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# ON CERTAIN INEQUALITIES FOR SOME REGULAR FUNCTIONS DEFINED ON THE UNIT DISC

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In this paper we obtain some inequalities for some regular functions $f$ defined on the unit disc. Our results include or improve several previous results.

## 1. Introduction.

Let $A(p)$ denote the class of function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are regular in the unit disc $E = \{z : |z| < 1\}$. If $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ and $h(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n$ belong to $A(p)$, we define the Hadamard product or convolution of $g$ and $h$ by $(g*h)(z) = z^p + \sum_{n=p+1}^{\infty} b_n c_n z^n$. $z \in E$. For $f \in A(p)$, define

$$D^{n+p-1} f(z) = f(z) * \left( \frac{z^p}{(1-z)^{n+p}} \right), \tag{1}$$

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where $n$ is any integer greater than $-p$. Then

$$(2) \qquad D^{n+p-1}f(z) = \left( \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!} \right).$$

It can be shown that (2) yields the following identity

$$(3) \qquad z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z).$$

In this note we give certain inequalities for $f \in A(p)$ which satisfies the condition

$$(4) \qquad \mathrm{Re}\{\frac{D^{n+p-1}f(z)}{z^p}\} > \alpha$$

and for the integral (5) of functions satisfying (4),

$$(5) \qquad F(z) = \frac{p+c}{z^c}\int_0^z u^{c-1}f(u)du.$$

For the existence of the integral in (5), the power represents the principal branch. These inequalities include or improve several results given by Bernardi [1], Goel and Sohi [2], Jack [3], Libera [4], Obradovic [5,6], Owa [7], Shukla and Kumar [8], Singh and Singh [9], Soni [10] and Strohacker [11].

To prove the inequalities, we need the following lemma of Jack [3].

LEMMA. *Let $w(z)$ be regular in the unit disc, with $w(0) = 0$. Then if $|w|$ attains its maximum value on the circle $|z| = r$ at a point $z_1$, we have*

$$z_1w'(z_1) = kw(z_1)$$

*where $k$ is real and $k \geqq 1$.*

## 2. Main results.

THEOREM 1. *Let $f \in A(p)$ for some $p \in N$ and satisfy the condition*

$$(6) \qquad Re\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\} > \alpha, \quad z \in E,$$

*for some integer $n$ greater than $-p$ and $\alpha < 1$, then*

$$Re\{\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}\} > \gamma(\alpha,n,p,c),$$

*where* $F$ *is defined as in (5),* $c+p>0$ , $c \geqq 2(1-\alpha)(n+p)-(p+1)$ *and*

$$\gamma(\alpha,n,p,c) = \{2(n-c)+2\alpha(p+n)-1+\sqrt{(2(c-n)+2\alpha(p+n)-1)^2+8(p+c)}\}/(4(p+n)) .$$

For $a=[2(c+p-1)(n+p-1)-1]/[2(n+p)(c+p-1)]$ and $c \geqq -p+2$ in Theorem 1, it is easy to obtain that $\gamma(\alpha,n,p,c) = (n+p-1)/(n+p)$ , thus Theorem 1 implies a result by Soni [10].

COROLLARY 1. *Let* $f \in A(p)$ *satisfy the condition*

$$Re\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \} > [2(c+p-1)(n+p-1)-1]/[2(n+p)(c+p-1)]$$

$z \in E$ , $p$ *a positive integer,* $n$ *any integer greater than* $-p$ *and* $c \geqq -p+2$.

*Then*

$$Re\{ \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} \} > (n+p-1)/(n+p) .$$

The second corollary which can be derived from Theorem 1 was proved by Obradovic [6]. Take $p = 1$, $n = 0$ and $\alpha < 1$ and note that

$$\frac{zf'(z)}{f(z)} = \frac{Df(z)}{f(z)} = -c + \frac{z^c f(z)}{\int_0^z u^{c-1} f(u)du}$$

and $\gamma(\alpha,0,1,c)+c = [2c+2a-1+\sqrt{(2c+2\alpha-1)^2+8(c+1)}]/4$ , so we get:

COROLLARY 2. *Let* $f \in S^*(a)$, $0 \leqq \alpha < 1$ , *and let* $c > max\{-1,-2\alpha\}$ , *then we have*

$$Re \frac{z^c f(z)}{\int_0^z u^{c-1} f(u)du} > [2c+2\alpha-1+\sqrt{(2c+2\alpha-1)^2+8(c+1)}]/4, \quad z \in E ,$$

*where* $S^*(\alpha) = \{f \in A(1) : Re[(zf'(z))/f(z)] > \alpha\}$ .

