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Analytical theory of homogeneous mean shear turbulence

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A compact nonlinear expression for the velocity spectra of homogeneous mean shear flow is derived by means of a simplified two-point closure. It applies to all scales and times. The derived equation can be viewed as a nonlinear extension of the linear, rapid-distortion-theory (RDT) equation. The principal simplification is to model the nonlinear pressure–strain rate as first-order in the spectral anisotropy: a spectral Rotta-equation. This simplified equation and its solution are expressed in terms of the RDT solution. That solution helps reveal the role of nonlinearity. An equation for the velocity spectrum is then obtained at all scales and times. A dominant characteristic predicted for nonlinear behaviour is that the turbulence energy grows exponentially with time, with the spectrum simultaneously moving to smaller and smaller wavenumbers. The nonlinear growth rate is determined. Other analytical predictions of the derived equation include: the conditions for self-similarity; local isotropy; various properties of mean shear flow, including characteristic energy, length and temporal growth scales; and a critique of perturbation theory. Comparisons are made with laboratory experiments and direct numerical simulations. Although the theory applies to all scales and times, including an exact expression for RDT, the calculations are focused on nonlinear behaviour at large times. Several approximations used in this work are examined.

Key words: homogeneous turbulence, Navier–Stokes equations, turbulence theory

1. Introduction

1.1. *Background and purpose*

Homogeneous turbulence subject to mean shear is a classical problem of turbulence. It is also among the simplest of real turbulent flows. These flows have been studied extensively by laboratory experiments for a half-century (e.g. Uberoi 1957; Rose 1966; Champagne, Harris & Corrsin 1970; Tavoularis & Corrsin 1981; Rohr *et al.* 1988, and many others), and, more recently, by direct numerical simulations (DNS) (Rogers, Moin & Reynolds 1986; Lee, Kim & Moin 1990; Gotoh & Kaneda 1991; Ishihara, Yoshida & Kaneda 2002). One early theory of such flows was formulated

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by Deissler (1961) and is sometimes referred to as rapid distortion theory (to be referred to as RDT). Nonlinearity was ignored. That theory is therefore limited to times and Reynolds numbers that are not large enough to trigger nonlinear cascade (Lee *et al.* 1990; Clark & Zemach 1995, and others). At the other extreme are two-point closure theories that treat mean shear as a perturbation of steady Kolmogorov turbulence at large Reynolds number: these studies include Leslie (1973), Rubinstein (1996), Ishihara *et al.* (2002) and Yoshida, Ishihara & Kaneda (2003). Such studies are generally confined to small scales (in the inertial range) and to large times. In contrast, Cambon, Jeandel & Mathieu (1981) derived a pragmatic simplification of the eddy-damped quasi-normal Markovian (EDQNM) closure (Orszag 1970; Pouquet *et al.* 1975). The predicted root-mean-square (RMS) statistics compared favourably with laboratory measurements, and has had other applications (e.g. Bos & Bertoglio 2007). An even simpler spectral model was developed by Besnard *et al.* (1996) and applied by (Clark & Zemach 1995). That model is more *ad hoc*, relying on Leith's (1967) diffusion approximation for spectral energy transfer in addition to an assumed spectral Rotta-type equation for the scalar pressure–strain rate. A review of work on homogeneous turbulence is given by Cambon & Sagaut (2008).

The situation then, to our knowledge, is that no expression yet exists for the velocity spectrum of homogeneous mean shear turbulence without limitations on time or wavenumber. Nor is it understood, by theory, how such turbulence would evolve in time when nonlinearity does become important. The purpose of our work is to derive an analytical, although approximate, nonlinear expression that determines the velocity spectrum and its temporal evolution at any scale and time.

Our method is based on a simplified two-point spectral closure. The principal simplification is to model the spectrum of the nonlinear pressure–strain rate (referred to as the PSR) to be first-order in the anisotropic velocity spectrum: a spectral Rotta-equation. That model PSR is *ad hoc*. This model is used in the interest of obtaining a computable solution of the fully anisotropic closure. No model is assumed for the spectra themselves. Such a PSR model was used previously by Besnard *et al.* (1996) for the same reason, but for scalar spectra. Here, it is used for vector spectra. It has the feature that, at the least, its average value is correct, i.e. the RMS of this model yields the original Rotta (1951) equation in real space, as extended by the present author (Weinstock 1982). The other approximation is to neglect the anisotropic part of the nonlinear energy transfer function. That part is small in comparison with the PSR term. Such a neglect (of energy transfer anisotropy) to simplify application of the direct interaction approximation (DIA) to anisotropic flow was originally suggested by Leslie (1973). A related anisotropic energy transfer approximation has been justified numerically (e.g. Herring 1974; Schumann & Herring 1976). A critique of our approximations is given in § 7.

With these approximations, the major part of our work is independent of previous closures. Only for the calculation of scalar energy spectra and numerical coefficients is the EDQNM closure used.

The derived closure equation may be viewed as a nonlinear extension of the RDT. Owing to its relative simplicity, the derived nonlinear equation is expressed in terms of the RDT equation, and its nonlinear solution is expressed in terms of the RDT solution. However, our focus is not on the RDT of which much is already known, but, rather, on its nonlinear extension.

2. Analytical closure for shear flows

To derive an equation for the spectral tensor $\boldsymbol{R}(\boldsymbol{k}, t)$ we begin with the Navier–Stokes equation. This derivation has been often been given (e.g. Townsend 1976; Cambon *et al.* 1981) except that, here, the PSR term is divided into isotropic and anisotropic parts. We repeat the derivation for the sake of completeness and to establish the nomenclature. The fluid velocity is first divided into a fluctuation part $\boldsymbol{u}$ and a mean part $\boldsymbol{U}$. An equation for $\boldsymbol{u}$ is then given by the following fluctuation part of the Navier–Stokes equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} + \boldsymbol{U}) \cdot \boldsymbol{\nabla} \boldsymbol{u} + \langle \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \rangle - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{U} - \frac{\boldsymbol{\nabla} p}{\rho_0} + \nu \nabla^2 \boldsymbol{u}, \tag{2.1}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity fluctuation at position $\boldsymbol{x}$ at time t, $p = p(\boldsymbol{x}, t)$ is the pressure fluctuation at $\boldsymbol{x}$ and t, ρ_0 is the fluid density assumed to be constant, ν is the molecular viscosity, $\boldsymbol{U} \equiv \boldsymbol{U}(\boldsymbol{x})$ denotes the mean velocity at $\boldsymbol{x}$, assumed constant in time, and the gradient $\boldsymbol{\nabla U}$ is assumed constant in space. Additionally, the angle brackets denote an ensemble average (as does an overbar). The Cartesian coordinate components of our system are denoted by x_1, x_2, x_3 so that $\boldsymbol{x} \equiv \{x_1, x_2, x_3\}$.

The Fourier transform of both sides of (2.1) gives

$$\frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t} = -\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{\nabla U} + \boldsymbol{k} \cdot (\boldsymbol{\nabla U})^{\mathrm{T}} \cdot \frac{\partial}{\partial \boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}} - (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})_{\boldsymbol{k}} - \frac{(\boldsymbol{\nabla} p)_{\boldsymbol{k}}}{\rho_0} - \nu k^2 \boldsymbol{u}_{\boldsymbol{k}}, \tag{2.2a}$$

where

$$\left.\begin{aligned} \boldsymbol{u}_{\boldsymbol{k}} &\equiv \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x}, t)\, \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \\ p_{\boldsymbol{k}} &\equiv \int \mathrm{d}\boldsymbol{x}\, p(\boldsymbol{x}, t)\, \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \\ (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})_{\boldsymbol{k}} &\equiv (2\Pi)^{-3} \int \mathrm{d}\boldsymbol{p}\, \boldsymbol{k} \cdot \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{p}} \boldsymbol{u}_{\boldsymbol{p}} \end{aligned}\right\} \tag{2.2b}$$

denote the Fourier transform of $\boldsymbol{u}(\boldsymbol{x}, t)$, $p(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}, t)$, respectively, and we use the condensed notation $\boldsymbol{u}_{\boldsymbol{k}} \equiv \boldsymbol{u}_{\boldsymbol{k}}(t)$ and $p_{\boldsymbol{k}} \equiv p_{\boldsymbol{k}}(t)$. For the Fourier transforms, it is assumed that the fluid volume of the system consists of a box with periodic boundaries or that V is so large that $\boldsymbol{u}, p$ and their first derivatives vanish at the boundaries. The superscript T in $(\boldsymbol{\nabla U})^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{\nabla U}$ (e.g. $\mathrm{T}\boldsymbol{kp} \equiv \boldsymbol{pk}$, $\mathrm{T}k_1 p_2 \equiv p_2 k_1$). The Cartesian directional components of $\boldsymbol{k}$ and $\boldsymbol{p}$ are denoted by k_1, k_2, k_3, and p_1, p_2, p_3, respectively. To arrive at (2.2*a*) use was made of the relation $\boldsymbol{U} \equiv \boldsymbol{x} \cdot \boldsymbol{\nabla U}$, where $\boldsymbol{\nabla U}$ is constant.

For our incompressible flow, the following expression for $p \equiv p(\boldsymbol{x}, t)$ is obtained from the divergence of both sides of (2.1):

$$\frac{\nabla^2 p}{\rho_0} = -\boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) - 2(\boldsymbol{\nabla u} : \boldsymbol{\nabla U}), \tag{2.3}$$

The Fourier transform of this expression can be written as

$$p_{\boldsymbol{k}} = p_{\boldsymbol{k}}^{L} + p_{\boldsymbol{k}}^{NL}, \tag{2.4a}$$

$$\frac{p_{\boldsymbol{k}}^{L}}{\rho_0} \equiv 2\mathrm{i} \boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{\nabla U} \cdot \boldsymbol{k}/k^2, \tag{2.4b}$$

$$\frac{p_{\boldsymbol{k}}^{NL}}{\rho_0} \equiv -(2\Pi)^{-3} \int \mathrm{d}\boldsymbol{p} \frac{\boldsymbol{kk}}{k^2} : \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{p}} \boldsymbol{u}_{\boldsymbol{p}}. \tag{2.4c}$$

Substitution of (2.4*a*) into (2.2*a*) yields the fluctuation equation (e.g. Cambon *et al.* 1981)

$$\frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t} = -\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{\nabla U} \cdot \left(\boldsymbol{I} - 2\frac{\boldsymbol{kk}}{k^2} \right) + \boldsymbol{k} \cdot (\boldsymbol{\nabla U})^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} - \left\lfloor (\boldsymbol{u} \cdot \boldsymbol{\nabla u})_{\boldsymbol{k}} + \frac{(\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}}}{\rho_0} \right\rfloor - \nu k^2 \boldsymbol{u}_{\boldsymbol{k}}, \tag{2.5}$$

where $\boldsymbol{I}$ is the identity matrix (it satisfies $\boldsymbol{I} \cdot \boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{u}_{\boldsymbol{k}}$) and p^{NL} denotes the nonlinear part of p. (We ask the readers' indulgence for our initial use of dyadic notation rather than coordinate indices. The former notation is convenient for our inclusion of general mean shear $\boldsymbol{\nabla U}$. Indices are afterward used for the case of transverse mean shear.) The velocity tensor $\boldsymbol{R}(\boldsymbol{k}, t)$ is related to the more commonly used coordinate component notation $R_{ij}(\boldsymbol{k}, t)$ by $R_{ij}(\boldsymbol{k}, t) \equiv \hat{\boldsymbol{x}}_i \cdot \boldsymbol{R}(\boldsymbol{k}, t) \cdot \hat{\boldsymbol{x}}_j$, where $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{x}}_j$ denote unit vectors along Cartesian coordinates x_i and x_j and $i, j = 1, 2$ or 3. Equivalently, one has $\boldsymbol{R}(\boldsymbol{k}, t) \equiv \sum_{i,j=1}^{3} R_{ij}(\boldsymbol{k}, t) \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$.

