertices is added to the representation as follows:

c. Labeled double edge—every double edge (turning edge) has a label, which represents the level (the location) of the rotating connection.

d. Reference vertices—the distance between each pair of connected gear wheels must be constant all the time, being maintained by a link or planet carrier. The vertex corresponding to such a link or a planet carrier is called in the literature (Freudenstein, 1971) a “transfer vertex.” In the terminology of this paper, the name “local reference vertex” is more suitable; thus it is highlighted by the gray color. In this representation, all the turning edges on one side of the local reference vertex are at the same level, and those on the opposite side of the local reference vertex are at a different level.

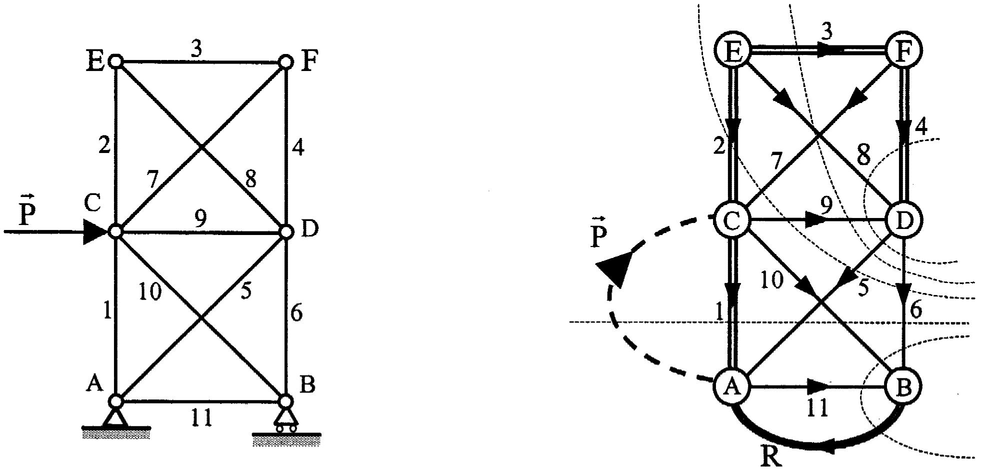
e. Labeled bold edge—every bold edge (gear engagement edge) has a label that represents the planet carrier (local reference vertex) that maintains the distance between the two corresponding gear wheels. In addition, the bold line is assigned a plus (minus) sign indicating that the engagement between the two gear wheels is internal (external).

f. Labeled gear wheel vertex—every vertex that corresponds to a gear wheel has a label that represents its center level.

Note, that Figure 18a is a standard engineering drawing for a gear system.

3. NEW CONNECTIONS BETWEEN ENGINEERING FIELDS

This section shows the application of MCA to obtaining novel relationships between engineering fields. These were



(a) (b)

$$
Q=\begin{array}{c|ccccccccccccc}
 & R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & P \\
\hline
R_x & \cos(270^\circ) & 0 & 0 & 0 & 0 & 0 & -\cos(270^\circ) & 0 & 0 & 0 & -\cos(315^\circ) & -\cos(0^\circ) & 0 \\
R_y & \sin(270^\circ) & 0 & 0 & 0 & 0 & 0 & -\sin(270^\circ) & 0 & 0 & 0 & -\sin(315^\circ) & -\sin(0^\circ) & 0 \\
1_x & 0 & \cos(270^\circ) & 0 & 0 & 0 & \cos(225^\circ) & \cos(270^\circ) & 0 & 0 & 0 & \cos(315^\circ) & 0 & \cos(0^\circ) \\
1_y & 0 & \sin(270^\circ) & 0 & 0 & 0 & \sin(225^\circ) & \sin(270^\circ) & 0 & 0 & 0 & \sin(315^\circ) & 0 & \sin(0^\circ) \\
2_x & 0 & 0 & \cos(270^\circ) & 0 & 0 & \cos(225^\circ) & \cos(270^\circ) & \cos(225^\circ) & 0 & -\cos(0^\circ) & 0 & 0 & 0 \\
2_y & 0 & 0 & \sin(270^\circ) & 0 & 0 & \sin(225^\circ) & \sin(270^\circ) & \sin(225^\circ) & 0 & -\sin(0^\circ) & 0 & 0 & 0 \\
3_x & 0 & 0 & 0 & \cos(0^\circ) & 0 & -\cos(225^\circ) & -\cos(270^\circ) & -\cos(225^\circ) & \cos(315^\circ) & \cos(0^\circ) & 0 & 0 & 0 \\
3_y & 0 & 0 & 0 & \sin(0^\circ) & 0 & -\sin(225^\circ) & -\sin(270^\circ) & -\sin(225^\circ) & \sin(315^\circ) & \sin(0^\circ) & 0 & 0 & 0 \\
4_x & 0 & 0 & 0 & 0 & \cos(270^\circ) & -\cos(225^\circ) & -\cos(270^\circ) & 0 & \cos(315^\circ) & \cos(0^\circ) & 0 & 0 & 0 \\
4_y & 0 & 0 & 0 & 0 & \sin(270^\circ) & -\sin(225^\circ) & -\sin(270^\circ) & 0 & \sin(315^\circ) & \sin(0^\circ) & 0 & 0 & 0
\end{array}
$$

(c)

$$
\mathbf{B}_M=\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 & P \\
7 & 0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 & 0 \\
10 & -0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1 & 0 \\
P & 0 & 0 & 0 & 0 & 1.414 & -1 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}
$$

(d)

$$
\mathbf{B}_{C'R}=\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 \\
7 & 0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 \\
10 & -0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1
\end{array}
$$

(e)

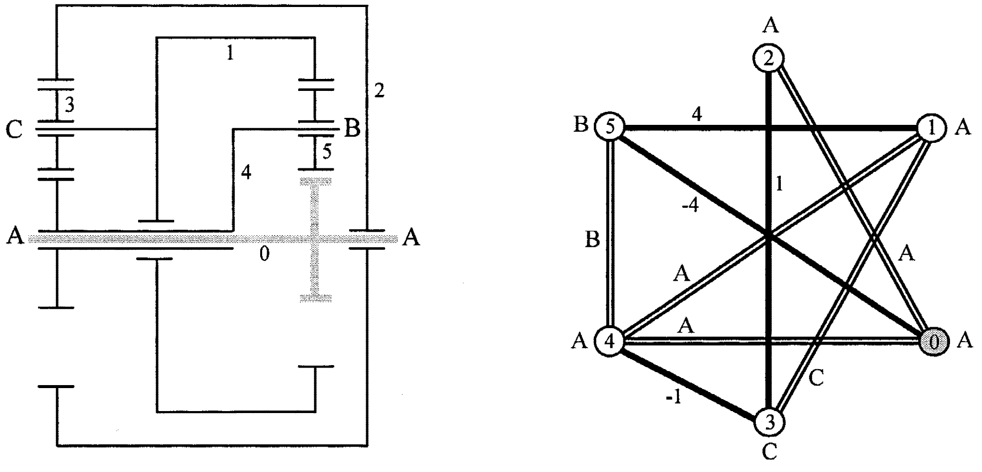
$$
\mathbf{B}_{PR}=\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 \\
P & 0 & 0 & 0 & 0 & -1.414 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

(f)

$$
\begin{pmatrix} 4.823 & 0.5 \\ 0.5 & 4.823 \end{pmatrix}\cdot\begin{pmatrix} F_7 \\ F_{10} \end{pmatrix}=-\begin{pmatrix} 0.707 \\ 3.411 \end{pmatrix}\cdot P
$$

(g)

Fig. 17. Example of analysis of indeterminate truss using RMR (Resistance Matroid Representation). (a) Indeterminate truss. (b) RGR of the truss. (c) Scalar cutset matrix of RGR of the truss. (d),(e),(f) Circuit matrix of RMR of the truss and its components. (g) Analysis equations based on RCM in RMR.



(a) (b)

Fig. 18. Example of a planetary mechanism and its graph. (a) The planetary mechanism. (b) Its line graph representation.

obtained through the mathematical relationships between the corresponding CR. More specifically, on the basis of the dualism between FGR and PGR, which will be developed in Section 3.1.1, a new mutual relationship between determinate trusses and mechanisms is established in Section 3.1.2. This relation postulates that for every determinate truss there exists a specific mechanism, called the "dual mechanism." In this mechanism, there is a link for every rod of the truss, such that the force in the rod is equal to the relative velocity of the corresponding link.

This innovation opens up a new avenue of research and practical applications, by creating favorable conditions for cooperation between structural and mechanical engineers. This cooperation enables us to use information, algorithms, and new technologies developed for one field in the other. One of the possible applications of that research direction is the development of a new design technique in engineering, shown in Section 7. The potential inherent in the innovations of this section can be appreciated also from its applications given in Section 6.5.

3.1. Duality between trusses and mechanisms based on the duality of their corresponding CR

3.1.1. Duality between flow and potential graphs

This section introduces the duality connection between the flow and potential graph representations from which the dualism between trusses and mechanisms will be later derived. FGR and PGR are shown to be dual by applying the following inference rules:

RULE 1:

IF: G is FGR, i.e. $G = G_F$
THEN: $\vec{\mathbf{Q}}(G_F)\cdot\vec{\mathbf{F}}(G_F) = \mathbf{0}$ (equation 1).

RULE 2

IF: G is PGR, that is, $G = G_\Delta$
THEN: $\vec{\mathbf{B}}(G_\Delta)\cdot\vec{\Delta}(G_\Delta) = \mathbf{0}$ (equation 2).

FACT 1: $\vec{Q}(G) = \vec{B}(G^*)$ (according to duality between graphs (Swamy & Thulasiraman, 1981)).

CONCLUSION 1: FACT 1 AND RULE 1 → $\vec{\mathbf{B}}(G_F^*)\cdot\vec{\mathbf{F}}(G_F) = \mathbf{0}$.

CONCLUSION 2: CONCLUSION 1 AND $\vec{\Delta}(G_F^*) = \vec{F}(G_F) \rightarrow \vec{\mathbf{B}}(G_F^*)\cdot\vec{\Delta}(G_F^*) = \vec{0}$.

