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FINITE SUM OF COMPOSITION OPERATORS ON FOCK SPACE

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Abstract

We investigate unbounded, linear operators arising from a finite sum of composition operators on Fock space. Real symmetry and complex symmetry of these operators are characterised.

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1. Introduction

Function composition is an operation that takes two functions f and φ and produces a function g such that $g(x) = f(\varphi(x))$. If f varies in a linear space of functions defined on the range of φ, then the mapping C_φ sending f into $f \circ \varphi$ is a linear transformation, called a *composition operator*. Although the composition operation is basic to mathematics and its studies have been pursued for a long time, the study of composition operators as a part of operator theory has a relatively short history, starting in the mid 1960s with Nordgren's paper [16].

The books by Cowen and MacCluer [4] and Shapiro [18] describe composition operators on Hardy and Bergman spaces. On Fock space, Carswell *et al.* [2] found that only the class of affine transformations $\varphi(z) = az + b, |a| \leq 1$ and $b = 0$ whenever $|a| = 1$ induces bounded composition operators. They characterised compactness of C_φ by the strict requirement $|a| < 1$. In [14], Le described all *bounded* composition operators that are normal. Later, the author [10] extended Le's result to *unbounded* composition operators and also obtained conditions that are more general than the bounded case.

Berkson [1] initiated the study of a sum of composition operators and this was taken further by Shapiro and Sundberg [19], who investigated the topological structure of the space of composition operators acting on Hardy space. For Fock spaces, Choe *et al.* [3] showed that a linear sum of two composition operators is bounded (respectively,

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compact) if and only if both composition operators are already bounded (respectively, compact). Other properties have been studied such as conditions for a sum to be bounded on Bloch spaces [12] or completely continuous on Hardy and Bergman spaces [5].

A second source of our study is the theory of complex symmetric operators initiated by Garcia and Putinar [8, 9]. A *complex symmetric operator* is an unbounded, linear operator $T : \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with the property that $T = CT^*C$, where C is an isometric involution (in short, *conjugation*) on $\mathcal{H}$. To indicate the dependence on conjugation, this case is called *C-selfadjoint*. A long list of well-known operators have been proven complex symmetric: normal operators, Hankel matrices and compressed Toeplitz operators (including the compressed shift). For more on complex symmetry, see [7–9].

Recently, there has been interest in the problem of classifying composition operators that are complex symmetric. The papers [6, 13] studied the problem on Hardy spaces of the unit disk corresponding to the conjugation

$$\mathcal{Q}f(z) = \overline{f(\overline{z})}.$$

The conjugation $\mathcal{Q}$ inspired the author and Khoi [11] to study this problem in Fock space.

In this paper, we are interested in unbounded, linear operators arising from the expression $\mathcal{E}(\varphi, d)f = \sum_{j=1}^{d} f \circ \varphi_j$, where the functions satisfy

$$\varphi_t \not\equiv \varphi_s \quad \text{whenever } t \neq s. \tag{1.1}$$

Our research is conducted on Fock space

$$\mathcal{F}^2 = \Big\{ f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ is entire with } \sum_{n=0}^{\infty} n!\,|f_n|^2 < \infty \Big\},$$

the reproducing kernel Hilbert space with the reproducing kernel $K_z^{[m]}(u) = u^m e^{u\overline{z}}$, equipped with the inner product

$$\langle f, g\rangle = \sum_{n=0}^{\infty} n!\, f_n \overline{g_n}, \quad \text{where } f(z) = \sum_{n=0}^{\infty} f_n z^n,\ g(z) = \sum_{n=0}^{\infty} g_n z^n.$$

The *maximal operator* corresponding to $\mathcal{E}(\varphi, d)$ over $\mathcal{F}^2$ is defined by

$$\mathrm{dom}(\mathcal{S}_{\varphi,d,\max}) = \{f \in \mathcal{F}^2 : \mathcal{E}(\varphi, d)f \in \mathcal{F}^2\},$$
$$\mathcal{S}_{\varphi,d,\max} f = \mathcal{E}(\varphi, d)f \quad \text{for all } f \in \mathrm{dom}(\mathcal{S}_{\varphi,d,\max}).$$

The operator $\mathcal{S}_{\varphi,d}$ is called a *nonmaximal operator* if $\mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max}$. We characterise the real symmetry and complex symmetry of $\mathcal{S}_{\varphi,d}$.

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2. Some initial properties

We list notation used in the present paper. Let $[1,d]_{\mathbb{Z}} = \{j \in \mathbb{Z} : 1 \le j \le d\}$. The symbol $\mathbb{C}_k[z]$ denotes the set of all polynomials with degree at most k and complex coefficients. For two unbounded operators A, B, writing $A \preceq B$ means that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and $Ax = Bx$ for $x \in \mathrm{dom}(A)$.