As is standard to obtain an equation for $\boldsymbol{R} \equiv \boldsymbol{R}(\boldsymbol{k}, t)$, or for $R_{ij}(\boldsymbol{k}, t)$, we post-multiply (2.5) by the complex-conjugate velocity $\boldsymbol{u}_{\boldsymbol{k}}^*$, pre-multiply the complex-conjugate of (2.5) by $\boldsymbol{u}_{\boldsymbol{k}}$, and then add the ensemble averages of the two resulting equations. The result, a condensed form of previous work (e.g. Townsend 1976; Cambon *et al.* 1981), is

$$\frac{\partial \boldsymbol{R}(\boldsymbol{k}, t)}{\partial t} = -\boldsymbol{L}_0(\boldsymbol{k}) \cdot \boldsymbol{R}(\boldsymbol{k}, t) - V^{-1}(1 + \mathrm{T}^*) \left\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla u})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* + \frac{(\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}}}{\rho_0} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle - \nu k^2 \boldsymbol{R}(\boldsymbol{k}, t), \tag{2.6a}$$

$$\boldsymbol{L}_0(\boldsymbol{k}) \cdot \boldsymbol{R} \equiv (1 + \mathrm{T}) \boldsymbol{R} \cdot \boldsymbol{\nabla U} \cdot \left(\boldsymbol{I} - 2\frac{\boldsymbol{kk}}{k^2} \right) - \boldsymbol{k} \cdot (\boldsymbol{\nabla U})^{\mathrm{T}} \cdot \frac{\partial}{\partial \boldsymbol{k}} \boldsymbol{R}. \tag{2.6b}$$

$$\boldsymbol{R}(\boldsymbol{k}, t) \equiv V^{-1} \left\langle \boldsymbol{u}_{\boldsymbol{k}}(t) \boldsymbol{u}_{\boldsymbol{k}}^*(t) \right\rangle. \tag{2.6c}$$

where $\boldsymbol{R} \equiv \boldsymbol{R}(\boldsymbol{k}, t)$ is the tensor velocity spectrum, V is the volume of the system, T^* denotes the complex conjugate of the transpose (e.g. $\mathrm{T}^* \boldsymbol{kp} \equiv \boldsymbol{p}^* \boldsymbol{k}^*$) and the tensor $\boldsymbol{L}_0(\boldsymbol{k})$ is a Liouville operator. (The components of $\boldsymbol{L}_0(\boldsymbol{k})$ for the case of transverse mean shear are the same as the tensor components $N_{ij}(\boldsymbol{k}, t)$ used by Leslie (1973) except for the transposed term.) The angle-bracketed term in (2.6*a*) is nonlinear. We note that deletion of this nonlinear term gives the RDT exactly.

Next, the superscripts I and A are defined to denote the isotropic and anisotropic parts of any correlation function. With that definition, the PSR term in (2.6*a*) is formally divided into isotropic and anisotropic parts as follows:

$$\left\langle \rho_0^{-1} (\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle = \left\langle \rho_0^{-1} (\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle^I + \left\langle \rho_0^{-1} (\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle^A, \tag{2.6d}$$

where the isotropic part is defined by

$$\left\langle \rho_0^{-1} (\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle^I \equiv (4\Pi)^{-1} \int \mathrm{d}\Omega \frac{1}{2} \left\langle \rho_0^{-1} (\boldsymbol{\nabla} p^{NL})_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}^* \right\rangle_{ii}. \tag{2.6e}$$

Here

$$\int \mathrm{d}\Omega \equiv \int_0^{2\Pi} \mathrm{d}\phi \int_0^{\Pi} \mathrm{d}\theta \, \sin\theta \tag{2.6f}$$

denotes the spherical integral with θ denoting the angle between $\boldsymbol{k}$ and the $\hat{\boldsymbol{x}}_2$-direction (so that $\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_2 = k\cos\theta$), ϕ denoting the azimuthal angle (i.e. the angle that the $\hat{\boldsymbol{x}}_3$-axis makes with the projection of $\boldsymbol{k}$ upon the $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_3$ plane).

The anisotropic part is formally defined by (2.6*d*) as the following difference:

$$\left\langle \rho_0^{-1}(\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*\right\rangle^A \equiv \left\langle \rho_0^{-1}(\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*\right\rangle - \left\langle \rho_0^{-1}(\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*\right\rangle^I. \tag{2.7}$$

Substitution of (2.6*d*) in (2.6*a*) yields the desired condensed form for $\boldsymbol{R}(\boldsymbol{k},t)$:

$$\frac{\partial \boldsymbol{R}(\boldsymbol{k},t)}{\partial t} = \underbrace{-\boldsymbol{L}_0(\boldsymbol{k})\cdot\boldsymbol{R}(\boldsymbol{k},t)}_{\text{Linear non-dissipation terms}} \underbrace{-\boldsymbol{\Phi}^A(\boldsymbol{k},t)}_{\text{Nonlinear pressure–strain rate}} \underbrace{-\boldsymbol{T}(\boldsymbol{k},t)}_{\text{Nonlinear advection}} \underbrace{-2\nu k^2\boldsymbol{R}(\boldsymbol{k},t)}_{\text{Dissipation}}, \tag{2.8a}$$

where

$$\boldsymbol{\Phi}^A(\boldsymbol{k},t) \equiv V^{-1}(1+\mathrm{T}^*)\left\langle (\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*/\rho_0\right\rangle^A, \tag{2.8b}$$

$$\boldsymbol{T}(\boldsymbol{k},t) \equiv V^{-1}(1+\mathrm{T}^*)\left[\left\langle (\boldsymbol{u}\cdot\nabla\boldsymbol{u})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*\right\rangle + \left\langle \rho_0^{-1}(\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*\right\rangle^I\right]. \tag{2.8c}$$

We note in (2.8*a*) that the isotropic PSR term $\left\langle (\nabla p^{NL})_{\boldsymbol{k}}\boldsymbol{u}_{\boldsymbol{k}}^*/\rho_0\right\rangle^I$ has been included within the energy transfer term $\boldsymbol{T}(\boldsymbol{k},t)$.

Thus far, (2.8*a*) differs from previous work in that the nonlinear term has been divided into two parts: the anisotropic part of the PSR term and an energy transfer term.

2.1. *The nonlinear terms* $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$ *and* $\boldsymbol{T}(\boldsymbol{k},t)$

To complete the closure of (2.8*a*), the two nonlinear terms $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$ and $\boldsymbol{T}(\boldsymbol{k},t)$ must be expressed in terms of $\boldsymbol{R}(\boldsymbol{k},t)$: to do this, we will use simplifying approximations as follows.

The PSR term $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$

A formal closure expression of $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$ is complicated, containing multiple integrals over time and over the wave vector components of all $\boldsymbol{R}(\boldsymbol{k},t)$. For arriving at a computable model, we assume that $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$ is first-order in spectral anisotropy – a spectral Rotta-equation (Rotta 1951) – given by

$$\boldsymbol{\Phi}^A(\boldsymbol{k},t) \approx -\varphi(\boldsymbol{k},t)\left[\boldsymbol{R}(\boldsymbol{k},t) - \boldsymbol{R}^I(\boldsymbol{k},t)\right], \tag{2.9a}$$

where $\boldsymbol{R}^I(\boldsymbol{k},t)$ denotes the isotropic part of $\boldsymbol{R}(\boldsymbol{k},t)$ defined by

$$\left.\begin{aligned} &\boldsymbol{R}^I(k,t) \equiv 2\Pi^2 k^{-2}E(\boldsymbol{k},t)\boldsymbol{P}(\boldsymbol{k}),\\ &\boldsymbol{P}(\boldsymbol{k}) \equiv \boldsymbol{I} - \boldsymbol{k}\boldsymbol{k}/k^2,\\ &E(k,t) \equiv \frac{k^2}{2\Pi^3}\int \mathrm{d}\Omega\, \tfrac{1}{2}R_{ii}(\boldsymbol{k},t) \quad \text{(summed on } i\text{)}. \end{aligned}\right\} \tag{2.9b}$$

Here, the coefficient $\varphi(k,t)$ is an eddy viscosity that may vary with k and t. It is determined next. Equation (2.9*a*) is the main approximation of our paper.

A closure expression for $\varphi(k,t)$ is derived in appendix A and is given there as follows:

$$\varphi(k,t) \approx \frac{1}{5\Pi}\int_0^\infty p^2\,\mathrm{d}p \int \mathrm{d}\Omega\, \Theta_{kpq}(t)\frac{k^4E(p,t)E(q,t)}{p^2q^2E(k,t)}\left[1 - \frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{k^2q^2}\right], \tag{2.9c}$$

$$\Theta_{kpq}(t) \equiv \frac{1-\exp\left(-t\left[\eta(k,t)+\zeta(p,t)+\zeta(q,t)+\nu(k^2+p^2+q^2)\right]\right)}{\eta(k,t)+\zeta(p,t)+\zeta(q,t)+\nu(k^2+p^2+q^2)}, \tag{2.9d}$$

$$\eta(k,t) \approx 0.36\left[\int_0^k \mathrm{d}p\, p^2 E(p,t)\right]^{1/2}, \tag{2.9e}$$

$$\zeta(k,t) \approx \eta(k,t), \tag{2.9f}$$

$$\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0. \tag{2.9g}$$

Here $\eta(k,t)$ notes the temporal decay rate of a response function (Kraichnan 1959, 1976), $\zeta(k,t)$ denotes the inverse correlation time of the velocity self-correlation function (e.g. Orszag 1970; Kraichnan 1976; André & Lesieur 1977) and the spherical integral is defined by (2.6*f*). A distinction between $\eta(k,t)$ and $\varphi(k,t)$ is that the former is a decay rate of a two-time spectrum whereas the latter is the decay rate of the (single-time) spectrum, itself. The two are proportional in the inertial range at large k, i.e. in that range (2.9*c*) reduces to $\varphi(k,t) \approx 2\eta(k,t) \approx 0.8k^{2/3}\varepsilon^{2/3}$, where ε denotes the energy dissipation rate.

The energy transfer function $\boldsymbol{T}(\boldsymbol{k},t)$

The other nonlinear quantity needed to complete (2.8*a*) is a closure expression for $\boldsymbol{T}(\boldsymbol{k},t)$. First, it is divided into isotropic and anisotropic parts as follows:

$$\left.\begin{aligned}
&\boldsymbol{T}(\boldsymbol{k},t) \equiv \boldsymbol{T}^I(k,t)+\boldsymbol{T}^A(\boldsymbol{k},t),\\
&\boldsymbol{T}^I(\boldsymbol{k},t) \equiv \frac{2\Pi^2}{k^2}T(k,t)\boldsymbol{P}(\boldsymbol{k})\\
&T(k,t) \equiv \frac{k^2}{(2\Pi)^3}\int \mathrm{d}\Omega\, \tfrac{1}{2}T_{ii}(\boldsymbol{k},t),
\end{aligned}\right\} \tag{2.10a}$$

where $\boldsymbol{T}^I(\boldsymbol{k},t)$ and $\boldsymbol{T}^A(\boldsymbol{k},t) \equiv \boldsymbol{T}(\boldsymbol{k},t)-\boldsymbol{T}^I(\boldsymbol{k},t)$ denote the isotropic and anisotropic parts of $\boldsymbol{T}(\boldsymbol{k},t)$ respectively, and $T(k,t)$ is the scalar energy transfer function. This anisotropic part is neglected in (2.8*a*) in comparison with $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$, an approximation required to arrive a computable model. Neglect of $\boldsymbol{T}^A(\boldsymbol{k},t)$ in order to simplify application of the DIA (direct interaction approximation, Kraichnan 1959) to anisotropic flow was originally suggested and used by Leslie (1973). With this approximation (2.10*a*) reduces to

$$\boldsymbol{T}(\boldsymbol{k},t) \approx \frac{2\Pi^2}{k^2}T(k,t)\boldsymbol{P}(\boldsymbol{k}). \tag{2.10b}$$

What remains now is the transfer function $T(k,t)$. A closure expression for it taken from the literature (e.g. Kraichnan 1959; Orszag 1970) is given by

$$T(k,t) = \frac{1}{2\Pi}\int_0^\infty p^2\,\mathrm{d}p \int \mathrm{d}\Omega\, \Theta_{kpq}(t)b(\boldsymbol{k},\boldsymbol{p})\frac{k^2E(p,t)}{p^2}\left[\frac{k^2}{q^2}E(q,t)-E(k,t)\right], \tag{2.10c}$$

$$b(\boldsymbol{k},\boldsymbol{p}) \equiv \frac{\left(1-\cos^2\theta_p\right)\left[1-2(p/k)\cos\theta_p+(p/k)^3\cos\theta_p\right]}{1+(p/k)^2-2(p/k)\cos\theta_p} \quad \text{(polar coordinates)}, \tag{2.10d}$$

where θ_p is the angle between $\boldsymbol{k}$ and $\boldsymbol{p}$ so that $\cos\theta_p \equiv \boldsymbol{k}\cdot\boldsymbol{p}/(kp)$, and the definition and limits of the spherical integral $\int \mathrm{d}\Omega$ are given by (2.6*f*). This expression for $T(k,t)$ is common to many closures, differing only in the decay time $\Theta_{kpq}(t)$. The decay time to be used in our work is given by (2.9*d*), which is found in the EDQNM closure

(Orszag 1970; Pouquet *et al.* 1975; Cambon *et al.* 1981). That decay time is consistent with a $k^{-5/3}$ scalar energy spectrum in the inertial range, but is *ad hoc*.

