THE REVIEW OF SYMBOLIC LOGIC
Volume 4, Number 4, December 2011

TRUTH WITHOUT CONTRA(DI)CTION

ELIA ZARDINI
Instituto de Investigaciones Filosóficas, Universidad Nacional Autónoma de México, and
Northern Institute of Philosophy, University of Aberdeen

Abstract. The concept of *truth* arguably plays a central role in many areas of philosophical theorizing. Yet, what seems to be one of the most fundamental principles governing that concept, i.e. the equivalence between '*'P'* is true' and '*P*', is inconsistent in full classical logic, as shown by the *semantic paradoxes*. I propose a new solution to those paradoxes, based on a principled revision of classical logic. Technically, the key idea consists in the rejection of the unrestricted validity of the structural principle of *contraction*. I first motivate philosophically this idea with the metaphysical picture of the states-of-affairs expressed by paradoxical sentences as being distinctively "*unstable*". I then proceed to demonstrate that the theory of truth resulting from this metaphysical picture is, in many philosophically interesting respects, surprisingly stronger than most other theories of truth endorsing the equivalence between '*'P'* is true' and '*P*' (for example, the theory vindicates the validity of the traditional laws of *excluded middle* and of *non-contradiction*, and also vindicates the traditional constraint of *truth preservation* on logical consequence). I conclude by proving a *cut-elimination* theorem that shows the consistency of the theory.

§1. Introduction. More than 2000 years of philosophical reflection about truth have barely altered the fact that the fully unrestricted *correlation rules*:[1]

T-introduction: '*'P'* is true' follows from[2] '*P*'
T-elimination: '*P*' follows from '*'P'* is true'

enjoy an extremely high plausibility in virtue of what seems to be a fundamental feature of our notion of truth.[3] It lies beyond the scope of this paper to offer an in-depth investigation

Received: November 8, 2010.

[1] Throughout, by '*law*' I mean a proposition accepted or rejected by a certain logical or truth-theoretic system (e.g., the law of excluded middle, or the law of T-necessity, see below in the text), by '*rule*' an argument accepted by such system (e.g., the rule of simplification, or the rule of T-introduction, see below in the text) and by '*metarule*' a conditional claim about rules accepted by such system (e.g., the metarule of universal generalization, or the metarule according to which logical consequence requires truth preservation, see below in the text). (In effect, I will follow common lore and treat laws as a special (zero-premise or zero-conclusion) case of rules—although from a more general point of view I'd be very suspicious of such identification, it will serve well for the purposes of this paper.) I will use '*principle*' as a catch-all phrase for laws, rules, and metarules (and, of course, for all the stuff to which that phrase is usually applied but which lies beyond the scope of logical and truth-theoretic systems). Finally, I will use '*rule of inference*' for the principles governing deductive systems (see Section §4).

[2] Throughout, I use 'follow from' and its relatives to denote the relation of logical consequence (broadly understood so as to encompass also the "logic of truth"), 'entail' and its relatives to denote the converse relation, and 'implication' and its relatives to denote the operation expressed by the conditional. 'Equivalence' and its relatives denote two-way entailment.

[3] This claim would require some fairly substantial qualifications, concerning, for example, *semantic context dependence* and the *modal metaphysics of language*. As such qualifications are however

doi:10.1017/S1755020311000177

of such feature, but let me at least cursorily indicate that, contrary to fashionable *deflationist* views, I regard it as grounded in a conception of truth as a property of *objective representational correctness*—more specifically, the property that a sentence has just in case *it describes things the way they are*. That seems an eminently interesting property for a sentence to have, and analogous properties applying to other kinds of truth bearers such as propositions, beliefs, and assertions seem indeed to be central to our self-understanding as inquirers seeking to represent an objective world.

Let us call a theory of truth validating fully unrestricted (suitable formalizations of) T-introduction and T-elimination over a *sufficiently rich language* (i.e., roughly, a language rich enough as to allow for the formulation of semantic paradoxes) 'a *naive theory of truth*'. In fact, the considerations just sketched support not only the fully unrestricted entailment claims constituting T-introduction and T-elimination, but also the two fully unrestricted conditional claims which are got by setting as antecedent the premise and as consequent the conclusion of those two rules and which constitute the *correlation laws*:

T-necessity: If P, 'P' is true
T-sufficiency: If 'P' is true, P

This is so because the considerations just given not only justify, say, that ' 'P' is true' is *assertable iff* 'P' is *assertable* (which, at least for some theorists, would be enough to support an entailment claim but not enough to support a conditional claim, see Field, 2008, pp. 162–164, for a brief discussion), but also justify that its being the case that 'P' is true is indeed a *necessary and sufficient condition* for its being the case that P—as I said, it is arguable that truth is that specific property of objective representational correctness that a sentence has *just in case* it describes things the way they are. And, to reinforce the point, that truth is a property of *objective* representational correctness seems to require just that: if it is the case that P, then, no matter whether it is assertable or not that P, that is sufficient for its being the case that 'P' is true, and if it is the case that 'P' is true, then, no matter whether it is assertable or not that 'P' is true, that is sufficient for its being the case that P. Let us call a theory of truth validating fully unrestricted (suitable formalizations of) T-necessity and T-sufficiency over a sufficiently rich language 'an *ingenuous theory of truth*'.

