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Infinitary affine proofs

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Even though the multiplicative–additive fragment of linear logic forbids structural rules in general, is does admit a bounded form of exponential modalities enjoying a bounded form of structural rules. The approximation theorem, originally proved by Girard, states that if full linear logic proves a propositional formula, then the multiplicative–additive fragment proves every bounded approximation of it. This may be understood as the fact that multiplicative–additive linear logic is somehow dense in full linear logic. Our goal is to give a technical formulation of this informal remark. We introduce a Cauchy-complete space of infinitary affine term-proofs and we show that it yields a fully complete model of multiplicative exponential polarised linear logic, in the style of Girard's ludics. Moreover, the subspace of finite term-proofs, which is a model of multiplicative polarised linear logic, is dense in the space of all term-proofs.

1. Introduction

1.1. *Approximating the contraction rule*

Mathematical truth is inexhaustible: a classical statement such as the Pythagorean theorem is no less true today than two millennia ago, inspite of having being used countless times in the meanwhile. This perennity of mathematical truth is reflected in formal logic by the provability of the implication $A \Rightarrow A \wedge A$ or, in more proof-theoretic terms, by the validity of the *contraction rule*

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} .$$

These are both ways of formalising that 'if A holds once, it holds twice' and, therefore, any finite number of times.

Linear logic (Girard 1987) shifts the focus from truth to proofs, i.e.to how truth is established. In this context, the multiplicity of use takes a different meaning: having one proof of A does not necessarily imply having two proofs of A, unless we take the two proofs to be identical copies of the original one. However, duplication is not for free and we must explicitly mention it by means of a modality, called *exponential*. Accordingly, the linear implication $A \multimap A \otimes A$ is not provable in general; instead, $!A \multimap !A \otimes !A$ is provable.

The exponential modalities (the modality ! mentioned above and its dual ?) are the distinguishing feature of linear logic with respect to other substructural logics refusing contraction. However, a less evident but perhaps more interesting feature is the interplay between the exponential layer of linear logic and its purely linear layer, known as its multiplicative–additive fragment. Here, albeit forbidden in their general form, structural rules still exist in a bounded form. Indeed, if $!A$ intuitively means 'A arbitrarily many times,' the multiplicative–additive fragment contains approximations of $!A$, namely

$$!_n A := \overbrace{(A \mathbin{\&} 1) \otimes \cdots \otimes (A \mathbin{\&} 1)}^{n}.$$

The formula $!_n A$ intuitively means 'A atmost n times' and is therefore a bounded form of $!A$. The existence of these approximations was observed by Girard since the introduction of linear logic:

Theorem 1 (approximation, Girard (1987)). Let $!_n A$ be defined as above and let $?_n A := (!_n A^\perp)^\perp$. Suppose that A is a provable propositional formula of full linear logic, containing m occurrences of $!$ and n occurrences of $?$. Then, for all $p_1, \ldots, p_m \in \mathbb{N}$, there exist $q_1, \ldots, q_n \in \mathbb{N}$ such that the formula A' obtained from A by replacing the ith occurrence of $!$ with $!_{m_i}$ and the jth occurrence of $?$ with $?_{n_j}$ is provable in the multiplicative–additive fragment.

The approximation theorem somehow says that multiplicative–additive linear logic is 'dense' in full linear logic. The aim of this paper is to take Theorem 1 seriously and show how the intuitive idea of density may be given a precise technical sense in the context of uniform spaces (a generalisation of metric spaces).

1.2. *The infinitary affine lambda-calculus*

The starting point is our recent work (Mazza 2012), in which the computational content (via Curry–Howard) of Theorem 1 is formalised: an affine λ-calculus is introduced and a metric is defined on terms, making the set of affine terms an incomplete metric space; the completion, which we call $\ell\Lambda^!_\infty$ in this paper, is endowed with a partial equivalence relation $\approx$ and it is proved that $\ell\Lambda^!_\infty/\approx$ is isomorphic to the usual λ-calculus (Corollary 10 below).

The shift from linearity to affinity allows us to replace $!_n A$ by the simpler formula $A^{\otimes n}$. This does not alter the moral of the result, since affine logic still refuses contraction and it is this latter rule which underlies perennity in logic. However, affinity is not merely a simplification: as we underline below, in a linear calculus reduction would not be continuous, a property which we do not use here but which is fundamental in other contexts (such as Mazza (2014)).

The relation $\approx$ accounts for the concept of *uniformity* (Girard 2001; Melliès 2004): the completion introduces infinitary terms which do not correspond to any algorithm (they rather correspond to *non-uniform algorithms*, such as those expressed by arbitrary circuit families (Vollmer 1999)). Only uniform infinitary terms correspond to usual λ-terms.

The infinitary affine calculus of Mazza (2012) is here endowed with primitives corresponding to affine logic, in particular dealing explicitly with the $\otimes$ connective and the $!$ modality. This generalised calculus is called $\ell\Lambda_\infty$, and its finite subcalculus is called $\ell\Lambda$. The calculus is endowed with a uniform-space structure, with two essential properties:

1. $\ell\Lambda_\infty$ is Cauchy-complete and $\ell\Lambda$ is dense in it;
2. one-step reduction, as a map on $\ell\Lambda_\infty$, is continuous.

An important technical difference with Mazza (2012) is the shift from metric spaces to more general uniform spaces. Although not strictly necessary to validate property 1 above or to formulate the isomorphism of Corollary 10, topological uniformity† has the benefit of making reduction continuous, a nice property which does not hold with the metric of Mazza (2012). Indeed, Proposition 8 of that paper erroneously claims that reduction is continuous, so we rectify that mistake here. Although we do not use property 2 here (nor did we in Mazza (2012)), as mentioned above continuity of reduction is fundamental for our results of Mazza (2014), which are also based on the calculus $\ell\Lambda_\infty$.

1.3. *The proof-theoretic perspective*

In order to give a formal content to the topological intuition underlying Theorem 1, we adopt the approach of Girard's ludics (Girard 2001). Ludics reconstructs logic from the interaction of *designs*, which are essentially generalised Böhm trees (Curien 1998). Two designs of opposite polarity generate a dynamics which is an abstraction of cut-elimination, or normalisation in the λ-calculus. If the result of this interaction is successful (in a certain sense), the designs are said to be *orthogonal*. Then, given a set of designs $\mathcal{E}$, one may define $\mathcal{E}^\perp$ as the set of all designs orthogonal to all designs of $\mathcal{E}$. A *type* is defined to be a set $\mathcal{A}$ of designs of the same polarity such that $\mathcal{A}^{\perp\perp} = \mathcal{A}$. Logic is then reconstructed by defining operations on designs and types which are shown to correspond, in the sense of denotational semantics, to logical rules and connectives.

In Girard (2001), it is proved that a polarised version of multiplicative–additive linear logic admits a fully complete denotational model in ludics. Here, we take as designs a subset of the infinitary affine terms of $\ell\Lambda_\infty$. The finite designs (those belonging to $\ell\Lambda$) roughly correspond to Girard's designs (or, better, to Terui's (2011) *computational designs*) and yield a model of polarised multiplicative linear logic. Instead, infinitary designs give a fully complete denotational semantics of the multiplicative–exponential fragment of Laurent's (2004) polarised linear logic (**MELLP**). Full completeness is based on a notion of *validity*, in which uniformity plays a crucial role. More precisely:

soundness: a **MELLP** proof π of A yields a valid design $[\![\pi]\!] \in [\![A]\!]$ which is stable under cut-elimination;

completeness: the $\approx$-equivalence classes of valid designs in $[\![A]\!]$ correspond to cut-free **MELLP** proofs of A.

Therefore, the novelty with respect to the results of Girard (2001) is that we are able to model non-linearity, i.e.the contraction rule. Although not unheard of (full completeness results for linear logic with exponentials already exist: let us cite Laurent (2004) for

† The term 'uniformity' is used in this paper with two completely unrelated meanings, one computational and one topological. Both uses are standard in their respective fields (e.g. Vollmer (1999) for computation, or Girard (2001) in a proof-theoretic context, and Bourbaki (1998) for topology) and we prefer not to introduce new terminology, also considering that there is little room for confusion.

polarised linear logic and Melliès (2005) for full linear logic, both based on games semantics), this is a further contribution of the paper, because it gives an alternative way of modelling exponentials in ludics, with respect to what was done by Basaldella and Faggian (2009).

Essentially, polarised linear logic is classical logic, seen through linear eyes (Girard 1991; Laurent and Regnier 2003). Therefore, our results may be succinctly formulated using an intriguing slogan: *classical logic is the uniform quotient of the completion of multiplicative affine logic*.

1.4. *Related work*

The idea of linear approximations was used implicitly or explicitly in the work of Ehrhard (2005) on differential linear logic and of Melliès *et al.* (2009) on denotational semantics of linear logic. Apart from the idea, the latter work is very different from ours because it is of categorical nature. We will return to it at the conclusion of this paper. On the contrary, Ehrhard's work has a syntactic counterpart, namely the Taylor expansion of λ-terms (Ehrhard and Regnier 2008) by means of resource λ-terms (Boudol 1993), which is very much related to our work. The main difference is the use of affinity in our setting, which yields simpler constructions, both at the syntactic level (no formal sums) and at the topological level (plain uniform spaces instead of topological vector spaces). On that note, it is interesting to observe how linearity seems to forbid a topology which allows both approximations in the spirit of Theorem 1 and the continuity of reduction. This is because approximations require the topology to be local, whereas in strictly linear calculi (like the resource λ-calculus) redexes are highly non-local. In this respect, affinity seems to be a mandatory choice.

We already mentioned the extension of ludics to exponential connectives by Basaldella and Faggian (2009). For a comparison, we may say that their approach is akin to the games semantics of Hyland and Ong (2000), whereas ours is closer to the work of Abramsky *et al.* (2000). We must mention that, in unpublished material contained in his PhD thesis, Basaldella (2008) did explore an Abramsky, Jagadeesan and Malacaria style definition of exponentials in ludics, which seems to be essentially equivalent to the one presented here. However, his results do not include full completeness and do not provide any insight on infinitary ludics being generated by the completion of a uniform space. On the other hand, the definition of computational uniformity (the PER $\approx$) is fairly standard in the context of Abramsky, Jagadeesan and Malacaria games, see for instance Melliès (2004).

For what concerns our infinitary affine calculus $\ell\Lambda_\infty$, it is similar to calculi considered by Kfoury (2000) and Melliès (2006). However, the motivations, technique and development of the first work are quite orthogonal with respect to ours, whereas the second, which is concerned with games semantics, only considers normal forms.

More generally, we are of course far from being the first to consider infinitary calculi. The investigation of infinitary rewriting was initiated by Dershowitz *et al.* (1991) and is still a growing research field. A survey on infinitary term rewriting systems and infinitary λ-calculi may be found in Chapter 12 of Terese (2003).

The work which is closest to the present one is the infinitary λ-calculus of Kennaway *et al.* (1997), or KKSdV for short. An immediate difference between KKSdV's work and our own is that we study terms which are possibly infinite in *width*, but are well-founded (or even finite) in height, whereas KKSdV's focus is exactly dual. This is because KKSdV are interested in studying the notion of infinitary rewriting, namely reductions of possibly infinite length, with no concern about affinity or duplication. On the other hand, our aim is to describe *finite* reductions in proofs (or λ-terms) by means of non-duplicating reductions. Other calculi which are similar, although not directly related, to ours may be found in Rodenburg (1998) (where terms with infinite width are considered) and Révész (1992) (where a λ-calculus with lists is introduced).