Finally, the completed closed equation for $\boldsymbol{R}(\boldsymbol{k},t)$ is obtained by substitution of (2.9*a*) and (2.10*b*) into (2.8*a*) with the following result:

$$\left[\frac{\partial}{\partial t} + \underset{\substack{\text{Linear}\\ \text{damping}}}{2\nu k^2} + \underset{\substack{\text{Nonlinear}\\ \text{damping}}}{\varphi(k,t)} + \underset{\substack{\text{Linear production}\\ +\text{Linear PSR}\\ +\text{Linear transport}}}{\boldsymbol{L}_0(\boldsymbol{k})\cdot}\right]\boldsymbol{R}(\boldsymbol{k},t) = \underset{\text{Nonlinear } k\text{-transfer}}{\frac{2\Pi^2}{k^2}\left[T(k,t)+\varphi(k,t)E(k,t)\right]\boldsymbol{P}(\boldsymbol{k})}. \tag{2.11a}$$

Equation (2.11*a*) is a nonlinear extension of the RDT. It is the principal result of our work. This equation determines the vector spectrum $\boldsymbol{R}(\boldsymbol{k},t)$ as a function of $E(k,t)$, since $\varphi(k,t)$ and $T(k,t)$ are functions of $E(\boldsymbol{k},t)$ given by (2.9*c*) and (2.10*c*), while $E(k,t)$, itself, is defined in terms of $\boldsymbol{R}(\boldsymbol{k},t)$ by

$$E(k,t) \equiv (2\Pi)^{-3}k^2\int \mathrm{d}\Omega\, R_{ii}(\boldsymbol{k},t)/2, \tag{2.11b}$$

so that (2.11*a*) and (2.11*b*) are a closed set of equations for $\boldsymbol{R}(\boldsymbol{k},t)$. They apply to all scales and times.

A formally exact solution of (2.11*a*) for $\boldsymbol{R}(\boldsymbol{k},t)$ in terms of $E(k,t)$ is given next for the special case of transverse mean shear.

3. Formal solution of the closure equation for $\boldsymbol{R}(\boldsymbol{k},t)$

3.1. *Solution of (2.11a) for* $\boldsymbol{R}(\boldsymbol{k},t)$

Although (2.11*a*) can now be solved for general mean shears, the purpose of this section is to solve it for the special case of a transverse mean shear flow given by $\boldsymbol{\nabla U} = (\partial U_1/\partial x_2)\hat{\boldsymbol{k}}_2\hat{\boldsymbol{k}}_1$, the flow for which, as previously mentioned, the temporal development seems clearest and has been investigated by experiments and DNS. Here, $\hat{\boldsymbol{k}}_1$ and $\hat{\boldsymbol{k}}_2$ denote unit vectors along Cartesian directions 1 and 2, the longitudinal and transverse and directions, respectively. For this shear the linear term $\boldsymbol{L}_0(\boldsymbol{k})\cdot\boldsymbol{R}(\boldsymbol{k},t)$ in (2.6*b*) reduces to

$$\boldsymbol{L}_0(\boldsymbol{k})\cdot\boldsymbol{R}(\boldsymbol{k},t) \equiv S\left[k_1\frac{\partial \boldsymbol{R}(\boldsymbol{k},t)}{\partial k_2} - (1+\mathrm{T})\boldsymbol{R}(\boldsymbol{k},t)\cdot\hat{\boldsymbol{k}}_2\left(\hat{\boldsymbol{k}}_1 - 2\frac{\boldsymbol{k}k_1}{k^2}\right)\right], \tag{3.1}$$

$$S \equiv \partial U_1/\partial x_2. \tag{3.2}$$

Upon substitution of (3.1) into the closure (2.11*a*) and then converting the result to coordinate notation we obtain

$$\left[\frac{\partial}{\partial t} - k_1 S\frac{\partial}{\partial k_2} + 2\nu k^2 + \varphi(k,t)\right]R_{ij}(\boldsymbol{k},t) = -S\left[\left(\delta_{i1} - 2\frac{k_i k_1}{k^2}\right)R_{j2}(\boldsymbol{k},t) + \left(\delta_{j1} - 2\frac{k_j k_1}{k^2}\right)R_{i2}(\boldsymbol{k},t)\right] + \frac{2\Pi^2}{k^2}\left[T(k,t)+\varphi(k,t)E(k,t)\right]P_{ij}(\boldsymbol{k}), \tag{3.3}$$

where δ_{ij} denotes the Kronecker delta. This expression consists of a set of coupled partial-differential equations for the individual spectral components $R_{ij}(\boldsymbol{k}, t)$ in terms of $E(k, t)$.

The simplest component equation of (3.3) is that for $R_{22}(\boldsymbol{k}, t)$ owing to its being uncoupled from the other equations. Being uncoupled, this equation is the key equation for our nonlinear set, as it was for the RDT. The other components $R_{ij}(\boldsymbol{k}, t)$ can afterwards be easily derived in terms of the solution of $R_{22}(\boldsymbol{k}, t)$.

3.2. *Solution of (3.3) for* $R_{22}(\boldsymbol{k}, t)$

To solve (3.3) for $R_{22}(\boldsymbol{k}, t)$ we first note that it has the form of a linear, first-order inhomogeneous partial differential equation for $R_{22}(\boldsymbol{k}, t)$ in the independent variables t and k_2. This equation is re-written as

$$\left[\frac{\partial}{\partial t} - k_1 S\frac{\partial}{\partial k_2} + \nu k^2 + \varphi(k, t) - 4S\frac{k_1 k_2}{k^2}\right] R_{22}(\boldsymbol{k}, t)$$
$$= \frac{2\Pi^2}{k^2}\left[T(k, t) + \varphi(k, t)E(k, t)\right] P_{22}(\boldsymbol{k}), \tag{3.4}$$

so that, the left side is the *homogeneous* part. A formal solution of (3.4) can be written down almost immediately in terms of the solution of the homogeneous part by integrating along the k_2-space trajectories of the differential operator $\partial/\partial t - k_1 S\partial/\partial k_2$; i.e. by recognizing that if $f(k_1, k_2, k_3, t)$ denotes any function of $\boldsymbol{k}$ at time t then the solution of the homogeneous equation

$$\left(\frac{\partial}{\partial t} - k_1 S_0\frac{\partial}{\partial k_2} + \nu k^2 + \varphi(k, t) - 4S\frac{k_1 k_2}{k^2}\right) f(k_1, k_2, k_3, t) = 0 \tag{3.5}$$

is given by the following (trajectory displaced) initial value:

$$f(k_1, k_2, k_3, t) = f(k_1, k_2 + k_1 St, k_3, 0)$$
$$\times \exp\left(-\int_0^t \mathrm{d}t_1 \left\{2\nu k^2(t - t_1) + \varphi\left[k(t - t_1), t_1\right] - 4S\frac{k_1 k_2 (t - t_1)}{k^2(t - t_1)}\right\}\right), \tag{3.6}$$

where the $\boldsymbol{k}(t)$ trajectory is defined by

$$\boldsymbol{k}(t) \equiv k_1\hat{\boldsymbol{x}}_1 + k_2(t)\hat{\boldsymbol{x}}_2 + k_3\hat{\boldsymbol{x}}_3, \tag{3.7a}$$
$$k_2(t) \equiv k_2 + tSk_1. \tag{3.7b}$$

Upon integrating both sides of (3.4) along the trajectory in k_2-space, we obtain the formal solution:

$$R_{22}(\boldsymbol{k}, t) = \underline{\text{Quasi-Linear}}$$
$$\left(\frac{k(t)}{k}\right)^4 R_{22}\left[\boldsymbol{k}(t), 0\right] \exp\left(-\int_0^t \mathrm{d}t_1 \left\{2\nu k^2(t_1) + \varphi\left[k(t_1), t - t_1\right]\right\}\right)$$
$$\underline{\text{Nonlinear}}$$
$$+\frac{2\Pi^2}{k^2}\int_0^t \mathrm{d}t_2 \frac{k^2(t_2)}{k^2} P_{22}\left[\boldsymbol{k}(t_2)\right]\left\{T\left[k(t_2), t - t_2\right]\right.$$
$$\left. + \varphi\left[k(t_2), t - t_2\right] E\left[k(t_2), t - t_2\right]\right\} \tag{3.8}$$

where $R_{22}(\boldsymbol{k}, 0)$ denotes the initial value of the spectrum (the spectrum at the time it is first observed), $\varphi[k(t-t_1), t_1]$ denotes $\varphi(k, t_1)$ evaluated at trajectory $k = k(t-t_1)$, and the integration variables were changed as in $\int_0^t \mathrm{d}t_1 \varphi[k(t-t_1), t_1] = \int_0^t \mathrm{d}t_1 \varphi[k(t_1), t-t_1]$. Additionally, we used the following integration:

$$\exp\left(\int_0^{t_2} \mathrm{d}t_1 4S\frac{k_1 k_2(t_1)}{k(t_1)^2}\right) = \left(\frac{k(t_2)}{k}\right)^4. \tag{3.9}$$

Equation (3.8) gives $R_{22}(\boldsymbol{k}, t)$ for any value of $S, \nu, t, \boldsymbol{k}$ and initial values in terms of $E(k, t)$ (since the quantities $\varphi(\boldsymbol{k}, t)$ and $T(k, t)$) are explicitly known functions of $E(k, t)$. The initial value $R_{22}[\boldsymbol{k}, 0]$ is arbitrary. The first term on the right-hand side of (3.8) is the same as the RDT expression of $R_{22}(\boldsymbol{k}, t)$ except for the additional, nonlinear damping $\varphi[k(t-t_1), t_1]$. This term is referred to as being 'quasi-linear' in keeping with the terminology used in theories of strong wave interactions and in plasma turbulence (e.g. Kadomtsev 1965). The second term on the right-hand side of (3.8) is purely nonlinear. It is new.