From these inference rules it follows that the dual graph of any FGR is a PGR, since the potential difference vector of the latter is identical to the flow vector of the former.

3.1.2. Duality between trusses and mechanisms

On the basis of the dualism connection between FGR and PGR given in the previous section, a new relation between determinate trusses and mechanisms is derived. This invention has been achieved by applying the following rules.

FACT 2: For every flow graph, there exists a dual potential graph and vice versa (conclusion 2).

FACT 3: Determinate trusses are isomorphic to flow graphs (Section 2.2.2).

FACT 4: Mechanisms are isomorphic to potential graphs (section 2.3.1).

CONCLUSION 3: FACT 2 AND FACT 3 AND FACT 4 → for every mechanism there exists a dual determinate truss and vice versa.

This reasoning is outlined in Figure 19.

Description of a dual mechanism. Let T be a statically determinate truss, $G_F(T)$ its FGR, and $G^*(T)$ be a PGR

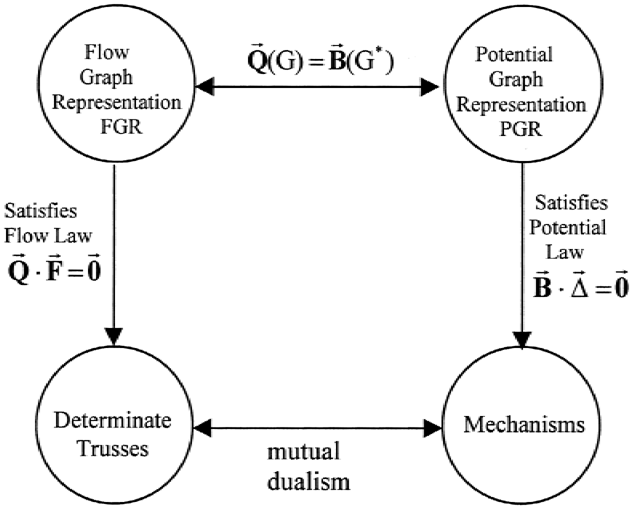


Fig. 19. Diagram explaining the derivation of the duality between determinate trusses and mechanisms.

dual to $G^*(T)$. Mechanism M is called a "mechanism dual to truss T" if $G^*(T)$ is its potential graph and for every edge e in $G(T)$ and its corresponding edge e′ in the dual graph $G^*(T)$, the equality (25) is satisfied.

$$\hat{r}(e) = \hat{v}(e'). \tag{25}$$

Table 5 summarizes the attributes of the duality between trusses and mechanisms. Figure 20 shows a four-bar mechanism (a) and its dual truss (c). More detail on the duality connection between trusses and mechanisms can be found in Shai (2001a).

4. META LAWS AND THEOREMS

As was mentioned in Section 2, each combinatorial representation contains combinatorial theorems called "embedded theorems," which have been thoroughly studied and investigated. The embedded theorems are actually meta-theorems that provide additional knowledge and enable us to derive engineering theorems exclusively from the representation. Therefore, when CR are used to represent an engineering problem, their embedded theorems become available as well. For example, when the Resistance Graph Representation was applied to represent indeterminate trusses (Section 2.4.6), its two analysis methods became available and were used. In this section, it is shown how theorems and methods in structural mechanics can be derived from a theorem embedded in RGR, called Tellegen's Theorem. Moreover, from the dualism law in RMR, a new proposition is deduced stating that displacement and force methods are actually dual methods (Shai, 1999, 2001b).

All these results show the potential inherent in applying MCA to allow, in the future, the derivation of new theorems and methods from the knowledge embedded in the CR.

4.1. Tellegen's theorem embedded in RGR

The theorem discussed in this section was developed by Professor B. D. H. Tellegen (Tellegen, 1952) and therefore bears his name. The main use made of this theorem nowadays is in electric circuit theory (Penfield et al., 1970; Chua et al., 1987). Since electric circuits are represented in MCA by RGR, it is concluded that this theorem can also be employed in other engineering systems represented with RGR. This is done in the current section. According to Section 2.4.6, indeterminate trusses are represented by RGR; therefore Tellegen's theorem, which is a meta-theorem in RGR, is applied, and engineering theorems and methods are derived.

THEOREM 2. *Tellegen's Theorem (combinatorial representations formulation):* Let G_F and G_Δ *be two isomorphic graphs, where the first is a flow graph and the second is a potential graph; then:*

$$\sum_{e}^{\text{all edges}} \vec{F}^{t}_{G_F}(e)\cdot\vec{\Delta}_{G_\Delta}(e) = 0. \tag{26}$$

■

The theorem deals with two isomorphic graphs, one of which, G_F, satisfies the flow law and the other, G_Δ, satisfies the potential law. It postulates that the sum of scalar prod-

Table 5. *The duality attributes*

In a mechanism link	In a truss rod
$\vec{\Delta}(e)$ – Relative velocity of link e.	$\vec{F}(e')$ – Force in rod e′.
Circuit.	Cutset.
Potential difference.	Flow.
Potential difference of edge e = relative linear velocity of the corresponding link $e = \vec{\Delta}(e) = V(e)\cdot v(e)$.	Flow in edge e = force acting in rod e = $\vec{F}(e) = F(e)\cdot r(e)$.
v(e) – relative linear velocity unit vector.	r(e) – unit vector in the rod direction.
$\omega_{i/0}$ – Angular velocity of link i = linear velocity/length.	$\frac{F_i}{L_i}$ – Force per unit length.

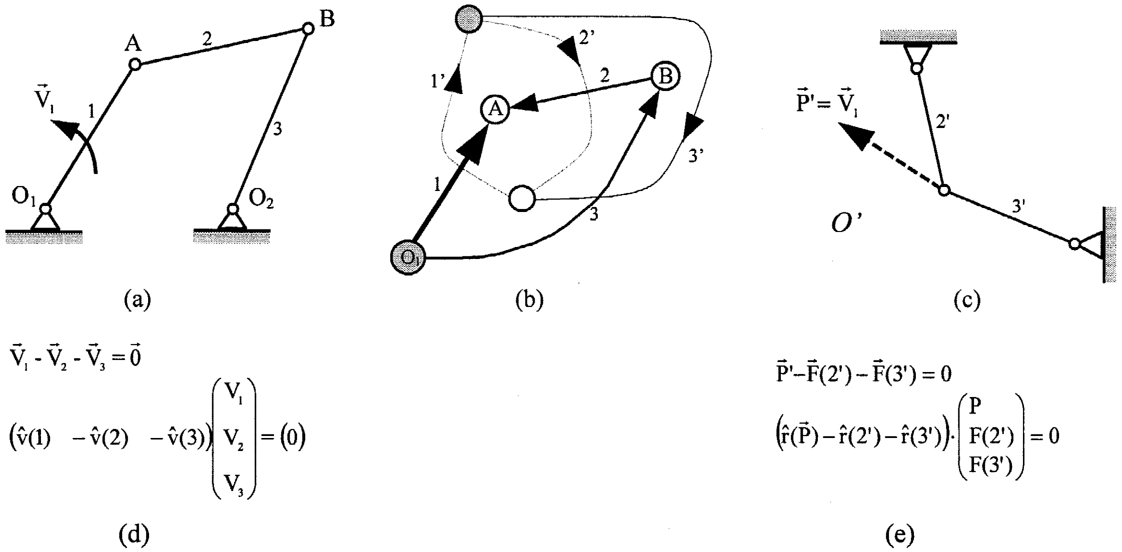


Fig. 20. Example of a mechanism, its dual truss, and the corresponding matrices. (a) The mechanism. (b) The potential graph and its dual (dashed line). (c) The dual truss. (d),(e) The corresponding matrices.

ucts between the flows in the edges of G_F and the potential differences in the corresponding edges of G_Δ is equal to zero.

4.1.1. *Explanation of Tellegen's Theorem using the electrical networks*

To facilitate understanding, Tellegen's theorem is first applied to electrical circuits. Consider two different electric circuits with the same topology shown in Figure 21a,b. Their corresponding graphs appear in Figure 21c,d. Analysis of the electrical systems gives the results shown in Figure 21 next to the edges in the corresponding graphs.

We can now choose the graph representing the system of (a) to be G_Δ and the graph representing the system of (b) to be G_F. These graphs satisfy the requirements of Theorem 2. Substituting the results into Eq. (26) confirms Tellegen's theorem:

$$\sum_{i=1}^{10} \Delta_i(G_\Delta)\cdot F_i'(G_F) = 0. \qquad (27)$$

4.1.2. *Applications of Tellegen's Theorem to structures*

The example that appears in the preceding section concerns one-dimensional systems. However, one can deduce from Eq. (26) that the theorem can be applied to the multidimensional systems as well.

The multidimensional trusses are represented by RGR; hence the multidimensional Tellegen's Theorem embedded in this representation can be employed in their analysis. The formulation of Tellegen's Theorem for trusses is as follows: Given two trusses with the same topology, the sum over multiplications of the forces in the first and the potential differences in the second is equal to zero.

When the angles of the rods of the truss are known, the following scalar formulation can be developed (Shai, 2000b):

$$\sum_{\substack{\text{rods of}\\ \text{the truss}}} F_i(G_F)\cdot D_i(G_\Delta) - \sum_{\substack{\text{external}\\ \text{forces}}} P_i(G_F)\cdot D_{Pi}(G_\Delta) = 0. \qquad (28)$$

This equation will be referred to as the "Multidimensional Tellegen's Theorem" for trusses.

4.1.3. *Deriving the method for analyzing joint displacement based on Tellegen's Theorem*

This section shows the derivation of the known equation for analyzing the displacement of a joint in a truss from the Multidimensional Tellegen's Theorem for trusses. To apply Tellegen's Theorem, PGR and FGR will be used. The steps for building the CR are the same as those explained in Section 2.2.2. An extra edge, called "control edge," is added. Its head vertex is the vertex whose displacement is to be analyzed, and the tail vertex is the reference vertex.