Proof of Theorem 1. Suppose $f \in A(p)$ satisfies the conditions in the theorem and write

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1-(2\gamma-1)w(z)}{1-w(z)} \tag{7}$$

where $\gamma = \gamma(\alpha,n,p,c)$ . By (3),

$$z(D^{n+p-1}F(z))' = (n+p)D^{n+p}F(z) - nD^{n+p-1}F(z)$$
$$= \left\{(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n\right\}D^{n+p-1}F(z) .$$

It is easy to use (5) and (7) to derive that

$$(8)\quad (c+p)D^{n+p}f(z) = z(D^{n+p}F(z))' + c(D^{n+p}F(z))$$
$$= z\left\{\frac{(1-(2\gamma-1)w(z))}{1-w(z)}D^{n+p-1}F(z)\right\}' + cD^{n+p}F(z)$$
$$= D^{n+p-1}F(z)\left\{\frac{2(1-\gamma)zw'(z)}{(1-w(z))^2}\right.$$
$$\left. + \frac{1-(2\gamma-1)w(z)}{1-w(z)}\left[(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n + c\right]\right\} ,$$

$$(c+p)D^{n+p-1}f(z) = z(D^{n+p-1}F(z))' + c(D^{n+p-1}F(z))$$
$$= D^{n+p-1}F(z)\left\{(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n+c\right\} .$$

From the above two identities, we conclude that

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{2(1-\gamma)zw'(z)}{(1-w(z))[p+c-((2\gamma-1)(n+p)+c-n]w(z)} + \frac{1-(2\gamma-1)w(z)}{1-w(z)} .$$

If $|w(z)| \not< 1$ , there exists $z_1 \in E$ , so that $|w(z_1)| = 1$ , then by Jack's lemma, there exists $k \geq 1$ , such that $z_1w'(z_1) = kw(z_1)$ .

Write $w(z_1) = u + iv$ and take the real part of the above identity. After some computations, one has

$$(9)\quad \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \alpha\right\}\Bigg|\ z = z_1$$
$$= \gamma - \alpha + 2(1-\gamma)k\ \mathrm{Re}\left\{\frac{u+iv}{(1-(u+iv))[p+c-((2\gamma-1)(n+p)+c-n)(u+iv)]}\right\}$$
$$= \gamma - \alpha + 2(1-\gamma)k\left\{\frac{(u-1+iv)(a-bu+ibv)}{2(1-u)((a-bu)^2+b^2v^2}\right\}$$

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$$= \gamma - \alpha + \left\{ \frac{-(1-\gamma)k(a+b)}{a^2 - 2abu + b^2} \right\}$$

where we write $a = p + c$, $b = (2\gamma - 1)(n+p) + c - n$ . Put

$$g(u) = (a+b)/(a^2 - 2abu + b^2) .$$

The condition $c \geqq 2(1-\alpha)(n+p) - (p+1)$ and the definition of $\gamma(\alpha,n,p,c)$ imply $b \geqq 0$ and $\gamma < 1$ , also $a = p + c > 0$ . Then $g(u)$ is increasing and thus $1/(a+b) = g(-1) \leqq g(u)$ . We have from (9) and $k \geqq 1$ that

$$\operatorname{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - \alpha \right\} \Bigg|_{z = z_1} \leqq \gamma - \alpha + \frac{-(1-\gamma)}{a+b}$$

$$= [2(n+p)\gamma^2 + (2(c-n) - 2\alpha(n+p) + 1)\gamma + 2\alpha(n-c) - 1]/(a+b) = 0 .$$

since $\gamma$ is a root of the polynomial

$$2(n+p)x^2 + (2(c-n) - 2\alpha(n+p) + 1)x + 2\alpha(n-c) - 1 = 0 .$$

This contradicts assumption (6), so the proof is completed.

We prove the remaining theorems for $p = 1$ only, the general case can be obtained by the same method.

THEOREM 2. *Suppose* $f \in A(1)$, $\alpha < 1 \leqq \beta$ , $n \in N_0$ *and*

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha$$

*then*

$$\operatorname{Re} \left( \frac{D^n f(z)}{z} \right)^{\frac{1}{2(n+1)(1-\alpha)\beta}} > \frac{\beta}{1+\beta}$$

*for* $z \in E$ .