3.3. *Formal solution of (3.3) for all components of $R_{ij}(\boldsymbol{k}, t)$*

Equation (3.3) can now be solved formally for all the components of $R_{ij}(\boldsymbol{k}, t)$ by successive time integrations of each component over trajectories in k_2-space. The result is

$$\begin{aligned} R_{ij}(\boldsymbol{k}, t) = {} & \underline{\text{Quasi-Linear}} \\ & Q_{ij}(\boldsymbol{k}, t) \\ & \underline{\text{Nonlinear}} \\ & + \frac{2\Pi^2}{k^2}\int_0^t \mathrm{d}t_2 A_{ij}(\hat{\boldsymbol{k}}, t_2)\left(T\left[k(t_2), t-t_2\right] + \varphi\left[k(t_2), t-t_2\right] E\left[k(t_2), t-t_2\right]\right) \\ & \times \exp\left(-\int_0^{t_2} \mathrm{d}t_1 \left\{\nu k^2(t_1) + \varphi\left[k(t_1), t-t_1\right]\right\}\right), \end{aligned} \tag{3.10a}$$

where the dimensionless coefficients $A_{ij}(\hat{\boldsymbol{k}}, t_2)$ are given by

$$\left.\begin{aligned} A_{11}\left(\hat{\boldsymbol{k}}, t_2\right) &\equiv \underbrace{\left(1 - \frac{k_1^2}{k(t_2)^2}\right)\frac{k^2}{k(t_2)^2}}_{\text{Term 1}} + 2\underbrace{\frac{k_1 k_2(t_2) k^2}{k(t_2)^4} J_{11}(\hat{\boldsymbol{k}}, t_2)}_{\text{Term 2}} + 2\underbrace{\frac{k_\perp^2 k(t_2)^2}{k^4} J_{11}^{(2)}(\hat{\boldsymbol{k}}, t_2)}_{\text{Term 3}}, \\ A_{22}\left(\hat{\boldsymbol{k}}, t_2\right) &\equiv \frac{k_\perp^2}{k^2}, \\ A_{33}(\hat{\boldsymbol{k}}, t_2) &\equiv \underbrace{\left(1 - \frac{k_3^2}{k(t_2)^2}\right)\frac{k^2}{k(t_2)^2}}_{\text{Term 1}} - 4\underbrace{\frac{k_3 k_2(t_2) k^2}{k(t_2)^4} J_{33}(\hat{\boldsymbol{k}}, t_2)}_{\text{Term 2}} + 8\underbrace{\frac{k_1^2 k_3^2 k_\perp^2}{k^6} J_{33}^{(2)}(\hat{\boldsymbol{k}}, t_2)}_{\text{Term 3}}, \\ \begin{Bmatrix} A_{12}\left(\hat{\boldsymbol{k}}, t_2\right) \\ A_{23}\left(\hat{\boldsymbol{k}}, t_2\right) \end{Bmatrix} &\equiv \begin{Bmatrix} P_{12}\left[\boldsymbol{k}(t_2)\right] \\ P_{23}\left[\boldsymbol{k}(t_2)\right] \end{Bmatrix} - \frac{k_\perp^2}{k^2}\begin{Bmatrix} J_{12}(\hat{\boldsymbol{k}}, t_2) \\ J_{23}(\hat{\boldsymbol{k}}, t_2) \end{Bmatrix}, \end{aligned}\right\} \tag{3.10b}$$

$$k_\perp^2 \equiv k_1^2 + k_3^2, \tag{3.10c}$$

$$\left.\begin{aligned}
J_{11}(\hat{\boldsymbol{k}}, t_2) &\equiv \int_0^{t_2} \mathrm{d}t_3 S\, H_{11}\left[\boldsymbol{k}(t_3)\right], \\
J_{33}(\hat{\boldsymbol{k}}, t_2) &\equiv \int_0^{t_2} \mathrm{d}t_3 S\, H_{13}\left[\boldsymbol{k}(t_3)\right], \\
J_{11}^{(2)}(\hat{\boldsymbol{k}}, t_2) &\equiv \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \frac{S^2 k^6 H_{11}\left[\boldsymbol{k}(t_3)\right] H_{11}\left[\boldsymbol{k}(t_4)\right]}{k(t_3)^4 k(t_4)^2}, \\
J_{33}^{(2)}(\hat{\boldsymbol{k}}, t_2) &\equiv \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \frac{S^2 k^8}{k(t_3)^4 k(t_4)^4}, \\
\begin{Bmatrix} J_{12}(\hat{\boldsymbol{k}}, t_2) \\ J_{23}(\hat{\boldsymbol{k}}, t_2) \end{Bmatrix} &\equiv \int_0^{t_2} \mathrm{d}t_3 S \frac{k^2}{k(t_3)^2} \begin{Bmatrix} H_{11}\left[\boldsymbol{k}(t_3)\right] \\ H_{13}\left[\boldsymbol{k}(t_3)\right] \end{Bmatrix},
\end{aligned}\right\} \tag{3.10d}$$

$$H_{ij}(\boldsymbol{k}) \equiv \delta_{ij} - 2\frac{k_i k_j}{k^2}. \tag{3.10e}$$

In addition, the time integrals $J_{ij}(\hat{\boldsymbol{k}}, t_2)$, $J_{11}^{(2)}(\hat{\boldsymbol{k}}, t_2)$ and $J_{33}^{(2)}(\hat{\boldsymbol{k}}, t_2)$ have all been evaluated analytically in terms of elementary functions of t_2, θ and ϕ (the polar and azimuthal angles of $\boldsymbol{k}$) by means of Mathematica software. The notation Term 1, 2, or 3 refers, respectively, to the number of time integrations of the coupled solution of (3.3). It can be seen from (3.10*b*) and (3.10*d*) that as $t_2 \to 0^+$ the factor $A_{ij}(\hat{\boldsymbol{k}}, t_2)$ reduces to the simple isotropic form

$$A_{ij}(\hat{\boldsymbol{k}}, t_2) \to \delta_{ij} - k_i k_j / k^2 \equiv P_{ij}(\boldsymbol{k}) \quad (t_2 \to 0^+). \tag{3.10f}$$

The quasi-linear term $Q_{ij}(\boldsymbol{k}, t)$ in (3.10*a*) can be expressed as follows:

$$Q_{ij}(\boldsymbol{k}, t) \equiv R_{ij}^{RDT}(\boldsymbol{k}, t)\, \exp\left\{-\int_0^t \varphi[k(t_1), t - t_1]\right\} \tag{3.10g}$$

where $R_{ij}^{RDT}(\boldsymbol{k}, t)$ denotes the RDT spectrum (e.g. Deissler 1961; Fox 1964; Lee, Piomelli & Reynolds 1986; Lee *et al.* 1990, and others). $Q_{ij}(\boldsymbol{k}, t)$ is seen to be the RDT spectrum multiplied by a nonlinear exponential damping. For convenience, and because the RDT term has previously been investigated, its components are relegated to the appendix B. However, it will be useful to give the trace of $Q_{ij}(\boldsymbol{k}, t)$ here as follows:

$$\begin{aligned}
Q_{ii}(\boldsymbol{k}, t) = \frac{k(t)^2}{k^2} R_{ii}[k(t), 0] &\left\{ \frac{k(t)^2}{k^2} + 1 + \frac{k_3^2 k^2}{k_1^2 (k_1^2 + k_3^2)} \right. \\
&\left. \times \left[\tan^{-1} \frac{k_2}{(k_1^2 + k_3^2)^{1/2}} - \tan^{-1} \frac{k_2 + S k_1(t)}{(k_1^2 + k_3^2)^{1/2}} \right]^2 \right\} \\
&\times \exp\left\{ -2\nu t \left[k^2 + S k_1 k_2 t + \tfrac{1}{3} S^2 k_1^2 t^2 \right] - \int_0^t \mathrm{d}t_1 \varphi[k(t_1), t - t_1] \right\}.
\end{aligned} \tag{3.10h}$$

Equation (3.10*a*) is a formally exact solution of (2.11*a*) for $R_{ij}(\boldsymbol{k}, t)$. It determines that spectrum for all scales and times as a function of $E(k, t)$. The latter spectrum is determined by (2.11*b*), a closed set of equations. Before solving this set of equations; we consider a qualitative feature.

3.4. *Quasi-linear and nonlinear terms*

Equation (3.10*a*) divides $R_{ij}(\boldsymbol{k}, t)$ in two parts with their two distinct temporal behaviours: the quasi-linear term $Q_{ij}(\boldsymbol{k}, t)$ which governs the early turbulence growth, the eventual diminution of growth being a consequence of the nonlinear damping quantity $\int_0^t \mathrm{d}t_1 \varphi[k(t_1), t - t_1]$ in (3.10*g*); and the second term which is nonlinear and governs the growth at large times. It is entirely new. Theoretically, then, (3.10*a*) implies a temporal transition from RDT-dominated growth to nonlinear growth.

The sequence in time from RDT to nonlinear cascade is expected intuitively. It is described theoretically by (3.10*a*). The transition time varies with k.

However, our objective in this subsection was not to investigate the RDT itself, but, rather, to point out that a time t exits beyond which nonlinearity is dominant.

3.5. *Closed equation for $E(k, t)$: completion of the solution for $R_{ij}(\boldsymbol{k}, t)$*

To complete the formal solution for $R_{ij}(\boldsymbol{k}, t)$ we obtain a closed equation for $E(k, t)$. This is accomplished by merely substituting (3.10*a*) into the right-hand side of (2.11*b*) to obtain

$$E(k,t) = \underbrace{\frac{1}{2}\overline{Q_{ii}(k,t)}}_{\text{Quasi-linear}} + \underbrace{\frac{1}{2}\int \frac{\mathrm{d}\Omega}{4\Pi}\int_0^t \mathrm{d}t_2 A_{ii}(\hat{\boldsymbol{k}}, t_2)\left\{T\left[k(t_2), t-t_2\right] + \varphi\left[k(t_2), t-t_2\right] E\left[k(t_2), t-t_2\right]\right\} \times \exp\left(-\int_0^{t_2} \mathrm{d}t_1 \left\{\nu k^2(t_1) + \varphi\left[k(t_1), t-t_1\right]\right\}\right)}_{\text{Nonlinear}}, \tag{3.11}$$

where $\overline{Q_{ii}(k,t)} \equiv k^2(2\Pi)^{-3}\int \mathrm{d}\Omega Q_{ii}(\boldsymbol{k}, t)$ is the integral of $Q_{ii}(\boldsymbol{k}, t)$ over a spherical shell of radius k.

This equation is closed for $E(k, t)$. Together, (3.10*a*) and (3.11) determine the details of transverse mean shear turbulence: (3.10*a*) gives $R_{ij}(\boldsymbol{k}, t)$ in terms of the scalar spectrum $E(k, t)$ and that spectrum is determined by solution of (3.11). The latter is the only differential equation that need be solved.

Numerical solutions of the nonlinear equations at large St (large Reynolds number)

Our numerical solution of (3.11) and (3.10*a*) makes use of the Liouville–Neumann method of successive approximations (e.g. Whittaker & Watson 1952). Briefly, the initial approximation for $E(k, t)$ was first required to satisfy the analytical solutions of (3.11) in three ranges: the inertial, dissipation and small-k ranges, and additionally to be equal to the derived value of $E(k, t)$ at a familiar integral scale $k_0(t)$ defined next. An algebraic form of $E(k, t)$ was initially assumed between the mentioned asymptotic ranges of k. This approximation for $E(k, t)$ was substituted into the right-hand side of (3.11) to obtain a second approximation for $E(k, t)$. The second approximation was then substituted into the right-hand side of (3.11) to obtain yet a third approximation and so on. An analitical solution is available on request.

The results of that solution are given in figures 1 and 2. In figure 1 is plotted the relative directional energy spectrum $E_{\alpha\alpha}(k, t)/E(k, t)$ versus $k/k_0(t)$ for $St = 60$, large enough for the nonlinear term to dominate in the energy-containing range. Here α denotes a directional index 1, 2 or 3. The directional kinetic energy spectrum $E_{\alpha\alpha}(k, t)$

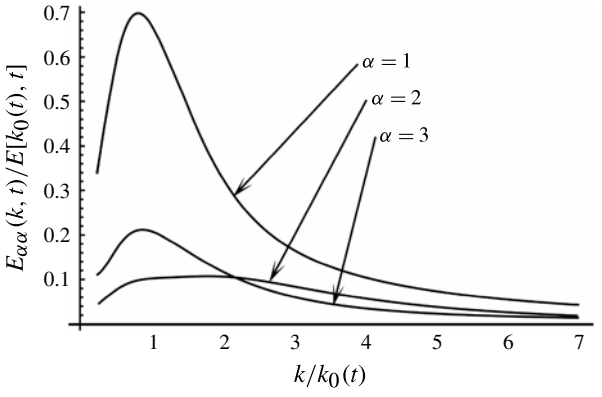

FIGURE 1. Graph of theoretically predicted $E_{\alpha\alpha}(k, t)/E[k_0(t), t]$ versus $k/k_0(t)$, for $\alpha = 1, 2, 3$ at $St = 60$ (Reynolds number of $Re_T \approx 100\,000$).