As I have defined it, a naive (ingenuous) theory of truth endorses *all* the instances of T-introduction and T-elimination (T-necessity and T-sufficiency) over a sufficiently rich language. However, in spite of the apparent centrality of unrestricted properties of objective representational correctness to our self-understanding as inquirers seeking to represent an objective world, such properties cannot possibly exist in a world governed by the principles of (full) *classical* logic.[4] Such is the lesson of the *semantic paradoxes*, the most famous of which, the *Liar* paradox, originally seems to go back to the Megarian Eubulides of Miletus (see Laertius, *De vitis*, 18.02; I will give concrete examples of the Liar and some other such

not relevant to the purposes of this paper, we can safely ignore them here (see Zardini, 2008, pp. 545–561, 2011a for a start on the issue concerning semantic context dependence). Relatedly, I am taking *sentences* to be the operative *truth bearers* (and be what the usual quotation environments refer to).

[4] There are consistent naive theories of truth that, in a sense, validate all classical laws and rules but not all classical metarules (e.g., the supervaluationist naive theory of McGee, 1991, which validates all classical laws and rules but does not validate reasoning by cases and some other related classical metarules). For these theories, there is a largely terminological question concerning their "classicality." For the purposes of this paper, it will be more convenient to leave them out of the extension of 'classical' and its relatives.

paradoxes in Sections 2.1 and 2.2). This is the problem we'll be concerned with. Without further argument, the philosophical point of view adopted in this paper will assume that the culprit is classical logic[5] and will focus on offering a new solution to the semantic paradoxes based on a principled weakening of the logic. It will do so by, after briefly motivating philosophically such weakening, developing a formal naive (and ingenuous) theory of truth, and proving its consistency.

The weakening in question consists in the failure, in the background logic of the theory, of the metarule of *contraction*, which very roughly says that if the premises $A_0, A_1, A_2 \ldots B, B \ldots$ entail the conclusion C, then $A_0, A_1, A_2 \ldots B \ldots$ already entail C (note that 'contraction' is often used in the literature on the semantic paradoxes in a related but importantly different sense, which I clarify in Footnote 42).[6] Contraction holds not only in classical logic, but in most logics proposed to deal with the semantic paradoxes and, as I will indicate in Sections 2.1 and 2.2, it is in fact implicitly used in the semantic paradoxes. I think it is precisely this implicit use of contraction that is the culprit for the semantic paradoxes and I will show that ridding truth of contraction suffices for ridding it of contradiction.[7]

At the end of the introduction, let me say a word on the scope and limits of the work in this paper. The aim, as I said, is to offer a new solution to the semantic paradoxes by, after briefly motivating philosophically the failure of contraction, developing a formal noncontractive naive theory of truth, and proving its consistency. The focus will thus be on demonstrating *the logical and truth-theoretic strength and the coherence of the theory*, especially in the surprisingly many respects of philosophically interesting strength in which it outperforms its naive rivals. As the reader will certainly notice, this focus will require a number of simplifications leaving many open problems, which I will list and briefly

[5] This has been extensively and variously argued, for example, by Priest, 2006a and Field, 2008. I'm sympathetic to the broad outlines of their treatments of this issue, but don't endorse many of the details. A vindication of the claim that classical theories are unacceptable requires a case-by-case consideration of the best classical theories, a task that is better left for other occasions (see López de Sa & Zardini, 2007, 2011; Zardini, 2008, 2011a for a start). I will say much more in Section §3 about some of the substantial advantages of the naive theory of truth presented in this paper over its (nonclassical) naive rivals, but there again the main purpose of the paper will be not so much to criticize these other theories as to develop a new theory with interesting distinctive features.

[6] Let me straight at the outset enter a *caveat* about restricting contraction. Obviously, any noncontractive logic cannot maintain that logical consequence can be (adequately represented by) a relation holding between *sets* of premises and conclusions (given that the set $\{A_0, A_1, A_2 \ldots B, B \ldots\}$ is identical with the set $\{A_0, A_1, A_2 \ldots B \ldots\}$). This raises very interesting and unfortunately badly underinvestigated issues in the philosophical interpretation of noncontractive logics. Although I will have something general to say about such interpretation in Section 2.3, in this paper I won't inquire further into this particular issue, and will rest content with the idea that the consequence relation does not hold between conclusion and premises *simpliciter*, but only between conclusion and premises *taken a certain (countable) number of times* (see Section 3.1 for a generalization of this point to a multiple-conclusion framework and for a standard way of representing all this mathematically using the theory of multisets).

[7] I don't know of anywhere in the literature where restriction of contraction is the key component in a philosophically motivated approach specifically focused on the semantic paradoxes. There are though a couple of logical and computer-science (rather than philosophical) traditions that have worked on the technical details of certain noncontractive logics, and have applied these to the set-theoretic (rather than semantic) paradoxes. The comparison of the naive theory of truth presented in this paper with the relevant work done in these traditions will however have to wait for another occasion.

comment on in Section §5. I do plan to overcome most of the limitations of this paper in future work, but even so restricted I hope that the present aim will appear to the reader to be worth the effort. Achieving it will certainly prove no trivial task (it will indeed keep us occupied for the rest of this long paper), and it is the necessary first step along a path leading to a new view on these millenary problems.

The rest of this paper is organized as follows. Section 2 starts by reviewing some of the most venerable semantic paradoxes; analyzing the crucial role played by contraction in various pieces of standard paradoxical reasoning, it introduces the idea of solving the paradoxes by restricting contraction and offers a sketch of a metaphysical picture that would make sense of such a restriction. Section 3 contains the development of a formal noncontractive naive theory of truth: it lays down the syntax of a language capable of a *minimum* of self-reference, proceeds to develop the noncontractive background logic of the theory (highlighting its various strengths against the background logics of its naive rivals) and concludes by developing the theory of truth proper (again, highlighting its various strengths against its naive rivals). Section 4 introduces a way of looking at the system developed so far as a deductive system; after some stage setting, this enables the proof of a cut-elimination theorem from which several consistency properties follow as straightforward applications. Section 5 concludes by listing and briefly commenting on some open problems lying ahead for future research.