2.1. Elementary observations. These observations are used in Lemmas 3.1 and 4.1.

LEMMA 2.1. *Let $\{C_j : j \in [1,d]_{\mathbb{Z}}\}$ and $\{D_j : j \in [1,d]_{\mathbb{Z}}\}$ be sets of distinct numbers. Let $\lambda_1, \ldots, \lambda_d, \gamma_1, \ldots, \gamma_d$ be nonzero complex numbers. If the equality*

$$\sum_{j=1}^{d} \lambda_j K_{C_j} = \sum_{j=1}^{d} \gamma_j K_{D_j}$$

holds, then for each $t \in [1,d]_{\mathbb{Z}}$ there exists a unique $s \in [1,d]_{\mathbb{Z}}$ such that

$$C_t = D_s, \quad \lambda_t = \gamma_s.$$

PROOF. Assume, towards a contradiction, that there is $t \in [1,d]_{\mathbb{Z}}$ for which $C_t \neq D_j$ for every $j \in [1,d]_{\mathbb{Z}}$. Set

$$f(z) = \prod_{\ell=1}^{d} (z - D_\ell) \cdot \prod_{\mu \neq t} (z - C_\mu).$$

It is clear that $f \in \mathcal{F}^2$ and $f(C_t) \neq 0$. We have

$$\overline{\lambda_t} f(C_t) = \Big\langle f, \sum_{j=1}^{d} \lambda_j K_{C_j} \Big\rangle = \Big\langle f, \sum_{j=1}^{d} \gamma_j K_{D_j} \Big\rangle = 0;$$

but this is impossible. Thus, there exists $s \in [1,d]_{\mathbb{Z}}$ such that $C_t = D_s$. Since the set $\{D_t : t \in [1,d]_{\mathbb{Z}}\}$ consists of distinct numbers, this number s is unique. Setting

$$g(z) = \prod_{\ell \neq s} (z - D_\ell) \cdot \prod_{\mu \neq t} (z - C_\mu),$$

$$\overline{\lambda_t} g(C_t) = \Big\langle g, \sum_{j=1}^{d} \lambda_j K_{C_j} \Big\rangle = \Big\langle g, \sum_{j=1}^{d} \gamma_j K_{D_j} \Big\rangle = \overline{\gamma_s} g(D_s) = \overline{\gamma_s} g(C_t),$$

which implies, as $g(C_t) \neq 0$, that $\lambda_t = \gamma_s$. □

2.2. Action on kernel functions. The next lemma describes how the adjoint $\mathcal{S}^*_{\varphi,d}$ acts on kernel functions.

LEMMA 2.2. *Let $\{\varphi_j : j \in [1,d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions; that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d}$ be the densely defined operator arising from the expression*

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$\mathcal{E}(\varphi, d)$. Then for every $z \in \mathbb{C}$, $K_z \in \operatorname{dom}(\mathcal{S}^*_{\varphi,d})$ and

$$\mathcal{S}^*_{\varphi,d} K_z = \sum_{j=1}^{d} K_{\varphi_j(z)}.$$

PROOF. For every $f \in \operatorname{dom}(\mathcal{S}_{\varphi,d})$,

$$\langle \mathcal{S}_{\varphi,d} f, K_z \rangle = \mathcal{S}_{\varphi,d} f(z) = \sum_{j=1}^{d} \langle f, K_{\varphi_j(z)} \rangle = \Big\langle f, \sum_{j=1}^{d} K_{\varphi_j(z)} \Big\rangle,$$

which gives the desired conclusion. □

2.3. Closed operators. As it turns out, the maximal operator is always closed.

PROPOSITION 2.3. *The maximal operator $\mathcal{S}_{\varphi,d,\max}$ is always closed on Fock space $\mathcal{F}^2$.*

PROOF. Let $(u_n) \subset \mathcal{F}^2$ and $u, v \in \mathcal{F}^2$ be such that $u_n \to u$ and $\mathcal{S}_{\varphi,d,\max} u_n \to v$ in $\mathcal{F}^2$. It follows that $u_n(z) \to u(z)$ and $\mathcal{S}_{\varphi,d,\max} u_n(z) \to v(z)$ for all $z \in \mathbb{C}$ and, consequently,

$$\mathcal{S}_{\varphi,d,\max} u_n(z) = \sum_{j=1}^{d} u_n(\varphi_j(z)) \to \sum_{j=1}^{d} u(\varphi_j(z)) \quad \text{for all } z \in \mathbb{C}.$$

Thus,

$$\sum_{j=1}^{d} u(\varphi_j(z)) = v(z) \quad \text{for all } z \in \mathbb{C}.$$

Since $v \in \mathcal{F}^2$, we conclude that $u \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max})$ and $\mathcal{S}_{\varphi,d,\max} u = v$. □

COROLLARY 2.4. *The maximal operator $\mathcal{S}_{\varphi,d,\max}$ is bounded on Fock space $\mathcal{F}^2$ if and only if its domain $\operatorname{dom}(\mathcal{S}_{\varphi,d,\max}) = \mathcal{F}^2$.*

2.4. Dense domain. In the next result, we characterise the maximal operator $\mathcal{S}_{\varphi,d,\max}$ when it is densely defined.