The flow and potential graphs are used in two different ways as follows. For the real potential graph G_Δ^R, the flow values in the source edges are the values of the external forces. In the control edge, we put a "potential difference measurement," which corresponds to a potential difference measuring device (e.g., voltmeter in electrical circuit), which is located between the end vertices of the corresponding edge. The R superscript over G indicates that the potential differences in it are due to the "real" external forces applied to the structure, and the Δ subscript indicates that the structure should satisfy only the potential law.

For the virtual flow graph G_F^V, all the source edges which correspond to the external forces are assigned zero flow

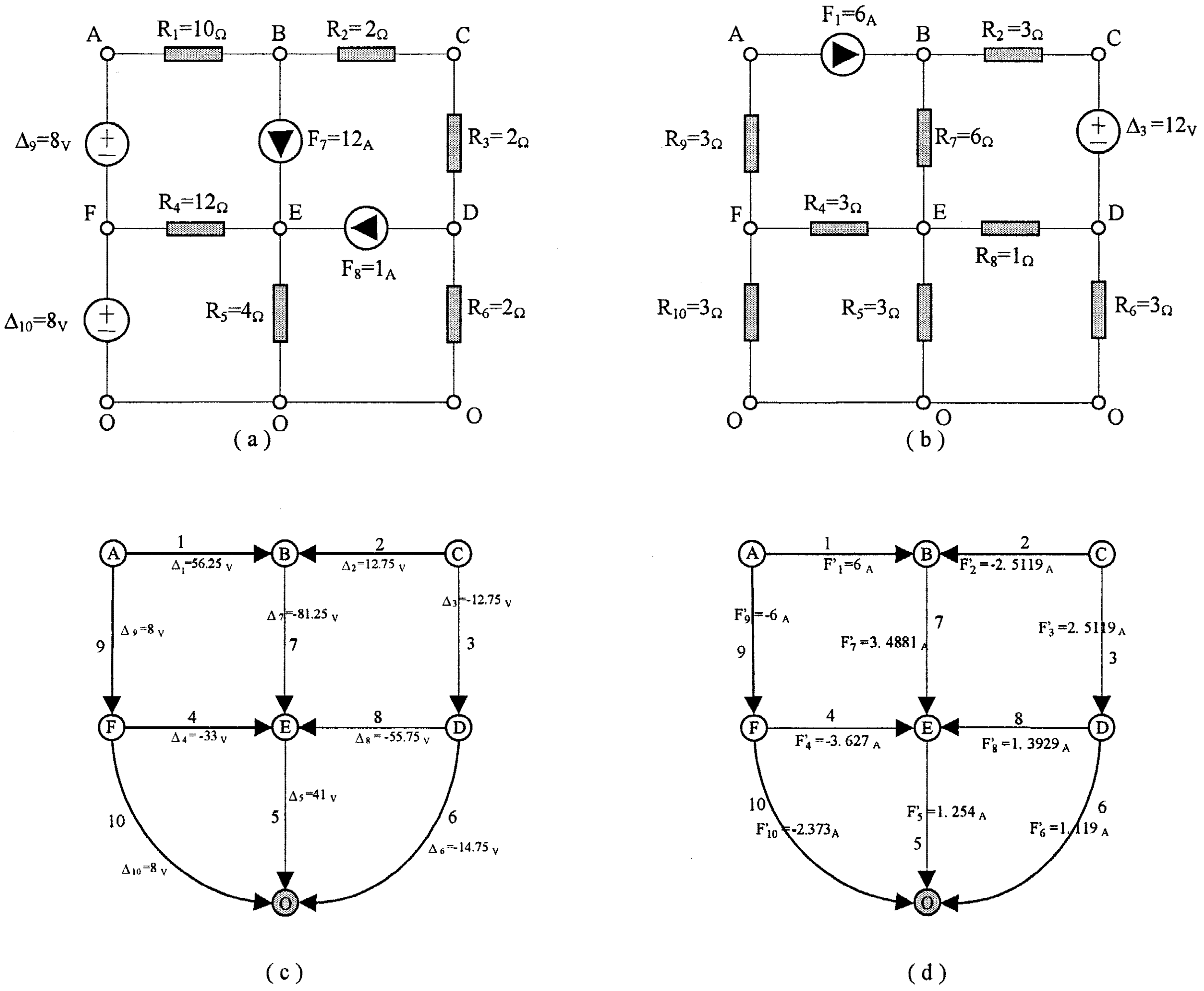


Fig. 21. Applying Tellegen's theorem to electrical circuits. (a),(b) Different electrical circuits possessing the same topology. (c) PGR of the circuit in a. (d) FGR of the circuit in b.

sources. One can think about it as a disconnection. In the control edge, a unit force is applied in the direction of the displacement that has to be measured. The V superscript over G indicates that the flows in the graph are not the real forces in the structure, but the forces due to a virtual external force applied to the structure, and the F subscript indicates that the structure should satisfy only the flow law.

Applying the Multidimensional Tellegen's Theorem [Eq. (28)] to the two graphs, gives

$$\sum_{\substack{\text{rods of}\\ \text{the truss}}} \mathrm{F}_i(\mathrm{G}_{\mathrm{F}}^{\mathrm{V}})\cdot \mathrm{D}_i(\mathrm{G}_{\Delta}^{\mathrm{R}}) - \sum_{\substack{\text{external}\\ \text{forces}}} 0\cdot \mathrm{D}_{\mathrm{P}i}(\mathrm{G}_{\Delta}^{\mathrm{R}}) - 1\cdot \mathrm{D}_{\text{control}}(\mathrm{G}_{\Delta}^{\mathrm{R}}) = 0. \tag{29}$$

From here, the well-known equation (West, 1993) for analyzing the displacement of a joint is derived:

$$\mathrm{D}_{\text{control}}(\mathrm{G}_{\Delta}^{\mathrm{R}}) = \sum_{\substack{\text{rods of}\\ \text{the truss}}} \frac{\mathrm{F}_i(\mathrm{G}_{\mathrm{F}}^{\mathrm{V}})\cdot \mathrm{F}_i(\mathrm{G}_{\Delta}^{\mathrm{R}})\cdot \mathrm{L}_i}{\mathrm{A}_i\cdot \mathrm{E}_i}. \tag{30}$$

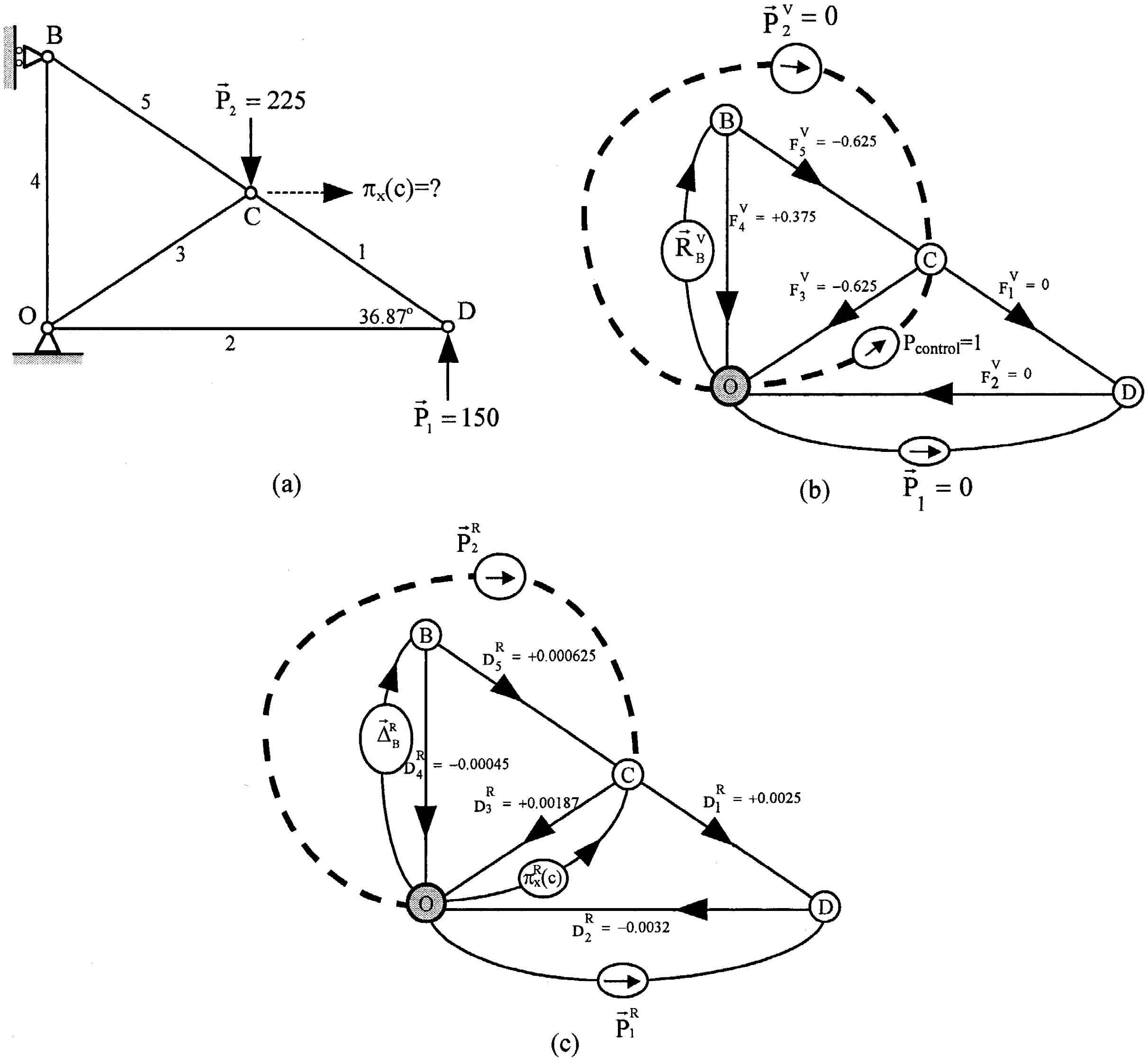
An example for applying Eq. (30) is given in Figure 22, where the horizontal displacement of joint c is to be computed.