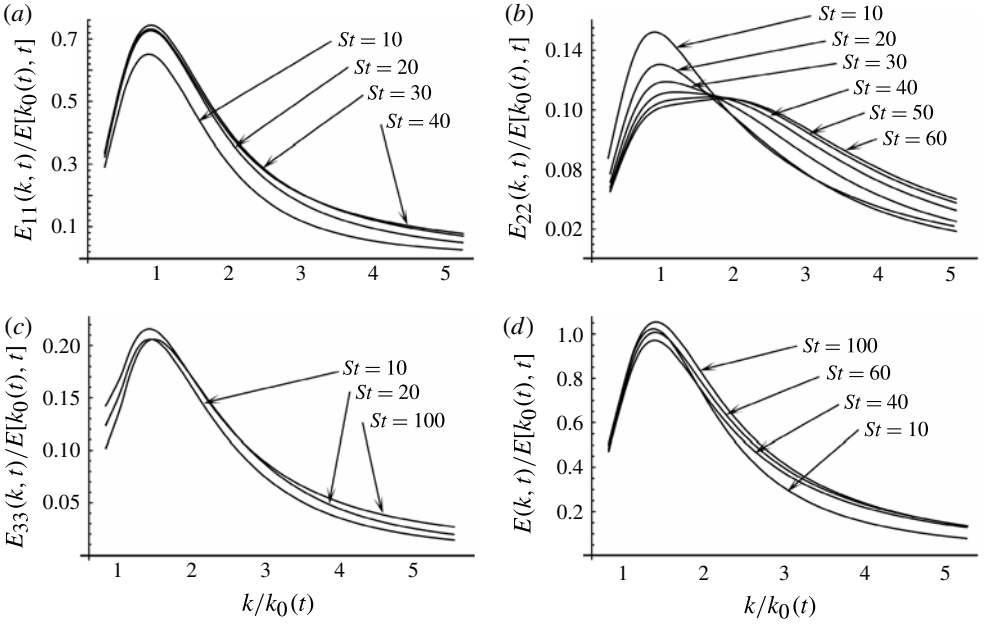

FIGURE 2. Graph of theoretical (*a*) $E_{11}(k, t)/E[k_0(t), t]$, (*b*) $E_{22}(k, t)/E[k_0(t), t]$, (*c*) $E_{33}(k, t)/E[k_0(t), t]$, and (*d*) $E(k, t)/E[k_0(t), t]$, versus $k/k_0(t)$ at various S_0t.

and integral scale $k_0(t)$ are defined by

$$\left.\begin{aligned} E_{\alpha\alpha}(k,t) &\equiv (2\Pi)^{-3}k^2 \int \mathrm{d}\Omega\, R_{\alpha\alpha}(\boldsymbol{k},t) \quad (\alpha = 1, 2 \text{ or } 3), \\ k_0(t) &\equiv \int \mathrm{d}k E(k,t) \Big/ \int \mathrm{d}k\, k^{-1}E(k,t) \quad \text{(integral wavenumber)}. \end{aligned}\right\} \tag{3.12a}$$

The values of S, ν and initial spectrum $E_{\alpha\alpha}(k, 0)$ used in our computation are taken from a DNS ('run' C128R of Rogers *et al.* 1986; $S = 28.3$, $\nu = 0.01$ and a 'top hat' initial spectrum). The value $St = 60$ corresponds to Reynolds number $Re_T \equiv \nu^{-1}\varepsilon(t)^{-1} q^4(t)$ of about 100 000 where $q^2(t) \equiv (3/2)\int \mathrm{d}k\, E(k, t)$ and $\varepsilon(t) \equiv 2\nu \int \mathrm{d}k\, k^2 E(k, t)$.

We note that for the large value of St considered here the RDT is confined to small k while most of the contribution to $\int \mathrm{d}k\, E(k, t)$ comes from the nonlinear terms, our nonlinear limit. Our reason for confining figure 1 to the range $0.4 \leqslant k/k_0(t) \leqslant 7$ is to emphasize the shape of the spectra in the energy-containing range, the range which is least known by theory when nonlinearity is dominant. The inertial–dissipation range is considered separately in § 5.2.

The computed theoretical Reynolds stresses $\langle u_i(t)u_j(t)\rangle \equiv \int \mathrm{d}k E_{ij}(k, t)$, roughly equal to the area under the curves in figure 2 for diagonal components, are given by

$$\frac{\langle u_\alpha(t)^2\rangle}{q^2(t)} = \{0.53, 0.18, 0.29\}, \quad St > 20, \tag{3.12b}$$

$$-\frac{\langle u_1(t)u_2(t)\rangle}{q^2(t)} = 0.11\frac{\left(1 + 5.7Re_T^{-1/2}\right)}{\left(1 + 3.2Re_T^{-1}\right)}. \tag{3.12c}$$

The predicted diagonal relative Reynolds stresses are within 10 % of the DNS (Rogers *et al.* 1986 for runs C128R, C128X and RR128 when $St > 6$); they are closer to the laboratory experiments of Harris, Graham & Corrsin (1977) and Tavoularis & Corrsin (1981). The off-diagonal relative stress in (3.12*c*) is 15 % smaller than the DNS runs but it, too, is closer to experiment.

In contrast to (3.12*b*), the DNS at lower Reynolds number (e.g. run C128V and C128W of Rogers *et al.* 1986) or with strong shear (Lee *et al.* 1990) resemble the RDT; e.g. they approach one-dimensionality of kinetic energy in the streamwise direction, as discussed further at the end of § 4.2.

With regard to the relative stress $-\langle u_1(t)u_2(t)\rangle/q^2(t)$, its slow decrease with increasing Re_T (or, equivalently, with increasing time) seen in (3.12*c*) is also found in DNS (Rogers *et al.* 1986; Lee *et al.* 1990). Other comparisons with DNS and laboratory experiments are given in §§ 4.2 and 6.1.

Various properties of shear flow are next determined by analytical solutions of (2.11*a*). They begin with conditions for self-similarity and local isotropy.

4. Self-similarity and derivation of energy-containing scales

4.1. *Self-similarity*

The purpose of this section is to derive conditions for which the spectrum $k^2 R_{ij}(\boldsymbol{k}, t)$ is self-similar: by self-similar we specifically mean that $k^2 R_{ij}(\boldsymbol{k}, t)$ varies with $\boldsymbol{k}$ solely in the ratio $\boldsymbol{k}/k_0(t)$.

To derive this behaviour in a *time-varying* turbulence the following self-similar form is first assumed for $E(k, t)$ (e.g. Lesieur & Shertzer 1978; Lesieur 1987):

$$E(k, t) = S^2 k_0(t)^{-3} G_1[k/k_0(t)] \quad \text{(range of } k \text{ to be determined)}, \tag{4.1a}$$

and, afterwards, its validity is determined by substitution in (2.11*a*), the defining equation of $R_{ij}(\boldsymbol{k}, t)$. For present purposes, the function $G_1[k/k_0(t)]$ need not be known except that it is dimensionless. To show that the nonlinear terms $\varphi(k, t)$ and $T(k, t)$ are themselves self-similar we first substitute (4.1*a*) into (2.9*c*) and (2.10*c*): the result of

those substitutions, followed by elementary changes of variables, can be expressed as

$$\varphi(k,t) = SG_2[k/k_0(t)], \tag{4.1b}$$

$$T(k,t) = S^3 k_0(t)^{-3} G_3[k/k_0(t)], \tag{4.1c}$$

where the quantities $G_2[k/k_0(t)]$ and $G_3[k/k_0(t)]$ are dimensionless and vary with k only in the ratio $k/k_0(t)$, within some range of k to be determined. For convenience, expressions for $G_2[k/k_0(t)]$ and $G_3[k/k_0(t)]$ are relegated to appendix C; they are not needed for present purposes. Substitution of (4.1*b*) and (4.1*c*) in (2.11*a*) results in

$$\left(\frac{\partial}{\partial(St)} + \frac{2\nu k^2}{S} + G_2\left[k/k_0(t)\right]\right) R_{ij}(\boldsymbol{k},t) + M_{nj}\left[\boldsymbol{k}/k_0(t)\right] R_{ni}(\boldsymbol{k},t)$$
$$= \frac{2\Pi^2}{k^2} S^2 k_0(t)^{-3} \left(G_3\left[k/k_0(t)\right] + G_2\left[k/k_0(t)\right] G_1\left[k/k_0(t)\right]\right) P_{ij}\left[\boldsymbol{k}/k_0(t)\right], \tag{4.1d}$$

where

$$M_{jn}(\boldsymbol{k}) \equiv \left[\delta_{nj} k_1 \frac{\partial}{\partial k_2} - (1+\mathrm{T})\left(\delta_{1j} - 2\frac{k_1 k_j}{k^2}\right)\delta_{n2}\right]. \tag{4.1e}$$

In addition, we used the identities $M_{jn}(\boldsymbol{k}) \equiv M_{jn}[\boldsymbol{k}/k_0(t)]$, $P_{ij}(\boldsymbol{k}) = P_{ij}[\boldsymbol{k}/k_0(t)]$ and $k_1 \partial/\partial k_2 \equiv [k_1/k_0(t)]\partial/\partial[k_2/k_0(t)]$.

It can be seen that, except for the dissipation range, the terms in (4.1*d*) depend on $\boldsymbol{k}$ only in the combination $\boldsymbol{k}/k_0(t)$, which includes $k/k_0(t)$. Therefore, outside the dissipation range, the solution of (4.1*d*) can be written in the form

$$k^2 R_{ij}(\boldsymbol{k},t) = S^2 k_0(t)^{-3} G_{ij}[\boldsymbol{k}/k_0(t)], \quad \text{(self-similar)} \quad (2\nu k^2 \ll S), \tag{4.1f}$$

where $G_{ij}[\boldsymbol{k}/k_0(t)]$ is dimensionless and varies with $\boldsymbol{k}$ in only the combination $\boldsymbol{k}/k_0(t)$. This self-similar equation was the goal of this section. For our purpose, the actual expression of $G_{ij}[\boldsymbol{k}/k_0(t)]$ is not needed; it can be obtained by solution of (3.10*a*). We emphasize that, owing to the temporal growth factor $k_0(t)^{-3}$ in (4.1*f*), $R_{ij}(\boldsymbol{k},t)$ varies with time independently of its variation with $\boldsymbol{k}/k_0(t)$.

The absence of self-similarity in the dissipation range confirms the same finding by Lesieur & Shertzer (1978) in their EDQNM calculation, albeit for a case of decaying turbulence. Other investigations of self-similarity include Kida & Murakami (1987), Besnard *et al.* (1996) and Clark & Zemach (1995) for decaying turbulence and Rubinstein & Clark (2005) for a forced turbulence.

An illustration of the evolution of spectra towards self-similarity is given in figure 2(*a–d*) in § 3.5. As was done for figure 1, the range of figure 2 is limited to $0.4 \leqslant k/k_0(t) \leqslant 5$ so as to emphasize the shape of the spectra in the energy-containing range.

As we have mentioned, the increase of turbulence energy with time is accompanied by the increase of Reynolds number. An approximate expression for that number is $Re_T \approx 70(S/\nu k_0(0)^2)\exp[0.11\,St]$, obtained by numerical solutions of (3.10*a*) and (3.11) for the nonlinear conditions of DNS (Rogers *et al.* 1986 Run C128R at St exceeding 6).

4.2. *Derivation of* $k_0(t)$ *and* $E[k_0(t), t]$

We have seen in the previous section that the spectrum varies in time as $k_0(t)^{-3}$. The purpose of this section is to derive analytically both $k_0(t)$ defined by (3.12*a*), and the spectrum $E[k_0(t), t]$, but limited to the case of transverse mean shear and asymptotically large St. The quantity $E[k_0(t), t]$ is a value of the spectrum near its maximum (e.g. see figure 2*d*). Being near the maximum it will help to estimate the scaling of the RMS turbulence velocity at all t; a scaling which, by itself, provides a direct comparison between theory and experiment.

