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Truncations of generalized shift-invariant systems

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Abstract. We provide conditions under which a generalized shift-invariant (GSI) system can be approximated by a GSI system for which the generators have compact support in the Fourier domain. The approximation quality will be measured in terms of the Bessel bound (upper frame bound) for the difference between the two GSI systems. In particular, this leads to easily verifiable conditions for a perturbation of a GSI system to preserve the frame property.

1 Introduction

For $a \in \mathbb{R}$, consider the translation operator T_a acting on $L^2(\mathbb{R})$ by $(T_a f)(x) = f(x-a)$. Recall that a *generalized shift-invariant system* (GSI system for short) is a system of functions of the form $\{T_{c_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}$, where J is a countable index set, $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R})$, and $\{c_j\}_{j\in J} \subset \mathbb{R}_+$. More information can be found in the papers [5, 7].

In applications of GSI systems, we typically need that the functions ϕ_j have compact support, either in the time domain or in the frequency domain. The purpose of this paper is to derive conditions under which a given GSI system $\{T_{c_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}$ can be approximated by a GSI system $\{T_{c_j k}\widetilde{\phi}_j\}_{j\in J, k\in\mathbb{Z}}$ with generators $\widetilde{\phi}_j$ having compact support in the Fourier domain. The *approximation quality* will be measured in terms of the Bessel bound (lower frame bound) for the system $\{T_{c_j k}(\phi_j - \widetilde{\phi}_j)\}_{j\in J, k\in\mathbb{Z}}$. Recall that $B > 0$ is a Bessel bound for $\{T_{c_j k}(\phi_j - \widetilde{\phi}_j)\}_{j\in J, k\in\mathbb{Z}}$ if

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}} |\langle f, T_{c_j k}(\phi_j - \widetilde{\phi}_j)\rangle|^2 \le B\|f\|^2, \forall f \in L^2(\mathbb{R}). \tag{1.1}$$

The rationale behind this condition is that it provides stability. For example, if the system $\{T_{c_j k}\phi\}_{j\in J, k\in\mathbb{Z}}$ is a frame, small values of the Bessel bound imply that $\{T_{c_j k}\widetilde{\phi}_j\}_{j\in J, k\in\mathbb{Z}}$ is also a frame, and that the synthesis operators and frame operators for the two systems are close (see [3] for a more detailed discussion).

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We will show that under suitable conditions, this Bessel bound can be controlled in terms of parameters associated with the given GSI system, and also be made arbitrarily small by appropriate choices of the approximating functions $\widetilde{\phi}_j$. This places our paper in the context of classical approximation theory for frames; indeed, assuming that the given GSI system is a frame, the results provide us with conditions ensuring that also the approximating GSI system will be a frame.

The paper is organized as follows. In the rest of the introduction, we set the stage by collecting results we need from the literature and providing further motivation. The new results appear in Section 2. First, in Section 2.1, we consider the case of a system $\{T_{ck}\phi\}_{k\in\mathbb{Z}}$ generated by a single function ϕ. The results in Section 2.1 are based on a decay condition on the Fourier transform $\widehat{\phi}$ of the function ϕ, and no relation between the parameter c and the function ϕ is required. In Section 2.2, we obtain general results for GSI systems $\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}$ by applying Section 2.1 to the functions ϕ_j "one by one." The advantage of the obtained results is that they are very general: they apply to any GSI system where the functions ϕ_j satisfy individual decay conditions, and still no relationship between ϕ_j and the parameters c_j is required. In Section 2.3, a different type of decay condition, involving as well the functions ϕ_j as the parameters c_j, is introduced. For GSI systems $\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}$ satisfying the introduced condition, we show how to obtain an alternative way of approximation, which simultaneously takes care of all the generators $\{\phi_j\}_{j\in J}$ in the GSI system. In particular, we prove that the results apply to wavelet-type systems. Finally, in Section 2.4, we analyze GSI systems of the same type, assuming now that the set of parameters $\{c_j\}_{j\in J}$ is relatively separated.

We define the Fourier transform of $f \in L^1(\mathbb{R})$ by

$$\widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\gamma} dx,$$

with the usual extension to $L^2(\mathbb{R})$. Our approach is based on the following result; it originally appeared in [4] as a generalization of a result in [6].

Lemma 1.1 *Consider a GSI system $\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}$, and assume that*

$$B := \sup_{\gamma\in\mathbb{R}} \sum_{j\in J}\sum_{k\in\mathbb{Z}} \frac{1}{c_j} |\widehat{\phi_j}(\gamma)\widehat{\phi_j}(\gamma - k/c_j)| < \infty. \tag{1.2}$$

Then $\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}$ is a Bessel sequence with bound B.

Given a GSI system $\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}$, the approximating GSI system will be defined by considering functions $\widetilde{\phi}_j$ that are defined via suitable truncations of the functions ϕ_j; that is, the Fourier transform of $\widetilde{\phi}_j$ will have the form $\widehat{\phi}_j\chi_{K_j}$ for some compact set $K_j \subset \mathbb{R}$. We will estimate the Bessel bound of the GSI system $\{T_{c_jk}(\phi_j - \widetilde{\phi}_j)\}_{j\in J,k\in\mathbb{Z}}$. Usually, the exact truncation will depend on the index j, as illustrated by the following example.