PROPOSITION 2.5. *Let $\mathcal{R}$ be the linear operator defined by*

$$\operatorname{dom}(\mathcal{R}) = \operatorname{span}\{K_x : x \in \mathbb{C}\}, \quad \mathcal{R} K_x = \sum_{j=1}^{d} K_{\varphi_j(x)}.$$

Then $\mathcal{S}_{\varphi,d,\max} = \mathcal{R}^*$. Moreover, the operator $\mathcal{S}_{\varphi,d,\max}$ is densely defined if and only if the operator $\mathcal{R}$ is closable.

PROOF. Let $u = \sum_{j=1}^{n} \lambda_j K_{x_j} \in \operatorname{dom}(\mathcal{R})$. For every $v \in \mathcal{F}^2$,

$$\langle \mathcal{R} u, v \rangle = \sum_{j=1}^{n} \sum_{t=1}^{d} \lambda_j \langle K_{\varphi_t(x_j)}, v \rangle = \sum_{j=1}^{n} \sum_{t=1}^{d} \lambda_j \overline{v(\varphi_t(x_j))} = \sum_{j=1}^{n} \lambda_j \overline{\mathcal{E}(\varphi, d) v(x_j)}.$$

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By the Riesz lemma, $v \in \mathrm{dom}(\mathcal{R}^*)$ if and only if there exists $\omega = \omega(v) > 0$ such that

$$|\langle \mathcal{R}u, v\rangle| \leq \omega\|u\| \quad \text{for all } u \in \mathrm{dom}(\mathcal{R})$$

or, equivalently,

$$\Big|\sum_{j=1}^{n} \mathcal{E}(\varphi, d)v(x_j)\overline{\lambda_j}\Big|^2 \leq \omega^2 \sum_{j,\ell=1}^{n} \lambda_j\overline{\lambda_\ell}K_{x_j}(x_\ell).$$

It was proven in [20] that this is equivalent to saying that $\mathcal{E}(\varphi, d)v \in \mathcal{F}^2$. Consequently, the domain $\mathrm{dom}(\mathcal{R}^*) = \mathrm{dom}(\mathcal{S}_{\varphi,d,\max})$ and

$$\langle \mathcal{R}u, v\rangle = \langle u, \mathcal{E}(\varphi, d)v\rangle = \langle u, \mathcal{S}_{\varphi,d,\max}v\rangle \quad \text{for all } u \in \mathrm{dom}(\mathcal{R}),\ v \in \mathrm{dom}(\mathcal{S}_{\varphi,d,\max}),$$

which yields $\mathcal{S}_{\varphi,d,\max} = \mathcal{R}^*$. The second assertion follows from [17, Proposition 1.8(i)]. □

3. Real symmetry

An unbounded, linear operator T is called *real symmetric* on a complex, separable Hilbert space if the equality $T = T^*$ holds; meaning that $\mathrm{dom}(T) = \mathrm{dom}(T^*)$ and $Tx = T^*x$ for $x \in \mathrm{dom}(T)$.

In this section, we are interested in how the real symmetry of $\mathcal{S}_{\varphi,d}$ impacts the function-theoretic properties of φ_j, for $j \in [1, d]_{\mathbb{Z}}$, and *vice versa*. For the necessary condition, we apply the real symmetry to kernel functions and this step leads to Lemma 3.1. As it turns out, the real symmetry significantly restricts the possible functions for the operator $\mathcal{S}_{\varphi,d}$. For the sufficient condition, we need a computation regarding the adjoint $\mathcal{S}^*_{\varphi,d}$.

LEMMA 3.1. *Let $\{\varphi_j : j \in [1, d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Suppose that*

$$\sum_{j=1}^{d} e^{x\overline{\varphi_j(z)}} = \sum_{j=1}^{d} e^{\varphi_j(x)\overline{z}} \quad \textit{for all } x, z \in \mathbb{C}. \tag{3.1}$$