4.1.4. *Deriving Betti's Law from a theorem embedded in RGR*

The previous section showed an example of derivation of a known method in structural mechanics from the knowledge embedded in RGR. In the current section it is shown that known theorems can also be derived from the embedded knowledge. This is demonstrated by deriving Betti's Law from Tellegen's Theorem.

Consider a truss and two different sets of external loads applied on it. The first set of external loads, $\vec{\mathbf{P}}_1$, causes joint displacements $\vec{\pi}_1$, internal forces $\vec{\mathbf{F}}_1$, and deformations $\vec{\mathbf{D}}_1$. The second set of external loads, $\vec{\mathbf{P}}_2$, causes joint displacements $\vec{\pi}_2$, internal forces $\vec{\mathbf{F}}_2$, and deformations $\vec{\mathbf{D}}_2$.

Since both sets of loads act on the same truss, and the forces (potential differences) satisfy the flow (potential) law, then according to Multidimensional Tellegen's Theorem [Eq. (28)], multiplication of forces from one set by the



$$\pi_x(c)^R = F_4^V \cdot D_4^R + F_5^V \cdot D_5^R + F_3^V \cdot D_3^R =$$
$$(0.375\cdot -0.00045) + (-0.625\cdot 0.000625) + (-0.625\cdot 0.001875) = -0.001731[m]$$

(d)

Fig. 22. Example of analyzing joint displacements by the Multidimensional Tellegen's theorem. (a) The truss. (b) The virtual flow graph G_F^V. (c) The real potential graph G_Δ^R. (d) Calculation of the displacement of joint c in the direction of the x axis.

potential differences from the other set is equal to zero, as follows:

$$(\vec{\mathbf{F}}_1^t\ \vec{\mathbf{P}}_1^t)\cdot\begin{pmatrix}\vec{\mathbf{D}}_2\\ -\vec{\pi}_{P2}\end{pmatrix} = \vec{0} \rightarrow \vec{\mathrm{F}}_1^t\cdot\vec{\mathbf{D}}_2 = \vec{\mathbf{P}}_1^t\cdot\vec{\pi}_{P2} \tag{31}$$

$$\vec{\mathbf{P}}_1^t\cdot\vec{\pi}_{P2} = \vec{\mathbf{F}}_1^t\cdot\vec{\mathbf{D}}_2 \overset{\substack{\text{resistance}\\ \text{relation}}}{=} \vec{\mathbf{F}}_1^t\cdot(\mathbf{R}\cdot\vec{\mathbf{F}}_2)$$
$$= (\vec{\mathbf{F}}_1^t\cdot\mathbf{R})\cdot\vec{\mathbf{F}}_2 \overset{\substack{\text{since R is}\\ \text{diagonal}}}{=} \vec{\mathbf{D}}_1^t\cdot\vec{\mathbf{F}}_2. \tag{32}$$

Another form of Tellegen's Theorem for the two graphs is

$$(\vec{\mathbf{F}}_2^t\ \vec{\mathbf{P}}_2^t)\cdot\begin{pmatrix}\vec{\mathbf{D}}_1\\ -\vec{\pi}_{P1}\end{pmatrix} = \mathbf{0} \rightarrow \vec{\mathbf{F}}_2^t\cdot\vec{\mathbf{D}}_1 = \vec{\mathbf{P}}_2^t\cdot\vec{\pi}_{P1}. \tag{33}$$

Combining the last two equations gives

$$\vec{\mathbf{P}}_1^t\cdot\vec{\pi}_{P2} = \vec{\mathbf{P}}_2^t\cdot\vec{\pi}_{P1}. \tag{34}$$

This is the reciprocity theorem or Betti's Law, which is well known in the literature (Hibbeler, 1984).

4.2. The relation between the Conductance Cutset Method (CCM) for analyzing indeterminate trusses and other known methods

In the current section it is shown that a known method in structural mechanics is derived from CCM. Many methods for analyzing indeterminate trusses have been reported in the literature; most of them are based on virtual work and minimum energy. In the displacement method (Hibbeler, 1984), each axis along which the joint is able to move is assigned a variable, which is designated: "unknown displacement." In the Conductance Cutset Method (CCM), the absolute potential of a vertex is the displacement of the vertex relative to the reference vertex.

In the example of Figure 23, since all the cutsets are such that they contain exactly one vertex in one of the two sides of the cutset, the cutset conductance matrix is equal to the incidence matrix, a well-known matrix in graph theory literature (Fenves & Branin, 1963; Deo, 1974). Therefore, the displacement matrix of the displacement method (Hibbeler, 1984) is the same as that obtained from the resistance graph, as shown in Figure 23.

To derive the displacement method, one starts with the incidence matrix, the rows of which are a linear combination of the vector cutset matrix rows. The flow law [Eq. (2)] can be written by using the incidence matrix as follows:

$$\vec{\mathbf{A}}\vec{\mathbf{F}}_{\mathrm{R}} = -\vec{\mathbf{A}}_{\mathrm{P}}\vec{\mathbf{P}}. \tag{35}$$

Since any resistance edge e = ⟨u,v⟩ satisfies Eq. 6, Eq. 35 becomes

$$\vec{\mathbf{A}}\mathbf{K}_{\mathrm{R}}\vec{\Delta}_{\mathrm{R}} = -\vec{\mathbf{A}}_{\mathrm{P}}\vec{\mathbf{P}}. \tag{36}$$

Due to Eq. (1), this can be written in matrix form:

$$\vec{\Delta}_{\mathrm{R}} = \vec{\mathbf{A}}^{t}\vec{\pi}, \tag{37}$$

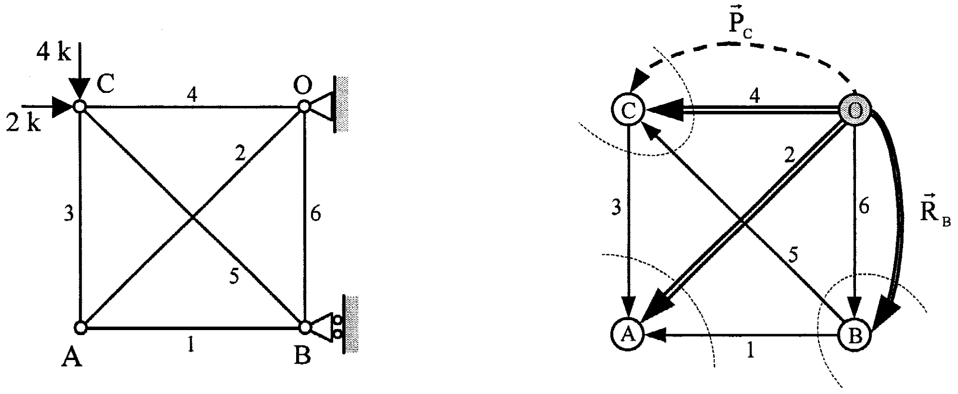
which gives us

$$\vec{\mathbf{A}}\mathbf{K}_{\mathrm{R}}\vec{\mathbf{A}}^{t}\vec{\pi} = -\vec{\mathbf{A}}_{\mathrm{P}}\vec{\mathbf{P}}. \tag{38}$$

The matrix $\vec{\mathbf{A}}\mathbf{K}_{\mathrm{R}}\vec{\mathbf{A}}^{t}$ is actually the "stiffness matrix," and the element $[\vec{\mathbf{A}}\mathbf{K}_{\mathrm{R}}\vec{\mathbf{A}}^{t}]_{ij}$ is the sum of the conductances of the rods that meet both joint i and joint j (in the case $i = j$ it equals the sum of conductances of all the rods meeting joint i).

5. A GLOBAL MULTIDISCIPLINARY PERSPECTIVE FOR INTEGRATED SYSTEMS

One of the immediate contributions made by MCA is making it possible to obtain a global perspective on various



(a)

(b)

$$\begin{pmatrix} \mathbf{K}_{\mathrm{R_B}} + \mathbf{K}_1 + \mathbf{K}_5 + \mathbf{K}_6 & -\mathbf{K}_1 & -\mathbf{K}_5 \\ -\mathbf{K}_1 & \mathbf{K}_2 + \mathbf{K}_1 + \mathbf{K}_3 & -\mathbf{K}_3 \\ -\mathbf{K}_5 & -\mathbf{K}_3 & \mathbf{K}_4 + \mathbf{K}_3 + \mathbf{K}_5 \end{pmatrix} \begin{pmatrix} \vec{\Delta}_{\mathrm{R_B}} \\ \vec{\Delta}_2 \\ \vec{\Delta}_4 \end{pmatrix} = -\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \cdot \vec{\mathbf{P}}$$

(c)

$$\begin{pmatrix} -0.035 & -0.1 & 0 & -0.035 & 0.035 \\ 0.135 + 0 \cdot (-0.035) & 0 & 0 & 0.035 & -0.035 \\ 0 + 0 \cdot (-0.1) & 0.135 & 0.035 & 0 & 0 \\ 0 + 0 \cdot (0) & 0.035 & 0.135 & 0 & -0.1 \\ 0.035 + 0 \cdot (-0.035) & 0 & 0 & 0.135 & -0.035 \\ -0.035 + 0 \cdot (0.035) & 0 & -0.1 & -0.035 & 0.135 \end{pmatrix} \cdot \begin{pmatrix} \Delta_{\mathrm{R_B}y} \\ \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{4x} \\ \Delta_{4y} \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathrm{P}_{\mathrm{Cx}} \\ \mathrm{P}_{\mathrm{Cy}} \end{pmatrix}$$

(d)

Fig. 23. Example of indeterminate truss analysis. (a) Indeterminate truss. (b) The corresponding graph. (c),(d) The corresponding vector and scalar equations.