2. The infinitary affine lambda-calculus

2.1. *Terms and reduction*

We fix two countably infinite sets $\mathcal{V}_l$ and $\mathcal{V}_{nl}$ of *linear* and *non-linear variables*, respectively, ranged over by a, b, c and x, y, z, respectively, and not including the symbol $\bot$. A *non-empty pattern* is either a linear variable a, or of the form $!x$ with $x \in \mathcal{V}_{nl}$, or of the form $\mathsf{p} \otimes \mathsf{q}$, where p and q are non-empty patterns sharing no variable. A *pattern*, also ranged over by p, q, is either a non-empty pattern or of the form $\_$ (the *empty pattern*).

Pre-terms are co-inductively generated as follows:

$$s, t, u ::= \bot \mid a \mid x_i \mid \lambda \mathsf{p}.t \mid tu \mid t \otimes u \mid * \mid \mathbf{u},$$

where $i \in \mathbb{N}$ is called an *index* and $\mathbf{u}$ denotes a function from $\mathbb{N}$ to pre-terms, which is referred to as a *box*. We use the notation $\langle \mathbf{u}(0), \mathbf{u}(1), \mathbf{u}(2), \ldots \rangle$ when we want to explicitly describe a box. When we write $\mathbf{u} = \langle u_0, \ldots, u_{n-1} \rangle$, we mean that $\mathbf{u}(i) = \bot$, for all $i \geqslant n$. By 'co-inductively' we mean that we allow pre-terms which may contain themselves as strict subterms, for instance solutions of the equation $t = \lambda a.t$.

The usual notion of free and bounded variable apply to pre-terms, with the remark that, in $\lambda !x.t$, all of $x_0, x_1, x_2, \ldots$ are bounded. We denote by $\mathrm{fv}(t)$ the set of free (linear or non-linear) variables of t. We also consider the standard notion of α-equivalence, which will be treated as equality in the rest of the paper. When needed, we will use *Barendregt's convention*, namely that binders in a pre-term t bind pairwise disjoint set of variables which are also disjoint from $\mathrm{fv}(t)$.

A pre-term may be seen as a labelled tree on the set $\Sigma := \{\bot, a, x_i, \lambda \mathsf{p}, @, \otimes, *, !\}$, with $a \in \mathcal{V}_l$, $x \in \mathcal{V}_{nl}$, $i \in \mathbb{N}$ and p ranging over patterns (so, inspite of our notation, Σ is an infinite set). The symbol @ is used for application and ! for boxes. By *tree* we mean a downward closed subset of $\mathbb{N}^*$ (finite sequences of natural numbers, with the prefix order) and by *labelled tree* we mean a function $t : \mathbb{N}^* \longrightarrow \Sigma$ such that the set

$$\mathrm{supp}\, t := \{\alpha \in \mathbb{N}^* \mid t(\alpha) \neq \bot\}$$

is a tree. A pre-term is then a labelled tree which further verifies that:

— if $t(\alpha) = *$ or $t(\alpha) = a$ or $t(\alpha) = x_i$ for some $a \in \mathcal{V}_l$ or $x \in \mathcal{V}_{nl}$ and $i \in \mathbb{N}$, then α is maximal in $\mathrm{supp}\, t$ ($*$ and occurrences of variables may only be the label of leaves);

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— if $t(\alpha) = \lambda\mathsf{p}$, then for all $n > 0$, $t(\alpha \cdot n) = \bot$ (abstractions have atmost one sibling);
— if $t(\alpha) = @$ or $t(\alpha) = \otimes$, then for all $n > 1$, $t(\alpha \cdot n) = \bot$ (applications and tensors have atmost two siblings).

Definition 1 (term). A *term* is a pre-term t such that

affinity: under Barendregt's convention, $t(\alpha) = a$, $t(\beta) = b$ and $a = b$ imply $\alpha = \beta$ (linear variables occur atmost once); similarly, $t(\alpha) = x_i$, $t(\beta) = x_j$ and $i = j$ imply $\alpha = \beta$ (distinct occurrences of non-linear variables have distinct indices);
boxes: for every box $\mathbf{u}$ of t, $\mathrm{fv}(\mathbf{u}) \subseteq \mathcal{V}_{\mathrm{nl}}$ (the free variables of boxes are non-linear);
well foundedness: the tree $\mathrm{supp}\, t$ is well founded.

We denote by $\ell\Lambda_\infty$, the set of all terms and by $\ell\Lambda$, the set of all finite terms, i.e.terms t such that $\mathrm{supp}\, t$ is a finite tree.

By definition, the strict subterm relation on terms is well-founded. This will allow us to define notions and prove facts by Noetherian induction.

We define $t \between \mathsf{p}$ (the term t *matches* the pattern p) as the smallest relation such that: $* \between \_$; $t \between a$ always holds; $\mathbf{u} \between !x$; if $t' \between \mathsf{p}$ and $t'' \between \mathsf{q}$, then $t' \otimes t'' \between \mathsf{p} \otimes \mathsf{q}$.

Given $u \between \mathsf{p}$, *substitution* $t[u/\mathsf{p}]$ is defined by simultaneous induction on t and p:

— $t[*/\_] := t$;
— $b[u/a] := \begin{cases} u & \text{if } a = b, \\ b & \text{if } a \neq b; \end{cases}$
— $x_i[\mathbf{u}/!x] := \mathbf{u}(i)$;
— $t[u' \otimes u''/\mathsf{p} \otimes \mathsf{q}] := t[u'/\mathsf{p}][u''/\mathsf{q}]$;
— $\bot[u/\mathsf{p}] := \bot$ and $*[u/\mathsf{p}] := *$;
— $(\lambda y.s)[u/\mathsf{p}] := \lambda y.t[u/\mathsf{p}]$, $(sv)[u/\mathsf{p}] := s[u/\mathsf{p}]v[u/\mathsf{p}]$ and $(s \otimes v)[u/\mathsf{p}] := s[u/\mathsf{p}] \otimes v[u/\mathsf{p}]$;
— $\mathbf{v}[u/\mathsf{p}] := \mathbf{v}'$ where $\forall i \in \mathbb{N}$, $\mathbf{v}'(i) := \mathbf{v}(i)[u/\mathsf{p}]$.

Definition 2 (reduction). *One-step reduction* is the smallest binary relation such that

— $(\lambda\mathsf{p}.t)u \to t[u/\mathsf{p}]$, whenever $u \between \mathsf{p}$;
— if $t \to t'$, then $\lambda x.t \to \lambda x, t'$, $tu \to t'u$, $ut \to ut'$, $t \otimes u \to t' \otimes u$ and $u \otimes t \to u \otimes t'$;
— if $\mathbf{u}(n) \to u'$ and $\mathbf{u}'$ is such that $\mathbf{u}'(n) = u'$ and $\mathbf{u}'(i) = \mathbf{u}(i)$ for all $i \neq n$, then $\mathbf{u} \to \mathbf{u}'$.

Reduction, denoted by $\to^*$, is the reflexive–transitive closure of $\to$.

Reduction is trivially confluent: indeed, $\to^=$ (reduction in atmost one step) satisfies the diamond property, because redexes have atmost one residue by reduction of another redex. It is also terminating on $\ell\Lambda$, because the size of finite terms shrinks under reduction. However, infinite reductions are possible in $\ell\Lambda_\infty$: if $\Delta := \lambda !x.x_0\langle x_1, x_2, x_3, \ldots\rangle$ and $\Omega := \Delta\langle\Delta, \Delta, \Delta, \ldots\rangle$, then $\Omega \to \Omega$.

2.2. *Topological considerations*

Remember how terms are particular functions $\mathbb{N}^* \longrightarrow \Sigma$. If we endow Σ with the discrete uniformity, the set of terms may be equipped (as a subspace of a function space) with the

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uniformity $\mathcal{FB}$ of uniform convergence on finitely branching trees, the set of which we denote by FBT. A family of entourages generating $\mathcal{FB}$ is given by

$$\mathcal{U}_\tau := \{(t,t') \in \ell\Lambda_\infty \times \ell\Lambda_\infty \mid t(\alpha) = t'(\alpha) \text{ for all } \alpha \in \tau\},$$

where $\tau \in$ FBT. A net $(t_\iota)_{\iota\in I}$ is Cauchy in the uniform space $(\ell\Lambda_\infty, \mathcal{FB})$ if, for every $\tau \in$ FBT, there exists $\kappa \in I$ such that, whenever $\iota, \iota' \geqslant \kappa$, t_ι and $t_{\iota'}$ agree on τ.

The induced topology, which we abusively still denote by $\mathcal{FB}$, is easily inferred: a neighbourhood basis for t is given by the sets $(V_\tau(t))_{\tau\in\mathrm{FBT}}$, where

$$V_\tau(t) := \{t' \in \ell\Lambda_\infty \mid t'(\alpha) = t(\alpha) \text{ for all } \alpha \in \tau\}.$$

Therefore, a net $(t_\iota)_{\iota\in I}$ converges to t, if t_ι eventually coincides with t on every finitely branching tree. A typical example is $\Delta_n := \lambda!x.x_0\langle x_1,\dots,x_n\rangle$, for which we have that $\lim \Delta_n = \Delta$.

The uniform structure defined above has two fundamental properties, proved below:

1. the space $(\ell\Lambda_\infty, \mathcal{FB})$ is Cauchy-complete, with $\ell\Lambda$ as a dense subset (Proposition 5);
2. one-step reduction is continuous (Proposition 7).

In the sequel, we denote by $\lambda\mathsf{p}.\ell\Lambda$, $\ell\Lambda$ @ $\ell\Lambda$, $\ell\Lambda \otimes \ell\Lambda$ and $!\ell\Lambda$, the sets of all finite terms of the form $\lambda\mathsf{p}.t$, tu, $t \otimes u$ and $\mathbf{u}$, respectively. All these sets are endowed with the subspace uniformity. We will also use the following notations:

— $\ell\Lambda \times_{\mathrm{aff}} \ell\Lambda$ is the set of all pairs of finite terms (t,u) such that tu is also a term (not just a pre-term), endowed with the subspace uniformity of the product uniformity on $\ell\Lambda \times \ell\Lambda$;

— $(\ell\Lambda)^{(\mathbb{N})}$ is the set $!\ell\Lambda$ endowed with the subspace uniformity of the product uniformity on $(\ell\Lambda)^{\mathbb{N}}$ (i.e.the set of *all* sequences of finite terms).

The maps $\lambda\mathsf{p} : \ell\Lambda \longrightarrow \lambda\mathsf{p}.\ell\Lambda$, @ : $\ell\Lambda \times_{\mathrm{aff}} \ell\Lambda \longrightarrow \ell\Lambda$ @ $\ell\Lambda$ and $\otimes : \ell\Lambda \times_{\mathrm{aff}} \ell\Lambda \longrightarrow \ell\Lambda \otimes \ell\Lambda$, defined by $t \mapsto \lambda\mathsf{p}.t$, $(t,u) \mapsto tu$ and $(t,u) \mapsto t \otimes u$, respectively, are obviously bijective. Furthermore, they and their inverses are all uniformly continuous.

Proposition 2 (isotropy). We have the following uniform homeomorphisms:

1. $\lambda\mathsf{p}.\ell\Lambda \cong \ell\Lambda$ via $\lambda\mathsf{p}$;
2. $\ell\Lambda$ @ $\ell\Lambda \cong \ell\Lambda \otimes \ell\Lambda \cong \ell\Lambda \times_{\mathrm{aff}} \ell\Lambda$ via @ and $\otimes$;
3. $!\ell\Lambda \cong (\ell\Lambda)^{(\mathbb{N})}$ via the identity.