§2. Paradox and contraction.

2.1. Lies. Let us see some indicative examples of how contraction is actually crucial in paradoxical reasoning. For the time being, I wish to leave things at a fairly informal level, with the familiar bits of standard notation plus the use of T[8] as a truth predicate and the use of lower-case Gothic letters as names of sentences (I will make the notation more precise and slightly revise it in Section 3.1). We start by considering the standard paradoxical reasoning using the instance of the law of excluded middle for a strengthened Liar sentence. Very roughly, a *strengthened Liar sentence* is a sentence saying of itself that it is not true: let us take as an example $\neg T\mathfrak{l}$, where $\mathfrak{l}$ is a name of $\neg T\mathfrak{l}$. The standard paradoxical reasoning using the instance of the law of *excluded middle* for $T\mathfrak{l}$ ($T\mathfrak{l} \vee \neg T\mathfrak{l}$) involves two subarguments, deriving a violation of the law of *noncontradiction* for $T\mathfrak{l}$ (rejection of $T\mathfrak{l} \wedge \neg T\mathfrak{l}$) from $T\mathfrak{l}$ and the same violation of the law of noncontradiction from $\neg T\mathfrak{l}$, respectively. *Reasoning by cases*, paradox would then ensue. Each subargument, however, crucially involves contraction. Let us look, for example, at the subargument deriving a violation of the law of noncontradiction from $T\mathfrak{l}$:[9]

$$\dfrac{\dfrac{\dfrac{}{T\mathfrak{l} \vdash T\mathfrak{l}}\text{ reflexivity} \qquad \dfrac{}{T\mathfrak{l} \vdash \neg T\mathfrak{l}}\ T\text{-elimination}}{T\mathfrak{l}, T\mathfrak{l} \vdash T\mathfrak{l} \wedge \neg T\mathfrak{l}}\ \wedge\text{-introduction}}{T\mathfrak{l} \vdash T\mathfrak{l} \wedge \neg T\mathfrak{l}}\text{ contraction}$$

8 Throughout, formal and semiformal symbols are understood autonymously to refer to themselves.

9 Throughout, I use $\vdash$ to refer to the contextually relevant consequence relation. Also, I will present the relevant pieces of reasoning in a *sequent-calculus* format rather than in a (philosophically more familiar) *natural-deduction* format. The former brings out the implicit use of contraction involved in paradoxical reasoning in a more immediate and dramatic fashion than the latter does. As is well known, if one were to analyze such pieces of reasoning using instead a natural-deduction format, the implicit use of contraction would be manifested in the multiple simultaneous discharge of different assumptions of the same sentence.

Even given both $T\wr \vdash T\wr$ and $T\wr \vdash \neg T\wr$, the intuitive metarule of $\wedge$-introduction requires to take the premises in both arguments as many times as they occur as premises in both arguments, which only yields $T\wr, T\wr \vdash T\wr \wedge \neg T\wr$.[10] *It is contraction* that allows us to go from that to $T\wr \vdash T\wr \wedge \neg T\wr$. In the absence of contraction, it is only $T\wr$ *taken twice* that entails a contradiction—$T\wr$ *in itself does not*, and so the paradoxical reasoning is blocked. For comparative purposes, notice that most naive theories of truth would rather block the paradoxical reasoning either at the step assuming the law of excluded middle (as happens in *analetheic* theories, see, e.g., Brady, 2006; Field, 2008), or at the step assuming the relevant version of the metarule of reasoning by cases (as happens in *supervaluationist* or *revision* theories, see, e.g., McGee, 1991; Gupta & Belnap, 1993), or at the step assuming the law of noncontradiction (as happens in *dialetheic* theories, see, e.g., Priest, 2006a; Beall, 2009). I find all these laws and the relevant version of the metarule very compelling, and I regard it as a major virtue of the naive theory of truth presented in this paper that, by restricting contraction instead, it validates all of them (see Section 3.1 for the details).

Of course, one needs neither the law of excluded middle nor the metarule of reasoning by cases to generate paradox. One could, for example, assume $T\wr$, derive by T-elimination $\neg T\wr$, infer from this derivation $\neg T\wr$ under no assumptions and then infer from it by T-introduction $T\wr$ as well, thereby violating the law of noncontradiction. The second last inference would be licensed by the (intuitionistically acceptable version of the) metarule of *reductio ad absurdum*. (This is incidentally why intuitionist logic has never been an option for a weakening of the logic able to deal with the semantic paradoxes.) But what is the argument for justifying this metarule? Very interestingly, the standard one crucially involves contraction. Let us consider it. We assume that a formula A entails its own negation and reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\text{ reflexivity} \qquad A \vdash \neg A}{A, A \vdash A \wedge \neg A}\wedge\text{-introduction}}{A \vdash A \wedge \neg A}\text{ contraction}}{\vdash A \to A \wedge \neg A}\to\text{-introduction}$$

where the formula on the right-hand side of the last line is in many systems equivalent and fully intersubstitutable with $\neg A$ (including the system I'll present and develop in Section §3). Once again, even given both $A \vdash A$ and $A \vdash \neg A$, the intuitive metarule of $\wedge$-introduction requires to take the premises in both arguments as many times as they occur as premises in both arguments, which only yields $A, A \vdash A \wedge \neg A$. *It is contraction* that allows us to go from that to $A \vdash A \wedge \neg A$. In the absence of contraction, it is only A *taken twice* that entails a contradiction—A *in itself does not*, and so the reasoning in favor of *reductio ad absurdum* is blocked. Notice that although A in itself does not *entail a contradiction*, it does *entail its own negation*: while these two things are lumped together in most logics, a noncontractive framework allows us to keep them nicely distinct (I will delve a bit more into this distinction in Section 3.1). For comparative purposes, notice also that most naive theories of truth would rather block the argument at the $\to$-introduction