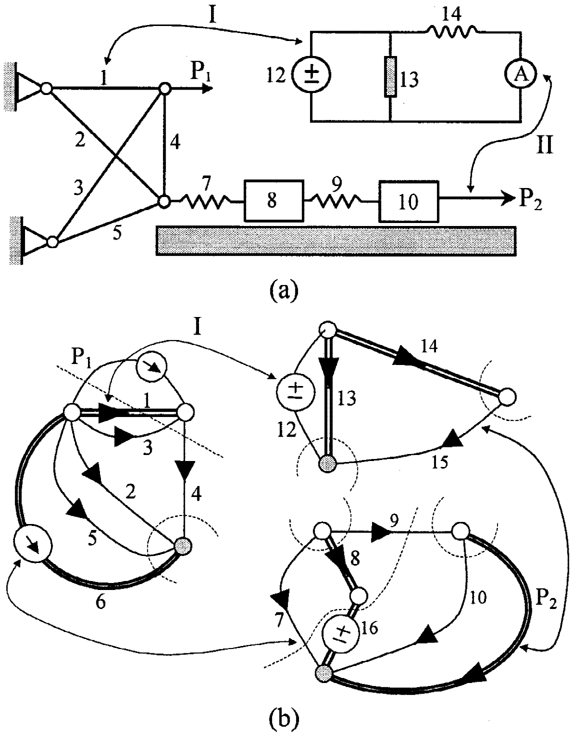


Fig. 24. Representing an integrated engineering system with RGR. (a) The integrated system. (b) Corresponding RGR.

disciplines. The general perspective is due to the fact that the same combinatorial representation is applied to different engineering fields. For example, Figure 24a shows a complex system composed of interacting dynamic, electric, and indeterminate truss elements. Even though the system contains different types of engineering elements, some of which have different coordinate systems, such as one-dimensional for electrical and hydraulic systems or two-dimensional for trusses, the integrated system is represented by one resistance graph as shown in Figure 24b. Therefore, all the different elements are dealt with in the same way when an MCA analysis algorithm is applied.

6. CHECKING THE VALIDITY OF ENGINEERING SYSTEMS ON THE BASIS OF THE COMBINATORIAL REPRESENTATIONS

This section shows a further contribution of MCA, which is the ability to check the validity of the engineering systems before applying to them the analysis process or starting to manufacture the products. The idea behind this issue is the same as the one behind all the rest of MCA applications that utilized the knowledge embedded in the CR. In the current section, this knowledge is applied to check whether there exists a contradiction between the representation of the engineering system and the rules and theorems embedded in it. The current section is concerned with checking the validity of truss topology and geometry, planetary gear systems, and geometric constrains in CAD systems.

6.1. Checking the topological validity of trusses

In Section 2.2.2, it was explained how to represent trusses by FGR, and the knowledge embedded in the CR was applied for truss analysis. In this section, it is shown how to use the properties of the CR to check the validity of the truss topological rigidity. This issue contributes to other engineering fields, such as checking the validity of mechanisms (Section 6.5.2) and the validity of the geometric constraint systems (Section 6.4).

Checking of the validity of a determinate truss is performed on its corresponding combinatorial representation—the FGR (Section 2.2). Whenever the analysis equations obtained from the flow graph are not soluble, this indicates that the truss represented by it is not stable. The word "rigidity" is used when referring to the truss structure without its supports, and "stability" for the one including the supports. To check the validity of truss supports as well, the following steps are to be performed.

Step 1. Create two extra vertices called X and Y and connect them by an edge.

Step 2. For every hinged support, create two edges connecting the vertex corresponding to the support, with the X and Y vertices.

Step 3. For every roller support, create an edge connecting the corresponding vertex with X (or Y) if the support is immobile on the horizontal (or vertical) plane. If the support is mobile on some inclined plane, create an additional vertex and connect it with edges to the vertices named X and Y and to the vertex corresponding to the support itself.

Note that edges representing applied loads do not affect the topological consistency of the graph, so they are to be removed from the graph when the validity is checked. Figure 25b shows the graph that corresponds to the truss of Figure 25a. Since the roller supports E and G are immobile on the Y coordinate, the corresponding two edges connect between Y and E and G. The same reasoning applies to the roller supports A and D.

6.1.1. Relevant theorems embedded in the FGR.

Most of the published literature on the subject of truss rigidity deals with determinate trusses (Laman, 1970).

There exists a fixed relation between the number of rods $e(G_F)$ and joints $v(G_F)$ in the graph G_F representing a rigid determinate truss, as follows:

$$e(G_F) = 2 \cdot v(G_F) - 3. \tag{38}$$

Maxwell (1864) proved that if the relation $e(G') \leq 2 \cdot v(G') - 3$ holds for every subgraph G' of G_F, then the corresponding determinate truss is rigid. About 100 years later, Laman (1970) proved that this condition is not only necessary, but is also sufficient.

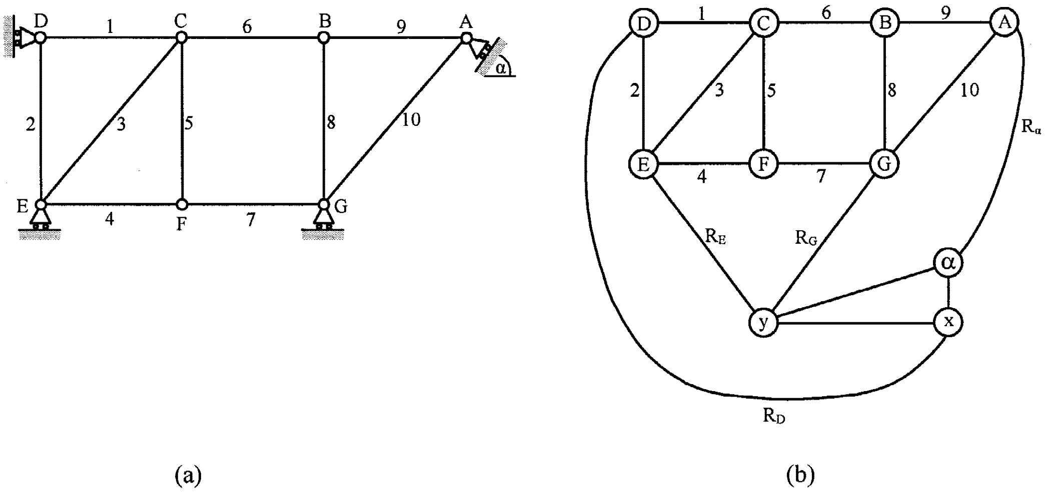


Fig. 25. Example for checking the stability of determinate truss using FGR. (a) A determinate truss. (b) The graph that represents it.

The connection between the rigidity of determinate and indeterminate trusses is established by the following theorems.

THEOREM 3. *Let G be a graph that corresponds to an indeterminate truss. Then, G is rigid if and only if there exists in G a connected rigid determinate subgraph* G′ *which includes all the vertices of G.* ■

Proof: If G′ is a determinate truss and is rigid, adding edges (rods) does not affect the property of rigidity. The inverse claim follows directly from the definition of an indeterminate truss. Suppose that in G there are *k* redundant rods, G is then said to have a redundancy degree of *k*. When deleting those k edges from G, the truss represented by G′ remains rigid and determinate. ■

The necessary and sufficient conditions for checking whether a determinate truss is topologically valid is given in the following theorem.

THEOREM 4. *(Lovasz & Yemini, 1982). A determinate truss is rigid if and only if when doubling each edge in turn in the corresponding graph, all the edges can be covered by two edge disjoint spanning trees.* ■

From this theorem one can derive the algorithm of Section 6.1.2 for checking the validity of the topology of determinate trusses.

6.1.2. *Algorithm for checking the validity of the graph of a determinate truss*

STEP 1. Build the graph corresponding to the truss as was explained in Section 3.1.

STEP 2. For every edge in the graph double the edge and search for two edge disjoint spanning trees using known algorithms (Swamy & Thulasiraman, 1981).

STEP 3. If step 2 is successful for every edge in the graph, then the graph topology is valid; otherwise it is not.

For example, Figure 26 shows a truss (a) and its corresponding graph (b). It can be proved to be rigid, since when doubling each edge in turn, it has two edge disjoint spanning trees covering all of its edges. Figure 26c shows an example of two edge disjoint spanning trees covering the graph when edge 1 is doubled. More details on checking the validity of determinate trusses can be found in Shai and Preiss (1999a).

6.2. Checking the validity of dynamic systems

A process similar to the above for trusses can be applied to a dynamic mass-spring-damper oscillator system. In this section, it will be shown that one can find a contradiction in the topological structure of a dynamic system with given initial conditions by analyzing its corresponding RGR. Given a dynamic system with initial conditions, there can be a solution only if its graph is consistent with the validity rules.

6.2.1. *The RGR of the dynamic system*

Using the information given in Table 3, the resistance graph corresponding to a dynamic system can be built. Including the information about the initial conditions of the dynamic system requires performing the following steps.

STEP 1. Every spring with initial tension will be represented by two parallel edges. On the basis of the superposition principle, one edge will represent the flow source with the value of the initial tension of the spring and the other will represent the flow (force) change in the spring caused by the changes of the dynamic system.