Proof. We consider directly point 2, and start with $@^{-1}$. We have to show that for all $\xi', \xi'' \in \mathbb{N}^{\mathbb{N}}$, there exists $\xi \in \mathbb{N}^{\mathbb{N}}$ such that $(tu, t'u') \in \mathcal{U}_\xi$ implies $(t,t') \in \mathcal{U}_{\xi'}$ and $(u,u') \in \mathcal{U}_{\xi''}$. It is easy to check that setting $\xi := 1 \cdot (\xi' \vee \xi'')$ meets the requirement. For @, we have $\xi = n \cdot \hat{\hat{\xi}}$ and must determine ξ', ξ'' to get the converse implication. It is easily seen that setting $\xi' := \xi'' := \hat{\hat{\xi}}$ is enough. The case of $\otimes$ is virtually identical. Point 1 is also similar.

Before proving point 3, let us remind that the uniformity on $(\ell\Lambda)^{(\mathbb{N})}$ is given by the following basis of entourages: we choose a finite $F \subset \mathbb{N}$, we choose one $\xi_i \in \mathbb{N}^{\mathbb{N}}$ for each $i \in F$, and we define the basic entourage $\mathcal{V}_{(\xi_i)_{i\in F}} := \{(\mathbf{u},\mathbf{u}') \mid \forall i \in F, (\mathbf{u}(i), \mathbf{u}'(i)) \in \mathcal{U}_{\xi_i}\}$. Now, in the direction from $!\ell\Lambda$ to $(\ell\Lambda)^{(\mathbb{N})}$, we need to prove that, for all $\{i_1,\dots,i_n\} \subset \mathbb{N}$ and for

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every $\xi_{i_1},\dots,\xi_{i_n} \in \mathbb{N}^{\mathbb{N}}$, there exists $\xi \in \mathbb{N}^{\mathbb{N}}$ such that $(\mathbf{u},\mathbf{u}') \in \mathcal{U}_\xi$ implies $(\mathbf{u},\mathbf{u}') \in \mathcal{V}_{\xi_{i_1},\dots,\xi_{i_n}}$. We define the integer $k := \max\{i_1,\dots,i_n\}$ and the sequence of integers $\xi' := \bigvee_{1\leqslant j\leqslant n} \xi_{i_j}$. We invite the reader to check that $\xi := k\cdot\xi'$ meets the requirement. The other direction is easier: we have ξ and we need to determine a finite $F \subset \mathbb{N}$ and $\xi_i \in \mathbb{N}^{\mathbb{N}}$ for each $i \in F$. Let $\xi = n\cdot\xi'$; then, the reader may check that it is enough to set $F := \{0,1,2,\dots,n\}$ and $\xi_i := \xi'$ for all $0 \leqslant i \leqslant n$. □

Corollary 3. A net $(t_\iota)_{\iota\in I}$ is Cauchy iff one of the following eventually holds:

1. $t_\iota = t$, where t is an occurrence of variable or $\bot$ or $*$;
2. $t_\iota \in \lambda \mathsf{p}.\ell\Lambda$ and $(\lambda\mathsf{p}^{-1}(t_\iota))_{\iota\in I}$ is Cauchy;
3. $t_\iota \in \ell\Lambda \mathbin{@} \ell\Lambda$ and both $(\pi_1(@^{-1}(t_\iota)))_{\iota\in I}$ and $(\pi_2(@^{-1}(t_\iota)))_{\iota\in I}$ are Cauchy;
4. $t_\iota \in \ell\Lambda \otimes \ell\Lambda$ and both $(\pi_1(\otimes^{-1}(t_\iota)))_{\iota\in I}$ and $(\pi_2(\otimes^{-1}(t_\iota)))_{\iota\in I}$ are Cauchy;
5. $t_\iota = \mathbf{u}_\iota$ and, for all $i \in \mathbb{N}$, $(\mathbf{u}_\iota(i))_{\iota\in I}$ is Cauchy.

Proof. That all terms are eventually of the same kind (variable, abstraction, application, etc.) is immediate. Then, case 1 is obvious, whereas the other cases follow from the isomorphisms with product uniformities (Proposition 2). □

In the following, we consider the functions $\Pi_i : !\ell\Lambda \longrightarrow \ell\Lambda$, for $i \in \mathbb{N}$, defined by $\Pi_i(\mathbf{u}) := \mathbf{u}(i)$. We also define the set of functions $\mathcal{F} := \{\lambda\mathsf{p}^{-1}, \pi_i \circ @^{-1}, \pi_i \circ \otimes^{-1}, \Pi_j \mid i \in \{1,2\},\ j \in \mathbb{N}\}$. Let $(t_\iota)_{\iota\in I}$, $(t'_\iota)_{\iota\in I}$ be two Cauchy nets. We write $(t_\iota)_{\iota\in I} \sqsubset (t'_\iota)_{\iota\in I}$ just if, eventually, $t_\iota = \varphi(t'_\iota)$ for some $\varphi \in \mathcal{F}$.

Lemma 4 (well foundedness). The relation $\sqsubset$ is well founded.

Proof. Assume the contrary, and let $(t^0_\iota)_{\iota\in I} \sqsupset (t^1_\iota)_{\iota\in I} \sqsupset (t^2_\iota)_{\iota\in I} \sqsupset \dots$ be an infinite descending chain. By definition, we have a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{F}$ such that, for all $n \in \mathbb{N}$, there exists $\kappa_n \in I$ such that, for all $\iota \geqslant \kappa_n$, $t^{n+1}_\iota = \varphi_n(t^n_\iota)$. Define $\iota_0 := \kappa_0$ and, inductively, let ι_{n+1} be an element of I which is above both κ_{n+1} and ι_n (such an element must exist because I is directed). We set $u_n := t^0_{\iota_n}$. Observe that, by construction, for arbitrary $n \in \mathbb{N}$, we have $\varphi_n \circ \cdots \circ \varphi_0(u_{n+1}) = t^{n+1}_{\iota_{n+1}} \neq \bot$. The latter disequality holds because, since $\iota_{n+1} \geqslant \kappa_{n+1}$, $t^{n+1}_{\iota_{n+1}}$ is in the domain of φ_{n+1}, which never contains $\bot$.

Let us now define $\xi \in \mathbb{N}^{\mathbb{N}}$ by setting $\xi(n)$ to be: 0 if $\varphi_n = \lambda\mathsf{p}^{-1}$; $i-1$ if $\varphi_n = \pi_i \circ @^{-1}$ or $\varphi_n = \pi_i \circ \otimes^{-1}$; and i if $\varphi_n = \Pi_i$. We denote by α_n the prefix of ξ of length $n+1$. We may prove by induction on n that $(\varphi_n \circ \cdots \circ \varphi_0(u_{n+1}))(\epsilon) = u_{n+1}(\alpha_n)$, for all $n \in \mathbb{N}$. But, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (because it is the subsequence of a Cauchy net) and therefore there exists $k \in \mathbb{N}$ such that, for all $i,j \geqslant k$ and for every $n \in \mathbb{N}$, $u_i(\alpha_n) = u_j(\alpha_n)$. In particular, $u_k(\alpha_n) = u_{n+1}(\alpha_n) \neq \bot$ for all $n \geqslant k$, which is absurd, because u_k is a finite term. □

Proposition 5. The space $(\ell\Lambda_\infty, \mathcal{F}B)$ is the Cauchy completion of $(\ell\Lambda, \mathcal{F}B)$.

Proof. We remind that the completion of a uniform space X is the set of Cauchy nets of X, quotiented under an equivalence relation which essentially says that two Cauchy nets have the same limit (see Bourbaki (1998) for the precise definition, which is not important here). Consider $\ell\Lambda$ endowed with the uniformity $\mathcal{S}$ of simple convergence, which is induced by the metric $d(t,t') = 2^{-|\alpha|}$, where α is the shortest sequence such that $t(\alpha) \neq t'(\alpha)$. By adapting well-known results (Arnold and Nivat 1980; Courcelle 1983;

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Kennaway *et al.* 1997), one may prove that the completion of the metric space $(\ell\Lambda, d)$ is the set T of pre-terms respecting the affinity and boxes conditions of Definition 1, which means that these are the canonical representatives of Cauchy nets of $(\ell\Lambda, d)$. Now, simple convergence is the same as uniform convergence on finite trees, so $\mathcal{S}$ is coarser, i.e.has more Cauchy nets than $\mathcal{FB}$. Therefore, if we denote by $\overline{\ell\Lambda}$ the underlying set of the completion of $(\ell\Lambda, \mathcal{FB})$, we have $\overline{\ell\Lambda} \subseteq T$. It is not hard to see that the relation $\sqsubset$ defined above on Cauchy nets is precisely the subterm relation on the pre-terms of T, so Lemma 4 allows us to conclude. □

Lemma 6 (substitution). Let $(u_\iota)_{\iota \in I}$ be a Cauchy net such that, eventually, $u_\iota \between \mathsf{p}$. In that case, if a net $(t_\iota)_{\iota \in I}$ is Cauchy, then so is $(t_\iota[u_\iota/\mathsf{p}])_{\iota \in I}$.

Proof. By well-founded induction on $(t_\iota)_{\iota \in I}$, using Corollary 3. □

Given $\alpha \in \mathbb{N}^*$, we define $R_\alpha : \ell\Lambda \longrightarrow \ell\Lambda$ by $R_\alpha(t) := t'$ if $t \to t'$ by reducing a redex at position α, or $R_\alpha(t) := t$ if no such reduction applies.

Proposition 7 (Cauchy continuity of reduction). For all $\alpha \in \mathbb{N}^*$, R_α is Cauchy continuous, hence continuous.

Proof. We remind that a function is Cauchy continuous if it preserves Cauchy nets. It is a stronger form of continuity. Let $(t_\iota)_{\iota \in I}$ be a Cauchy net, and let $t'_\iota := R_\alpha(t_\iota)$. We need to show that $(t'_\iota)_{\iota \in I}$ is Cauchy. We do this by induction on the length of α. We start with the inductive case, which is easy. Suppose $\alpha = n \cdot \alpha'$. The proof depends on the value of n, but it is similar in all cases. To give the idea, we let $n = 1$. Now, if we are not in one of cases 3, 4 or 5 of Corollary 3, then eventually $t_\iota(\alpha) = \bot$ and the result is vacuously true. Again, to show the idea, we pick case 4, i.e.eventually $t_\iota = t'_\iota t''_\iota$. By Corollary 3, both $(t'_\iota)_{\iota \in I}$ and $(t''_\iota)_{\iota \in I}$ are Cauchy; by the induction hypothesis, $(R_{\alpha'}(t''_\iota))_{\iota \in I}$ is Cauchy, so $(t'_\iota R_{\alpha'}(t''_\iota))_{\iota \in I}$ is Cauchy (again by Corollary 3). But observe that, for all u, v, $R_\alpha(uv) = uR_{\alpha'}(v)$, hence we are done.

So we only need to prove the Cauchy continuity of R_ϵ. We start by defining, for every pattern p, a finite $A(\mathsf{p}) \subset \mathbb{N}^*$ as follows. First, we set $A^-(\_) := A^-(a) := A^-(x) := \{\epsilon\}$ and $A^-(\mathsf{p} \otimes \mathsf{q}) := 0 \cdot A^-(\mathsf{p}) \cup 1 \cdot A^-(\mathsf{q})$; then, we let $A(\mathsf{p}) := 1 \cdot A^-(\mathsf{p})$. Now, observe that t is a redex $(\lambda\mathsf{p}.u)v$ precisely if: $t(\epsilon) = @$; $t(0) = \lambda$; for all $\alpha' \in A(\mathsf{p})$, $t(\alpha')$ has a suitable value depending on p and α' (for instance: if $\mathsf{p} = a$, there is no requirement; if $\mathsf{p} = x$, then we must have $t(1) = !$; for more complex patterns, we have at least $t(1) = \otimes$ and the rest depends on the pattern). The essence of the above discussion is that a pattern p induces $\alpha_1, \ldots, \alpha_k \in \mathbb{N}^*$ and $\sigma_1, \ldots, \sigma_k \in \Sigma$ such that t is a redex of pattern p iff $t(\alpha_i) = \sigma_i$ for all $1 \leqslant i \leqslant k$. Therefore, by extending each α_i arbitrarily to get $\xi_i \in \mathbb{N}^{\mathbb{N}}$ and by setting $\xi := \bigvee_{1 \leqslant i \leqslant k} \xi_i$, we have that α_i is a prefix of $\xi_i \leqslant \xi$ for all $1 \leqslant i \leqslant k$ and, whenever $(t, t') \in \mathcal{U}_\xi$, t is a redex iff t' is.