Example 1.2 Consider the GSI system $\{T_{c_jk}\phi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, where $\phi(x) := e^{-x^2}$ and $c_j = 1$ for all $j \in \mathbb{N}$; that is, the system is an infinite repetition of the shift-invariant

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system $\{T_k\phi\}_{k\in\mathbb{Z}}$. Note that $\widehat{\phi}(\gamma)=\sqrt{\pi}e^{-\pi^2\gamma^2}$. Applying (1.1), in order to approximate the GSI system $\{T_{c_jk}\phi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ with a truncated GSI system $\{T_{c_jk}\widetilde{\phi}_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, within a given Bessel bound ε, we need to find functions $\widetilde{\phi}_j$ such that

$$\sup_{\|f\|=1}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}}|\langle f, T_k(\phi-\widetilde{\phi}_j)\rangle|^2\le\varepsilon. \tag{1.3}$$

The natural way to obtain (1.3) would be to approximate each system $\{T_k\phi\}_{k\in\mathbb{Z}}$ individually. In other words, for each $j\in\mathbb{N}$, we will consider a function $\widetilde{\phi}_j$ defined via $\widehat{\widetilde{\phi_j}}:=\widehat{\phi}_j\chi_{[-K_j,K_j]}$, and show that $K_j>0$ can be chosen such that

$$\sup_{\|f\|=1}\sum_{k\in\mathbb{Z}}|\langle f, T_k(\phi-\widetilde{\phi}_j)\rangle|^2\le\varepsilon/2^j. \tag{1.4}$$

Note that by Lemma 1.1,

$$\begin{aligned}\sup_{\|f\|=1}\sum_{k\in\mathbb{Z}}|\langle f, T_k(\phi-\widetilde{\phi}_j)\rangle|^2 &\le \sup_{\gamma\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\left|(\widehat{\phi}-\widehat{\widetilde{\phi_j}})(\gamma)(\widehat{\phi}-\widehat{\widetilde{\phi_j}})(\gamma-k)\right|\\ &\le \sup_{\gamma\in\mathbb{R}}\left|(\widehat{\phi}-\widehat{\widetilde{\phi_j}})(\gamma)\right|\left(\sup_{\gamma\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\left|(\widehat{\phi}-\widehat{\widetilde{\phi_j}})(\gamma-k)\right|\right)\\ &\le \sqrt{\pi}e^{-\pi^2K_j^2}\left(\sum_{k=0}^{\infty}\left|\widehat{\phi}(K_j+k)\right|+\sum_{k=0}^{\infty}\left|\widehat{\phi}(-K_j-k)\right|\right)\\ &\le 2\pi e^{-\pi^2K_j^2}\sum_{k=0}^{\infty}e^{-\pi^2(K_j+k)^2}\\ &\le 2\pi e^{-\pi^2K_j^2}\sum_{k=0}^{\infty}e^{-\pi^2(K_j^2+2kK_j)}\le\frac{2\pi e^{-2\pi^2K_j^2}}{1-e^{-2\pi^2K_j}}.\end{aligned}$$

Now, fix any $K\ge\max\{\ln(32/\varepsilon),\ln 32\}$ and take $K_j:=\sqrt{Kj/(2\pi^2)}$. A direct calculation shows that then (1.4) holds. Hence, we have obtained the desired estimate in (1.3).

Note that for this construction to work, it is essential that the support size of ϕ_j in the Fourier domain is allowed to depend on $j\in\mathbb{N}$; in fact, it is necessary that $K_j\to\infty$ as $j\to\infty$.

Let us finally remind the reader about a classical perturbation result (see, e.g., Corollary 22.1.5 in [2]), which shows the relevance of the results in the current paper in the frame context: it shows that a perturbation of a frame with a sufficiently small Bessel bound again yields a frame. For notational convenience, we formulate the result in the setting of a frame in a general Hilbert space.

Lemma 1.3 *Assume that $\{f_k\}_{k\in I}$ is a frame in the Hilbert space $\mathcal{H}$, with frame bounds A,B. Consider any sequence $\{g_k\}_{k\in I}$ in $\mathcal{H}$, and assume that $\{f_k-g_k\}_{k\in I}$ is a Bessel sequence with Bessel bound $R<A$. Then $\{g_k\}_{k\in I}$ is a frame with frame bounds $A(1-\sqrt{R/A})^2, B(1+\sqrt{R/B})^2$.*

It is known that perturbations in the sense of Lemma 1.3 preserve other key features, e.g., the excess of the frame.

2 The results

Motivated by the considerations in Example 1.2, we will apply the following convention in the entire paper.