[10] The point in the text can be buttressed by observing that it should be uncontroversial that the general metarule of $\wedge$-introduction is to the effect that, if $A \vdash B$ holds and $C \vdash D$ holds, $A, C \vdash B \wedge D$ holds. But then, in the specific case where $A \vdash B$ holds, $C \vdash D$ holds and A is identical with C (so that in fact $A \vdash B$ holds and $A \vdash D$ holds), that metarule *by itself* only yields $A, A \vdash B \wedge D$. No matter how obvious the move might seem, a further nontrivial assumption would thus be needed to go from that to $A \vdash B \wedge D$.

step. I find that metarule very compelling, and I regard it as a major virtue of the naive theory of truth presented in this paper that, by restricting contraction instead, it validates it (see Section 3.1 for the details).

2.2. Curries. Consideration of *reductio ad absurdum* has naturally led to consideration of conditionals, and consideration of conditionals naturally leads to consideration of a second kind of semantic paradox: *Curry*'s (see Curry, 1942). Let us consider the standard paradoxical reasoning with a Curry sentence for B. Very roughly, a *Curry sentence* for B is a sentence saying of itself that, if it is true, B: let us take as an example $T\mathfrak{c} \to B$, where $\mathfrak{c}$ is a name for $T\mathfrak{c} \to B$. The standard paradoxical reasoning with $\mathfrak{c}$ involves a subargument deriving B from $T\mathfrak{c}$, inferring from this derivation, by $\to$-introduction, $T\mathfrak{c} \to B$ under no assumptions, then inferring from the latter, by T-introduction, $T\mathfrak{c}$ and finally inferring from all this, by *modus ponens*, B. The subargument, however, crucially involves contraction. Let us look at it:

$$\dfrac{\dfrac{\dfrac{}{T\mathfrak{c} \vdash T\mathfrak{c}}\,\text{reflexivity} \qquad \dfrac{}{B \vdash B}\,\text{reflexivity}}{T\mathfrak{c}, T\mathfrak{c} \to B \vdash B}\,\to\text{-elimination} \qquad \dfrac{}{T\mathfrak{c} \vdash T\mathfrak{c} \to B}\,T\text{-elimination}}{\dfrac{T\mathfrak{c}, T\mathfrak{c} \vdash B}{T\mathfrak{c} \vdash B}\,\text{contraction}}\,\text{transitivity}$$

Even given $T\mathfrak{c}, T\mathfrak{c} \to B \vdash B$, T-elimination, and transitivity only yield $T\mathfrak{c}, T\mathfrak{c} \vdash B$.[11] *It is contraction* that allows us to go from that to $T\mathfrak{c} \vdash B$. In the absence of contraction, it is only $T\mathfrak{c}$ *taken twice* that entails B—$T\mathfrak{c}$ *in itself does not*, and so the paradoxical reasoning is blocked. Again, for comparative purposes, notice that most naive theories of truth would rather block the argument at the $\to$-introduction step (thus prompting the same critical comment as the one made in Section 2.1).

2.3. The idea of solving the paradoxes by restricting contraction. There are, of course, many other semantic paradoxes involving the notion of truth, some of which are interestingly different from the Liar paradox and Curry's paradox. I think that all of these crucially involve contraction, at one step or another (Theorem 4.9 and its corollaries can be taken as a proof of this claim, at least for those paradoxes expressible in the system developed in Section §3). I also think that it is actually a very worthwhile and instructive enterprise, for some of these paradoxes, to analyze exactly where contraction is involved in them, as I just did for the Liar paradox and Curry's paradox. However, those further analyses are better left for another occasion.

I now rather want to pursue the thought—suggested by this brief survey—that, as contraction seems to be such a crucial ingredient in the generation of the semantic paradoxes,

[11] Again, the point in the text can be buttressed by observing that it should be uncontroversial that the general metarule of transitivity which is here relevant is to the effect that, if $A, B \vdash C$ holds and $D \vdash B$ holds, $A, D \vdash C$ holds. But then, in the specific case where $A, B \vdash C$ holds, $D \vdash B$ holds and A is identical with D (so that in fact $A, B \vdash C$ holds and $A \vdash B$ holds), that metarule *by itself* only yields $A, A \vdash C$. No matter how obvious the move might seem, a further nontrivial assumption would thus be needed to go from that to $A \vdash C$. (More precisely, what the general metarule of transitivity which is here relevant becomes in the case where A is identical with D is a version of what is known as '*cumulative transitivity*', a metarule which is particularly useful in the study of nonmonotonic logics, see Gabbay, 1985. The metarule of cumulative transitivity is usually stated in the literature on those logics with the stronger consequent to the effect that $A \vdash C$ holds, but that's exactly because, in that literature, contraction is usually implicitly assumed.)