To obtain $k_0(t)$ and $E[k_0(t), t]$, we substitute $k_0(t)$ for k everywhere in (3.11) and, in addition, take the asymptotic limit $St \sim \infty$ (which corresponds to asymptotically large Reynolds number for growing turbulence). For that scale and time, both dissipation and $\overline{Q_{ii}[k_0(t), t]}$ can be neglected. Additionally, for that scale and time one can show that $k(t_2) \approx k$ for the main part of the t_2-integration in (3.11). Upon making those approximations (3.11) reduces to

$$E[k_0(t), t] = \int_0^t \mathrm{d}t_2 C_1(t, t_2) E[k_0(t - t_2), t - t_2] \exp(-t_2 \varphi[k_0(t), t]) \quad (St \sim \infty), \tag{4.2}$$

where, for convenience of notation, we defined $C_1(t, t_2)$ by

$$C_1(t, t_2) \equiv (T[k_0(t), t]/E[k_0(t), t] + \varphi[k_0(t), t]) \int \frac{\mathrm{d}\Omega}{4\Pi} \frac{A_{ii}(\hat{\boldsymbol{k}}, t_2)}{2}. \tag{4.3}$$

Equation (4.2) is not difficult to solve since $\varphi[k_0(t), t]/S$ and $T[k_0(t), t]/E[k_0(t), t]$ are both constant in time; i.e.

$$\varphi[k_0(t), t]/S = \text{constant} \quad (St \sim \infty), \tag{4.4a}$$

$$T[k_0(t), t]/E[k_0(t), t]/S = \text{constant} \quad (St \sim \infty). \tag{4.4b}$$

These self-similarity relations for $\varphi(k, t)$ and $T(k, t)$ follow from (4.1*b*) and (4.1*c*). Owing to these constancies, a solution of (4.2) for $E[k_0(t), t]$ is an exponential in time, which can be written as follows:

$$E[k_0(t), t] = E[k_0(t_0), t_0] \exp[aS(t - t_0)] \quad (St > St_0 \gg 1), \tag{4.5}$$

$$k_0(t) = k_0(t_0) \exp[-3^{-1} aS(t - t_0)], \tag{4.6}$$

where t_0 is an arbitrary large time and (4.6) follows from (4.5) and (4.1*a*). The growth rate a in (4.5) is determined implicitly by the integral

$$\int_{t_0}^{t \to \infty} \mathrm{d}t_2 C_1(t, t_2) \exp[-(aS + \varphi[k_0(t), t])t_2] = 1 \quad (St \sim \infty). \tag{4.7}$$

We emphasize that a approaches its asymptotic value when t is asymptotically large; more precisely, when $t \gg 1/\varphi[k_0(t), t]$. The following value of this growth rate was calculated by numerical solution of (4.7) combined with (2.9*c*), (2.10*c*), (3.11) and the definition of $k_0(t)$ in (3.12*a*):

$$a = 0.17 \quad (St \sim \infty), \tag{4.8}$$

a calculation for which the EDQNM closure expressions for $\varphi(k, t)$ and $T(k, t)$ (given by (2.9*c*) and (2.10*c*)) are used.

The energy spectrum in (4.5) at $k_0(t)$ can be expressed with (4.1*a*) by

$$\left.\begin{aligned} E[k_0(t), t] &= G_1[1] S^2 \exp[0.11(t - t_0)] \\ k_0(t) &= k_0(t_0) \exp[-0.057 S(t - t_0)], \end{aligned}\right\} \quad (St \sim \infty), \tag{4.9}$$

where $G_1[1] \equiv G_1[k_0(t)/k_0(t)]$ is a numerical constant. The value of this constant is calculated by numerical solution of (3.11) combined with the definition of $k_0(t)$ in (4.6), which results in

$$G_1[1] \approx 0.53. \tag{4.10}$$

We note from (4.1*a*) and (4.6) that the spectrum at any k can be written as

$$E(k, t) = \exp(0.17St)\, G_1[k/k_0(t)]. \tag{4.11}$$

This equation states that the energy spectrum $E(k, t)$ grows exponentially with time as $\exp(0.17St)$ while, simultaneously, the spectrum evolves self-similarly to smaller and smaller k; the latter evolution is due to the decrease of $k_0(t)$ in time. The exponential growth is limited only by the size of the system: the largest eddy may not exceed that size.

Let us digress to show that (4.5) conforms to observations of turbulence growth.

Universal scalings implied by (4.5) *and* (4.6)

An observed temporal scaling can be obtained by combining the relation $(3/2)u_0(t)^2 \equiv \int \mathrm{d}k E(k, t) \approx 2k_0(t)E[k_0(t), t]$ with (4.9) so as to arrive at

$$k_0(t)^2 u_0(t)^2/S^2 \approx 4G[1]/3 \approx 0.71 \quad (\text{large } t). \tag{4.12}$$

This self-similarity has been reached in the experiments of Tavoularis & Corrsin (1981) and Rohr *et al.* (1988).

The quantitative growth rate of $u_0(t)$ is obtained by substitution of (4.6) in (4.12) to yield

$$u_0(t) = u_0(t_0) \exp[0.057S(t - t_0)] \quad (St > St_0 \gg 1). \tag{4.13}$$

Such exponential growth is observed experimentally by Tavoularis (1985) at $Re_T \approx 10^4$ and is consistent with measurements of Rohr *et al.* (1988), although for less than a decade in time for both experiments. A comparison of theory with experimental results of Tavoularis (1985) is given in figure 3. Theory and experiment are close, perhaps fortuitously so. An explanation of the small difference in the two measurements is that their Reynolds numbers are not quite large enough to reach the asymptotic limit of theory.

Exponential growth of turbulence energy is also observed in DNS of Rogers *et al.* (1986), runs C128R, C128U and C128X. The observed DNS energy growth rate of $2a/3 = 0.125$ when $6 \leqslant St \leqslant 14$ fairly well confirms the theoretical $2a/3 \approx 0.115$ given by (4.13).

In contrast, DNS for the particular runs C128V and C128W of Rogers *et al.* (1986) exhibit a much smaller growth rate. The difference between the different sets of runs is, we surmise, that the latter two behave linearly whereas the former (e.g. run C128R) behaves practically nonlinearly for $St > 6$. For example, $\langle u_2(t)u_2(t)\rangle$, $\langle u_3(t)u_3(t)\rangle \ll \langle u_1(t)u_1(t)\rangle$ in runs C128V and C128W. This approach toward one-dimensionality of kinetic energy was found in the high-shear DNS of Lee *et al.* (1990). It was also predicted by Fox (1964).

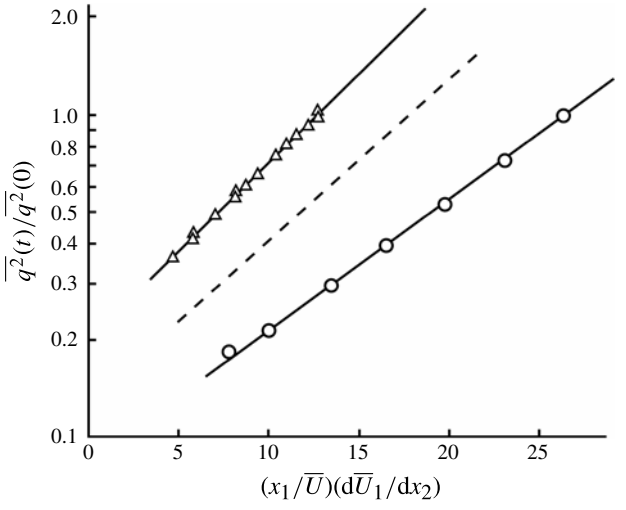

FIGURE 3. $q(t)^2/q(0)^2$ versus S_0t: –o–o– and –△–△–, experiment (Tavoularis 1985); – – –, theory.

5. Local isotropy and the inertial–dissipation range

5.1. *Local isotropy*

The previous section, § 4, considered the solution of (2.11*a*) in the energy-containing range. The purpose of this section is to determine the solution of (2.11*a*) for the inertial and dissipation ranges, including a condition for local isotropy.

To derive that condition we first substitute (2.6*b*) into (2.11*a*), and then rearrange terms to obtain

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{k},t) = &\left[1+\frac{T(k,t)/E(k,t)-2\nu k^2}{2\nu k^2+\varphi(k,t)}\right]\boldsymbol{R}^I(\boldsymbol{k},t)\\ &-\frac{(1+\mathrm{T})\boldsymbol{R}(\boldsymbol{k},t)\cdot\boldsymbol{\nabla U}\cdot(\boldsymbol{I}-2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})}{2\nu k^2+\varphi(k,t)}\\ &-\frac{1}{2\nu k^2+\varphi(k,t)}\left[\frac{\partial}{\partial t}-k\cdot(\boldsymbol{\nabla U})^{\mathrm{T}}\cdot\frac{\partial}{\partial k}\right]\boldsymbol{R}(\boldsymbol{k},t).\end{aligned}\tag{5.1}$$

Equation (5.1) is still formally exact. For the approach to local isotropy, we take k sufficiently large to satisfy $|\partial U_j/\partial x_i|/[2\nu k^2+\varphi(k,t)]\ll 1$ in (5.1), where $|\ |$ denotes the absolute value, and so obtain

$$\boldsymbol{R}(\boldsymbol{k},t)=\left[1+\frac{\mathrm{T}(k,t)/E(k,t)-2\nu k^2}{2\nu k^2+\varphi(k,t)}\right]\boldsymbol{R}^I(\boldsymbol{k},t)\quad\left(\frac{|\partial U_j/\partial x_i|}{2\nu k^2+\varphi(k,t)}\ll 1\right).\tag{5.2}$$

Taking the spherical integral of the trace of both sides of (5.2) followed by use of (2.9*b*) leads to the established relation (e.g. Kraichnan 1959, 1965; Lesieur 1987) for isotropic turbulence.

$$T(k,t)=-2\nu k^2E(k,t)\quad(\text{large } k).\tag{5.3}$$

Substitution of (5.3) back into (5.2) gives the local isotropy relation

$$\boldsymbol{R}(\boldsymbol{k},t) \to \boldsymbol{R}^I(\boldsymbol{k},t) \quad \begin{pmatrix} \text{local isotropy} \\ 2\nu k^2 + \varphi(k,t) \gg |\partial U_j/\partial x_i| \end{pmatrix}. \tag{5.4}$$

The approach to isotropy can be seen from (5.1) to be slow in the inertial range, the deviation from isotropy varying as $k^{-2/3}$ since, in that range, $\varphi(k,t) \approx 0.8k^{2/3}\varepsilon^{2/3}$.

5.2. *Expressions for $E(k,t)$ and $\varepsilon(t)$ at large k*

Returning to (5.3), we note that it is a closed equation for $E(k,t)$ (e.g. Kraichnan 1959, 1965) since $T(k,t)$ is a function of $E(k,t)$ given by (2.10*c*). It applies to the inertial–dissipation range. A previous approximate analytical solution of (5.3) (e.g. Kolmogorov 1941; Kraichnan 1965, 1959, and others), confirmed by DNS (e.g. Chen *et al.* 1993; She *et al.* 1993, and others), is given by

$$E(k,t) \approx 1.5\varepsilon(t)^{2/3}k^{-5/3}(1 + [k/k_d(t)]^5)\exp[-7.1k/k_d(t)] \quad (\text{large } k), \tag{5.5}$$

where $k_d(t) \equiv [\varepsilon(t)/\nu^3]^{1/4}$ is the Kolmogorov dissipation wavenumber and the numerical coefficients 5 and 7.1 were determined by the DNS of Chen *et al.* (1993). Implicit in this relation is that $E(k,t)$ grows with time proportional to $\varepsilon(t)^{2/3}$.

An expression for $\varepsilon(t)$ is derived by combination of the inertial-range form of (5.5) with the inertial-range form of (4.1*a*): the latter form is given, approximately, by

$$E(k,t) \approx G[1]S^2k_0(t)^{-3}[k/k_0(t)]^{-5/3} \quad (\text{inertial range}). \tag{5.6}$$

Combination of (5.6) with (5.5), together with use of $G[1] = 0.53$, gives the dissipation rate as

$$\varepsilon(t) \approx 0.22S^3k_0(t)^{-2} \quad (\text{large } Re_T \text{ and } St). \tag{5.7}$$

This relation predicts the value of $\varepsilon(t)$ in transverse mean shear flow at very large St. Comparison of (5.7) with related DNS expressions by Rogers *et al.* (1986) and by (Lee *et al.* 1990) is made in the next section.