The above means that, by virtue of the Cauchy property, either eventually t_ι is a redex, or eventually none of t_ι is. In the latter case, $(t'_\iota)_{\iota \in I}$ is trivially Cauchy. In the former case, eventually $t_\iota = (\lambda\mathsf{p}.s_\iota)u_\iota$ and $u_\iota \between \mathsf{p}$. Therefore, eventually $t'_\iota = s_\iota[u_\iota/\mathsf{p}]$. Now, by Corollary 3, $(s_\iota)_{\iota \in I}$ and $(u_\iota)_{\iota \in I}$ are both Cauchy, so we conclude by Lemma 6. □

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We conclude this section by observing that the space $(\ell\Lambda_\infty, \mathcal{FB})$ is not metrisable, i.e.the uniform structure cannot be defined by means of a metric. This may be seen by showing that the induced topology is not first-countable. Let $t \in \ell\Lambda_\infty$ and $(U_i)_{i\in\mathbb{N}}$ be a countable family of open neighbourhoods of t. We will show that there exists an open neighbourhood V of t that is not generated by the family, i.e.such that $U_i \nsubseteq V$ for all $i \in \mathbb{N}$.

We start by observing that, by virtue of $(V_\tau(t))_{\tau\in\mathrm{FBT}}$ being a neighbourhood basis of t, we have a family $(\rho_i)_{i\in\mathbb{N}}$ of finitely branching trees such that $V_{\rho_i}(t) \subseteq U_i$ for all $i \in \mathbb{N}$. Since each ρ_i is finitely branching, there certainly exists $m_i \in \mathbb{N}$ such that, for all $\alpha \in \rho_i$ and $j \in \mathbb{N}$, $|\alpha| = i$ and $j \geqslant m_i$ imply $\alpha \cdot j \notin \rho_i$ (we denote by $|\alpha|$ the length of the sequence α). In other words, m_i strictly bounds the 'maximum branching index' of the immediate descendants of ρ_i at level i (the root being at level 0). We define $\rho \in \mathrm{FBT}$ to be such that each node at level i has exactly $m_i + 1$ immediate descendants (indexed by $0, \ldots, m_i$). Now, it is not hard to define a sequence of terms $(t_i)_{i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$:

— $t_i(\beta) = t(\beta)$ for all $\beta \in \rho_i$;

— there exists $\alpha \in \mathbb{N}^*$ such that $|\alpha| = i$ and $t_i(\alpha \cdot m_i) \neq t(\alpha \cdot m_i)$.

In other words, t_i and t coincide on ρ_i but differ on a position which is the immediate successor of a node of level i of branching index m_i.

By construction, $V_\rho(t)$ is the open neighbourhood V we were seeking: it is an open neighbourhood of t and yet, by definition, for all $i \in \mathbb{N}$ we have $U_i \nsubseteq V_\rho(t)$, because $t_i \in V_{\rho_i}(t) \setminus V_\rho(t)$.

Inspite of not being metrisable, the topology $\mathcal{FB}$ is sequential, i.e.sequences (rather than general nets) are enough to describe it. In fact, it may be proved that $(\ell\Lambda_\infty, \mathcal{FB})$ is a Fréchet–Urysohn space (Vial 2014).

2.3. *Computational uniformity and the isomorphism with the full λ-calculus*

The set $\ell\Lambda_\infty$ is uncountable, it contains 'infinite algorithms.' These are akin to so-called *non-uniform algorithms*, which are families consisting of a different finite algorithm for each different size of the input (Vollmer 1999). In this section, which is a synthesis of Mazza (2012), we impose a uniformity constraint and recover the usual λ-calculus.

Non-uniformity arises in $\ell\Lambda_\infty$ because boxes are arbitrary sequences. Now, a box is morally the argument of a non-linear variable x, which may appear with several (even infinitely many) occurrences of the form x_i. During an ordinary, uniform computation, the argument of x would be a plain term u, which is duplicated and substituted to each x_i. Here, affinity excludes duplication, which is why we have to resort to boxes. However, the fact that boxes are arbitrary means that x_i and x_j may be substituted with completely different terms. To ensure that computation is uniform, we only consider boxes of the form $\langle u_0, u_1, u_2, \ldots\rangle$ where each u_i is morally a copy of the same term u, i.e.u_i and u_j only differ in the indices of their free non-linear variables. For instance, if u is closed (and uniform), then $\langle u, u, u, \ldots\rangle$ will be uniform. The terms Δ and Ω defined above are typical examples of uniform terms.

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Definition 3 (uniformity). We define *reindexing equivalence*, denoted by $\approx$, as the smallest partial equivalence relation on $\ell\Lambda_\infty$ such that:

— $* \approx *$, $a \approx a$ and $x_i \approx x_j$ for all $i, j \in \mathbb{N}$;
— if $t \approx t'$ and $u \approx u'$, then $\lambda x.t \approx \lambda x.t'$, $tu \approx t'u'$ and $t \otimes u \approx t' \otimes u'$;
— if $\mathbf{u}(i) \approx \mathbf{u}'(j)$ for all $i, j \in \mathbb{N}$, then $\mathbf{u} \approx \mathbf{u}'$.

A term t is *uniform* if $t \approx t$. We denote by $\mathrm{u}\Lambda_\infty$ the set of uniform terms.

All subterms of a uniform term are uniform. Hence, $\perp$ cannot appear in any uniform term. In particular, no finite term containing a box is uniform. Also note that uniform terms are necessarily of finite height. Another interesting remark is that the quotient space $\ell\Lambda_\infty/\approx$ is discrete (we leave the easy verification to the reader).

Uniformity is not preserved under reduction. For example, if u is a closed uniform term such that $u \to u'$, then $\langle u, u, u, \ldots\rangle \to \langle u', u, u, \ldots\rangle$, which has no reason to be uniform. However, we do know that $\langle u', u', u', \ldots\rangle$ is uniform; the idea then is to define a notion of reduction which allows infinitely many parallel steps, so as to preserve uniformity.

Definition 4 (infinitary reduction). We define the relations $\Rightarrow_k$ on $\mathrm{u}\Lambda_\infty$, with $k \in \mathbb{N}$, as follows:

— $(\lambda \mathsf{p}.t)u \Rightarrow_0 t[u/\mathsf{p}]$ whenever $u \between \mathsf{p}$;
— if $t \Rightarrow_k t'$, then $\lambda \mathsf{p}.t \Rightarrow_k \lambda \mathsf{p}.t'$, $tu \Rightarrow_k t'u$, $ut \Rightarrow_k ut'$, $t \otimes u \Rightarrow_k t' \otimes u$ and $u \otimes t \Rightarrow_k u \otimes t'$;
— if $\mathbf{u}(0) \Rightarrow_k u_0'$, by uniformity the 'same' reduction may be performed in all $\mathbf{u}(i)$, $i \in \mathbb{N}$, obtaining the term u_i'. If we define $\mathbf{u}'(i) = u_i'$ for all $i \in \mathbb{N}$, we set $\mathbf{u} \Rightarrow_{k+1} \mathbf{u}'$.

We denote by $\Rightarrow$ the union of all $\Rightarrow_k$, for $k \in \mathbb{N}$.

For instance, if $I = \lambda x.x_0$, and $t = I\langle I, I, I, \ldots\rangle$, we have

$$a\langle y_0\langle t, t, \ldots\rangle, y_1\langle t, t, \ldots\rangle, \ldots\rangle \Rightarrow_2 a\langle y_0\langle I, I, \ldots\rangle, y_1\langle I, I, \ldots\rangle, \ldots\rangle.$$

Note that $\Rightarrow_k$ is infinitary iff $k > 0$. Indeed, $\Rightarrow_0$ is *shallow reduction* (a strongly confluent extension of head reduction) and is *not* infinitary.

Proposition 8. Let $t \in \mathrm{u}\Lambda_\infty$. Then:

— $t \Rightarrow t'$ implies $t' \in \mathrm{u}\Lambda_\infty$;
— furthermore, for all $u \approx t$, $u \Rightarrow u' \approx t'$.

Proof. An immediate adaptation of Proposition 14 in Mazza (2012). □

In what follows, we denote by Λ the set of usual λ-terms, ranged over by M, N. We denote by $\to_\beta$ usual β-reduction. The set of λ-calculus variables is assumed to be $\mathcal{V}_{\mathrm{n}l}$. Let us define $\ell\Lambda_\infty^!$ to be the subset of $\ell\Lambda_\infty$ generated as follows:

$$t ::= x_i \mid \lambda!x.t \mid t\mathbf{u}.$$

Essentially, $\ell\Lambda_\infty^!$ is the image of Λ under Girard's translation of intuitionistic logic in linear logic. We denote by $\mathrm{u}\Lambda_\infty^!$ the set of uniform terms of $\ell\Lambda_\infty^!$.

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We fix an injection $\ulcorner\cdot\urcorner : \mathbb{N}^* \to \mathbb{N}$. For all $\alpha \in \mathbb{N}^*$, we define by induction the family of maps $[\![\cdot]\!]_\alpha : \Lambda \to \mathrm{u}\Lambda^!_\infty$ and the map $(\!|\cdot|\!) : \mathrm{u}\Lambda^!_\infty \to \Lambda$, as follows:

$$[\![x]\!]_\alpha := x_{\ulcorner\alpha\urcorner} \qquad (\!|x_i|\!) := x \quad (\text{for all } i \in \mathbb{N})$$
$$[\![\lambda x.M]\!]_\alpha := \lambda !x.[\![M]\!]_\alpha \qquad (\!|\lambda !x.t|\!) := \lambda x.(\!|t|\!)$$
$$[\![MN]\!]_\alpha := [\![M]\!]_{\alpha\cdot 0}\langle [\![N]\!]_{\alpha\cdot 1}, [\![N]\!]_{\alpha\cdot 2}, [\![N]\!]_{\alpha\cdot 3}, \ldots\rangle \qquad (\!|t\mathbf{u}|\!) := (\!|t|\!)(\!|\mathbf{u}(0)|\!).$$

For example, if $M := (\lambda x.xx)(\lambda z.z)$, then

$$[\![M]\!]_\epsilon = (\lambda !x.x_{\ulcorner 00\urcorner}\langle x_{\ulcorner 01\urcorner}, x_{\ulcorner 02\urcorner}, x_{\ulcorner 03\urcorner}, \ldots\rangle)\langle \lambda !z.z_{\ulcorner 1\urcorner}, \lambda !z.z_{\ulcorner 2\urcorner}, \lambda !z.z_{\ulcorner 3\urcorner}, \ldots\rangle,$$

which is obviously uniform, and if $I := \lambda !z.z_0$, we have $[\![M]\!]_\epsilon \approx \Delta\langle I, I, I, \ldots\rangle$.

The *applicative depth* of a redex in the λ-calculus is defined by induction: $(\lambda x.M)N$ is at applicative depth 0; if a redex is at applicative depth k in M, then its applicative depth is k in $\lambda x.M$ and MN, and $k+1$ in NM. In the following, we write $M \to_{\beta k} M'$ to denote the fact that $M \to_\beta M'$ by reducing a redex at applicative depth k (e.g.$\to_{\beta 0}$ is a mild generalisation of head reduction).