Standing convention: Given $\phi_j \in L^2(\mathbb{R})$ and any $K_j > 0$, define the function $\widetilde{\phi}_j \in L^2(\mathbb{R})$ by

$$\widehat{\widetilde{\phi}_j} := \widehat{\phi}_j \chi_{[-K_j, K_j]}. \tag{2.1}$$

In the entire paper, we will use the short notation $\Phi := \{T_{c_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}$ and $\widetilde{\Phi} := \{T_{c_j k}\widetilde{\phi}_j\}_{j\in J, k\in\mathbb{Z}}$. Furthermore, we will denote the (optimal) Bessel bound for $\{T_{c_j k}(\phi_j - \widetilde{\phi}_j)\}_{j\in J, k\in\mathbb{Z}}$ by $B(\Phi, \widetilde{\Phi})$.

With this notation, our goal is to estimate $B(\Phi, \widetilde{\Phi})$ in terms of parameters related to the given GSI system Φ. In particular, for the considered GSI systems, we will provide explicit values for the cutoff points K_j ensuring that $B(\Phi, \widetilde{\Phi})$ stays below a desired threshold.

2.1 The case $\{T_{ck}\phi\}_{k\in\mathbb{Z}}$

We will first consider the case of a (shift-invariant) system generated by a single function ϕ, i.e., a system of the form $\Phi = \{T_{ck}\phi\}_{k\in\mathbb{Z}}$ for some $\phi \in L^2(\mathbb{R})$ and some $c > 0$.

Theorem 2.1 *For a shift-invariant system $\Phi = \{T_{ck}\phi\}_{k\in\mathbb{Z}}$, assume that there exist $E > 0$ and $\sigma > 0$ such that*

$$|\widehat{\phi}(\gamma)| \leq \frac{E}{1 + |\gamma|^{1+\sigma}}, \ \gamma \in \mathbb{R}. \tag{2.2}$$

Let $K \geq \sigma/c$ and consider a function $\widetilde{\phi}$ defined via $\widehat{\widetilde{\phi}} = \widehat{\phi}\chi_{[-K,K]}$. Then, with $\widetilde{\Phi} = \{T_{ck}\widetilde{\phi}\}_{k\in\mathbb{Z}}$,

$$B(\Phi, \widetilde{\Phi}) \leq \frac{4E^2}{\sigma K^{1+2\sigma}}. \tag{2.3}$$

Proof By Lemma 1.1,

$$\begin{aligned} B(\Phi, \widetilde{\Phi}) &\leq \sup_{\gamma\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \left| c^{-1}(\widehat{\phi} - \widehat{\widetilde{\phi}})(\gamma)(\widehat{\phi} - \widehat{\widetilde{\phi}})(\gamma - k/c) \right| \\ &\leq \left(\sup_{\gamma\in\mathbb{R}} \left| c^{-1}(\widehat{\phi} - \widehat{\widetilde{\phi}})(\gamma) \right| \right) \left(\sup_{\gamma\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \left| (\widehat{\phi} - \widehat{\widetilde{\phi}})(\gamma - k/c) \right| \right). \end{aligned}$$

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Let $u(\gamma) := E/(1+|\gamma|^{1+\sigma})$. Then

$$\begin{aligned} B(\Phi,\widetilde{\Phi}) &\le \frac{E}{cK^{1+\sigma}}\left(\sum_{k=0}^{\infty} u(K+k/c) + \sum_{k=0}^{\infty} u(-K-k/c)\right) \\ &\le \frac{E}{cK^{1+\sigma}}\left(\sum_{k=0}^{\infty} \frac{2E}{(K+k/c)^{1+\sigma}}\right) \\ &\le \frac{2E^2}{cK^{1+\sigma}}\left(\frac{1}{K^{1+\sigma}} + c\int_K^{\infty} \frac{1}{x^{1+\sigma}}dx\right) \\ &\le \frac{2E^2}{cK^{1+\sigma}}\left(\frac{1}{K^{1+\sigma}} + \frac{c}{\sigma K^{\sigma}}\right) \\ &= \frac{2E^2}{\sigma K^{1+2\sigma}}\left(\frac{\sigma}{cK}+1\right). \\ &\le \frac{4E^2}{\sigma K^{1+2\sigma}}, \end{aligned}$$

where we used $K \ge \sigma/c$ in the last inequality. ∎

2.2 The general case $\{T_{c_j k}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$

The approach in Section 2.1 gives a natural way of approximating a general GSI system $\{T_{c_j k}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, subject to appropriate decay conditions on each of the generators ϕ_j. Indeed, applying Theorem 2.1 "one by one" to the functions ϕ_j, we obtain the following theorem.