Proof. Suppose $f$ satisfies the conditions in the theorem, let $\gamma = \beta/(1+\beta)$ and let $w(z)$ be a regular function such that

(10) $$\left( \frac{D^n f(z)}{z} \right)^{\frac{1}{2(n+1)(1-\alpha)\beta}} = \frac{1 - (2\gamma - 1)w(z)}{1 - w(z)}$$

then $w(0) = 0$ . The theorem will follow if we can show that $|w(z)| < 1$ in $E$ .

Now by differentiating (10) logarithmically, we get

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$$\frac{-(2\gamma-1)zw'(z)}{1-(2\gamma-1)w(z)} - \frac{-zw'(z)}{1-w(z)} = \frac{1}{2(1-\alpha)\beta}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - 1\right\} ,$$

where we use identity (3) for $p = 1$ , so

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{4(1-\alpha)\beta(1-\gamma)zw'(z)}{[1-w(z)][1-(2\gamma-1)w(z)]} + 1 . \tag{11}$$

If $|w(z)| \not< 1$ in $E$ , by Jack's lemma, there exist $z_1 \in E$ and a real $k \geqq 1$ , such that $|w(z_1)| = 1 \geqq |w(z)|$ , $\forall |z| \leqq |z_1| = r < 1$ , and $z_1 w'(z_1) = kw(z_1)$ . Let $w(z_1) = u + iv$ , then

$$\begin{aligned}\mathrm{Re}\,&\{z_1 w'(z_1)/[(1-w(z_1))(1-(2\gamma-1)w(z_1))]\} \\ &= -\gamma k/[2(2\gamma^2 - 2\gamma + 1 - (2\gamma-1)u)] .\end{aligned} \tag{12}$$

Put
$$g(u) = 1/[2\gamma^2 - 2\gamma + 1 - (2\gamma-1)u]$$

$$g(-1) = 1/(2\gamma^2),\ g(1) = 1/(2(1-\gamma)^2) .$$

$\gamma \geqq 1/2$ implies $g(u)$ is an increasing function of $u$ ,

$$1/(2\gamma^2) \leqq g(u) \leqq 1/(2(1-\gamma)^2) .$$

Applying (11) and (12), one has

$$\begin{aligned}\mathrm{Re}\,&\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - \alpha\right\}\Bigg|_{z=z_1} \\ &=[4(1-\alpha)\beta(1-\gamma)(-\gamma k)]/[2(2\gamma^2-2\gamma+1-(2\gamma-1)u)] + 1 - \alpha \\ &= 1 - \alpha - 2(1-\alpha)\beta(1-\gamma)\gamma k g(u) \\ &\leqq 1 - \alpha - (1-\alpha)\beta(1-\gamma)/\gamma = 0\end{aligned}$$

which contradicts the assumption. Thus the theorem is proved.

Replacing $n$ by $n+1$ and $\alpha = 1/2,\ \beta = 1$ in Theorem 2, we obtain:

COROLLARY 3. *If* $f \in A(1),\ z \in E$ *and*

$$\mathrm{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} > 1/2 ,\ \text{then}\ \mathrm{Re}\left(\frac{D^{n+1}f(z)}{z}\right)^{\frac{1}{n+2}} \quad 1/2 .$$

This is Theorem 3 of [9] by Singh and Singh. Under the condition of

Corollary 3, taking $n = 0$, we have the known result of Strohacker [11], that is, $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ implies $\operatorname{Re}\{\sqrt{f'(z)}\} > 1/2$. By considering $n = 0$, $\beta = 1$ in Theorem 2, one obtains:

COROLLARY 4. *If* $f \in A(1)$ *and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{, then } Re\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} > 1/2 \text{, } z \in E \text{.}$$

This is Theorem 2 of Jack [3]. Recently Obradovic [5] proved the following result which can be derived from Theorem 2 by taking $n = 0$ and $\beta = 1/(2(1-\alpha))$.

COROLLARY 5. *If* $f \in A(1)$ *and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{, then } Re\left(\frac{f(z)}{z}\right) > 1/(3-2\alpha), \; z \in E \text{.}$$

Note that $f \in K(\alpha)$, denoting the class

$$\{f \in A(1) : \operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha\},$$

is equivalent to $\operatorname{Re}(D^2 f/Df(z)) > (1+\alpha)/2$. So replacing $\alpha$ by $(1+\alpha)/2$, $n$ by $1$ and $\beta$ by $1/(1-\alpha)$ in Theorem 2, one obtains Corollary 6 which was proved by Obradovic and Owa [7].