Then the functions φ_j have the form

$$\varphi_j(z) = A_j z, \quad j \in [1, d]_{\mathbb{Z}}, \tag{3.2}$$

where the coefficients satisfy the two conditions

$$\{A_j : j \in [1, d]_{\mathbb{Z}}\} = \{\overline{A_j} : j \in [1, d]_{\mathbb{Z}}\} \tag{3.3}$$

and

$$A_t \neq A_s \quad \textit{for all } t \neq s. \tag{3.4}$$

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PROOF. *Step 1:* We show that for each $k \in [1,d]_{\mathbb{Z}}$,

$$\sum_{j=1}^{d} \overline{\varphi_j(z)}^k e^{x\overline{\varphi_j(z)}} = \sum_{j=1}^{d} \Big(\sum_{t=1}^{k} \omega_{t,j,k}(x)\overline{z}^t \Big) e^{\varphi_j(x)\overline{z}} \quad \text{for all } x, z \in \mathbb{C}, \tag{3.5}$$

where

$$\begin{aligned} \omega_{1,j,1}(x) &= \varphi_j'(x) && \text{if } j \in [1,d]_{\mathbb{Z}},\\ \omega_{1,j,k+1}(x) &= \omega_{1,j,k}'(x) && \text{if } j \in [1,d]_{\mathbb{Z}},\\ \omega_{t,j,k+1}(x) &= \omega_{t,j,k}'(x) + \varphi_j'(x)\omega_{t-1,j,k}(x) && \text{if } t \in [2,k]_{\mathbb{Z}} \text{ and } j \in [1,d]_{\mathbb{Z}},\\ \omega_{k+1,j,k+1}(x) &= \varphi_j'(x)\omega_{k,j,k}(x) && \text{if } j \in [1,d]_{\mathbb{Z}}. \end{aligned}$$

Indeed, differentiating (3.1) with respect to the variable x,

$$\sum_{j=1}^{d} \overline{\varphi_j(z)} e^{x\overline{\varphi_j(z)}} = \sum_{j=1}^{d} \varphi_j'(x)\overline{z} e^{\varphi_j(x)\overline{z}} = \sum_{j=1}^{d} \omega_{1,j,1}(x)\overline{z} e^{\varphi_j(x)\overline{z}},$$

which means that equality (3.5) holds when $k = 1$. Now suppose that (3.5) holds for k and consider $k+1$. We have

$$\sum_{j=1}^{d} \overline{\varphi_j(z)}^k e^{x\overline{\varphi_j(z)}} = \sum_{j=1}^{d} \Big(\sum_{t=1}^{k} \omega_{t,j,k}(x)\overline{z}^t \Big) e^{\varphi_j(x)\overline{z}},$$

which implies, after differentiating with respect to the variable x, that

$$\begin{aligned} \sum_{j=1}^{d} \overline{\varphi_j(z)}^{k+1} e^{x\overline{\varphi_j(z)}} &= \sum_{j=1}^{d} \Big(\sum_{t=1}^{k} \omega_{t,j,k}'(x)\overline{z}^t + \varphi_j'(x)\overline{z} \sum_{t=1}^{k} \omega_{t,j,k}(x)\overline{z}^t \Big) e^{\varphi_j(x)\overline{z}} \\ &= \sum_{j=1}^{d} \Big(\sum_{t=1}^{k+1} \omega_{t,j,k+1}(x)\overline{z}^t \Big) e^{\varphi_j(x)\overline{z}}. \end{aligned}$$

Step 2: We claim that $\varphi_j(0) = 0$ for every $j \in [1,d]_{\mathbb{Z}}$. Assume, towards a contradiction, that $\varphi_t(0) \neq 0$ for some $t \in [1,d]_{\mathbb{Z}}$. Set

$$\Omega_t = \{j : \varphi_j(0) = \varphi_t(0)\}, \quad f(z) = z \prod_{j \notin \Omega_t} (z - \varphi_j(0)).$$

Letting $x = 0$ in (3.1), we find that $\sum_{j=1}^{d} K_{\varphi_j(0)} = d = dK_0$ and so

$$0 = d\langle f, K_0 \rangle = \sum_{j=1}^{d} \langle f, K_{\varphi_j(0)} \rangle = \Big(\sum_{j\in\Omega_t} + \sum_{j\notin\Omega_t} \Big) f(\varphi_j(0)) = |\Omega_t| f(\varphi_t(0)),$$

where the notation $|\Omega_t|$ stands for the cardinality of Ω_t; but this is impossible.

Step 3: We claim that

$$\sum_{1\le j_1<j_2<\cdots<j_k\le d} \prod_{\ell=1}^{k} \varphi_{j_\ell}(z) = (-1)^k q_{d-k}(z) \quad \text{for some } q_m \in \mathbb{C}_{d-m}[z]. \tag{3.6}$$

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Let $k \in [1, d]_{\mathbb{Z}}$. Evaluating (3.5) at $x = 0$ and using $\varphi_j(0) = 0$ from Step 2 gives $\sum_{j=1}^{d} \varphi_j(z)^k = p_k(z)$ for some $p_k \in \mathbb{C}_k[z]$. Denote the left-hand side of (3.6) by $\widetilde{q}_k(z)$. By Newton's identities [15],

$$k\widetilde{q}_k(z) = \sum_{j=1}^{k-1} (-1)^{j-1} \widetilde{q}_{k-j}(z) p_j(z).$$

By induction on k, we have $\widetilde{q}_k \in \mathbb{C}_k[z]$. Setting $q_{d-k}(z) = (-1)^k \widetilde{q}_k(z)$ gives (3.6).