Theorem 2.2 *For a GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, assume that for $j\in\mathbb{N}$, there exist $E_j > 0$ and $\sigma_j > 0$ such that*

$$|\widehat{\phi}_j(\gamma)| \le \frac{E_j}{1+|\gamma|^{1+\sigma_j}}, \ \gamma\in\mathbb{R}.$$

Then

$$B(\Phi,\widetilde{\Phi}) \le \sum_{j=1}^{\infty} \frac{4E_j^2}{\sigma_j K_j^{1+2\sigma_j}}. \tag{2.4}$$

In particular, if we for any given $\varepsilon > 0$ take

$$K_j \ge \max\left\{\frac{\sigma_j}{c_j}, \left(\frac{4E_j^2 2^j}{\varepsilon\sigma_j}\right)^{\frac{1}{1+2\sigma_j}}\right\}, \tag{2.5}$$

then

$$B(\Phi,\widetilde{\Phi}) \le \varepsilon.$$

Proof For $j\in\mathbb{N}$, let $\Phi_j := \{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}}$ and $\widetilde{\Phi_j} := \{T_{c_j k}\widetilde{\phi}_j\}_{k\in\mathbb{Z}}$. Then, by (2.5) and Theorem 2.1,

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$$B(\Phi_j,\widetilde{\Phi}_j) \le \frac{4E_j^2}{\sigma_j K_j^{1+2\sigma_j}}. \tag{2.6}$$

This immediately leads to (2.4). Finally, due to (2.5), we have that $B(\Phi_j,\widetilde{\Phi}_j) \le \frac{\varepsilon}{2^j}$ and hence $B(\Phi,\widetilde{\Phi}) \le \varepsilon$, as desired. ■

We will now provide an application of Theorem 2.2 to a Gabor-type system. For this purpose, we need the following elementary result.

Lemma 2.3 *Let $k \in \mathbb{R}$. Then*

$$\frac{1}{1+|\gamma-k|^2} \le \frac{2(1+k^2)}{1+|\gamma|^2}, \quad \forall \gamma \in \mathbb{R}.$$

Proof Let $f(\gamma) := 2(1+k^2)(1+(\gamma-k)^2) - (1+\gamma^2)$. It is enough to show that $f(\gamma) \ge 0$ for $\gamma \in \mathbb{R}$; this follows from

$$\begin{aligned} f(\gamma) &= 2 + 2k^2 + 2(\gamma-k)^2 + 2k^2(\gamma-k)^2 - 1 - \gamma^2 \\ &\ge 1 + 2k^2 + 2(\gamma-k)^2 - \gamma^2 \\ &= (\gamma-2k)^2 + 1 > 0. \end{aligned}$$

■

Example 2.4 Let $a, b > 0$ and assume that $g \in L^2(\mathbb{R})$ satisfies

$$|\widehat{g}(\gamma)| \le \frac{E}{1+|\gamma|^2}, \ \gamma \in \mathbb{R},$$

for some $E > 0$. Consider a GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, where $c_j = a$ for all $j \in \mathbb{N}$ and

$$\phi_j = \begin{cases} E_{\frac{(1-j)b}{2}} g, & j \in 2\mathbb{N}-1, \\ E_{\frac{jb}{2}} g, & j \in 2\mathbb{N}. \end{cases}$$

Then we see $\{T_{c_j k}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}} = \{T_{na}E_{mb}g\}_{m,n\in\mathbb{Z}}$. By Lemma 2.3, if $j \in 2\mathbb{N}-1$, then

$$\left|\widehat{\phi_j}(\gamma)\right| = \left|T_{\frac{(1-j)b}{2}}\widehat{g}(\gamma)\right| \le \frac{E}{1+|\gamma-\frac{(1-j)b}{2}|^2} \le \frac{2E(1+(\frac{(1-j)b}{2})^2)}{1+|\gamma|^2};$$

if $j \in 2\mathbb{N}$, then

$$\left|\widehat{\phi_j}(\gamma)\right| = \left|T_{\frac{jb}{2}}\widehat{g}(\gamma)\right| \le \frac{E}{1+|\gamma-\frac{jb}{2}|^2} \le \frac{2E(1+(\frac{jb}{2})^2)}{1+|\gamma|^2}.$$

A direct calculation shows that for $j \in \mathbb{N}$,

$$\left|\widehat{\phi_j}(\gamma)\right| \le \frac{2E(1+(\frac{(1-j)b}{2})^2+(\frac{jb}{2})^2)}{1+|\gamma|^2} \le \frac{E(2+(jb)^2)}{1+|\gamma|^2}.$$

Choose $E_j := E(2+(jb)^2)$. Let $\varepsilon > 0$ and for $j \in \mathbb{N}$, take K_j as in (2.5) with $\sigma_j = 1$. By Theorem 2.2, $B(\Phi,\widetilde{\Phi}) \le \varepsilon$, as desired.

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At first glance, it seems cumbersome that the result in Theorem 2.2 is obtained by considering each system $\Phi_j := \{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}}$ individually and requiring "better and better approximations for increasing j." However, to the best understanding of the authors, this is the only way a general result just based on decay conditions on ϕ_j can be obtained. Indeed, the general setting of a GSI system is very broad and as illustrated already by Example 1.2 the functions ϕ_j and the parameters c_j are in general unrelated, forcing us to make individual choices for the parameters K_j. In the next section, we will show that more universal ways of choosing the parameters K_j can be obtained by imposing decay conditions on the functions ϕ_j that also involve the parameters c_j.

2.3 A condition relating $\{c_j\}_{j\in J}$ and $\{\phi_j\}_{j\in J}$

In this section, we continue the analysis of a GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}}$, but we apply a different decay condition which involves as well the generators ϕ_j as the parameters c_j. We begin with a general result, which does not yet specify how to choose the "cutoff" points K_j in order to obtain a desired approximation. In the subsequent examples, we will see how to choose these parameters in a number of concrete cases, e.g., for wavelet-type examples.