Proposition 9. For all $M \in \Lambda$, $t \in \mathrm{u}\Lambda^!_\infty$, and $\alpha \in \mathbb{N}^*$:

1. $(\!|[\![M]\!]_\alpha|\!) = M$;
2. $[\![(\!|t|\!)]\!]_\alpha \approx t$;
3. $M \to_{\beta k} M'$ implies $[\![M]\!]_\alpha \Rightarrow_k t' \approx [\![M']\!]_\alpha$;
4. $t \Rightarrow_k t'$ implies $(\!|t|\!) \to_{\beta k} (\!|t'|\!)$.

Proof. See Theorem 19 in Mazza (2012). □

Corollary 10. In the Curry–Howard sense, $(\Lambda, \to_\beta)$ is isomorphic to $(\ell\Lambda^!_\infty/\!\approx, \Rightarrow)$.

Note that, as we remarked after Definition 3, the topology induced on Λ from the quotient topology on $\ell\Lambda^!_\infty/\!\approx$ is discrete. This shows that our topology is very different from the familiar Scott topologies used in denotational semantics (such as the topology of Böhm trees). In fact, our topology is meaningful only for approximating λ-terms by means finite affine terms. Since finite terms disappear in the quotient (they are not uniform), the resulting topology is trivial.

3. The proof-theoretic perspective

3.1. *A non-locative version of ludics*

We start by adding to $\ell\Lambda_\infty$ the constants $\mho$ (convergence, called *daimon* in Girard (2001)) and Ω (divergence), and we denote by $\ell\Lambda^{\mho,\Omega}_\infty$ the resulting set of terms.† The notion of uniformity (Definition 3) applies to $\ell\Lambda^{\mho,\Omega}_\infty$ by adding $\mho \approx \mho$ and $\Omega \approx \Omega$.

Definition 5 (normal form). We define the relation $t \Downarrow v$ as the smallest such that:

— $\bot \Downarrow \bot$, $* \Downarrow *$, $a \Downarrow a$, $x_i \Downarrow x_i$ and $\mho \Downarrow \mho$;
— if, for all $i \in \mathbb{N}$, $\mathbf{u}(i) \Downarrow \mathbf{v}(i)$, then $\mathbf{u} \Downarrow \mathbf{v}$;

† We will not make any topological consideration in this section, observe however that $\ell\Lambda^{\mho,\Omega}_\infty$ is uniformly homeomorphic to $\ell\Lambda_\infty$: occurrences of $\mho$ and Ω behave like occurrences of non-linear variables.

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— if $t \Downarrow v$ and $t' \Downarrow v'$, then $\lambda \mathsf{p}.t \Downarrow \lambda \mathsf{p}.t'$ and $t \otimes t' \Downarrow v \otimes v'$; in case v is not an abstraction, then we also have $tt' \Downarrow vv'$; instead, if $v = \lambda \mathsf{p}'.s$, then $tt' \Downarrow v''$ provided that $v' \between \mathsf{p}'$ and $s[v'/\mathsf{p}'] \Downarrow v''$.

Given $t \in \ell\Lambda_\infty^{\mho,\Omega}$, we define $\mathrm{n}f(t) := v$ if $t \Downarrow v$ or $\mathrm{n}f(t) := \Omega$ if there is no such v.

Note that, if $\mathrm{n}f(u)$ does not match p, then $\mathrm{n}f((\lambda \mathsf{p}.t)u) = \Omega$.

Lemma 11. Let $t, u \in \ell\Lambda_\infty^{\mho,\Omega}$.

1. If $t \to t'$ and $\mathrm{n}f(t) \neq \Omega$, then $\mathrm{n}f(t) = \mathrm{n}f(t')$.
2. If $u \between \mathsf{p}$ and $\mathrm{n}f(t[u/\mathsf{p}]) \neq \Omega$, then $\mathrm{n}f(t[u/\mathsf{p}]) = \mathrm{n}f(\mathrm{n}f(t)[\mathrm{n}f(u)/\mathsf{p}])$.

Proof. Both points are established by straightforward inductions: point 1 on the derivation that $t \Downarrow v$ for some v; point 2 on t and the derivation that $t[u/\mathsf{p}] \Downarrow v$. □

An *exponential pattern of arity n* is a pattern of the form $!x^1 \otimes \cdots \otimes !x^n$, by which we mean the pattern $\_$ if $n = 0$. We use $\vec{\mathsf{p}}, \vec{\mathsf{q}}, \vec{\mathsf{r}}$ to range over exponential patterns. Note that if $\vec{\mathsf{p}}, \vec{\mathsf{q}}$ are exponential patterns, then so is $\vec{\mathsf{p}} \otimes \vec{\mathsf{q}}$, with the convention that $\vec{\mathsf{p}} \otimes \_ = \_ \otimes \vec{\mathsf{p}} = \vec{\mathsf{p}}$.

An *exponential tensor of arity n* is a term of the form $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n$, by which we mean the term $*$ if $n = 0$. We use $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ to range over exponential tensors. Note that if $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are exponential tensors, then so is $\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}$, with the convention that $\vec{\mathbf{u}} \otimes * = * \otimes \vec{\mathbf{u}} = \vec{\mathbf{u}}$.

Definition 6 (design). A *design* is a term of $\ell\Lambda_\infty^{\mho,\Omega}$ generated as follows:

$$t^- ::= \bot \mid \lambda \vec{\mathsf{p}}.t^+ \qquad \text{(negative designs)},$$
$$t^+ ::= \mho \mid a\vec{\mathbf{u}}^- \mid x_i\vec{\mathbf{u}}^- \qquad \text{(positive designs)},$$

where $\vec{\mathbf{u}}^-$ ranges over exponential tensors whose boxes contain negative designs. Observe that the definition is inductive, which means that designs have a finite *height* (the height of the underlying tree). They also have a finite *depth*, which is the maximum nesting of boxes. We denote by $\ell\Lambda_\infty^{des}$ the set of all designs.

The *arity* of $\lambda\vec{\mathsf{p}}.t$ (resp. $a\vec{\mathbf{u}}$ or $x_i\vec{\mathbf{u}}$) is the arity of $\vec{\mathsf{p}}$ (resp. $\vec{\mathbf{u}}$).

In the following, we fix a linear variable z. A design t is *atomic,* if it is negative and $\mathrm{fv}(t) = \varnothing$ or if it is positive and either $t = \mho$ or $t = z\vec{\mathbf{u}}$ with the designs in $\vec{\mathbf{u}}$ atomic.

Note that designs, by construction, are normal. Our finite designs are more or less equivalent to a non-locative (what Girard (2001) calls 'spiritual') version of the usual designs of ludics, restricted to multiplicative constructions. Indeed, our finite designs basically coincide with normal linear multiplicative *computational designs* of Terui (2011). Finite designs may be used to give a syntactic model of a polarised version of multiplicative linear logic. We will see that our infinitary designs, inspite of being affine, are capable of also modelling the non-linear features of linear logic, i.e.the contraction rule.

Definition 7 (orthogonality). Let t, u be atomic designs of opposite polarity, with t positive. We say that they are *orthogonal*, and we write $t \perp u$ (or $u \perp t$, i.e.the relation is symmetric), if $\mathrm{n}f(t[u/z]) = \mho$. Given a set of atomic designs $\mathcal{E}$, define

$$\mathcal{E}^\perp = \{t' \in \ell\Lambda_\infty^{des} \mid \forall t \in \mathcal{E},\ t' \perp t\}.$$

Note that we trivially have $\maltese \perp u$ for every negative u. By contrast, no negative design is orthogonal to all positive designs. Indeed, for $t \perp u$ to hold with u negative and $t \neq \maltese$, t and u must have the same arity.

The operation of taking the orthogonal of a set of designs is easily seen to have the following standard properties: $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$; $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{B}^{\perp} \subseteq \mathcal{A}^{\perp}$; hence, $\mathcal{A}^{\perp\perp\perp} = \mathcal{A}^{\perp}$.

Definition 8 (type). A *pre-type* (*ethics* in Girard (2001)) is a non-empty set of atomic designs of the same polarity. A *type* (*behaviour* in Girard (2001)) is a pre-type $\mathcal{A}$ such that $\mathcal{A}^{\perp\perp} = \mathcal{A}$; its polarity is the polarity of the designs it contains.

It is immediate to check that the set of all negative atomic designs $\top$ and $\mathbf{0} := \{\maltese\}$ are both types, orthogonal of each other. A *proper type* is a type which is neither $\top$ nor $\mathbf{0}$. If $\mathcal{A}$ is a proper type, and $t, u \in \mathcal{A}$ both different from $\maltese$, it is easy to check that t and u have the same arity. This is said to be the *arity* of $\mathcal{A}$. Furthermore, observe that the arity of $\mathcal{A}^{\perp}$ is the same as that of $\mathcal{A}$.

An important example of proper positive type is $\mathbf{1} = \{z*\}^{\perp\perp}$, of arity 0. This is easily seen to be equal to $\{\maltese, z*\}$. In fact, given a negative atomic design t, the only way that $t*$ reduces to $\maltese$ is that $t = \lambda\_.\maltese$ (the arities must match, and t must be closed, because it is atomic). So we have $\{z*\}^{\perp} = \{\lambda\_.\maltese\}$ and the remarks following Definition 7 allow us to conclude. Note that the daimon is necessarily present in all positive types.

Definition 9 (tensor product). Let t, u be positive atomic designs. We define

$$t \boxtimes u = \begin{cases} z(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) & \text{if } t = z\vec{\mathbf{u}} \text{ and } u = z\vec{\mathbf{v}}; \\ \maltese & \text{if one of } t, u \text{ is equal to } \maltese. \end{cases}$$

Let $\mathcal{A}, \mathcal{B}$ be positive types. We define

$$\mathcal{A} \boxtimes \mathcal{B} = \{t \boxtimes u \mid t \in \mathcal{A},\ u \in \mathcal{B}\},$$
$$\mathcal{A} \otimes \mathcal{B} = (\mathcal{A} \boxtimes \mathcal{B})^{\perp\perp}.$$

Note that, if $\mathcal{A}$ and $\mathcal{B}$ are both proper of arity m and n, respectively, then $\mathcal{A} \otimes \mathcal{B}$ is proper of arity $m + n$.

Lemma 12 (internal completeness of tensor product). For all positive types $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \otimes \mathcal{B} = \mathcal{A} \boxtimes \mathcal{B}$.