STEP 2. Every mass with initial velocity will be represented by two serial edges. On the basis of the superposi-

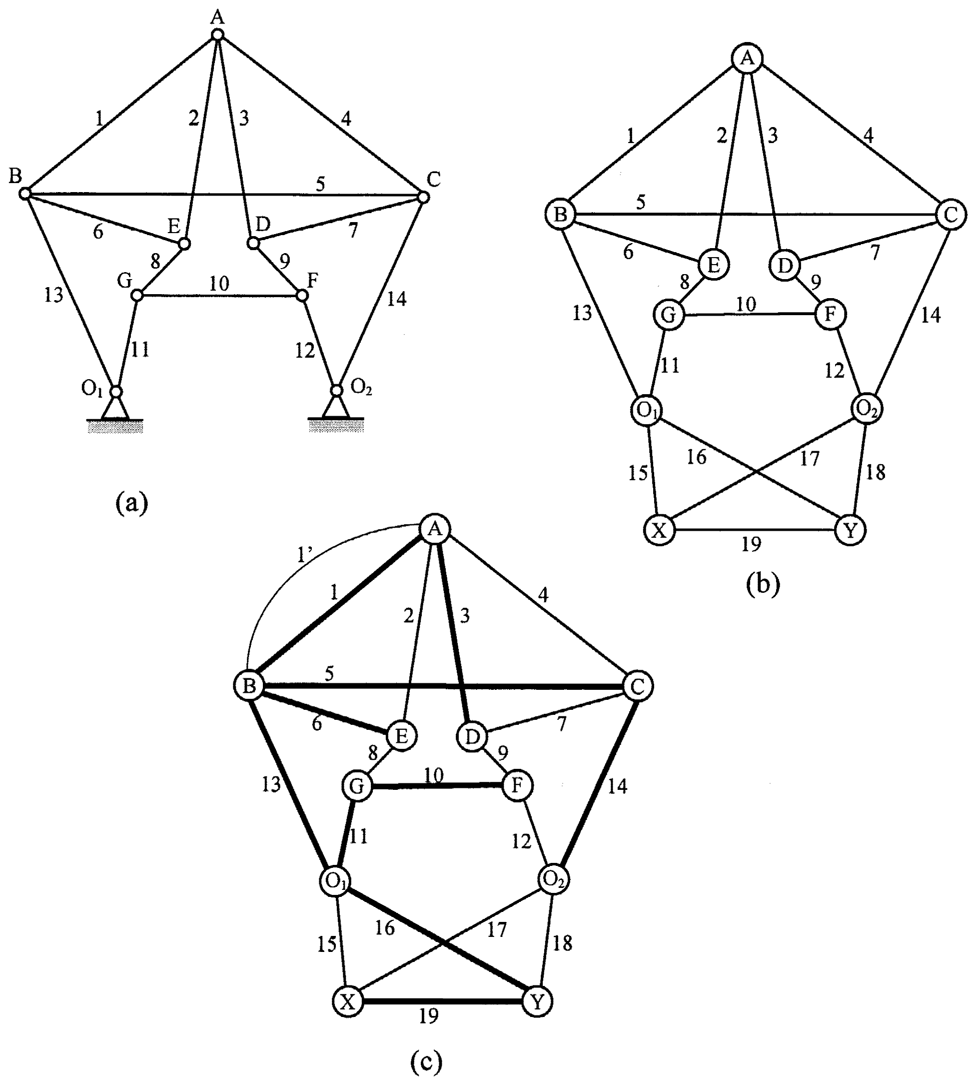


Fig. 26. Example of a proof that a determinate truss is stable. (a) The truss. (b) The corresponding graph. (c) Two edge disjoint spanning trees when doubling edge 1.

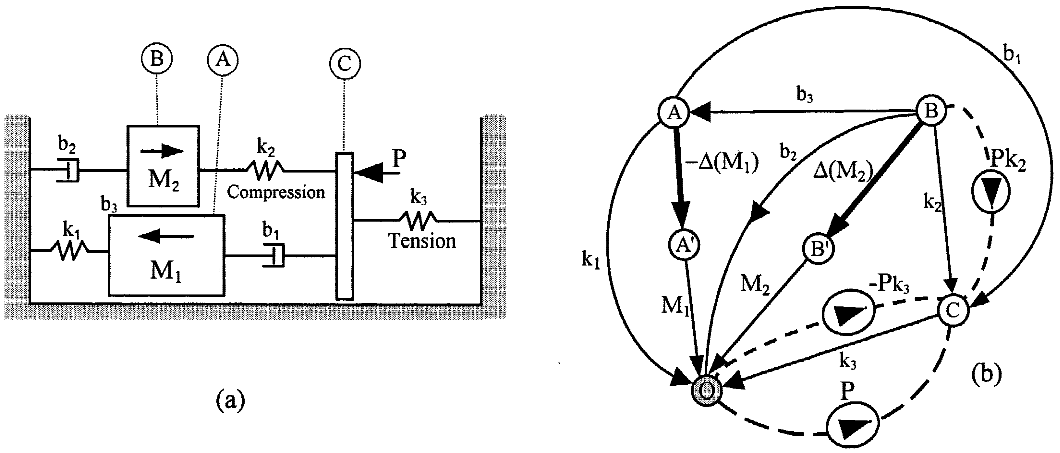


Fig. 27. Representing a dynamic system with RGR. (a) A dynamic system. (b) Its corresponding graph.

tion principle, one edge represents the source of potential difference with a value equal to the initial velocity value of the mass, and the other represents the change in potential (velocity) with time in the dynamic system. An example of RGR representing a dynamic system appears in Figure 27.

6.2.2. The method for checking the validity of the dynamic system with initial conditions

For the Flow Law and the Potential Law to be satisfied, one has to verify the following two validity checking rules.

Validity rule of cutsets: There should be no cutset containing only flow sources (bold dashed edges).

Validity rule of circuits: There should be no circuit containing only potential sources (bold solid edges).

The reason for these restrictions is derived from the property of the source edge. For example, if there were a cutset of only flow source edges, the sum of flows over a cutset might not be equal to zero, in contradiction with the Flow Law. A similar reason holds for the potential difference source edges.

An example of a dynamic system graph which contradicts the cutset validity rule is shown in Figure 28. The contradiction occurs because the graph of Figure 28 has a cutset with only bold dashed lines. For such a graph, the initial forces applied to junction C may not satisfy the flow law.

6.3. Checking the validity of a planetary gear system using the LGR

The work reported in this section employs the possibility that the domain knowledge will consist of the topological validity rules of the graph. Therefore, the process of checking the validity of planetary gear systems becomes a process of checking whether there exists a contradiction between the domain knowledge and the graph representation of the given system.

6.3.1. Topological validity rules of the graph

Part of the embedded properties in the graph representation of the planetary gear system, given below, is based on Erdman (1993), who published a set of necessary conditions which were used by him for a different purpose: mechanism synthesis. This paper uses this knowledge to deduce the validity of the system.

Rule 1: Planetary gear system is a kinematic chain → There is no circuit formed exclusively by turning edges.

Rule 2: Circuit of turning edges → locked mechanism OR kinematic chain with degree of freedom greater that 1.

Rule 3: Every link has at least one element around which it rotates → every vertex is incident to at least one turning edge.

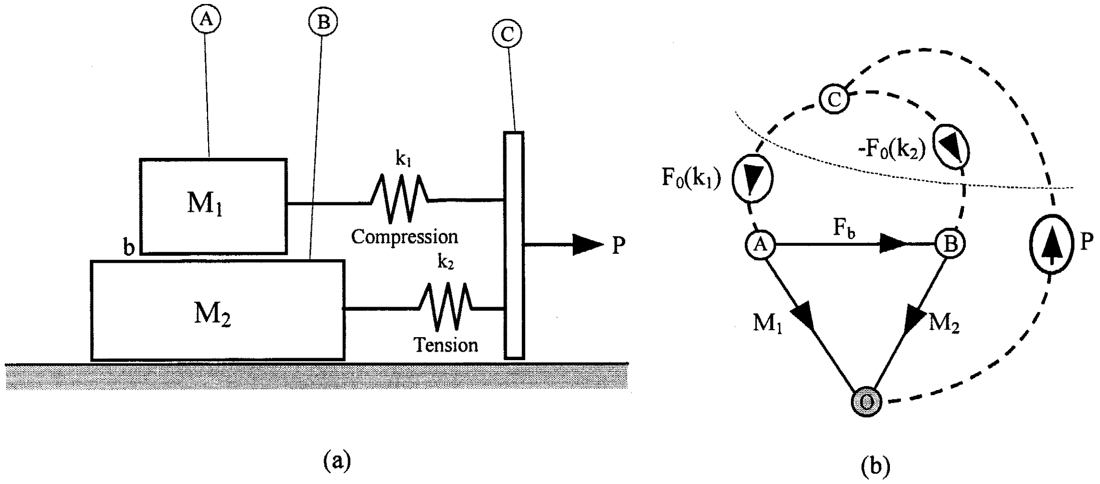
Rule 4: The distance between each pair of engaged gears should be preserved during the system operation → the subgraph of the turning edges forms a connected subgraph.

Topological conclusion 1: Rule 1 AND Rule 2 → the turning edges constitute a spanning tree.

Rule 5: Each gear pair is located on a different turning edge level AND the distance between the centers should be maintained constant → there is one and only one planet carrier in each fundamental circuit defined by an edge corresponding to a gear pair.

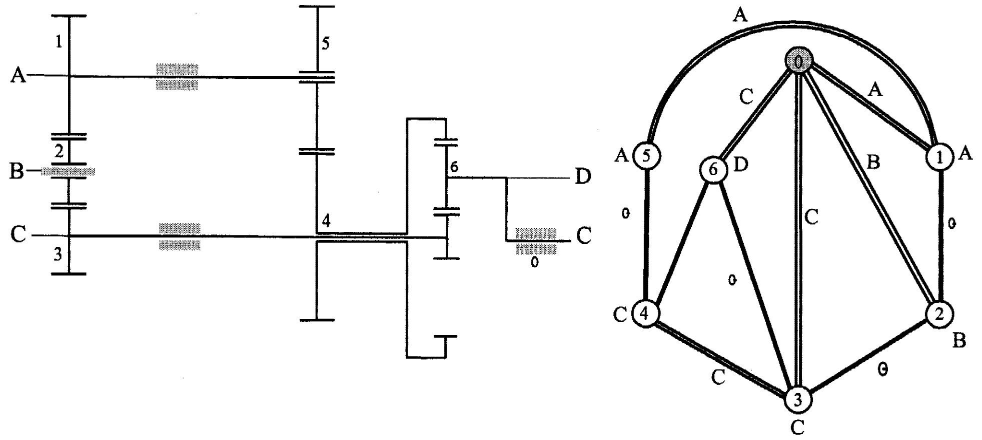
Topological conclusion 2: Rule 5 AND (planet carrier = local reference vertex) → in each fundamental circuit, there is one and only one local reference vertex.