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

there is some hope to tame these by restricting that principle. The bulk of this paper, in Sections §3 and §4, is in effect devoted to vindicating that hope. But what is the intuitive rationale for restricting contraction? What is it about the state-of-affairs expressed by a sentence that explains its failure to contract? Although the answers to these and related questions deserve an extended treatment that lies beyond the scope of this paper (which is focused instead on demonstrating the logical and truth-theoretic strength and the coherence of a noncontractive naive theory of truth), it seems in order that, before the start of the more formal development in Section §3, I at least give a rough idea of where I think such answers lie. As the tone of my second question should already suggest, I believe that in attempting to find these answers one has to step out of the abstract realm of *formal* theories of truth and engage in some concrete *metaphysics* of truth.[12]

As far as the naive theory of truth presented in this paper is concerned, I think that the key to understanding what it is about the state-of-affairs expressed by a sentence that explains its failure to contract is given by thinking of that state-of-affairs as distinctively *unstable*. I conceive of instability as the *metaphysical* property attaching to states-of-affairs exemplification of which *causes* the exemplification of the *logical* property of failing to contract attaching to the corresponding sentences. Let me thus explain how I understand the former property and its exemplification by states-of-affairs by taking $\neg T\ell$ as a paradigmatic example. By T-introduction, if the state-of-affairs expressed by $\neg T\ell$ obtained, it would lead to the state-of-affairs expressed by $T\ell$. The instability of the former state-of-affairs consists in the fact that, since the state-of-affairs it would lead to is *incompatible* with it, to the extent that the latter obtained the former would not—although it is precisely the obtaining of the former that would lead to the obtaining of the latter. (Conversely, by T-elimination, if the state-of-affairs expressed by $T\ell$ obtained, it would lead to the state-of-affairs expressed by $\neg T\ell$. The instability of the former state-of-affairs consists in the fact that, since the state-of-affairs it would lead to is *incompatible* with it, to the extent that the latter obtained the former would not—although it is precisely the obtaining of the former that would lead to the obtaining of the latter.) The situation is helpfully contrasted with that involving *stable* states-of-affairs, for which we can take 'Snow is black' as a paradigmatic example. By the rule of addition (see Theorem 3.9), if the state-of-affairs expressed by 'Snow is black' obtained, it would lead to the state-of-affairs expressed by 'Either snow is black or grass is green'. The stability of the former state-of-affairs consists in the fact that, to the extent that the latter state-of-affairs obtained, the former would too—if it is precisely the obtaining of the former that would lead to the obtaining of the latter (for one, their logical relationships constitute no bar to such stability, since the two states-of-affairs are *compatible*). Stable states-of-affairs, if they obtained, would *co-obtain with all of their consequences*; unstable states-of-affairs would *not*.

Thus, the behavior of an unstable state-of-affairs as the one expressed by $\neg T\ell$ resembles the behavior of *physical* states: both kinds of states, if they obtained, would lead to other states with which they would not co-obtain—although it is precisely the obtaining of the former states that would lead to the obtaining of the latter states. For example, in the case of physical states, the state in which one moving object is about to enter into collision with

12 More generally, over and above the specifics of the project undertaken in this paper, I believe that the interaction between formal theories of truth and more traditional theories concerning themselves with the metaphysics of truth is a very welcome and productive trend in the contemporary debate (see, e.g., Maudlin, 2004, for a paradigmatic case where a certain formal theory of truth is ultimately justified by a certain substantial metaphysics of truth).

another moving object, if it obtained, would typically lead to a state where the direction and velocity of the two objects are different from and hence incompatible with the original ones. Hence, the former state would lead to a state with which it would not co-obtain—although it is precisely the obtaining of the former that would lead to the obtaining of the latter.

In the case of physical states, one state "leading to" another state can in fact be interpreted as a *temporal* transition from a time t_0 to another time t_1, and the former state not co-obtaining with the latter state can in fact be interpreted as the former state obtaining at t_0 and ceasing to obtain at t_1 (and as the latter state not obtaining at t_0 and starting to obtain at t_1). The temporal dimension is of course absent in the case of an unstable state-of-affairs as the one expressed by $\neg T\wr$, but there too we can discern a broadly analogous structure of "*stages of truth evaluation*," with the transition from one stage to the other being governed by T-introduction (for evaluations of *truth*) and (the contrapositive of) T-elimination (for evaluations of *untruth*). That is in effect the structure generated by the relation of *semantic grounding* playing such a crucial role in the informal motivation for the construction of Kripke, 1975. In Kripke's case, the monotonicity of the construction determines however that the structure so generated is *cumulative*, in the sense that states-of-affairs obtaining at one stage obtain at all later stages (later stages can only *expand* earlier stages); in our case, on the contrary, the inclusion of an unstable state-of-affairs as the one expressed by $\neg T\wr$ determines that the structure so generated is *dialectical*, in the sense that states-of-affairs obtaining at one stage may fail to obtain at a later stage (later stages can *revise* earlier stages).[13]

I assume that the states-of-affairs expressed by paradoxical sentences are indeed unstable in the way I've just indicated. I think that this assumption is independently plausible and, once it has been made, it can be claimed that *it is exactly such instability that causes failures of contraction*. For, given that $\neg T\wr, \neg T\wr \vdash T\wr \wedge \neg T\wr$ plausibly holds (for the reasons given in Footnote 10), contraction implies that $\neg T\wr \vdash T\wr \wedge \neg T\wr$ holds, but that in turn requires that, if the state-of-affairs expressed by $\neg T\wr$ obtained, it would lead to the state-of-affairs expressed by $T\wr$ *and would co-obtain with it*, which cannot be given the instability of the state-of-affairs expressed by $\neg T\wr$. Failure of contraction is the logical symptom of an underlying unstable metaphysical reality.[14]

The discussion in Section 2.1 might have left the reader partially unsatisfied: granted, paradox with $\neg T\wr$ is blocked because contraction fails, but what is then the *status* of $\neg T\wr$ according to the naive theory of truth presented in this paper?[15] The previous metaphysical picture is valuable also because it offers the beginnings of an answer to that query. The naive theory of truth presented in this paper will in effect validate the laws of bivalence and contravalence (see Corollaries 3.26 and 3.27), and so $\neg T\wr$ will simply be *either true-only or false-only*: thus, very importantly, *no status intermediate between truth-only and falsity-only need be introduced*. Moreover, that plausibly implies that either the state-of-affairs expressed by $\neg T\wr$ obtains or the state-of-affairs expressed by its contradictory obtains. However, whichever obtains, *it only obtains unstably*—at the next stage of truth

[13] I am here appropriating and pushing into a certain direction some themes of the *revision-theoretic* tradition, especially as these are developed and understood by Herzberger, 1982.