Proof. We start by observing that $\mathcal{A} \boxtimes \mathbf{0} = \mathbf{0} \boxtimes \mathcal{B} = \mathbf{0} = \mathbf{0}^{\perp\perp}$, so we may assume that both $\mathcal{A}$ and $\mathcal{B}$ are proper. By the remarks following Definition 7, it is enough to prove the inclusion $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{A} \boxtimes \mathcal{B}$. So let $s \in \mathcal{A} \otimes \mathcal{B}$. If $s = \maltese$, we are done, because $\maltese \in \mathcal{A} \cap \mathcal{B}$ and $\maltese \otimes \maltese = \maltese$. So suppose $s = z\vec{\mathbf{w}}$. We actually know that the arity of s is $m + n$, where m is the arity of $\mathcal{A}$ and n is the arity of $\mathcal{B}$. Therefore, we may write $\vec{\mathbf{w}} = \vec{\mathbf{u}} \otimes \vec{\mathbf{v}}$ and $s = t \boxtimes u$, with $t = z\vec{\mathbf{u}}$ and $u = z\vec{\mathbf{v}}$. So all that is left to do is prove that $t \in \mathcal{A}$ and $u \in \mathcal{B}$. We will prove the first statement, the second being perfectly symmetric. Let $t' \in \mathcal{A}^{\perp}$. We must have $t' = \lambda\vec{\mathsf{p}}.v$ with $\vec{\mathsf{p}}$ of arity m. We define $s' := \lambda\vec{\mathsf{p}} \otimes \vec{\mathsf{q}}.v$, with $\vec{\mathsf{q}}$ of arity n binding variables not appearing in v. We contend that $s' \in (\mathcal{A} \boxtimes \mathcal{B})^{\perp}$. Indeed, let $t_0 = z\vec{\mathbf{u}}' \in \mathcal{A}$ and $u_0 = z\vec{\mathbf{v}}' \in \mathcal{B}$. We have $s'(\vec{\mathbf{u}}' \otimes \vec{\mathbf{v}}') \to v[\vec{\mathbf{u}}'/\vec{\mathsf{p}}] =: v'$. But we also have $t'\vec{\mathbf{u}}' \to v'$, so $\mathrm{n}f(v') = \maltese$, because $t' \perp t_0$. This proves that $s' \perp (t_0 \boxtimes u_0)$ and, by genericity of t_0 and u_0,

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that $s' \in (\mathcal{A} \boxtimes \mathcal{B})^\perp$, as claimed. Consider now $t[t'/z] = t'\vec{\mathbf{u}}$; this reduces to $v[\vec{\mathbf{u}}/\vec{\mathsf{p}}]$, which is also the reduct of $s[s'/z]$, which has normal form $\mho$, because $s \in (\mathcal{A} \boxtimes \mathcal{B})^{\perp\perp}$ by hypothesis. Hence $t \perp t'$ and, by genericity of t', $t \in \mathcal{A}$, as desired. □

An immediate consequence of Lemma 12 is that the tensor product is associative and has neutral element **1**. It is also commutative, but only upto isomorphism.

Definition 10 (exponentiation). Let $\mathcal{A}$ be a proper negative type. We define

$$\dagger\mathcal{A} = \{z\mathbf{u} \mid \forall i \in \mathbb{N},\, \mathbf{u}(i) \in \mathcal{A}\} \cup \{\mho\}$$
$$!\mathcal{A} = (\dagger\mathcal{A})^{\perp\perp}.$$

Note that, independently of the arity of $\mathcal{A}$, $!\mathcal{A}$ is always proper of arity 1.

Lemma 13 (internal completeness of exponentiation). For every proper negative type $\mathcal{A}$, $!\mathcal{A} = \dagger\mathcal{A}$.

Proof. Again, we only have to prove the inclusion $!\mathcal{A} \subseteq \dagger\mathcal{A}$, so let $t \in !\mathcal{A}$. If $t = \mho$, we trivially have $t \in \dagger\mathcal{A}$ by definition. Hence, since $!\mathcal{A}$ is of arity 1, we may suppose $t = z\mathbf{u}$, and it is enough to prove that $\mathbf{u}(i) \in \mathcal{A}$ for all $i \in \mathbb{N}$. Let $u' = z\vec{\mathbf{v}} \in \mathcal{A}^\perp$, and let $s := \lambda x.x_i\vec{\mathbf{v}}$, with $x \notin \mathrm{fv}(\vec{\mathbf{v}})$. We contend that $s \in (\dagger\mathcal{A})^\perp$. Indeed, let $\mathbf{w}$ be an arbitrary box of terms in $\mathcal{A}$. We have $\mathrm{n}f(s\,\mathbf{w}) = \mathrm{n}f(\mathbf{w}(i)\vec{\mathbf{v}}) = \mathrm{n}f(u'[\mathbf{w}(i)/z]) = \mho$ because $\mathbf{w}(i) \perp u'$. Now, note that $t[s'/z] \to \mathbf{u}(i)\vec{\mathbf{v}} = u'[\mathbf{u}(i)/z]$, which has normal form $\mho$ because $t \in (\dagger\mathcal{A})^{\perp\perp}$ by hypothesis. Therefore, $\mathbf{u}(i) \perp u'$ and, by generality of u' and i, $\mathbf{u}(i) \in \mathcal{A}$ for all $i \in \mathbb{N}$. □

3.2. *A fully complete model of multiplicative exponential polarised linear logic*

The formulas of (atom-free) *multiplicative exponential polarised linear logic* (**MELLP**, Laurent (2004)) are defined as follows:

$$A^+, B^+ ::= 1 \mid A^+ \otimes B^+ \mid\ !A^-,$$
$$A^-, B^- ::= \perp \mid A^- \parr B^- \mid\ ?A^+.$$

Formulas of the form A^+ are *positive*, those of the form A^- are *negative*. Linear negation $(\cdot)^\perp$ is defined as usual through De Morgan laws, and it exchanges $1/\perp$, $\otimes/\parr$ and $!/?$ (so the negation of a positive formula is negative, and vice versa). Propositional atoms are excluded for simplicity; we believe that the main ideas underlying the model are better conveyed in their absence, without missing any important technical ingredient.

A *polarised sequent* is an expression of the form $\Gamma \vdash \Sigma$ where Γ and Σ are finite (unordered) sequences of positive formulas, with Σ consisting of at most one formula. The sequent calculus of **MELLP** is given in Figure 1. The usual presentation (Laurent 2004) manipulates one-sided sequents containing at most one positive formula, and has explicit structural rules (weakening and contraction) on all negative formulas. Our two-sided formulation with implicit structural rules is obviously equivalent, and is best fit for our purposes. For technical reasons, we also consider the system $\mathbf{MELLP}_\mho$, in which there is a *daimon* rule that allows to prove any sequent.

$$\frac{}{\Gamma \vdash \Sigma}\ \text{dai} \qquad \frac{\Gamma \vdash A \quad A, \Gamma \vdash \Sigma}{\Gamma \vdash \Sigma}\ \text{cut}$$

$$\frac{\Gamma \vdash \Sigma}{1, \Gamma \vdash \Sigma}\ 1\vdash \qquad \frac{}{\Gamma \vdash 1}\ \vdash 1 \qquad \frac{A, B, \Gamma \vdash \Sigma}{A \otimes B, \Gamma \vdash \Sigma}\ \otimes\vdash \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \otimes B}\ \vdash\otimes$$

$$\frac{!A^\perp, \Gamma \vdash A}{!A^\perp, \Gamma \vdash}\ !\vdash \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash !A^\perp}\ \vdash!$$

Fig. 1. The sequent calculus of $\mathbf{MELLP}_\mho$. The exchange rule is implicit. The sequent calculus of **MELLP** is obtained by removing the rule (dai).

The *interpretation* of a formula A, denoted by $[\![A]\!]$, is a type defined by induction on A, using the operations introduced in Section 3.1:

$$\begin{aligned} [\![1]\!] &= \mathbf{1} & [\![\bot]\!] &= \mathbf{1}^\perp \\ [\![A \otimes B]\!] &= [\![A]\!] \otimes [\![B]\!] & [\![A \parr B]\!] &= ([\![A]\!]^\perp \otimes [\![B]\!]^\perp)^\perp . \\ [\![!A]\!] &= ![\![A]\!] & [\![?A]\!] &= (![\![A]\!]^\perp)^\perp \end{aligned}$$

Note that the interpretation of a formula is always proper.

We now decorate the calculus of Figure 1 with terms of $\ell\Lambda_\infty^{\mho,\Omega}$. A sequent $C_1, \ldots, C_n \vdash \Sigma$ will be replaced by a judgment $\vec{\mathsf{p}}_1 : C_1, \ldots, \vec{\mathsf{p}}_n : C_n \vdash t : \Sigma$, where:

1. the arity of each $\vec{\mathsf{p}}_j$ is equal to that of $[\![C_j]\!]$;
2. $\mathrm{fv}(t)$ is contained in $\vec{\mathsf{p}}_1, \ldots, \vec{\mathsf{p}}_n$, with the possible exception of z when $\Sigma \neq \varnothing$;
3. if $\Sigma \neq \varnothing$, then t is either $\mho$ or Ω or of the form $z\vec{\mathbf{u}}$.

The last requirement allows us to extend Definition 9. Given t, u as in point 3 above, we define $t \boxtimes u$ as follows:

— if one of t, u is Ω, then $t \boxtimes u := \Omega$;
— if none of t, u is Ω but one of them is $\mho$, then $t \boxtimes u := \mho$;
— otherwise, $t = z\vec{\mathbf{u}}$ and $u = z\vec{\mathbf{v}}$, and we set $t \boxtimes u := z(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})$.

Finally, we fix infinitely many injections $\iota_n : \mathbb{N} \to \mathbb{N}$, $n \in \mathbb{N}$, whose ranges are pairwise disjoint and, given a term t, we define $\iota_n(t)$ to be the term obtained by replacing each free occurrence of non-linear variable x_i with $x_{\iota_n(i)}$. Also, if $x \in \mathrm{fv}(t)$, we define $t^{x{+}{+}}$ to be the term obtained from t by replacing every occurrence x_i with x_{i+1}.

With the above definitions in place, the decoration of $\mathbf{MELLP}_\mho$ proofs is given in Figure 2. Now, given a $\mathbf{MELLP}_\mho$ proof π of $C_1, \ldots, C_n \vdash \Sigma$, by applying the above decoration we obtain a judgment $\vec{\mathsf{p}}_1 : C_1, \ldots, \vec{\mathsf{p}}_n : C_n \vdash t : \Sigma$. Then, we define the *interpretation* of π as $[\![\pi]\!] := \lambda\vec{\mathsf{p}}_1 \otimes \cdots \otimes \vec{\mathsf{p}}_n.t$.

We will prove later that the interpretation of a $\mathbf{MELLP}_\mho$ proof is actually a design, of the following kind.

Definition 11 (functional design, function types). A *functional design* is a negative design of the form $\lambda\vec{\mathsf{p}}.t$ where t is atomic (hence, $\mathrm{fv}(\lambda\vec{\mathsf{p}}.t) \subseteq \{z\}$).

Let $\mathcal{A}, \mathcal{B}$ be positive types, with $\mathcal{A}$ proper. We define the set $\mathcal{A} \vdash \mathcal{B}$ as follows:

— $\mathcal{A} \vdash \mathbf{0}$ is defined as $\mathcal{A}^\perp$;

$$\frac{}{\Gamma \vdash \mho : \Sigma}\ \text{dai} \qquad \frac{\Gamma \vdash t : A \quad \vec{\mathsf{p}} : A, \Gamma \vdash u : \Sigma}{\Gamma \vdash \mathrm{nf}(\iota_0(t)[\iota_1(\lambda \vec{\mathsf{p}}.u)/z]) : \Sigma}\ \text{cut}$$

$$\frac{\Gamma \vdash t : \Sigma}{\_ : 1, \Gamma \vdash t : \Sigma}\ 1\vdash \qquad \frac{}{\Gamma \vdash * : 1}\ \vdash 1$$

$$\frac{\vec{\mathsf{p}} : A, \vec{\mathsf{q}} : B, \Gamma \vdash t : \Sigma}{\vec{\mathsf{p}} \otimes \vec{\mathsf{q}} : A \otimes B, \Gamma \vdash t : \Sigma}\ \otimes\vdash \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \iota_0(t) \boxtimes \iota_1(u) : A \otimes B}\ \vdash\otimes$$

$$\frac{!x : !A^\perp, \Gamma \vdash t : A}{!x : !A^\perp, \Gamma \vdash (t^{x++})[x_0/z]}\ !\vdash \qquad \frac{\vec{\mathsf{p}} : A, \Gamma \vdash t}{\Gamma \vdash z\langle \iota_0(\lambda \vec{\mathsf{p}}.t), \iota_1(\lambda \vec{\mathsf{p}}.t), \iota_2(\lambda \vec{\mathsf{p}}.t), \ldots \rangle : !A^\perp}\ \vdash !$$

Fig. 2. The decoration of $\mathbf{MELLP}_\mho$ proofs.