Rule 6: The geometric center of a gear wheel and its local center of rotation must coincide → in each fun-



(a) (b)

Fig. 28. Example of a dynamic system which is not valid.



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The system is not valid because there is a contradiction with  rule 4.
In circuits {6,0,3} and {6,0,3,4} there is no local reference vertex.
The level of the turning edge (06) is 'c' while the level of vertex 6
is 'd', which contradicts rule 6.

The explanation to the user is: The connection between wheels 6 and 3
is not legal because the distance between their centers is zero.
The same problem occurs with the connection between wheels 6 and 4.
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Fig. 29. Example of checking the validity of planetary gear system, with the computer program output.

damental circuit, the levels of the vertex representing a gear wheel and the turning edge incident to it must be identical.

Example of deducing the validity of planetary gear system. A computer program for checking the validity of planetary gear systems that is based on this representation has been developed (Polomodov & Gershon, 1995; Preiss & Shai, 1996). An example of its output to a test case with a verbal explanation to the user is given in Figure 29. In addition, it is possible to arrange the computer program to advise the designer what to change in the gear kinematic chain in order to make it valid.

6.4. Checking the validity of constraint systems in CAD using the flow graph representation (FGR)

One of the main topics in CAD system research is the problem of checking whether a geometric constraints system is well defined (valid). In other words, to determine whether the given geometric form is uniquely and validly defined. It was found, according to Owen (1996) and Hoffmann (1995) that such a constraint system can be represented by a special graph. As will be shown later, in the terminology adopted in MCA, such a graph is actually FGR since it possesses the same properties. The steps for representing the geometric constraint system by a flow graph are as follows. Each element of the geometric system, such as line, point, arc, etc., is represented in the graph by a vertex and each constraint by an edge. The edge connecting two vertices corresponds to the constraint imposed on the corresponding two elements.

THEOREM 5. *Geometric constraint system is valid (well constrained) if and only if its corresponding graph is rigid in two-dimensional space.* ■

Proof: To clarify the proof, suppose that all the graphic elements are points, infinite lines, and circles. For each of these elements, exactly two parameters are needed in order to locate them in the plane. According to the construction of the graph (Hoffmann, 1995), each element is represented by a vertex and each constraint by an edge. Thus the number of the parameters needed to describe a geometric constrained system or any of its subsystems is equal to $2 \cdot v(G)$. To determine all these parameters, one needs to have the same number of independent constraints minus three (number three is dictated by freedom of location in a plane of the whole system). A necessary condition for the system to be uniquely defined is that this number should be equal to the number of constraints. Since each constraint is represented by an edge in the graph, the necessary condition becomes

$$2 \cdot v(G) - 3 = e(G). \tag{39}$$

For this condition to become sufficient, equality (39) is to be valid not only for G as one whole, but also for any subgraph of G.

On the basis of Section 6.1, the graph possessing such a quality is actually an FGR. A more detailed proof can be found in Owen (1996) and Hoffmann (1995). ■

Hence, the process of checking the validity of a geometric constraint system is as follows: 1) build the FGR representing the geometric constrained system; 2) check the rigidity of the graph using methods explained in Section 6.1.

Moreover, since FGR is dual to PGR (Section 3.1.1) and the latter corresponds to a mechanism, instead of checking the rigidity of the FGR, one can check the mobility of the mechanism which is dual to the truss represented by the FGR.

Consider for example, the geometric constraints system of Figure 30. In the geometric constraint system presented in Figure 30a there are eight elements: four straight lines and four points: 1,2,3,4 and A,B,C,D. There are 13 constraints: 8 for the interconnection between the elements designated in the graph by *i*; 1 for the distance between points C and D; 2 for the angle between lines; 2 for the distance between points and lines. Checking whether the given data defines a well-constrained geometric system in Figure 30a requires first building the corresponding flow graph. Each element is represented by a vertex and each constraint by an edge. The graph corresponding to the problem given in Figure 30a appears in Figure 30b.

As was explained above, checking the rigidity of a truss can be performed by employing one of the two methods given in Section 6.1, for instance, the two edge disjoint spanning trees method.

It can be verified that the graph in Figure 30b is not rigid, that is, the geometric constraint system in Figure 30a is not well defined. This conclusion is easily derived through the dual mechanism shown in Figure 30c. This mechanism is locked in the given position, since the continuations of links meet at the same point.

6.5. Employing the connections between the CR in checking the validity of engineering systems

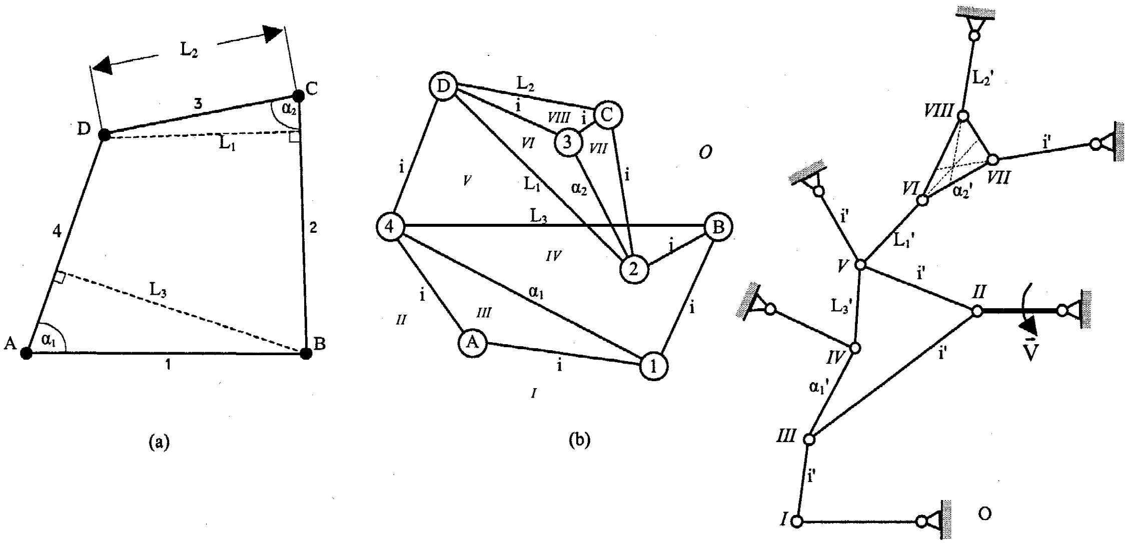
One of the main contributions of MCA is the ability that it provides to use knowledge from one field in the other on the basis of the connections between the CR. This ability is used in this section to turn the validity checking of mechanism mobility into checking the stability of its dual truss and vice versa. This new way was made possible by applying the knowledge and theorems from machine theory to structural analysis and vice versa.

6.5.1. Using the duality relation to check the stability of determinate trusses

On the basis of the mutual dualism between trusses and mechanisms (Section 3.1.2), one can deduce the following necessary and sufficient rule for checking stability of trusses and mobility of mechanisms.

Dualism validity rule: Determinate truss is valid if and only if its dual mechanism is valid, or in other words: determinate truss is stable if and only if its dual mechanism is mobile.

Hence, instead of checking the stability of a truss directly, one can build its dual mechanism and the problem will become a problem of checking the mobility of a mechanism. In many cases, checking the mobility of mechanisms can be carried out quickly by applying known theorems and algorithms from mechanism and machine theory.



(a) (b) (c)

Fig. 30. Geometric constraints system and the corresponding graph. (a) Geometric constraints system. (b) Corresponding graph. (c) The dual mechanism.

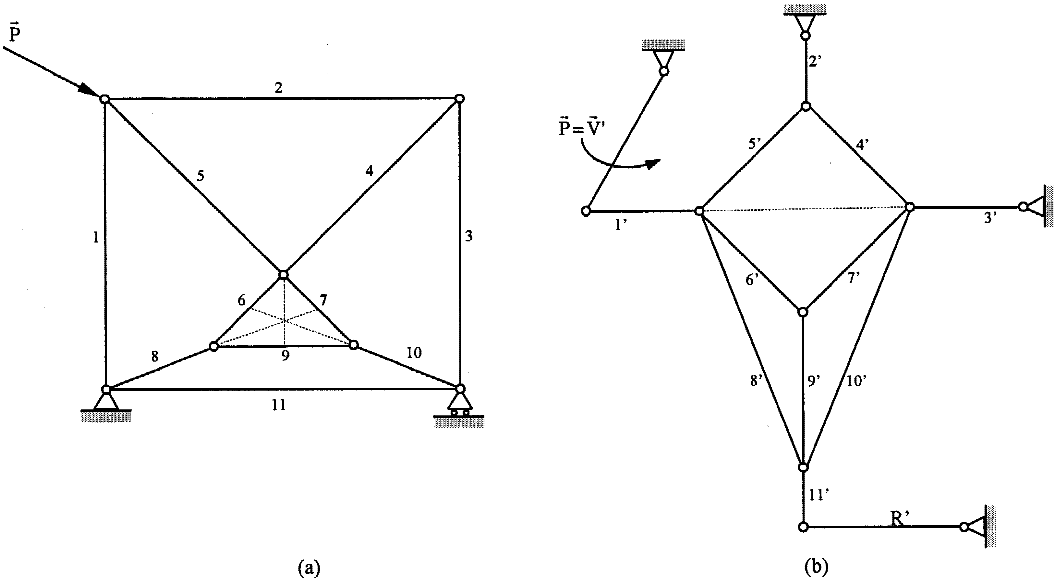


Fig. 31. Example of checking the validity of a truss by utilizing its dual mechanism. (a) A truss. (b) Its dual mechanism.

Consider as an example the truss presented in Figure 31a. While it is not easy to verify its stability, its dual mechanism in Figure 31b is obviously stuck since links $1'$ and $3'$ are located on the same line. Therefore, the original truss is not valid, that is, not stable.