COROLLARY 6. *If* $f \in K(\alpha)$, *then* $Re\sqrt{f'(z)} > 1/(2-\alpha)$.

THEOREM 3. *Suppose* $Re\, c > -1$, $\alpha < 1$, $n \in N_0 = N \cup \{0\}$, $f \in A(1)$ *and satisfies*

$$Re\, \{\frac{D^{n+1}f(z)}{z}\} > \alpha, \text{ for } z \in E, \text{ then } Re\, \{\frac{D^{n+1}F(z)}{z}\} > \gamma(\alpha,c)\text{,}$$

*for* $z \in E$, *where* $\gamma(\alpha,c) = \{\alpha + Re[1/(2(c+1))]\}/\{1+Re[1/(2(c+1))]\}$ *and* $F(z)$ *is defined as in* (5) *for* $p = 1$.

Proof. As in Theorem 1, we assume the function $f$ satisfies the conditions in the theorem and write

$$\frac{D^{n+1}F(z)}{z} = \frac{1-(2\gamma-1)w(z)}{1-w(z)} \tag{13}$$

where $\gamma = \gamma(\alpha, c)$ and $w(z)$ a regular function in $E$, then $w(0) = 0$. It is sufficient to show that $|w(z)| < 1$, $\forall\, z \in E$.

Taking the logarithmic derivative of (13), one obtains

(14) $$\frac{z(D^{n+1}F(z))'}{D^{n+1}F(z)} - 1 = \left\{\frac{2(1-\gamma)zw'(z)}{[1-w(z)][1-(2\gamma-1)w(z)]}\right\}.$$

From (13), (14) and (8) with $p = 1$, we have

(15) $$(c+1)\left\{\frac{D^{n+1}f(z)}{z}\right\} = (D^{n+1}F(z))' + c\,\frac{D^{n+1}F(z)}{z}$$

$$= \frac{2(1-\gamma)zw'(z)}{(1-w(z))^2} + (1+c)\left\{\frac{1-(2\gamma-1)w(z)}{1-w(z)}\right\} .$$

If $|w(z)| \nless 1$, there exists $z_1 \in E$, so that $|w(z)| \leqq |w(z_1)| = 1$, for all $z \in E$, then by Jack's lemma, there exists $k \geqq 1$, such that

$$z_1w'(z_1) = kw(z_1) .$$

Write $w(z_1) = u + iv$ so that $z_1w'(z_1)/(1-w(z_1))^2 = -k/(2(1-u)$ and take the real part of (15). After some computations we have

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{z} - \alpha\right\}\Bigg|_{z=z_1} = \frac{-(1-\gamma)k}{1-u}\cdot \operatorname{Re}\left\{\frac{1}{c+1}\right\} + \gamma - \alpha$$

$$\leqq \frac{-(1-\gamma)}{2}\cdot \operatorname{Re}\left\{\frac{1}{c+1}\right\} + \gamma - \alpha$$

$$= \gamma\left\{1 + \frac{1}{2}\operatorname{Re}\left(\frac{1}{c+1}\right)\right\} - \alpha - \frac{1}{2}\operatorname{Re}\left(\frac{1}{c+1}\right)$$

$$= 0 ,$$

which contradicts the assumption. So $|w(z)| < 1$, for $z \in E$. Thus we have proved the theorem.

From (3) and (8),

$$\frac{D^{n+2}F(z)}{z} = (c+1)\,\frac{D^{n+1}f(z)}{z} + (n+1-c)\,\frac{D^{n+1}F(z)}{z}$$

A few computations lead to:

**COROLLARY 7.** *If $-1 < c \leqq n+1$ and under the conditions of Theorem 3, then we have*

(16) $$\operatorname{Re}\frac{D^{n+2}F(z)}{z} > \alpha + [(n+1-c)(1-\alpha)]/[(n+2)(2c+3)] .$$

This result improves the estimations of theorem 3.3 of Shukla and Kumar [8].

If $-1 < c \leq n+1$ and $\alpha = -(n+1-c)/[(c+1)(2n+5)] > -1/2(c+1))$, (16) implies $\mathrm{Re}\{D^{n+2}F(z)/z\} > 0$. From this result, considering real $c$ and $\alpha$ replaced by $0$ and $-1/(2(c+1))$ respectively in Theorem 3, we derive the following two Corollaries which are theorems 2 and 4 obtained by Singh and Singh in [9].