— if $\mathcal{B}$ is proper, $\mathcal{A} \vdash \mathcal{B}$ is the set of all functional designs s such that, for all $t \in \mathcal{A}$ and $u' \in \mathcal{B}^\perp$, $nf(t[s[u'/z]/z]) = \mho$.

Note that the arity of the designs in $\mathcal{A} \vdash \mathcal{B}$ is always equal to the arity of $\mathcal{A}$. Additionally, if $\mathcal{A} = \mathbf{1}$, then $t \in \mathbf{1} \vdash \mathcal{B}$ iff $t = \lambda\_.u$ with $u \in \mathcal{B}$, so $\mathbf{1} \vdash \mathcal{B}$ is essentially the same as $\mathcal{B}$.

We may now interpret a polarised sequent $\Gamma \vdash \Sigma$ as follows: if $\Gamma = C_1, \ldots, C_n$, we set $[\![\Gamma]\!] := [\![C_1]\!] \otimes \cdots \otimes [\![C_n]\!]$ (with $[\![\Gamma]\!] := \mathbf{1}$ in case $n = 0$); if Σ consists of a formula B, then $[\![\Sigma]\!] := [\![B]\!]$, otherwise $[\![\Sigma]\!] := \mathbf{0}$; then, we define $[\![\Gamma \vdash \Sigma]\!] := [\![\Gamma]\!] \vdash [\![\Sigma]\!]$.

Interpretations are designs but not all designs are interpretations of **MELLP** proofs. To characterise them, we introduce *validity*:

Definition 12 (valid design). A design is *valid* (*winning* in Girard (2001)) if it is uniform and daimon-free, i.e.it does not contain $\mho$ as subterm.

Validity induces a notion of *truth* on types: a type is *true* if it contains a valid design. This notion is non-contradictory: at most one of $\mathcal{A}$ and $\mathcal{A}^\perp$ may be true (this is because, given $t \in \mathcal{A}$ and $t' \in \mathcal{A}^\perp$, for $t \perp t'$ to hold one of the two must contain the subterm $\mho$, since it cannot be created through reduction). As pointed out by Girard (2001), the result that follows may be seen as a more general reformulation of the standard properties relating truth (in all models) and provability:

soundness: if A is provable, then $[\![A]\!]$ is true;
completeness: if $[\![A]\!]$ is true, then A is provable.

Theorem 14 (full completeness for MELLP).

1. For every **MELLP** proof π of $\Gamma \vdash A$, $[\![\pi]\!]$ is a valid design in $[\![\Gamma \vdash A]\!]$, and if $\pi \to \pi'$ by cut-elimination, then $[\![\pi]\!] = [\![\pi']\!]$.
2. Furthermore, for every $\approx$-equivalence class of valid designs of $[\![\Gamma \vdash A]\!]$, there exists a cut-free **MELLP** proof π of $\Gamma \vdash A$ such that $[\![\pi]\!]$ is a representative of that class.

The soundness part (1) of Theorem 14 is essentially the cut-elimination theorem for **MELLP**, which is standard (Laurent 2004). Indeed, one starts by defining the *degree* $\deg R$ and the *height* $h(R)$ of a cut rule R as the complexity of the cut formula and the distance from the rule to the leaves of the proof, respectively. Then, one proceeds by induction on

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the multiset $\mu(\pi)$ containing all pairs $(\deg R, h(R))$ for R ranging over the cut rules of π. If this multiset is empty, then π is cut-free and the result is obtained easily by induction on the last rule. Otherwise, there is always a cut rule R in π which may be transformed to obtain a proof π' such that $[\![\pi']\!] = [\![\pi]\!]$ (this is immediate from the definitions) and $\mu(\pi') < \mu(\pi)$, so the induction hypothesis allows us to conclude.

The completeness part (2) is an immediate consequence of Theorem 15 below. In order to prove it, we need an auxiliary definition:

Definition 13 (rank). Let u be a design and let $x \in \mathrm{fv}(u) \cap \mathcal{V}_{\mathrm{n}l}$. The *rank* of x in u is the non-negative integer defined as follows, by induction on u:

- $\mathrm{rk}_x(\mho) = \mathrm{rk}_x(\bot) = 0$;
- $\mathrm{rk}_x(v(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n)) = \delta + \sum_{j=1}^{n} \mathrm{rk}_x(\mathbf{u}_j(0))$, where $\delta = 1$ if $v = x_i$ for some i, or $\delta = 0$ otherwise;
- $\mathrm{rk}_x(\lambda\vec{\mathsf{p}}.t) = \mathrm{rk}_x(t)$.

Let $u = \lambda !x^1 \otimes \cdots \otimes !x^n.t$ be a negative design. Its *rank* is defined by $\mathrm{rk}(u) = \sum_{j=1}^{n} \mathrm{rk}_{x^j}(t)$.

The rank is defined for every (negative, non $\bot$) design, but it only makes sense for uniform designs: in that case, $\mathrm{rk}_x(u)$ is the number of free occurrences of x in the λ-term $(\!|u|\!)$.

Theorem 15 (full completeness for $\mathbf{MELLP}_{\mho}$). Let $s \in [\![\Gamma \vdash \Sigma]\!]$ be uniform. Then, there exists a cut-free proof π of $\mathbf{MELLP}_{\mho}$ such that $[\![\pi]\!] \approx s$ (where $\approx$ is reindexing equivalence, introduced in Definition 3).

Proof. The proof is by induction on the triple (h, r, d), where: h is the height of s, $r = \mathrm{rk}(s)$, and d is the size of $\Gamma \vdash \Sigma$, i.e.the total number of symbols in Γ and Σ. Triples are ordered lexicographically: $(h', r', d') < (h, r, d)$ iff $h' < h$, or $h' = h$ and $r' < r$, or $h' = h$, $r' = r$ and $d' < d$. So we suppose that the theorem holds for all $s' \in [\![\Gamma' \vdash \Sigma']\!]$ such that $(h', r', d') < (h, r, d)$, and we prove it for s and $\Gamma \vdash \Sigma$.

Let us start with the case $s = \lambda\vec{\mathsf{r}}.\mho$, in which we do not need the induction hypothesis: the proof π consisting of a single (dai) rule trivially satisfies the requirement. Then, we may suppose that $s = \lambda\vec{\mathsf{r}}.t$ with $t \neq \mho$. The proof splits into two cases, depending on whether Σ is empty or not.

Let Σ be empty. By definition, $s \in [\![\Gamma]\!]^{\bot}$, with $[\![\Gamma]\!]$ proper. Let n be its arity. The case $n = 0$ does not apply, because then we would have $s = \lambda\_.\mho$, contrarily to our assumptions. Then, we have $\Gamma = C, \Delta$, with $C \neq 1$, and $s = \lambda\vec{\mathsf{p}} \otimes \vec{\mathsf{q}}.x_i\vec{\mathbf{u}}$, with $\vec{\mathsf{p}}$ and $\vec{\mathsf{q}}$ matching the arities of $[\![C]\!]$ and $[\![\Delta]\!]$, respectively. Moreover, x is bound; we may assume without loss of generality that x appears in $\vec{\mathsf{p}}$ (otherwise, if x is in $\vec{\mathsf{q}}$, we simply permute the formulas in Γ). We have two cases:

- $C = A \otimes B$: trivially, $s \in [\![A, B, \Delta \vdash]\!]$, and $h' = h$, $r' = r$ and $d' < d$, so by the induction hypothesis we have a proof π' of conclusion $A, B, \Delta \vdash$, to which we apply a $(\otimes \vdash)$ rule and conclude;
- $C = !A^{\bot}$: $[\![C]\!]$ is of arity 1, so we actually have $s = \lambda !x \otimes \vec{\mathsf{q}}.x_i\vec{\mathbf{u}}$. We contend that $\lambda !x \otimes \vec{\mathsf{q}}.z\vec{\mathbf{u}} \in [\![!A^{\bot}, \Delta \vdash A]\!]$. Note that this satisfies $h' = h$ and $r' < r$, so it is enough to conclude: the induction hypothesis gives us a proof π' to which we apply a $(! \vdash)$ rule, obtaining a proof π such that $[\![\pi]\!] \approx s$, as desired. We need to check that, for all

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$z(\mathbf{v} \otimes \vec{\mathbf{w}}) \in [\![!A^\perp]\!] \otimes [\![\Delta]\!]$ and $t' \in [\![A]\!]^\perp$, $\mathrm{n}f((\lambda!x \otimes \vec{\mathsf{q}}.t'\vec{\mathbf{u}})(\mathbf{v} \otimes \vec{\mathbf{w}})) = \mho$. Observe that, if we define $\mathbf{v}'(i) := t'$ and $\mathbf{v}'(j) := \mathbf{v}(j)$ for $j \neq i$, we obviously have $z\mathbf{v}' \in [\![!A^\perp]\!]$, hence $z(\mathbf{v}' \otimes \vec{\mathbf{w}}) \in [\![!A^\perp]\!] \otimes [\![\Delta]\!]$. But $s(\mathbf{v}' \otimes \vec{\mathbf{w}})$ and $(\lambda!x \otimes \vec{\mathsf{q}}.t'\vec{\mathbf{u}})(\mathbf{v} \otimes \vec{\mathbf{w}})$ reduce to the same term, so we conclude because $s \in ([\![!A^\perp]\!] \otimes [\![\Delta]\!])^\perp$.

Let now Σ consist of the formula S. We have three cases:

— $S = 1$: by hypothesis we have, for all $t \in [\![\Gamma]\!]$ and $u' \in \mathbf{1}^\perp$, $\mathrm{n}f(t[s[u'/z]/z]) = \mho$. But $\mathbf{1}^\perp = \{\lambda\_.\mho\}$, which forces $s = \lambda\vec{\mathsf{p}}.z*$, with $\vec{\mathsf{p}}$ matching the arity of $[\![\Gamma]\!]$. Then, π is the proof consisting of a single $(\vdash 1)$ rule.

— $S = A \otimes B$: for similar reasons as above, we must have $s = \lambda\vec{\mathsf{p}}.z(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})$, with $\vec{\mathsf{p}}$ matching the arity of $[\![\Gamma]\!]$ and $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ matching the arity of $[\![A]\!], [\![B]\!]$, respectively. Given $z\vec{\mathbf{w}} \in [\![\Gamma]\!]$, if we define $\vec{\mathbf{u}}' := \vec{\mathbf{u}}[\vec{\mathbf{w}}/\vec{\mathsf{p}}]$ and $\vec{\mathbf{v}}' := \vec{\mathbf{v}}[\vec{\mathbf{w}}/\vec{\mathsf{p}}]$, the hypothesis $s \in [\![\Gamma \vdash A \otimes B]\!]$ gives us, for all $r' \in ([\![A]\!] \otimes [\![B]\!])^\perp$, $\mathrm{n}f(s[r'/z]\vec{\mathbf{w}}) = \mathrm{n}f(r'(\vec{\mathbf{u}}' \otimes \vec{\mathbf{v}}')) = \mho$. From this, using Lemma 11, we deduce that $z(\mathrm{n}f(\vec{\mathbf{u}}') \otimes \mathrm{n}f(\vec{\mathbf{v}}')) \in [\![A]\!] \otimes [\![B]\!]$, so by Lemma 12, $z\,\mathrm{n}f(\vec{\mathbf{u}}') \in [\![A]\!]$ and $z\,\mathrm{n}f(\vec{\mathbf{v}}') \in [\![B]\!]$. Therefore, if we define $s' := \lambda\vec{\mathsf{p}}.z\vec{\mathbf{u}}$ and $s'' := \lambda\vec{\mathsf{p}}.z\vec{\mathbf{v}}$, then for all $z\vec{\mathbf{w}} \in [\![\Gamma]\!]$, for all $t' \in [\![A]\!]^\perp$ and for all $u' \in [\![B]\!]^\perp$, using Lemma 11 again we have that $\mathrm{n}f(s'[t'/z]\vec{\mathbf{w}})$ and $\mathrm{n}f(s''[u'/z]\vec{\mathbf{w}})$ are both equal to $\mho$, showing that $s' \in [\![\Gamma \vdash A]\!]$ and $s'' \in [\![\Gamma \vdash B]\!]$. For both of these we have $h' = h'' = h$, $r', r'' \leqslant r$ and $d', d'' < d$, so the induction hypothesis applies and gives us two proofs π', π'' of $\Gamma \vdash A$ and $\Gamma \vdash B$, respectively, to which it is enough to apply a $(\vdash \otimes)$ rule to obtain the desired π.