6. Spectral anisotropy at large k, comparison with a perturbation theory and dimentionless coefficients

The purpose of this section is derive an analytical expression for the spectral anisotropy at large k. Comparisons are afterwards made with Leslie's (1973) perturbation theory.

It is convenient to begin this derivation with (5.1), which is equivalent to (2.11*a*). In the formal solution of (5.1) the operator $[\partial/\partial t - k \cdot (\boldsymbol{\nabla U})^{\mathrm{T}} \cdot \partial/\partial \boldsymbol{k}]$ displaces $\boldsymbol{k}$ along a trajectory $\boldsymbol{k}(t_2)$, as is shown explicitly in (3.10*a*) for the case of transverse mean shear. Moreover, this t_2-integral of displacement is found (by means of pedestrian expansions of the integrand of (3.10*a*) in powers of t_2) to approach zero as $S/[2\nu k^2 + \varphi(k,t)]$ with increasing k. The operator term can therefore be neglected in (5.1) when k is large, sufficiently large that $[2\nu k^2 + \varphi(k,t)] \gg S$. This proportionality of the operator term to $S^2/[2\nu k^2 + \varphi(k,t)]^2$ is also found for general $\boldsymbol{\nabla U}$. Generally, then, (5.1) reduces to

$$\boldsymbol{R}(\boldsymbol{k},t) = \boldsymbol{R}^I(\boldsymbol{k},t) - \frac{(1+\mathrm{T})\,\boldsymbol{R}(\boldsymbol{k},t)\cdot\boldsymbol{\nabla U}\cdot\left(\boldsymbol{I} - 2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\right)}{2\nu k^2 + \varphi(k,t)} \quad (\text{large } k), \tag{6.1}$$

where we also substituted (5.3) to eliminate $T(k,t)/E(\boldsymbol{k},t) - 2\nu k^2$ in the $\boldsymbol{R}^I(\boldsymbol{k},t)$ of (5.1) term. A further simplification we make is to replace $\boldsymbol{R}(\boldsymbol{k},t)$ by $\boldsymbol{R}^I(\boldsymbol{k},t)$ in the second term on the right of (6.1), justified since $\boldsymbol{R}^A(\boldsymbol{k},t)$ is of the order $\boldsymbol{\nabla U}/[2\nu k^2 + \varphi(k,t)]$. With that approximation (6.1) reduces to

$$\boldsymbol{R}(\boldsymbol{k},t) = \boldsymbol{R}^I(\boldsymbol{k},t) - \frac{2\mathit{\Pi}^2}{k^2}\frac{(1+\mathrm{T})\boldsymbol{P}(\boldsymbol{k})\cdot\boldsymbol{\nabla}\bar{\boldsymbol{U}}\cdot(\boldsymbol{I}-2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})}{2\nu k^2+\varphi(k,t)}E(k,t) \quad \text{(large } k\text{)}. \tag{6.2}$$

This equation determines $\boldsymbol{R}^A(\boldsymbol{k},t)$ to first-order in mean shear $\boldsymbol{\nabla U}$. It corresponds to small-shear perturbation theory (Leslie 1973; Ishihara *et al.* 2002, and others), but is more general since it applies to the dissipation range and to arbitrary $\boldsymbol{\nabla U}$ and, most significantly, is not limited to stationary spectra. Additionally, it can be seen that the shear need not be small for (6.2): all that is required is that k be sufficiently large.

For the case of transverse mean shear (6.2) reduces to

$$R_{ij}(\boldsymbol{k},t) = R^I_{ij}(\boldsymbol{k},t) - \frac{2\mathit{\Pi}^2}{k^2}S\left[P_{i2}(\boldsymbol{k})\left(\delta_{j1}-2\frac{k_jk_1}{k^2}\right) + P_{j2}(\boldsymbol{k})\left(\delta_{i1}-2\frac{k_ik_1}{k^2}\right)\right]\frac{E(k,t)}{2\nu k^2+\varphi(\boldsymbol{k},t)} \quad \begin{pmatrix}\text{transverse shear}\\ \text{at large } k\end{pmatrix}. \tag{6.3}$$

This relation is next compared with a previous perturbation theory.

6.1. *Comparison with a perturbation theory*

The purpose of this section is to compare Leslie's (1973) perturbation theory at large k with our theory: specifically, to compare (15.36) of Leslie with our (6.3). To facilitate this comparison, (15.36) of Leslie is written here as follows:

$$R^A_{ij}(\boldsymbol{k},t)_{(L)} = S\left[\left\{-P_{i1}(\boldsymbol{k})P_{j2}(\boldsymbol{k}) - \frac{P_{i2}(\boldsymbol{k})}{2}\frac{k_1k_j}{k^2}\right\}\frac{q^{(0)}(k,0)}{\eta+\zeta} + \frac{P_{ij}(\boldsymbol{k})}{2}\frac{k_1k_2}{k^2}\left\{\frac{k}{\eta+\zeta}\frac{\partial q^{(0)}(k,0)}{\partial k} - \frac{kq^{(0)}(k,0)}{(\eta+\zeta)^2}\frac{\partial\zeta}{\partial k}\right\}\right], \tag{6.4}$$

where $R^A_{ij}(\boldsymbol{k},t)_{(L)}$ denotes Leslie's expression for the anisotropic part of $R_{ij}(\boldsymbol{k},t)$, $q^0(k,t) \equiv 2\mathit{\Pi}^2E(k,0)k^{-2}$, $E(k,0)$ denotes the stationary Kolmogorov spectrum and the eddy quantities η and ζ are defined in (2.9*d*).

Leslie's equation, (6.4), has the same variation with k as does our (6.3) in the inertial range, taking into account that η and ζ are approximately equal to $0.5\varphi(k,t)$ in that range. However, their respective numerical coefficients do differ a little.

Another difference is that perturbation theory contains the derivative $\partial\zeta/\partial k$ whereas (6.3) does not. We do not know whether this derivative term is valid. Its absence in our theory might be due to the assumptions of our model PSR term. A refinement of our PSR model might explain this difference but is beyond the scope, and would defeat the purpose, of our present paper; it will be considered in a future work. In any event, the magnitude of this derivative term is small, much smaller than the other terms in (6.4) and in (6.3).

The stationary restriction of (6.4) as well as of previous perturbation theories may be appropriate to small scales in flows other than homogeneous mean shear. Our theory is not subject to that restriction.

Among the agreements of perturbation theory (6.4) with (6.3) in the inertial range are: the predicted stress $\langle u_1(\boldsymbol{x},t)u_2(\boldsymbol{x},t)\rangle = (2\mathit{\Pi})^{-3}\int \mathrm{d}\boldsymbol{k}R_{12}(\boldsymbol{k},t)_{(L)}$ is non-zero and

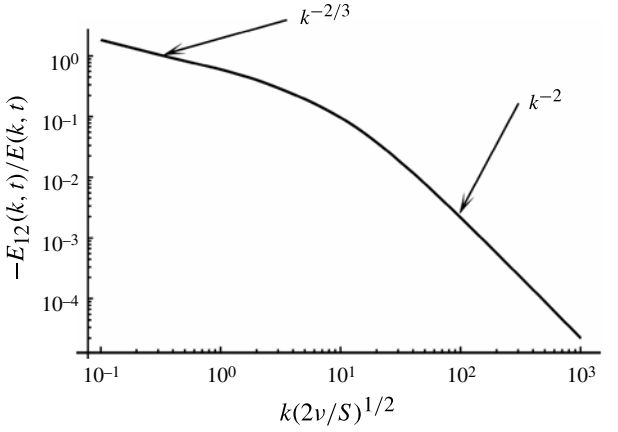

FIGURE 4. Relative scalar stress spectrum $-E_{12}(k,t)/E(k,t)$ versus $k(2\nu/S_0)^{1/2}$ displaying the inertial-range $k^{-2/3}$ behaviour and the dissipation-range k^{-2} behaviour at $S_0t = 100$.

sizable since $R^A_{12}(\boldsymbol{k},t)_{(L)}$ is symmetric in k_1 and k_2; the scalar stress spectrum $E_{12}(k,t)$ varies as $k^{-7/3}$ in the inertial range; and local isotropy is approached relatively slowly.

6.2. *Spectral power laws and dimensionless coefficients*

Equation (6.2) almost immediately gives the following asymptotic power laws for the relative scalar stress spectrum $E_{12}(k,t)/E(k,t)$, where $E_{12}(k,t) \equiv (2\Pi)^{-3}k^2\int \mathrm{d}\Omega R_{12}(\boldsymbol{k},t)$:

$$\frac{E_{12}(k,t)}{E(k,t)} \propto \begin{cases} -(S\varepsilon^{-1/3})k^{-2/3} & \text{(inertial range)} \\ -(S/\nu)k^{-2} & \text{(dissipation range)} \end{cases} \tag{6.5}$$

taking into account that (2.9*c*) reduces in the inertial range to

$$\varphi(k,t) \approx 0.8k^{2/3}\varepsilon^{1/3} \quad \text{(inertial range)}, \tag{6.6}$$

and where the far dissipation range is described by $2\nu k^2 \gg \varphi(k,t)$. These k behaviours are illustrated in figure 4. To our knowledge, the power law in the dissipation range has not been derived previously. Here, it can be seen from (6.2) to apply to all $\boldsymbol{\nabla U}$.

6.2.1. *The Lumley coefficient*

In Lumley's (1967) expression for the scalar stress spectrum, given by $E_{12}(k,t) \approx -CS\varepsilon^{1/3}k^{-7/3}$, the dimensionless coefficient C was undetermined. However, its numerical value was calculated by DNS of Ishihara *et al.* (2002). This coefficient is easily obtained analytically from (6.3) by taking the spherical integral over both sides: the result is

$$E_{12}(k,t) = -\frac{4}{15}S\frac{E(k,t)}{2\nu k^2 + \varphi(k,t)}. \tag{6.7}$$

In the inertial range (where $E(k,t) = 1.5\varepsilon(t)^{2/3}k^{-5/3}$, $\varphi(k,t) \approx 0.8k^{2/3}\varepsilon(t)^{1/3}$ and $2\nu k^2$ is negligible) this relation reduces to Lumley's expression

$$E_{12}(k,t) = -0.5S\varepsilon(t)^{1/3}k^{-7/3} \quad \text{(inertial range)} \tag{6.8}$$

and predicts its constant to be 0.5

This theoretical value is close to that obtained by DNS (Ishihara *et al.* 2002). The latter work also finds a second dimensionless constant for the Reynolds stress spectra.

6.2.2. *Ratio of stress to energy*

The ratio of stress to kinetic energy is of special interest because it has been widely measured and is fairly well known. These quantities are defined, respectively, by $\langle u_1(t)u_2(t)\rangle \equiv (2\Pi)^{-3}\int_0^\infty \mathrm{d}kE_{12}(k,t)$ and $\langle u_i(t)u_i(t)\rangle/2 \equiv (2\Pi)^{-3}\int_0^\infty \mathrm{d}kE_{ii}(k,t)/2$. A heuristic, rough estimate of their ratio can be quickly obtained from (6.7) and (5.5) by extrapolating both relations to $k = k_0(t)$ and cutting off both k-integrals at $k_0(t)$. The result is

$$\frac{\langle u_1(t)u_2(t)\rangle}{\langle u_i(t)u_i(t)\rangle/2} \approx 0.28 \quad \begin{pmatrix}\text{extrapolated}\\ \text{spectra}\end{pmatrix}. \tag{6.9}$$

This value is fortuitously close to the measured value of ~0.3.

A less heuristic value for the ratio of stress to energy can be obtained by numerical solution of (3.10*a*) and (3.11) for all k. The ratio found in this way varies from 0.3 to 0.27 as Re_T increases from 300 to 800. These values are about 15 % less than the ratios obtained by DNS, but closer to laboratory values (e.g. Tavoularis & Corrsin 1981).