With these preparations in place, we prove the conclusion of the lemma as follows. By Vieta's formulas, $\varphi_1(z), \ldots, \varphi_d(z)$ are solutions of the algebraic equation

$$X^d + \sum_{j=0}^{d-1} q_j(z) X^j = 0.$$

Hence, for $|z| > R$ sufficiently large, we can find a constant $C = C(R)$ with the property that

$$|X|^d \le C|z|^d \Big(\sum_{j=0}^{d-1} |X|^j \Big) \le C|z|^d (1 + |X|)^{d-1}.$$

By Liouville's theorem, this means that the functions $\varphi_1, \ldots, \varphi_d$ are polynomials with degrees at most d. Since $\varphi_t(0) = 0$, we can write $\varphi_t(z) = z g_t(z)$, where $g_t \in \mathbb{C}_{d-1}[z]$. Since the product

$$\varphi_1(z) \cdots \varphi_d(z) = z^d \prod_{j=1}^{d} g_j(z)$$

lies in $\mathbb{C}_d[z]$, the polynomials g_j are constants, giving the required form for the φ_j in (3.2). Substituting these forms back into (3.1) gives

$$\sum_{j=1}^{d} e^{x\overline{A_j z}} = \sum_{j=1}^{d} e^{A_j x \bar{z}} \quad \text{for all } x, z \in \mathbb{C}.$$

Consequently, taking into account the explicit form of the kernel functions,

$$\sum_{j=1}^{d} K_{A_j}(u) = \sum_{j=1}^{d} K_{\overline{A_j}}(u) \quad \text{for all } u \in \mathbb{C}.$$

Now we can apply Lemma 2.1 to get (3.3). Finally, (3.4) follows from (1.1). □

The next result concerns real symmetric operators with maximal domains.

THEOREM 3.2. *Let $\{\varphi_j : j \in [1, d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d,\max}$ be the maximal operator arising from the expression $\mathcal{E}(\varphi, d)$. Then the operator $\mathcal{S}_{\varphi,d,\max}$ is real symmetric if and only if the functions φ_j have the form (3.2) with conditions (3.3) and (3.4).*

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PROOF. Suppose that the operator $\mathcal{S}_{\varphi,d,\max}$ is real symmetric. This implies, in particular, that

$$\mathcal{S}^*_{\varphi,d,\max}K_z(x) = \mathcal{S}_{\varphi,d,\max}K_z(x) \quad \text{for all } x, z \in \mathbb{C}$$

and, by Lemma 3.1, we get the necessary condition.

For the sufficient condition, take the functions as in the statement of the theorem. A computation shows that $K_z \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max})$ and moreover

$$\mathcal{S}_{\varphi,d,\max}K_z(x) = \sum_{j=1}^{d} K_{\overline{A_j}z}(x) \quad \text{for all } z, x \in \mathbb{C}.$$

First, we prove the inclusion

$$\mathcal{S}^*_{\varphi,d,\max} \preceq \mathcal{S}_{\varphi,d,\max}. \tag{3.7}$$

Indeed, for $u \in \operatorname{dom}(\mathcal{S}^*_{\varphi,d,\max})$ and $z \in \mathbb{C}$,

$$\begin{aligned}
\mathcal{S}^*_{\varphi,d,\max}u(z) &= \langle \mathcal{S}^*_{\varphi,d,\max}u, K_z\rangle = \langle u, \mathcal{S}_{\varphi,d,\max}K_z\rangle \\
&= \Big\langle u, \sum_{j=1}^{d} K_{\overline{A_j}z}\Big\rangle = \Big\langle u, \sum_{j=1}^{d} K_{A_j z}\Big\rangle \quad \text{(by (3.3))} \\
&= \sum_{j=1}^{d} u(A_j z) = \mathcal{S}_{\varphi,d,\max}u(z).
\end{aligned}$$

Next, we show that equality occurs in (3.7), meaning that

$$\langle \mathcal{S}_{\varphi,d,\max}g, h\rangle = \langle g, \mathcal{S}_{\varphi,d,\max}h\rangle \quad \text{for all } g, h \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max}).$$

Take arbitrary $g(z) = \sum_{n=0}^{\infty} g_n z^n, h(z) = \sum_{n=0}^{\infty} h_n z^n \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max})$ with Taylor coefficients $g_n, h_n \in \mathbb{C}$. We have

$$\mathcal{S}_{\varphi,d,\max}g(z) = \sum_{j=1}^{d}\sum_{n=0}^{\infty} g_n A_j^n z^n = \sum_{n=0}^{\infty} g_n\Big(\sum_{j=1}^{d} A_j^n\Big)z^n.$$

Since (z^n) is an orthogonal basis,

$$\langle \mathcal{S}_{\varphi,d,\max}g, h\rangle = \sum_{n=0}^{\infty} g_n\overline{h_n}\Big(\sum_{j=1}^{d} A_j^n\Big)\|z^n\|^2 = \sum_{n=0}^{\infty} g_n\overline{h_n}\Big(\sum_{j=1}^{d} \overline{A_j}^n\Big)\|z^n\|^2 = \langle g, \mathcal{S}_{\varphi,d,\max}h\rangle. \qquad \square$$

The next result relaxes the domain to show that the real symmetry cannot be separated from the maximality of the domain.