[14] I believe that this connection between instability in the metaphysics and failure of contraction in the logic holds promise of affording a new understanding of the implicit logic at work in some prominent aspects of the dialectical tradition, but that is a matter better left for another occasion.

[15] Thanks to Greg Restall for pressing this point.

evaluation, it leads to the opposite state-of-affairs with which it cannot co-obtain and, at the next further stage of truth evaluation, that in turn leads back to the original state-of-affairs with which it cannot co-obtain, thus giving rise to an endless atemporal cycle. Hence, it is wrong to ask which one "obtains," if one means which one *stably* obtains (which is what I think one would usually mean by 'obtain' and its likes in this kind of question), for, in that sense, neither does: whichever unstably obtains, the truth in its entirety consists rather in the endless atemporal self-removal of one state-of-affairs in favor of its opposite and *vice versa*. Thus, the previous metaphysical picture helps to understand how the fact that $\neg T\mathfrak{l}$ is simply either true-only or false-only (and that either the state-of-affairs expressed by $\neg T\mathfrak{l}$ obtains or the state-of-affairs expressed by its contradictory obtains) does not really imply any disturbing *asymmetry* between $\neg T\mathfrak{l}$ and its contradictory (and between the states-of-affairs they express).

I think that this underlying metaphysical picture is in itself very appealing and holds potential for capturing the strong albeit vague intuition of "dynamicity" that is naturally associated with paradoxical sentences. Of course, the picture raises in turn several pressing questions, for example, whether and how it can be generalized beyond the case of $\neg T\mathfrak{l}$ so as to cover *all* cases of failure of contraction required by the naive theory of truth presented in this paper, and how best one can understand a *dynamic* process that is also supposed to be *atemporal*. These and many other questions do need to be addressed, but clearly their treatment lies beyond the scope of this paper. It was here sufficient to give at least a rough idea of my favored way of making philosophical sense of failure of contraction.[16]

§3. A noncontractive naive theory of truth.

3.1. The background logic. It is time to develop a formal noncontractive naive theory of truth. For reasons that will become apparent in this section and Section 3.2, I will call such theory '**IKT**$^{\omega}$' and its background logic '**IK**$^{\omega}$'. We will work with a standard first-order language $\mathscr{L}^1$ without identity (in particular, $\mathscr{L}^1$ has the 2ary connective $\otimes$ for conjunction and the 2ary connective $\oplus$ for disjunction).[17] We pick designated individual constants to serve as canonical names of all sentences in the language; if an individual constant is the canonical name of a sentence A, we will refer to that individual constant by '$\ulcorner A \urcorner$' or by a lower-case Gothic letter. We assume to have paradoxical sentences in the language, such as, for example, those denoted by $\mathfrak{l}$ and $\mathfrak{c}$ in Section §2. We also pick a designated 1ary predicate constant to serve as a truth predicate, and denote it with 'T' (as I have announced in Section 2.1, I am now making the notation more precise and slightly revising it).

---

[16] Thanks to Dan López de Sa, Andreas Pietz, Carlos Romero Castillo, and Sven Rosenkranz for especially stimulating discussions of the metaphysical picture presented in this section.

[17] In the formal development of **IKT**$^{\omega}$, I will use $\otimes$ for conjunction and $\oplus$ for disjunction to make salient their tight relationships with the "*multiplicative*" connectives tensor and par of linear logics and the "*intensional*" connectives fusion and fission of relevant logics. I stress however that with $\otimes$ and $\oplus$ I mean to give a theory of our *informal notions of conjunction and disjunction* (as they occur, e.g., in informal presentations of the semantic paradoxes), notions that, outside of the formal development of **IKT**$^{\omega}$, I myself express in this paper with the more usual $\wedge$ and $\vee$. At root, and anticipating a bit, the reason why my $\otimes$ and $\oplus$ are much better candidates for a theory of our informal notions of conjunction and disjunction than the usual multiplicative or intensional connectives consists in the presence of *monotonicity* in **IKT**$^{\omega}$, which "*extensionalizes*" the behavior of $\otimes$ and $\oplus$, making them obey, for example, the rules of simplification and addition (see Theorem 3.9; see also Footnotes 32 and 34 for discussions of related issues). Thanks to Aaron Cotnoir for pressing me on this.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

Given some crucial features of $\mathbf{IK}^{\omega}$, a *multiple-conclusion* framework will be necessary in order to give an adequate treatment of disjunction. Moreover, given the failures of contraction and as already intimated in Footnote 6, we will have to think of logical consequence as (being adequately represented by)[18] a relation holding not between *sets* of sentences (i.e., between a set of premises and a set of conclusions), but between *multisets* of sentences (i.e., between a multiset of premises and a multiset of conclusions). In turn, multisets are informally like sets except that they are sensitive to the number of times that a member occurs in them. For the purposes of this paper, it will suffice to focus attention on multisets that discriminate number of occurrences[19] only within the domain of the *countable*.[20] We make this informal notion mathematically precise with the following definitions:[21]

DEFINITION 3.1. *A multiset of well-formed formulae (wffs) of $\mathscr{L}^1$ is a function whose domain is the set of wffs of $\mathscr{L}^1$ ($WFF_{\mathscr{L}^1}$) and whose range is $\omega + 1$. We form pairwise and countable combinations of multisets using cardinal summation:*

- $\Gamma, \Delta(\varphi) := \Gamma(\varphi) + \Delta(\varphi)$;[22]
- $\bigsqcup_{0 \leq i < \omega} (\Gamma_i)(\varphi) := \sum_{0 \leq i < \omega} (\Gamma_i(\varphi))$.