6.2.3. *Shear rate parameter*

The shear rate parameter Sq^2/ε is a measure of the shear strength that has been determined by DNS (e.g. Rogers *et al.* 1986; Lee *et al.* 1990). For our nonlinear case of large St, it is obtained by combination of (4.12) with (5.7) with the result

$$Sq^2/\varepsilon \approx 10 \quad (\text{Large } St \text{ and } Re_T). \tag{6.10}$$

This value is fairly close to the DNS runs RR128, C128R and C128X for $St \geqslant 8$; the DNS values vary from ~9 to 11 for $8 \leqslant St \leqslant 14$ peaking at ~11. Theoretically, the (nonlinear) shear rate parameter is time-invariant at large St as are (4.12) and (5.7) from which it is derived.

Much larger values of this parameter were found in the large shear DNS of Lee *et al.* (1990). A possible explanation for this difference is that DNS of Lee *et al.* (1990) is nearly linear whereas the mentioned DNS runs of Rogers *et al.* (1986) are nearly nonlinear.

7. Approximations of the theory

Sources of errors or uncertainties in the theory are: (*a*) modelling of the nonlinear PSR as first-order in the anisotropic part of the vector spectrum; (*b*) neglect of mean shear for the PSR model; (*c*) neglect of $\boldsymbol{T}^A(\boldsymbol{k},t)$; (*d*) use of the formal EDQNM closure (Orszag 1970) for $T(k,t)$. An assessment of these uncertainties in our work is as follows:

(*a*) A Rotta-like model of the nonlinear pressure–strain rate is used as a first approximation for the sake of obtaining a computable model. This is our main approximation. Other authors (e.g. Cambon *et al.* 1981; Clark & Zemach 1995; Besnard *et al.* 1996) have previously used this model for the same reason. The plausibility of this model is that it conforms to the physical considerations given by Rotta (1951) for his original expression in real space, and, relatedly, correctly reduces to that expression when both sides are integrated over all $\boldsymbol{k}$ (Weinstock 1982). Nevertheless, it remains *ad hoc* for individual values of $\boldsymbol{k}$. It is most uncertain for $k/k_0 \ll 1$.

(*b*) Neglect of mean shear in the PSR coefficient $\varphi(k,t)$ may cause the theory to be too nonlinear at early times where shear terms are still comparable to, or exceed, nonlinear terms.

(*c*) One justification of neglect of $\boldsymbol{T}^A(\boldsymbol{k},t)$ is that it is small in comparison with $\boldsymbol{\Phi}^A(\boldsymbol{k},t)$ in (2.8*a*)): the latter dominates the anisotropy of the turbulence. A related anisotropy approximation has been verified numerically (e.g. Herring 1974; Schumann & Herring 1976). Neglect of $\boldsymbol{T}^A(\boldsymbol{k},t)$ was originally suggested and used by Leslie (1973) to simplify the application of the DIA to anisotropic flow.

(*d*) The accuracy of the EDQNM for shear flow turbulence is not established for general scales. Its accuracy may be estimated by comparison with experiment. Such a comparison made by Cambon *et al.* (1981) found the EDQNM in good agreement with measurements of stationary turbulence. Additionally, an EDQNM prediction of the Rotta coefficients by the present author (Weinstock 1982) agreed with DNS by Rogers *et al.* (1986).

8. Summary and discussion

(*a*) Our principal result is (2.11*a*), a compact nonlinear equation for the velocity spectrum of mean shear turbulence at all scales and times. This equation can be viewed as a nonlinear extension of the RDT.

(*b*) The main simplification was to model the spectrum of the nonlinear PSR as first-order in the anisotropic velocity spectrum: a spectral Rotta equation. This simplification was made for the sake of arriving at a computable model. It is *ad hoc*, but plausibility arguments are given in § 7. The coefficient $\varphi(k,t)$ of this Rotta-like equation was derived rationally.

(*c*) Equation (2.11*a*) determines the temporal evolution of turbulence from the initial state onwards. Initially, an RDT term emerges that governs the early evolution of turbulence. The initial growth is followed by steady nonlinear exponential growth that continues indefinitely, restricted only by the size of the system.

The nonlinear terms cause cascade to smaller scales while the RDT terms produce larger scales. A theoretical 'transition' time predicts when both terms become competitive. That time varies with wavenumber. This work focuses on conditions for which nonlinear terms dominate the energy of the system: very large St and Reynolds numbers.

(*d*) The solution of (2.11*a*) is obtained for the case of transverse mean shear. That solution is given by (3.10*a*). It consists of a formally exact expression for $R_{ij}(\boldsymbol{k},t)$ in terms of $E(k,t)$, while $E(k,t)$ itself is determined by (3.11), a closed integral equation. Both numerical and analytical solutions of the latter equation are given.

The analytical solution is available on request.

(*e*) The velocity spectrum is proven to be self-similar for k^{-1} larger than dissipation wavenumbers; i.e. when k satisfies $2\nu k^2 > S$ in the case of transverse shear.

Local isotropy is proven to occur when k is large enough to satisfy $\varphi(k,t) + 2\nu k^2 \gg |\partial U_j \partial x_i|$, the inertial and dissipation range.

(*f*) Analytical expressions are derived for the total kinetic energy, the integral scale, the growth rate of turbulence spectra and various coefficients; e.g. these expressions show that the scalar spectra $E(k,t)$ and integral length scale $k_0(t)^{-1}$ grow exponentially in time as $\exp(0.17St)$ and $\exp(0.057St)$, respectively. Simultaneously with growth, the spectrum shifts to smaller and smaller wavenumber. Such growth was observed in laboratory experiments and in DNS but was limited to less than a decade in time.

Theory predicts that this growth continues indefinitely, subject only to the size of the system.

(*g*) Equation (2.11*a*) asymptotically reduces to a simple expression for $R_{ij}(\boldsymbol{k}, t)$ in the inertial–dissipation ranges for all $\boldsymbol{\nabla U}$ and time t. This asymptotic equation is given by (6.3). The *relative* scalar stress spectra are predicted to vary as $k^{-2/3}$ in the inertial range, as found previously, and to vary as k^{-2} in the dissipation range. The latter behaviour has not been previously derived so far as we know.

(*h*) Comparisons are made between our theory and the perturbation theory of Leslie (1973) for large k. The stress spectra of the two theories agree qualitatively for large k but differ a little in their respective numerical coefficients. This difference may be due to the assumed PSR model of our work. A discussion is given in § 6. The principal differences are that the perturbation theory is limited to large wavenumbers and to an assumed stationary condition whereas the present theory shows energy grows in time. Because of its stationary condition, perturbation theory may be confined to small scales in inhomogeneous flows.

Appendix A. Equation for $\varphi(k, t)$

To derive (2.9*c*) for $\varphi(k, t)$ we substituted the following definition of the directional *scalar* kinetic energy spectrum:

$$E_{\alpha\alpha}(k, t) \equiv (2\Pi)^{-3} k^2 \int \mathrm{d}\Omega R_{\alpha\alpha}(\boldsymbol{k}, t), \tag{A 1}$$

into the integrand on the right-hand side of (27) of Weinstock (1982) and substituted $(2\Pi)^{-3} \int \mathrm{d}\boldsymbol{k}\varphi(k, t)[R_{\alpha\alpha}(\boldsymbol{k}, t) - R^I_{\alpha\alpha}(\boldsymbol{k}, t)]$ for $2A^N_{\alpha\alpha}$ in the left-hand side of that equation, where the directional index α equals 1, 2 or 3. Additionally, we used $\eta(k, t)$ and $\zeta(k, t)$ instead of $k\langle u(r, t)u(r, t)\rangle^{1/2}$; the latter is the small-k limit of the inverse correlation time whereas the former is an inertial-range expression. The wavevector notation of Weinstock was changed from $\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2$ to $\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}$, respectively,

Appendix B. The components of $R^{RDT}_{ij}[\boldsymbol{k}, t]$

The components of the RDT spectrum $R^{RDT}_{ij}[\boldsymbol{k}, t]$ can be written as follows:

$$\begin{aligned} R^{RDT}_{ij}(\boldsymbol{k}, t) \equiv \Bigg(& R_{ij}(\boldsymbol{k}, 0) \left\{ \frac{k(t)^4}{k^4}\delta_{i2}\delta_{j2} + \frac{k(t)^2}{k^2}\delta_{i2}(\delta_{j1} + \delta_{j3}) + \delta_{i1}\delta_{j1} + \delta_{i3}\delta_{j3} \right\} \\ & - S \int_0^t \mathrm{d}t_2 \left\{ 2\frac{\boldsymbol{k}(t)^2}{k^2}(H_{11}[\boldsymbol{k}(t_2)]R_{12}[\boldsymbol{k}(t), 0]\delta_{i1}\delta_{j1} \right. \\ & + H_{13}[\boldsymbol{k}(t_2)]R_{23}[\boldsymbol{k}(t), 0]\delta_{i3}\delta_{j3}) \\ & \left. + \frac{k(t)^4}{k^2 k(t_2)^2}(H_{11}[\boldsymbol{k}(t_2)]\delta_{i1}\delta_{j2} + H_{13}[\boldsymbol{k}(t_2)]\delta_{i2}\delta_{j3})R_{22}[\boldsymbol{k}(t), 0] \right\} \\ & + 2S^2 \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \frac{k(t)^4}{k(t_2)^2 k(t_3)^2} \{H_{11}[\boldsymbol{k}(t_3)]P_{12}[\boldsymbol{k}(t_2)]\delta_{i1}\delta_{j1} \\ & + H_{13}[\boldsymbol{k}(t_3)]P_{23}[\boldsymbol{k}(t_2)]\delta_{i3}\delta_{j3}\}R_{22}[\boldsymbol{k}(t), 0] \Bigg) \exp\left(- \int_0^{t_2} \mathrm{d}t_1 \nu k^2(t_1) \right). \end{aligned} \tag{B 1}$$

Appendix C. The dimensionless functions $G_2[k/k_0(t)]$ and $G_3[k/k_0(t)]$

The functions $G_2[k/k_0(t)]$ and $G_3[k/k_0(t)]$ are straightforwardly obtained by substitution of (4.1*a*,*b*,*c*) in (2.9*c*) and (2.10*c*) to obtain

$$G_2[k/k_0(t)] \equiv 0.5 \int \frac{\mathrm{d}\boldsymbol{p}}{k_0(t)^3} \left(\frac{\Theta_{kpq}(t)}{S}\right) \frac{k^4/k_0(t)^4}{q_2/k_0(t)^2} \frac{k_0(t)^2}{k^2+p^2} \times G_1[p/k_0(t)] \times \frac{G_1[q/k_0(t)]}{G_1[k/k_0(t)]} \times \left[1 - \frac{(\boldsymbol{q}\cdot\boldsymbol{k})^2}{q^2k^2}\right] \tag{C 1a}$$

$$G_3[k/k_0(t)] = \frac{1}{2\Pi} \int \frac{\mathrm{d}\boldsymbol{p}}{k_0(t)^3} \frac{\Theta_{kpq}(t)}{S} b\left(\frac{\boldsymbol{k}}{k_0(t)}, \frac{\boldsymbol{p}}{k_0(t)}\right) \frac{k^2/k_0(t)^2}{p^2/k_0(t)^2} \times \left[\frac{k^2/k_0(t)^2}{q^2/k_0(t)^2} G_1\left[\frac{q}{k_0(t)}\right] - G_1\left[\frac{k}{k_0(t)}\right]\right], \tag{C 1b}$$

$$\frac{\Theta_{kpq}(t)}{S} = 0.36 \left\{ \sqrt{\int_0^{k/k_0(t)} \mathrm{d}yy^2 G_1[y]} + \sqrt{\int_0^{p/k_0(t)} \mathrm{d}yy^2 G_1[y]} + \sqrt{\int_0^{q/k_0(t)} dyy^2 G_1[y]} \right\} + \text{dissipation terms of order } \nu k^2/S, \tag{C 2}$$

where use was made of the identity $b(\boldsymbol{k},\boldsymbol{p}) = b(\boldsymbol{k}/k_0(t), \boldsymbol{p}/k_0(t))$. Equations (C1) and (C 2) confirm that $G_2[k/k_0(t)]$ and $G_3[k/k_0(t)]$ vary with k only in the combination $k/k_0(t)$.

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