— $S = !A^\perp$: again, we have $s = \lambda\vec{\mathsf{r}}.z\mathbf{u}$, with $\vec{\mathsf{r}}$ matching the arity of $[\![\Gamma]\!]$. Given $z\vec{\mathbf{w}} \in [\![\Gamma]\!]$, if we define $\mathbf{u}' := \mathbf{u}[\vec{\mathbf{w}}/\vec{\mathsf{r}}]$, similar reasoning as above (using Lemma 11) gives us $z\,\mathrm{n}f(\mathbf{u}') \in [\![!A^\perp]\!]$. Then, by Lemma 13, $\mathrm{n}f(\mathbf{u}'(i)) \in [\![A^\perp]\!]$ for all $i \in \mathbb{N}$, which means that $\mathbf{u}'(i) = \lambda\vec{\mathsf{p}}.v_i'$ with $\vec{\mathsf{p}}$ matching the arity of $[\![A]\!]$, so $\mathbf{u}(i) = \lambda\vec{\mathsf{p}}.v_i$, with $v_i' = v_i[\vec{\mathbf{w}}/\vec{\mathsf{r}}]$. Now, take any $i \in \mathbb{N}$, and define $s' := \lambda\vec{\mathsf{p}} \otimes \vec{\mathsf{r}}.v_i$. For a generic $z(\vec{\mathbf{v}} \otimes \vec{\mathbf{w}}) \in [\![A]\!] \otimes [\![\Gamma]\!]$, we have $s'(\vec{\mathbf{v}} \otimes \vec{\mathbf{w}}) \to v_i'[\vec{\mathbf{v}}/\vec{\mathsf{p}}]$, but this latter is also the reduct of $\mathbf{u}'(i)\vec{\mathbf{v}}$, and we know that $\mathrm{n}f(\mathbf{u}'(i)\vec{\mathbf{v}}) = \mho$. We have thus proved that $s' \in [\![A, \Gamma \vdash]\!]$, and since the height of s' is strictly smaller than the height of s, we may apply the induction hypothesis, yielding a proof π' to which we apply a $(\vdash\, !)$ rule to obtain the desired π. Indeed, note that the choice of i is irrelevant: by uniformity, $v_i \approx v_j$ for all i, j, so in any case we obtain $[\![\pi]\!] \approx s$.

□

4. Further work: the categorical perspective

We already mentioned that Melliès *et al.* (2009) used the idea underlying the approximation theorem to provide an explicit formula for constructing the free commutative comonoid in certain symmetric monoidal categories. This offers a categorical viewpoint on our work and yields a potentially interesting research direction.

Melliès, Tabareau and Tasson's construction starts with a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ such that the free copointed object on every object A exists, is denoted by $A^\bullet$ and its canonical projection by $\pi^A : A^\bullet \longrightarrow 1$.

We start by defining $A^{\leqslant n}$ to be the equaliser, if it exists, of the $n!$ parallel isomorphisms $(A^\bullet)^{\otimes n} \longrightarrow (A^\bullet)^{\otimes n}$ obtained from the symmetry of $\mathcal{C}$.

Now, by the universal property of equalisers on the morphism π^A, we know that there is a canonical projection $\pi_n^A : A^{\leqslant n+1} \longrightarrow A^{\leqslant n}$, for all $n \in \mathbb{N}$. Then, we define $!A$ to be the limit, if it exists, of the diagram

$$1 \xleftarrow{\pi^A} A^{\bullet} \xleftarrow{\pi_1^A} A^{\leqslant 2} \xleftarrow{\pi_2^A} A^{\leqslant 3} \xleftarrow{\pi_3^A} \cdots .$$

The main result of Melliès *et al.* (2009) is that, under certain hypotheses of commutation with the tensor, $!A$ is the free commutative comonoid on A. It is known that, in a $*$-autonomous category with finite products, the existence of the free commutative comonoid on every object yields a denotational model of full linear logic (a result due to Lafont, see Melliès's survey in Curien *et al.* (2010)). Therefore, the above result provides a way of building, under certain conditions, models of full linear logic starting from models of its multiplicative–additive fragment.

Our approach suggests swapping the two steps of Melliès, Tabareau and Tasson's construction: first one computes a projective limit, then one equalises. This follows our procedure for building a model of **MELLP**: we first complete the space $\ell\Lambda$ to obtain $\ell\Lambda_\infty$, then we introduce uniformity and obtain **MELLP** proofs as a uniform quotient.

More in detail, we start by defining $p_n^A : (A^{\bullet})^{\otimes n+1} \longrightarrow (A^{\bullet})^{\otimes n}$ as the morphism obtained by composing $id_{(A^{\bullet})^{\otimes n}} \otimes \pi^A$ with the iso $(A^{\bullet})^{\otimes n} \otimes 1 \cong (A^{\bullet})^{\otimes n}$. Then, we define A^{ω} as the limit (if it exists) of the following diagram, which we call $\mathcal{D}$:

$$1 \xleftarrow{\pi^A} A^{\bullet} \xleftarrow{p_1^A} (A^{\bullet})^{\otimes 2} \xleftarrow{p_2^A} (A^{\bullet})^{\otimes 3} \xleftarrow{p_3^A} \cdots .$$

At this point, if we suppose that the above limit commutes with the tensor, i.e.that, for all $n \in \mathbb{N}$, $(A^{\omega})^{\otimes n}$ is the limit of the n-dimensional version of $\mathcal{D}$, which we call $\mathcal{D}^n$, then it is easy to see that A^{ω} is a cone for every $\mathcal{D}^n$. Therefore, we have canonical morphisms $\varphi_n : A^{\omega} \longrightarrow (A^{\omega})^{\otimes n}$. We may then ask $!A$ to be the equaliser of all combinations of φ_n with the symmetries of $\mathcal{C}$. Ongoing joint work with Pellissier (2014) shows that this is indeed an alternative rephrasing of Melliès, Tabareau and Tasson's construction and is doubtlessly an interesting topic of further research.

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References

Abramsky, S., Jagadeesan, R. and Malacaria, P. (2000). Full abstraction for PCF. *Information Computation* **163** (2) 409–470.

Arnold, A. and Nivat, M. (1980). The metric space of infinite trees. Algebraic and topological properties. *Fundamenta Informaticae* **3** (4) 445–476.

Basaldella, M. (2008). *On Exponentials in Ludics*, PhD thesis, University of Siena.

Basaldella, M. and Faggian, C. (2009). Ludics with repetitions (exponentials, interactive types and completeness). In: *Proceedings of LICS* 375–384.

Boudol, G. (1993). The lambda-calculus with multiplicities. *Technical report* 2025, INRIA Sophia-Antipolis.

Bourbaki, N. (1998). *General Topology: Chapters 1–4*, Springer.

Courcelle, B. (1983). Fundamental properties of infinite trees. *Theoretical Computer Science* **25** (2) 95–169.

Curien, P. (1998). Abstract böhm trees. *Mathematical Structures in Computer Science* **8** (6) 559–591.

Curien, P.-L., Herbelin, H., Krivine, J.-L. and Melliès, P.-A. (2010). *Interactive Models of Computation and Program Behavior*, American Mathematical Society.

Dershowitz, N., Kaplan, S. and Plaisted, D. A. (1991). Rewrite, rewrite, rewrite, rewrite, rewrite,. . . . *Theoretical Computer Science* **83** (1) 71–96.

Ehrhard, T. (2005). Finiteness spaces. *Mathematical Structures in Computer Science* **15** (4) 615–646.

Ehrhard, T. and Regnier, L. (2008). Uniformity and the Taylor expansion of ordinary lambda-terms. *Theoretical Computer Science*, 403(2–3) 347–372.

Girard, J.-Y. (1987). Linear logic. *Theoretical Computer Science* **50** (1) 1–102.

Girard, J.-Y. (1991). A New Constructive Logic: Classical Logic. *Mathematical Structures in Computer Science* **1** (3) 255–296.

Girard, J.-Y. (2001). Locus solum. *Mathematical Structures in Computer Science* **11** (3) 301–506.

Hyland, M. and Ong, L. (2000). On full abstraction for PCF: I, II and III. *Information Computation* **163** (2) 285–408.

Kennaway, R., Klop, J. W., Sleep, R. and de Vries, F.-J. (1997). Infinitary lambda calculus. *Theoretical Computer Science*. **175** (1) 93–125.

Kfoury, A. J. (2000). A linearization of the lambda-calculus and consequences. *Journal of Logic and Computation* **10** (3) 411–436.

Laurent, O. (2004). Polarized games. *Annals of Pure and Applied Logic* **130** (1–3) 79–123.

Laurent, O. and Regnier, L. (2003). About translations of classical logic into polarized linear logic. In: *Proceedings of LICS* 11–20.

Mazza, D. (2012). An infinitary affine lambda-calculus isomorphic to the full lambda-calculus. In: *Proceedings of LICS* 471–480.

Mazza, D. (2014). Non-uniform polytime computation in the infinitary affine lambda-calculus. In: *Proceedings of ICALP, Part II* 305–317.

Melliès, P.-A. (2004). Asynchronous games 1: A group-theoretic formulation of uniformity. Technical Report PPS//04//06//n°31, Preuves, Programmes et Systèmes.

Melliès, P.-A. (2005). Asynchronous games 4: A fully complete model of propositional linear logic. In: *Proceedings of LICS* 386–395.

Melliès, P.-A. (2006). Asynchronous games 2: The true concurrency of innocence. *Theoretical Computer Science* **358** (2–3) 200–228.

Melliès, P.-A., Tabareau, N. and Tasson, C. (2009). An explicit formula for the free exponential modality of linear logic. In: *Proceedings of ICALP* 247–260.

Pellissier, L. (2014). *Another Formula for the Free Exponential Modality in Linear Logic*. Mémoire de Master 2, ENS Cachan.

Révész, G. E. (1992). A list-oriented extension of the lambda-calculus satisfying the Church–Rosser theorem. *Theoretical Computer Science* **93** (1) 75–89.

Rodenburg, P. H. (1998). Termination and confluence in infinitary term rewriting. *Journal of Symbolic Logic* **63** (4) 1286–1296.

Terese (2003). *Term Rewriting Systems*, Cambridge Tracts in Theoretical Computer Science volume 55, Cambridge University Press.

Terui, K. (2011). Computational ludics. *Theoretical Computer Science* **412** (20) 2048–2071.

Vial, P. (2014). *Structures uniformes en lambda-calcul polyadique affine*. Mémoire de Master 2, Université Paris 7.

Vollmer, H. (1999). *Introduction to Circuit Complexity - A Uniform Approach*, Texts in Theoretical Computer Science, Springer.