We write a list of sentences enclosed by square brackets for the multiset containing, for every φ, as many occurrences of φ as there are in the list and set $\Gamma, \varphi := \Gamma, [\varphi]$ (including the case where Γ is the empty multiset, i.e., the multiset with no positive occurrences, which we denote with '$\oslash$'). We also understand talk of multiset membership and its likes to be restricted to positive occurrences.

---

18 The parenthetical qualification hints at the fact that the *informal* notion of logical consequence would rather seem to be a notion of a relation holding between (certain finely structured) *pluralities*, just like the relation of being more numerous than informally holds between Andy, Bill, and Charlie on the one side and Dave and Emmie on the other side, although it is usually adequately represented by a relation holding between the sets {Andy, Bill, Charlie} and {Dave, Emmie} (as I have hinted at in Footnote 6, in the case of $\mathbf{IK}^{\omega}$ the pluralities should be so finely structured as to discriminate how many times something is one of them). I will ignore such niceties in what follows.

19 This notion of occurrence extends naturally the standard notion of occurrence as a relation holding between *linguistic types* (broadly understood), which is also used in this paper (as when it is said, e.g., that a variable occurs twice in a formula). I trust that no confusion will arise as to which of these two notions is meant in a particular context.

20 While crucial for the development of $\mathbf{IK}^{\omega}$, this inclusion of the countably infinite is actually nonstandard for investigations of multisets, which are typically restricted to multisets that discriminate number of occurrences only within the domain of the *finite* (see, e.g., Blizard, 1989, which nevertheless offers a useful survey of the literature on and a systematic development of the (standard) theory of multisets).

21 Comments analogous to those in Footnote 18 apply to the relation between the informal notion of a multiset as just explained in the text and the mathematically precise notion to be introduced in Definition 3.1. So there are in effect *two* levels of modeling: we model logical consequence—informally, a relation holding between (certain finely structured) pluralities—as a relation holding between multisets, and we in turn model multisets—informally, collections sensitive to the number of times that a member occurs in them—as standard set-theoretic functions with the properties described in Definition 3.1.

22 Throughout, 'Γ' and 'Δ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over the set of multisets of members of $WFF_{\mathscr{L}^1}$; 'φ', 'ψ', and 'χ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over $WFF_{\mathscr{L}^1}$.

With the notion of a multiset thus in place, we are in a position also to define the informal notion of a logic in a mathematically precise way:

DEFINITION 3.2. *A logic* $\mathbf{L}$ *for* $\mathscr{L}^1$ *is any subset of* $\omega + 1^{WFF_{\mathscr{L}^1}} \times \omega + 1^{WFF_{\mathscr{L}^1}}$.

We can then proceed to specify the logic $\mathbf{IK}^{\omega}$ as the smallest logic containing as axiom the structural rule:[23]

$$\frac{}{\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi}\ \text{I}$$

closed under the structural metarules:

$$\frac{\Gamma_0 \vdash_{\mathbf{IK}^{\omega}} \Delta}{\Gamma_0, \Gamma_1 \vdash_{\mathbf{IK}^{\omega}} \Delta}\ \text{K-L} \qquad \frac{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta_0}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1}\ \text{K-R}$$

$$\frac{\Gamma_0 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \varphi \qquad \Gamma_1, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta_1}{\Gamma_0, \Gamma_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1}\ \text{S}$$

and under the operational metarules:

$$\frac{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi}{\Gamma, \neg\varphi \vdash_{\mathbf{IK}^{\omega}} \Delta}\ \neg\text{-L} \qquad \frac{\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \neg\varphi}\ \neg\text{-R}$$

$$\frac{\Gamma, \varphi, \psi \vdash_{\mathbf{IK}^{\omega}} \Delta}{\Gamma, \varphi \otimes \psi \vdash_{\mathbf{IK}^{\omega}} \Delta}\ \otimes\text{-L} \qquad \frac{\Gamma_0 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \varphi \qquad \Gamma_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_1, \psi}{\Gamma_0, \Gamma_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1, \varphi \otimes \psi}\ \otimes\text{-R}$$

$$\frac{\Gamma_0, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta_0 \qquad \Gamma_1, \psi \vdash_{\mathbf{IK}^{\omega}} \Delta_1}{\Gamma_0, \Gamma_1, \varphi \oplus \psi \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1}\ \oplus\text{-L} \qquad \frac{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi, \psi}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi \oplus \psi}\ \oplus\text{-R}$$

$$\frac{\Gamma_0 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \varphi \qquad \Gamma_1, \psi \vdash_{\mathbf{IK}^{\omega}} \Delta_1}{\Gamma_0, \Gamma_1, \varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1}\ \to\text{-L} \qquad \frac{\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta, \psi}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi \to \psi}\ \to\text{-R}$$

$$\frac{\Gamma, \varphi_{\upsilon_0/\xi}, \varphi_{\upsilon_1/\xi}, \varphi_{\upsilon_2/\xi} \dots \vdash_{\mathbf{IK}^{\omega}} \Delta}{\Gamma, \forall\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \Delta}\ \forall\text{-L}$$