THEOREM 3.3. *Let $\{\varphi_j : j \in [1, d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d}$ be the operator arising from the expression $\mathcal{E}(\varphi, d)$. Then the following assertions are equivalent.*

(1) *The operator $\mathcal{S}_{\varphi,d}$ is real symmetric.*

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(2) *The operator $\mathcal{S}_{\varphi,d}$ satisfies both the following conditions:*

(a) *the operator $\mathcal{S}_{\varphi,d}$ is maximal, that is, $\mathcal{S}_{\varphi,d} = \mathcal{S}_{\varphi,d,\max}$;*

(b) *the functions φ_j have the form in (3.2) with conditions (3.3) and (3.4).*

PROOF. The implication (2)$\Longrightarrow$(1) follows directly from Theorem 3.2. We prove the reverse implication (1)$\Longrightarrow$(2) as follows. Suppose that the operator $\mathcal{S}_{\varphi,d}$ is real symmetric. Since $\mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max}$, it follows from [17, Proposition 1.6] that

$$\mathcal{S}^*_{\varphi,d,\max} \preceq \mathcal{S}^*_{\varphi,d} = \mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max}.$$

In particular,

$$\mathcal{S}^*_{\varphi,d,\max} K_z(x) = \mathcal{S}_{\varphi,d,\max} K_z(x) \quad \text{for all } z, x \in \mathbb{C}.$$

Lemma 3.1 yields (2b). Hence, by Theorem 3.2, the operator $\mathcal{S}_{\varphi,d,\max}$ is real symmetric. Then (2a) follows from

$$\mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max} = \mathcal{S}^*_{\varphi,d,\max} \preceq \mathcal{S}^*_{\varphi,d} = \mathcal{S}_{\varphi,d}. \qquad \square$$

4. Complex symmetry

In this section, we describe precisely when the functions $\varphi_j, j \in [1, d]_{\mathbb{Z}}$ ensure that the operator $\mathcal{S}_{\varphi,d}$ is complex symmetric corresponding to the conjugation

$$\mathcal{Q}_\omega f(z) = \overline{f(\omega\overline{z})}.$$

As a consequence, we obtain the interesting fact that real symmetry implies complex symmetry; namely, if the operator $\mathcal{S}_{\varphi,d}$ is real symmetric, then it is complex symmetric with respect to $\mathcal{Q}_\omega$.

We start this section with an algebraic observation that is analogous to Lemma 3.1. We include a proof for the sake of completeness.

LEMMA 4.1. *Let $\omega \in \mathbb{T}$. Suppose that condition (1.1) holds. If*

$$\sum_{j=1}^{d} e^{\overline{\omega}x\varphi_j(\omega\overline{z})} = \sum_{j=1}^{d} e^{\varphi_j(x)\overline{z}} \quad \text{for all } x, z \in \mathbb{C}, \tag{4.1}$$

then the functions φ_j have the form (3.2) with condition (3.4).

PROOF. Letting $x = 0$ in (4.1),

$$\sum_{j=1}^{d} K_{\overline{\varphi_j(0)}} = d = dK_0.$$

Using arguments similar to those used in Step 2 of Lemma 3.1, we also have $\varphi_j(0) = 0$ for $j \in [1, d]_{\mathbb{Z}}$.

For $k \in [1, d]_{\mathbb{Z}}$, differentiating (4.1) k times with respect to the variable x and then evaluating it at the point $x = 0$ gives $\sum_{j=1}^{d} \varphi_j(z)^k = p_k(z)$ for some $p_k \in \mathbb{C}_k[z]$. By an

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inductive argument, we can find polynomials $q_m \in \mathbb{C}_{d-m}[z]$ for which

$$\sum_{1\le j_1<j_2<\cdots<j_k\le d} \prod_{\ell=1}^{k} \varphi_{j_\ell}(z) = (-1)^k q_{d-k}(z).$$

By Vieta's formulas, $\varphi_1(z), \ldots, \varphi_d(z)$ are solutions of the equation

$$X^d + \sum_{j=0}^{d-1} q_j(z) X^j = 0.$$

Hence, for $|z| > R$ large enough, $|X|^d \le C|z|^d(1+|X|)^{d-1}$, which means that the functions $\varphi_1, \ldots, \varphi_d$ are polynomials with degrees at most d. Since $\varphi_t(0) = 0$, we can write $\varphi_t(z) = z g_t(z)$, where $g_t \in \mathbb{C}_{d-1}[z]$. Since $\varphi_1(z)\cdots\varphi_d(z) \in \mathbb{C}_d[z]$, the polynomials g_j are constant. Hence, the functions φ_j have the form in (3.2). Condition (3.4) follows directly from (1.1). □

Now we use Lemma 4.1 to characterise maximal operators that are complex symmetric corresponding to the conjugation $\mathcal{Q}_\omega$.