Theorem 2.5 *For a GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}}$, assume that there exist $E > 0$ and $\sigma > 0$ such that for $j \in J$,*

$$\left|c_j^{-1/2}\widehat{\phi}_j(\gamma/c_j)\right| \leq \frac{E}{1+|\gamma|^{1+\sigma}},\ \gamma\in\mathbb{R}. \tag{2.7}$$

Consider now a truncated GSI system $\widetilde{\Phi} = \{T_{c_j k}\widetilde{\phi}_j\}_{j\in J,k\in\mathbb{Z}}$ as in the general setup, and let $J_i = \{j\in J \mid K_j \leq K_i\}$ for $i \in J$. Then

$$B(\Phi,\widetilde{\Phi}) \leq 2E^2\left(\sup_{j\in J}\frac{1}{\sigma(c_jK_j)^\sigma}\left(\frac{\sigma}{c_jK_j}+1\right)\right)\left(\sup_{i\in J}\frac{1}{(K_i)^{1+\sigma}}\sum_{k\in J_i}\frac{1}{(c_k)^{1+\sigma}}\right). \tag{2.8}$$

Proof For notational convenience, let $p_j(x) := \phi_j(x) - \widetilde{\phi}_j(x)$ for $j \in J$ and $u(\gamma) := E/(1+|\gamma|^{1+\sigma})$. We first note that $\widehat{p}_j(\gamma) = \widehat{\phi}_j(\gamma)\chi_{[-K_j,K_j]^c}(\gamma)$, and

$$\begin{aligned} B(\Phi,\widetilde{\Phi}) &\leq \sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}\left|c_j^{-1}\widehat{p}_j(\gamma)\widehat{p}_j(\gamma-k/c_j)\right| \\ &\leq \left(\sup_{\gamma\in\mathbb{R},j\in J}\sum_{k\in\mathbb{Z}}\left|c_j^{-1/2}\widehat{p}_j(\gamma-k/c_j)\right|\right)\left(\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\left|c_j^{-1/2}\widehat{p}_j(\gamma)\right|\right). \end{aligned} \tag{2.9}$$

We now proceed with a number of estimates that finally yields the result.

Estimate of $\sum_{k\in\mathbb{Z}}\left|c_j^{-1/2}\widehat{p}_j(\gamma-k/c_j)\right|$: Fix $\gamma\in\mathbb{R}$ and $j \in J$. Then the contribution from $\gamma - k/c_j$ hitting the interval $]-\infty,-K_j]\cup[K_j,\infty[$ is at most

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$$\sum_{k\in\mathbb{Z}}\left|c_j^{-1/2}\widehat{p_j}(\gamma-k/c_j)\right|\le\sum_{k=0}^{\infty}u(c_jK_j+k)+\sum_{k=0}^{\infty}u(-c_jK_j-k)$$
$$\le 2\sum_{k=0}^{\infty}\frac{E}{\left(c_jK_j+k\right)^{1+\sigma}}$$
$$\le\frac{2E}{(c_jK_j)^{1+\sigma}}+\int_{c_jK_j}^{\infty}\frac{2E}{t^{1+\sigma}}dt$$
$$=2E\left(\frac{1}{(c_jK_j)^{1+\sigma}}+\frac{1}{\sigma(c_jK_j)^{\sigma}}\right)$$
$$=\frac{2E}{\sigma(c_jK_j)^{\sigma}}\left(\frac{\sigma}{c_jK_j}+1\right). \tag{2.10}$$

This leads to

$$\sup_{\gamma\in\mathbb{R},j\in J}\sum_{k\in\mathbb{Z}}\left|c_j^{-1/2}\widehat{p_j}(\gamma-k/c_j)\right|\le\sup_{j\in J}\frac{2E}{\sigma(c_jK_j)^{\sigma}}\left(\frac{\sigma}{c_jK_j}+1\right). \tag{2.11}$$

Estimate of $\sum_{j\in J}\left|c_j^{-1/2}\widehat{p_j}(\gamma)\right|$ **:** Fix $\gamma\in\mathbb{R}\setminus\{0\}$. Note that $\widehat{p_j}(\gamma)=\widehat{\phi_j}(\gamma)\chi_{[-K_j,K_j]^c}(\gamma)$. If $|\gamma|\le K_j$ for all $j\in J$, then $\widehat{p_j}(\gamma)=0$. Now, assume that there exists $j\in J$ such that $K_j<|\gamma|$. Choose $j_0\in J$ such that $K_{j_0}=\max_{j\in J}\{K_j|\,K_j<|\gamma|\}$. Note that $J_{j_0}=\{j\in J|\,K_j\le K_{j_0}\}$. Then we have

$$\sum_{j\in J}\left|c_j^{-1/2}\widehat{p_j}(\gamma)\right|\le\sum_{j\in J_{j_0}}u(c_j\gamma)\le\sum_{j\in J_{j_0}}u(c_jK_{j_0})$$
$$\le\sum_{j\in J_{j_0}}^{\infty}\frac{E}{\left(c_jK_{j_0}\right)^{1+\sigma}}=\frac{E}{(K_{j_0})^{1+\sigma}}\sum_{j\in J_{j_0}}\frac{1}{(c_j)^{1+\sigma}}.$$