6.5.2. *Checking the mobility of mechanisms using structural analysis*

The previous section adopted the principle that a truss is valid if and only if its dual mechanism is also valid. The principle was used in order to check the stability of trusses, by checking the mobility of their dual mechanisms instead. This subsection demonstrates the second possibility: it checks the mobility of a mechanism by means of the stability of its dual truss.

Consider, for example, the mechanism presented in Figure 32a. It is difficult, even for experts in this field, to decide whether the mechanism in Figure 32a is mobile or stuck. On the other hand, its dual truss, presented in Figure 32b, obviously possesses redundancy in its right part,

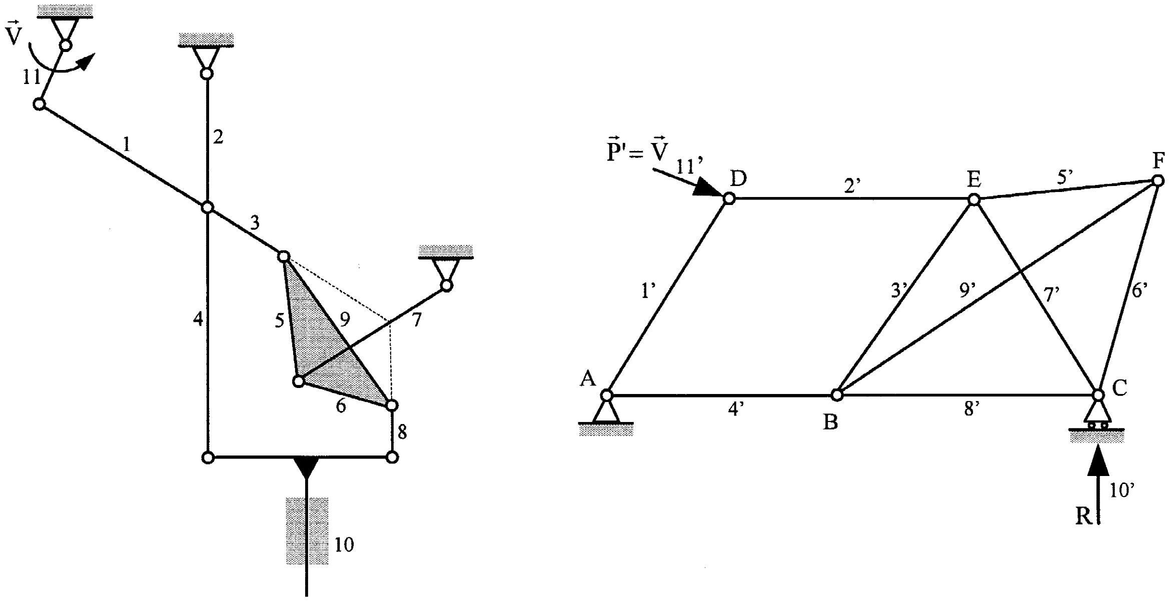
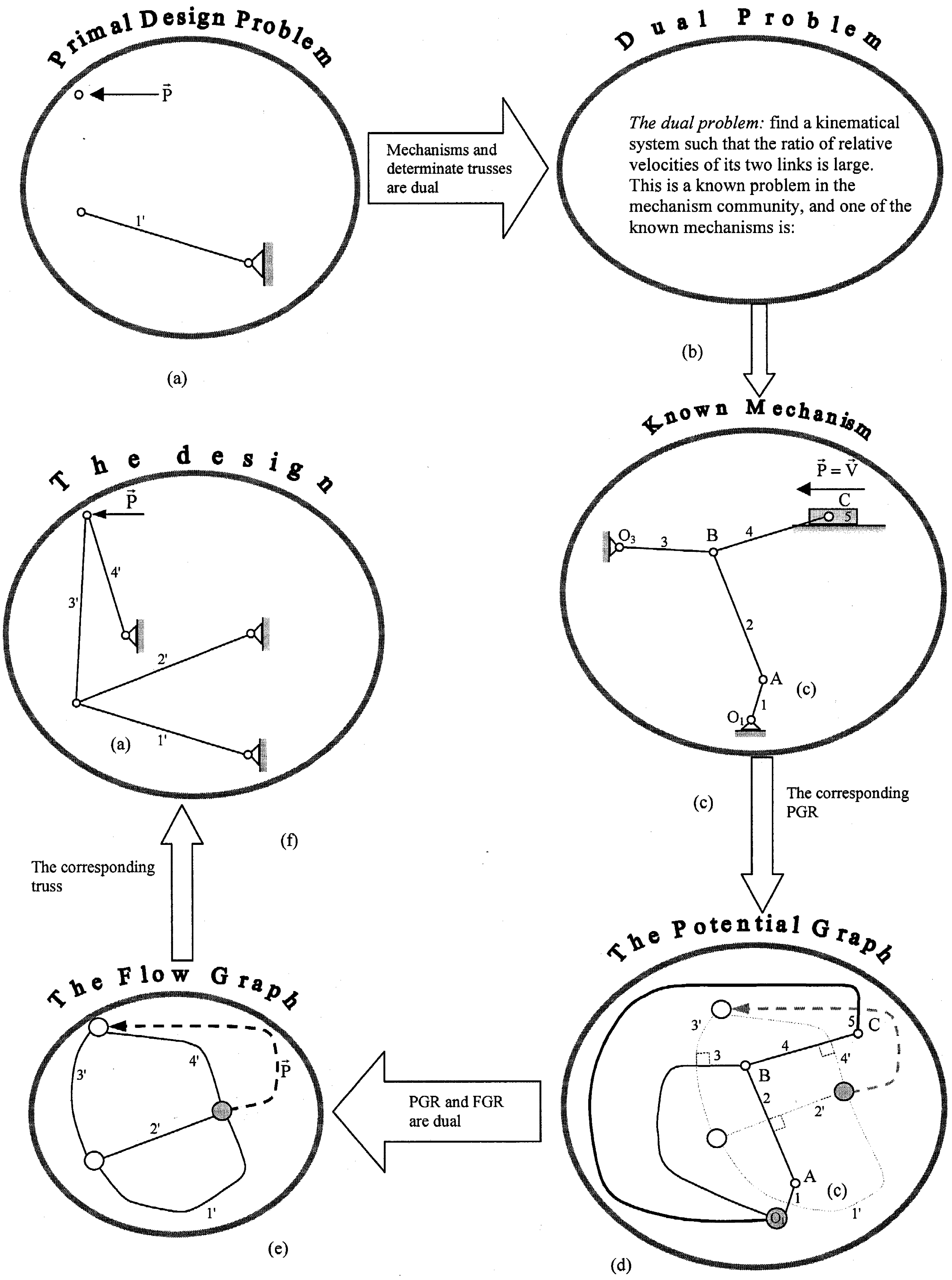


Fig. 32. Checking the validity of a mechanism by checking the validity of its dual truss. (a) A mechanism. (b) Its dual truss.



The dual problem: find a kinematical system such that the ratio of relative velocities of its two links is large. This is a known problem in the mechanism community, and one of the known mechanisms is:

Fig. 33. Employing truss-mechanism duality in design—diagram showing the systematic creativity technique applied to a design problem.

thus its left part lacks rods and hence the whole truss is unstable. Therefore, the original mechanism of Figure 32a is not valid, that is, stuck.

7. USING MCA FOR DESIGN

This section introduces a new application of this approach—to design. Development of this topic has started lately, and only a brief glance into its implementation is introduced in this section.

As was explained above, the ability to share knowledge between different fields is achieved by exploiting the connections between the individual CR. This section shows a new application of this property, which is developing a new technique in design. The example given here is based on using the connection between trusses and mechanisms, established in Section 3.

The main idea behind this approach lies in the fact that when a mechanism has a special property, its dual truss should possess the same property, and vice versa. This idea is demonstrated in the following small example.

Suppose one needs to design a truss, such that when a small force is applied to one of its joints, a magnified force is produced in a specific rod. Applying the approach transforms the problem into a problem of creating the dual mechanism.

The process starts with looking for a known mechanism which has similar velocity characteristics, namely, a mechanism that for a small relative velocity in its driving link produces in its other link a magnified relative velocity. One of many mechanisms satisfying this requirement is presented in Figure 33c. The relative velocity of link 1 of this mechanism is considerably larger than that of the link 5. After building the PGR (Fig. 33d) representing the mechanism and its dual FGR (Fig. 33e), the truss we were seeking is produced by reconstruction from the FGR (Fig. 33f).

According to the duality property, the truss possesses the same force characteristics as the velocity characteristics of the mechanism, that is, a small external force F causes a much greater force in the rod 1. Note, that the solution to the design problem was obtained through applying deterministic steps, thus giving rise to a new direction in systematic creativity.

8. CONCLUSIONS

This paper has introduced the idea behind the MCA (Multidisciplinary Combinatorial Approach), which was implemented as follows: First, Combinatorial Representations (CR) were developed and the properties of each and their interrelations were thoroughly investigated. Afterwards, these CR were applied to represent engineering problems from different fields, which gave rise to interesting results. Some of these results appeared in this paper.

This paper has shown that representing engineering problems by CR enables us to get a general perspective on different engineering fields. Moreover, new relations between engineering fields have been derived. This issue has been demonstrated by introducing a new connection between mechanisms and trusses, which had been derived from the relation between their corresponding CR: PGR and FGR. The theorems and methods embedded in the CR have been found to be valuable both for theoretical research and practical applications. From the theoretical point of view, they enable derivation of theorems and methods in engineering. For instance, known methods, such as the displacement method in structures, have been proven to be special cases of the methods embedded in RGR. On this basis, new connections between known methods have been derived. From the practical point of view, this enables application of knowledge, algorithms, and methods from one field in another.

The knowledge embedded in the representations has been applied to analyze trusses and to check the validity of dynamic systems, planetary gear systems, trusses, and geometric constraint systems in CAD. In addition, this paper has introduced a general perspective that enables representation of integrated multidisciplinary systems as one whole and means to deal with them in a unified way.

The concept of MCA has reaffirmed the postulate that when encountering a difficult problem, an effective solution strategy is to change its representation so as to make its solution transparent.

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