THEOREM 4.2. *Let $\{\varphi_j : j \in [1,d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d,\max}$ be the maximal operator arising from the expression $\mathcal{E}(\varphi, d)$. Then the operator $\mathcal{S}_{\varphi,d,\max}$ is $\mathcal{Q}_\omega$-selfadjoint if and only if the functions are of the form in (3.2) with condition (3.4).*

PROOF. Suppose that the operator $\mathcal{S}_{\varphi,d,\max}$ is $\mathcal{Q}_\omega$-selfadjoint. This implies, in particular, that

$$\mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega K_z(x) = \mathcal{S}_{\varphi,d,\max} K_z(x) \quad \text{for all } x, z \in \mathbb{C}$$

and the necessary condition follows from Lemma 4.1.

For the sufficient condition, take functions as in the statement of the theorem. A computation shows that $K_z \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max})$ and moreover

$$\mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} \mathcal{Q}_\omega K_z(x) = \sum_{j=1}^{d} K_{A_j z}(x) \quad \text{for all } z, x \in \mathbb{C}.$$

First, we prove that

$$\mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega \preceq \mathcal{S}_{\varphi,d,\max}. \tag{4.2}$$

Indeed, for $u \in \operatorname{dom}(\mathcal{S}^*_{\varphi,d,\max})$ and $z \in \mathbb{C}$,

$$\begin{aligned}\mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega u(z) &= \langle \mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega u, K_z\rangle = \langle u, \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} \mathcal{Q}_\omega K_z\rangle \\ &= \Big\langle u, \sum_{j=1}^{d} K_{A_j z}\Big\rangle = \sum_{j=1}^{d} u(A_j z) = \mathcal{S}_{\varphi,d,\max} u(z).\end{aligned}$$

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Next, we show that equality occurs in (4.2), meaning that

$$\langle \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} g, h\rangle = \langle \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} h, g\rangle \quad \text{for all } g, h \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max}).$$

Take arbitrary $g(z) = \sum_{n=0}^{\infty} g_n z^n, h(z) = \sum_{n=0}^{\infty} h_n z^n \in \operatorname{dom}(\mathcal{S}_{\varphi,d,\max})$ with Taylor coefficients $g_n, h_n \in \mathbb{C}$. We have

$$\mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} g(z) = \sum_{j=1}^{d}\sum_{n=0}^{\infty} \overline{g_n A_j^n \omega^n} z^n = \sum_{n=0}^{\infty} \overline{g_n \omega^n}\Big(\sum_{j=1}^{d} \overline{A_j}^n\Big) z^n.$$

Since (z^n) is an orthogonal basis,

$$\langle \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} g, h\rangle = \sum_{n=0}^{\infty} \overline{g_n h_n \omega^n}\Big(\sum_{j=1}^{d} \overline{A_j}^n\Big)\|z^n\|^2 = \langle \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} h, g\rangle. \qquad \square$$

As with real symmetry, a complex symmetric operator must be maximal.

THEOREM 4.3. *Let $\{\varphi_j : j \in [1,d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d}$ be the operator arising from the expression $\mathcal{E}(\varphi,d)$. Then the following assertions are equivalent.*

(1) *The operator $\mathcal{S}_{\varphi,d}$ is $\mathcal{Q}_\omega$-selfadjoint.*
(2) *The operator $\mathcal{S}_{\varphi,d}$ satisfies both the following conditions:*
   - (a) *the operator $\mathcal{S}_{\varphi,d}$ is maximal, that is, $\mathcal{S}_{\varphi,d} = \mathcal{S}_{\varphi,d,\max}$;*
   - (b) *the functions φ_j have the form in (3.2) with condition (3.4).*

PROOF. The implication (2)$\Longrightarrow$(1) follows directly from Theorem 4.2. We prove the reverse implication (1)$\Longrightarrow$(2) as follows. Suppose that the operator $\mathcal{S}_{\varphi,d}$ is $\mathcal{Q}_\omega$-selfadjoint. Since $\mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max}$, it follows from [17, Proposition 1.6] that

$$\mathcal{S}^*_{\varphi,d,\max} \preceq \mathcal{S}^*_{\varphi,d} = \mathcal{Q}_\omega \mathcal{S}_{\varphi,d} \mathcal{Q}_\omega \preceq \mathcal{Q}_\omega \mathcal{S}_{\varphi,d,\max} \mathcal{Q}_\omega.$$