This leads to

$$\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\left|c_j^{-1/2}\widehat{p_j}(\gamma)\right|\le\sup_{i\in J}\frac{E}{(K_i)^{1+\sigma}}\sum_{j\in J_i}\frac{1}{(c_j)^{1+\sigma}}. \tag{2.12}$$

The result now follows from (2.9) together with (2.11) and (2.12). ■

We will now consider a number of concrete manifestations of Theorem 2.5. We first prove that for the case $c_j=j$, the decay condition (2.7) implies that we can choose a cutoff point K that is independent of $j\in J$.

Example 2.6 Consider a GSI system $\Phi=\{T_{c_jk}\phi_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, where $\{c_j\}_{j\in\mathbb{N}}=\{j\}_{j\in\mathbb{N}}$. Assume that ϕ_j, $j\in\mathbb{N}$, satisfies (2.7) with $E>0$ and $\sigma=1$. Let $K\ge1$ and take $\widetilde{\phi}_j\in L^2(\mathbb{R})$ as in (2.1) with $K_j=K$ for $j\in\mathbb{N}$. Then

$$J_i=\{j\in\mathbb{N}|\,K_j\le K_i\}=\mathbb{N},\ i\in\mathbb{N}.$$

By Theorem 2.5, we have

$$B(\Phi,\widetilde{\Phi}) \leq 2E^2\left(\sup_{j\in\mathbb{N}}\frac{1}{jK}\left(\frac{1}{jK}+1\right)\right)\left(\sup_{i\in\mathbb{N}}\frac{1}{K^2}\sum_{k\in\mathbb{N}}\frac{1}{k^2}\right)$$
$$= 2E^2\left(\frac{1}{K}\left(\frac{1}{K}+1\right)\right)\left(\frac{1}{K^2}\sum_{k\in\mathbb{N}}\frac{1}{k^2}\right) \leq \frac{2\pi^2E^2}{3K^3},$$

where we used $K \geq 1$ and $\sum_{k\in\mathbb{N}} 1/k^2 = \pi^2/6$ in the last inequality.

In the next example, we consider the case where $c_j = a^j$ for some $a \neq 1$. The result will be formulated for $a > 1$, but obviously we obtain a corresponding result for $a < 1$ by "running through the system in the opposite order." The example covers classical wavelet-type examples. Indeed, defining the scaling operator $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $(D_a f)(x) = a^{1/2} f(ax)$, the wavelet system $\{D_{a^j} T_k \phi\}_{j,k\in\mathbb{Z}}$ equals the GSI system $\{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}}$, where $c_j = a^{-j}$ and $\phi_j = D_{a^j}\phi$. In this particular case, the decay conditions stated in condition (2.7) reduce to the single condition

$$|\widehat{\phi}(\gamma)| \leq \frac{E}{1+|\gamma|^{1+\sigma}},\ \gamma \in \mathbb{R}. \tag{2.13}$$

Note that this condition appeared already in (2.2).

Example 2.7 Consider a GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$, where $\{c_j\}_{j\in\mathbb{Z}} = \{a^{-j}\}_{j\in\mathbb{Z}}$ for some $a > 1$. Assume that ϕ_j, $j \in \mathbb{Z}$, satisfies (2.7) with $E > 0$ and $\sigma = 1$. Let $K > 0$ and take $\widetilde{\phi}_j \in L^2(\mathbb{R})$ as in (2.1) with $K_j = a^j K$ for $j \in \mathbb{Z}$. Then

$$J_i = \{j \in \mathbb{Z} | K_j \leq K_i\} = \{j \in \mathbb{Z} | j \leq i\},\ i \in \mathbb{Z}.$$

By Theorem 2.5, we have

$$B(\Phi,\widetilde{\Phi}) \leq 2E^2\left(\sup_{j\in\mathbb{Z}}\frac{1}{K}\left(\frac{1}{K}+1\right)\right)\left(\sup_{i\in\mathbb{Z}}\frac{1}{(a^iK)^2}\sum_{k=-\infty}^{i}\frac{1}{(a^{-k})^2}\right)$$
$$\leq \frac{4E^2}{K}\left(\frac{1}{K^2}\sum_{k=0}^{\infty}\frac{1}{(a^2)^k}\right) = \frac{4E^2a^2}{K^3(a^2-1)}.$$

This example shows that Theorem 2.5 generalizes previous results from the literature (see [1]).

2.4 Relatively separated sets $\{c_j\}_{j\in J}$

In this section, we will continue the analysis of GSI systems $\Phi = \{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}}$ satisfying the decay condition (2.7); however, we will apply different assumptions on the set $\{c_j\}_{j\in J}$.

Recall that a set $\{c_j\}_{j\in J} \subset \mathbb{R}_+$ is *separated* if $\inf_{i,j\in J, i\neq j} |c_i - c_j| > 0$. The set $\{c_j\}_{j\in J}$ is *relatively separated* if it is a finite union of separated sets.