COROLLARY 8. *If* $f \in M_n(0)$ *, then (i)* $\forall c > -1$ , $F \in M_n(0)$ *; (ii) for* $-1 < c \leq n+1$ , $F \in M_{n+1}(0)$, *where* $n \in N_0$ *and*

$$M_n(\alpha) = \{f \in A(1) : Re[D^{n+1}f(z)/z] > \alpha\ ,\ z \in E\}\ .$$

COROLLARY 9. *If* $f \in A(1)$ *and* $Re\ [D^{n+1}f(z)/z] > -1/(2(c+1))$, $c > -1$ *, then* $F \in M_n(0)$ .

Since $\gamma(\alpha,c) = \alpha + \{(1-\alpha)\mathrm{Re}[1/(2(c+1))]\}/\{1+\mathrm{Re}[1/(2(c+1))]\} > \alpha$, Theorem 3 may be rewritten as: If $f \in M_n(\alpha)$ , then $F \in M_n(\gamma(\alpha,c)) \subset M_n(\alpha)$ which implies the theorem 2 of Goel and Sohi [2]. When $n = \alpha = 0$ , we have Bernardi's result [1]: If $\mathrm{Re}f'(z) > 0$ , then $\mathrm{Re}F'(z) > 0$ . If $c = 1$ , we have a result of Libera [4].

COROLLARY 10. *If* $f \in A(1)$, $c > -1$, $0 \leq \alpha < 1$ *and*

$$Re[D^{n+1}f(z)/z] > \alpha - (1-\alpha)/(2(1+c)) \quad \text{then} \quad F \in M_n(\alpha)\ .$$

This is Theorem 3 in [2] which can be derived from our Theorem 3 by taking $\alpha = \alpha - (1-\alpha)/(2(1+c))$.

Obradovic [5,6] recently gave the following two results which can also be obtained from Theorem 3 by taking $n = c = \alpha = 0$ and $n = -1$ respectively. Note that $D^0f(z) = f(z)$ .

COROLLARY 11. *Let* $f \in A(1)$ *, then* $Re\{f'(z)\} > 0$ *implies* $Re\{f(z)/z\} > 1/3$ .

COROLLARY 12. *Let* $f \in A(1)$ , $\alpha < 1$ *and* $c > -1$ *, then for* $z \in E$ $Re\{f(z)/z\} > \alpha$ *implies*

$$Re\left\{\frac{c+1}{z^{c+1}}\int^{z} t^{c-1}f(t)dt\right\} > \alpha + (1-\alpha)/(3+2c)\ .$$

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We state the following theorem which is proved by a similar method. It improves a result of Goel and Sohi [2].

THEOREM 4. *Suppose* $\alpha < 1$, $f \in A(1)$ *and* $\mathrm{Re}[D^{n+2}f(z)/z] > \alpha$, *then*

$$[D^{n+1}f(z)/z] > [2(n+2)\alpha + 1]/(2n+5), n \in N_0 .$$

## References


[1] S. D. Bernardi, "Convex and starlike functions", *Trans. Amer. Math. Soc.* **135** (1969), 429-446.

[2] R. M. Goel and N. S. Sohi, "Subclasses of univalent functions", *Tamkang J. Math.* **11** (1980), 77-81.

[3] I. S. Jack, "Functions starlike and convex of order α", *J. London Math. Soc.* (2) **3** (1971), 469-474.

[4] R. J. Libera, "Some classes of regular univalent functions", *Proc. Amer. Math. Soc.*, **16** (1965), 755-758.

[5] M. Obradovic, "Estimates of the real part of $f(z)/z$ for some classes of univalent functions", *Mat. Vesnik* **36** (4) (1984), 266-270.

[6] M. Obradovic, "On certain inequalities for some regular functions in $|z| < 1$", *Internat. J. Math. and Math. Sci.* **8** (1985), 677-681.

[7] M. Obradovic and S. Owa, "On some results of convex functions of order α", *Mat. Vesnik*, (to appear).

[8] S. L. Shukla and V. Kumar, "Univalent functions defined by Ruscheweyh derivatives", *Internat. J. Math. and Math. Sci.*, **6** (1983), 483-486.

[9] S. Singh and R. Singh, "New subclasses of univalent functions", *Indian Math. Soc.*, **43** (1979), 149-159.

[10] A. K. Soni, "Generalizations of $p$-valent functions via the Hadamard product", *Internat. J. Math. and Math. Sci.* **5** (1982) 289-299.

[11] E. Strohacker, "Beitrage zur theorie der schlichten funkitone", *Math. Z.*, **37** (1933), 256-280.



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