Thus, $\mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega \preceq \mathcal{S}_{\varphi,d,\max}$. In particular,

$$\mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega K_z(x) = \mathcal{S}_{\varphi,d,\max} K_z(x) \quad \text{for all } z, x \in \mathbb{C}.$$

Lemma 4.1 yields (2b). Hence, by Theorem 4.2, the operator $\mathcal{S}_{\varphi,d,\max}$ is $\mathcal{Q}_\omega$-selfadjoint. Then (2a) follows from

$$\mathcal{S}_{\varphi,d} \preceq \mathcal{S}_{\varphi,d,\max} = \mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d,\max} \mathcal{Q}_\omega \preceq \mathcal{Q}_\omega \mathcal{S}^*_{\varphi,d} \mathcal{Q}_\omega = \mathcal{S}_{\varphi,d}. \qquad \square$$

COROLLARY 4.4. *Let $\{\varphi_j : j \in [1,d]_{\mathbb{Z}}\}$ be a set of distinct, entire functions, that is, condition (1.1) holds. Let $\mathcal{S}_{\varphi,d}$ be the operator arising from the expression $\mathcal{E}(\varphi,d)$. If the operator $\mathcal{S}_{\varphi,d}$ is real symmetric, then it is $\mathcal{Q}_\omega$-selfadjoint for every $\omega \in \mathbb{T}$.*

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References

[1] E. Berkson, 'Composition operators isolated in the uniform operator topology', *Proc. Amer. Math. Soc.* **81** (1981), 230–232.

[2] B. J. Carswell, B. D. MacCluer and A. Schuster, 'Composition operators on the Fock space', *Acta Sci. Math. (Szeged)* **69**(3–4) (2003), 871–887.

[3] B. R. Choe, K. H. Izuchi and H. Koo, 'Linear sums of two composition operators on the Fock space', *J. Math. Anal. Appl.* **369**(1) (2010), 112–119.

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions* (CRC Press, Boca Raton, FL, 1995).

[5] N. Q. Dieu and S. Ohno, 'Complete continuity of linear combinations of composition operators', *Arch. Math.* **94**(1) (2010), 67–72.

[6] S. R. Garcia and C. Hammond, 'Which weighted composition operators are complex symmetric?', *Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation: 22nd Int. Workshop in Operator Theory and its Applications, IWOTA 11, Sevilla, Spain, 2011* (eds. M. C. Boiso, H. Hedenmalm, M. A. Kaashock, A. M. Rodríguez and S. Treil) (Birkhäuser, Basel, 2014), 171–179.

[7] S. R. Garcia, E. Prodan and M. Putinar, 'Mathematical and physical aspects of complex symmetric operators', *J. Phys. A* **47**(35) (2014), Article no. 353001.

[8] S. R. Garcia and M. Putinar, 'Complex symmetric operators and applications', *Trans. Amer. Math. Soc.* **358**(3) (2006), 1285–1315.

[9] S. R. Garcia and M. Putinar, 'Complex symmetric operators and applications. II', *Trans. Amer. Math. Soc.* **359**(8) (2007), 3913–3931.

[10] P. V. Hai, 'Unbounded weighted composition operators on Fock space', *Potential Anal.* **53**(1) (2020), 1–21.

[11] P. V. Hai and L. H. Khoi, 'Complex symmetry of weighted composition operators on the Fock space', *J. Math. Anal. Appl.* **433**(2) (2016), 1757–1771.

[12] T. Hosokawa, P. J. Nieminen and S. Ohno, 'Linear combinations of composition operators on the Bloch spaces', *Canad. J. Math.* **63**(4) (2011), 862–877.

[13] S. Jung, Y. Kim, E. Ko and J. E. Lee, 'Complex symmetric weighted composition operators on $H^2(D)$', *J. Funct. Anal.* **267**(2) (2014), 323–351.

[14] T. Le, 'Normal and isometric weighted composition operators on the Fock space', *Bull. Lond. Math. Soc.* **46**(4) (2014), 847–856.

[15] D. G. Mead, 'Newton's identities', *Amer. Math. Monthly* **99**(8) (1992), 749–751.

[16] E. A. Nordgren, 'Composition operators', *Canad. J. Math.* **20** (1968), 442–449.

[17] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, 265 (Springer, Dordrecht, 2012).

[18] J. H. Shapiro, *Composition Operators and Classical Function Theory* (Springer, New York, 1993).

[19] J. H. Shapiro and C. Sundberg, 'Isolation amongst the composition operators', *Pacific J. Math.* **145**(1) (1990), 117–152.

[20] F. H. Szafraniec, 'The reproducing kernel property and its space: the basics', in: *Operator Theory* (ed. D. Alpay) (Springer, Basel, 2015), 3–30.

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