Let us now consider a relatively separated set $\{c_j\}_{j\in J} \subset \mathbb{R}_+$ and a corresponding GSI system $\Phi = \{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}}$; that is, for some $N \in \mathbb{N}$, we can write

$$\{T_{c_j k}\phi_j\}_{j\in J,k\in\mathbb{Z}} = \bigcup_{\ell=1}^{N} \{T_{c_j^{\ell} k}\phi_j^{(\ell)}\}_{j\in J^{(\ell)},k\in\mathbb{Z}}, \tag{2.14}$$

where $\{c_j^{(\ell)}\}_{j\in J^{(\ell)}}$ is separated for each $\ell \in \{1,2,\dots,N\}$. Let δ be a joint *separation constant,* meaning that for each $\ell \in \{1,2,\dots,N\}$, we have $|c_i^{(\ell)} - c_j^{(\ell)}| \geq \delta$ for $i \neq j$. This setup implies that for each $\ell \in \{1,2,\dots,N\}$, any interval of the form $[n\delta, (n+1)\delta[$, $n = 0,1,2,\dots$, contains *at most* one parameter $c_i^{(\ell)}$. For technical reasons, we will now consider an *enlarged* GSI system where each such interval contains *precisely* one parameter $c_i^{(\ell)}$.

Definition 2.8 Consider a GSI system Φ of the form (2.14), with separation constant $\delta > 0$. The corresponding enlarged GSI system $\{T_{c_j k}\phi_j\}_{j\in\widetilde{J},k\in\mathbb{Z}}$ is defined by adding parameters c_j, $j \in \widetilde{J}\setminus J$ and corresponding functions ϕ_j as follows:

(i) If for some $\ell \in \{1,2,\dots,N\}$ an interval of the form $[(n-1)\delta, n\delta[$, $n \in \mathbb{N}$, does not contain any parameter c_i^{ℓ}, define a parameter c_j, $j \in \widetilde{J}\setminus J$, by adding the point $n\delta - \delta/2$.

(ii) For $j \in \widetilde{J}\setminus J$, let $\phi_j := 0$.

Formulated in words, the enlarged GSI system contains all the elements of the given GSI system, plus a number of added zero function. We immediately obtain the following consequence of this observation.

Lemma 2.9 *Consider a GSI system Φ of the form (2.14), with separation constant $\delta > 0$. Then the corresponding enlarged set of parameters $\{c_j\}_{j\in\widetilde{J}}$ is relatively separated with separation constant $\delta/2$. The GSI systems $\{T_{c_j k}(\phi_j - \widetilde{\phi}_j)\}_{j\in J,k\in\mathbb{Z}}$ and $\{T_{c_j k}(\phi_j - \widetilde{\phi}_j)\}_{j\in\widetilde{J},k\in\mathbb{Z}}$ have identical Bessel bounds.*

We will now prove that for a GSI system of the form (2.14), with separation constant $\delta > 0$, our decay condition implies that a uniform cutoff point can be chosen for the functions ϕ_j, $j \in J$.

Theorem 2.10 *Consider a GSI system of the form (2.14), with separation constant $\delta > 0$. Let $\delta_0 = \min_{j\in J}\{\delta/2, c_j\}$. Assume that ϕ_j, $j \in J$, satisfies (2.7) with $E > 0$ and $\sigma > 0$, that is,*

$$\left| c_j^{-1/2}\widehat{\phi}_j(\gamma/c_j)\right| \leq \frac{E}{1+|\gamma|^{1+\sigma}},\ \gamma \in \mathbb{R}.$$

Let $K \geq \sigma/\delta_0$ and for each $j \in \mathbb{N}$, consider a function $\widetilde{\phi}_j$ defined via $\widehat{\widetilde{\phi}_j} := \widehat{\phi}_j\chi_{[-K,K]}$. Then

$$B(\Phi,\widetilde{\Phi}) \leq \frac{4NE^2}{\sigma(\delta_0 K)^{1+2\sigma}}\left(2 + \frac{1}{2\delta_0}\right). \tag{2.15}$$

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Proof Consider the corresponding enlarged GSI system $\bigcup_{\ell=1}^{N}\{T_{c_j^{(\ell)}k}\phi_j^{(\ell)}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ with separation constant $\delta/2$. For the first interval, instead of $[0,\delta/2[$, if $\delta_0<\delta/2$, then we take $[\delta_0,\delta/2[$; if $\delta_0=\delta/2$, then we take $\{\delta/2\}$. We then note that $c_1^{(\ell)}\in[\delta_0,\delta/2]$ and $c_k^{(\ell)}\in[(k-1)\delta/2,k\delta/2[$ for $k=2,3,\dots$. Fix $\ell\in\{1,2,\dots,N\}$ and let $\Phi^{(\ell)}:=\{T_{c_j^{(\ell)}k}\phi_j^{(\ell)}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ and $\widetilde{\Phi}^{(\ell)}:=\{T_{c_j^{(\ell)}k}\widetilde{\phi}_j\}_{j\in\mathbb{N},k\in\mathbb{Z}}$. By Theorem 2.5, we have

$$
\begin{aligned}
B(\Phi^{(\ell)},\widetilde{\Phi}^{(\ell)}) &\le 2E^2\left(\sup_{j\in\mathbb{N}}\frac{1}{\sigma(c_j^{(\ell)}K)^{\sigma}}\left(\frac{\sigma}{c_j^{(\ell)}K}+1\right)\right)\left(\sup_{i\in\mathbb{N}}\frac{1}{K^{1+\sigma}}\sum_{k\in\mathbb{N}}\left(\frac{1}{c_k^{(\ell)}}\right)^{1+\sigma}\right)\\
&\le 2E^2\left(\frac{1}{\sigma(\delta_0K)^{\sigma}}\left(\frac{\sigma}{\delta_0K}+1\right)\right)\left(\frac{1}{K^{1+\sigma}}\left(\frac{1}{\delta_0^{1+\sigma}}+\sum_{k=2}^{\infty}\left(\frac{2}{(k-1)\delta}\right)^{1+\sigma}\right)\right)\\
&\le \frac{4E^2}{\sigma(\delta_0K)^{\sigma}}\left(\frac{1}{K^{1+\sigma}}\left(\frac{1}{\delta_0^{1+\sigma}}+\left(\frac{2}{\delta}\right)^{1+\sigma}\sum_{k=1}^{\infty}\frac{1}{k^{1+\sigma}}\right)\right)\\
&\le \frac{4E^2}{\sigma(\delta_0K)^{\sigma}}\left(\frac{1}{K^{1+\sigma}}\left(\frac{1}{\delta_0^{1+\sigma}}+\left(\frac{2}{\delta}\right)^{1+\sigma}\left(1+\frac{1}{\delta}\right)\right)\right)\\
&\le \frac{4E^2}{\sigma(\delta_0K)^{\sigma}}\left(\frac{1}{K^{1+\sigma}}\left(\frac{1}{\delta_0^{1+\sigma}}+\frac{1}{\delta_0^{1+\sigma}}\left(1+\frac{1}{2\delta_0}\right)\right)\right)\\
&= \frac{4E^2}{\sigma(\delta_0K)^{1+2\sigma}}\left(2+\frac{1}{2\delta_0}\right).
\end{aligned}
$$

Since $B(\Phi,\widetilde{\Phi})\le\sum_{\ell=1}^{N}B(\Phi^{(\ell)},\widetilde{\Phi}^{(\ell)})$, (2.15) holds. ∎

Example 2.11 Let $N\in\mathbb{N}$. Consider a GSI system

$$\{T_{c_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}}=\bigcup_{\ell=1}^{N}\{T_{c_j^{(\ell)}k}\phi_j^{(\ell)}\}_{j\in\mathbb{N},k\in\mathbb{Z}},$$

where $c_j^{(\ell)}\in[j,j+1[$ for $j\in\mathbb{N}$ and $\ell\in\{1,2,\dots,N\}$. Assume that $\phi_j^{(\ell)}$, $j\in\mathbb{N}$ and $\ell\in\{1,2,\dots,N\}$, satisfies (2.7) with $E>0$ and $\sigma=1$. That is,

$$\left|\left(c_j^{(\ell)}\right)^{-1/2}\widehat{\phi_j^{(\ell)}}\left(\gamma/c_j^{(\ell)}\right)\right|\le\frac{E}{1+|\gamma|^{1+\sigma}},\ \gamma\in\mathbb{R}.$$

Let $K\ge1$ and take $\widetilde{\phi}_j\in L^2(\mathbb{R})$ as in (2.1) with $K_j=K$ for $j\in\mathbb{N}$. Then $J_i=\mathbb{N}$, $i\in\mathbb{N}$. Fix $\ell\in\{1,2,\dots,N\}$. By Theorem 2.5, we have

$$
\begin{aligned}
B(\Phi^{(\ell)},\widetilde{\Phi}^{(\ell)}) &\le 2E^2\left(\sup_{j\in\mathbb{N}}\frac{1}{c_j^{(\ell)}K}\left(\frac{1}{c_j^{(\ell)}K}+1\right)\right)\left(\sup_{i\in\mathbb{N}}\frac{1}{K^2}\sum_{k\in\mathbb{N}}\frac{1}{(c_k^{(\ell)})^2}\right)\\
&\le 2E^2\left(\sup_{j\in\mathbb{N}}\frac{1}{jK}\left(\frac{1}{jK}+1\right)\right)\left(\sup_{i\in\mathbb{N}}\frac{1}{K^2}\sum_{k\in\mathbb{N}}\frac{1}{k^2}\right)\\
&\le \frac{2\pi^2E^2}{3K^3}.
\end{aligned}
$$

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Hence,

$$B(\Phi, \widetilde{\Phi}) \leq \sum_{\ell=1}^{N} B(\Phi^{(\ell)}, \widetilde{\Phi}^{(\ell)}) \leq \frac{2\pi^2 N E^2}{3K^3}.$$

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