$$\frac{\Gamma_0 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \varphi_{\upsilon_0/\xi} \qquad \Gamma_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_1, \varphi_{\upsilon_1/\xi} \qquad \Gamma_2 \vdash_{\mathbf{IK}^{\omega}} \Delta_2, \varphi_{\upsilon_2/\xi} \dots}{\bigsqcup_{0 \leq i < \omega} (\Gamma_i) \vdash_{\mathbf{IK}^{\omega}} \bigsqcup_{0 \leq i < \omega} (\Delta_i), \forall\xi\varphi}\ \forall\text{-R}$$

$$\frac{\Gamma_0, \varphi_{\upsilon_0/\xi} \vdash_{\mathbf{IK}^{\omega}} \Delta_0 \qquad \Gamma_1, \varphi_{\upsilon_1/\xi} \vdash_{\mathbf{IK}^{\omega}} \Delta_1 \qquad \Gamma_2, \varphi_{\upsilon_2/\xi} \vdash_{\mathbf{IK}^{\omega}} \Delta_2 \dots}{\bigsqcup_{0 \leq i < \omega} (\Gamma_i), \exists\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \bigsqcup_{0 \leq i < \omega} (\Delta_i)}\ \exists\text{-L}$$

$$\frac{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi_{\upsilon_0/\xi}, \varphi_{\upsilon_1/\xi}, \varphi_{\upsilon_2/\xi} \dots}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \exists\xi\varphi}\ \exists\text{-R}$$

(where φ_{τ_0/τ_1} is the result of substituting τ_0 for all free occurrences of τ_1 in φ, with τ_0 being free for τ_1 in φ, and where '$\upsilon_0, \upsilon_1, \upsilon_2 \dots$' and its likes denote a designated complete enumeration of the set of closed terms of $\mathscr{L}^1$ and its likes).[24]

---

[23] I will take a rule or metarule to be *structural* if it does not refer to any particular member of the vocabulary of $\mathscr{L}^1$. Otherwise, I will say that the rule or metarule is *operational*.

[24] Throughout, 'τ' and 'υ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over the set of terms of $\mathscr{L}^1$; 'ξ' is used as a metalinguistic variable ranging over the set of variables of $\mathscr{L}^1$.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

Emphatically, we do *not* include the structural metarules of contraction in $\mathbf{IK}^{\omega}$:

$$\frac{\Gamma, \varphi, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta}{\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta}\text{W-L} \qquad \frac{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi, \varphi}{\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \varphi}\text{W-R}$$

In fact, as Corollary 4.14 to Theorem 4.9 will show, not only are W-L and W-R *absent from the defining principles* of $\mathbf{IK}^{\omega}$ (and $\mathbf{IKT}^{\omega}$): they are simply not *admissible* in $\mathbf{IK}^{\omega}$ (or $\mathbf{IKT}^{\omega}$)—that is, neither system is closed under them.

Literature on substructural logics, and in particular on linear logics, typically considers also operators governed by *additive* (or *context-sharing*) rather than *multiplicative* (or *context-free*) metarules, where the latter are pairs of metarules in which (in the multipremise*[25] metarule) the wffs that explicitly occur in the premises* are allowed to be in *different* contexts and these are *not* contracted in the conclusion*, while the former are pairs of metarules in which (in the multipremise* metarule) the wffs that explicitly occur in the premises* are required to be in the *same* context and these are *contracted* in the conclusion*.[26] Thus, for example, also the following metarules for a conjunctive connective $\overset{A}{\wedge}$ and for a disjunctive connective $\overset{A}{\vee}$ are considered:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \overset{A}{\wedge} \psi \vdash \Delta}\overset{A}{\wedge}\text{-L}^0 \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \overset{A}{\wedge} \psi \vdash \Delta}\overset{A}{\wedge}\text{-L}^1 \qquad \frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \overset{A}{\wedge} \psi}\overset{A}{\wedge}\text{-R}$$

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \overset{A}{\vee} \psi \vdash \Delta}\overset{A}{\vee}\text{-L} \qquad \frac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \varphi \overset{A}{\vee} \psi}\overset{A}{\vee}\text{-R}^0 \qquad \frac{\Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \overset{A}{\vee} \psi}\overset{A}{\vee}\text{-R}^1$$

Moreover, to the best of my knowledge, when it comes to the quantifiers[27] the above-mentioned literature exclusively considers something along the lines of the following metarules for a universal quantifier $\overset{A}{\forall}$ and for a particular quantifier $\overset{A}{\exists}$:

$$\frac{\Gamma, \varphi_{\tau/\xi} \vdash \Delta}{\Gamma, \overset{A}{\forall} \xi\varphi \vdash \Delta}\overset{A}{\forall}\text{-L} \qquad \frac{\Gamma \vdash \Delta, \varphi_{\tau/\xi}}{\Gamma \vdash \Delta, \overset{A}{\forall} \xi\varphi}\overset{A}{\forall}\text{-R}$$

$$\frac{\Gamma, \varphi_{\tau/\xi} \vdash \Delta}{\Gamma, \overset{A}{\exists} \xi\varphi \vdash \Delta}\overset{A}{\exists}\text{-L} \qquad \frac{\Gamma \vdash \Delta, \varphi_{\tau/\xi}}{\Gamma \vdash \Delta, \overset{A}{\exists} \xi\varphi}\overset{A}{\exists}\text{-R}$$

(where in $\overset{A}{\forall}$-R and $\overset{A}{\exists}$-L τ does not occur free in either Γ or Δ).

25 To avoid confusion, I reserve unstarred 'premise', 'conclusion', and their likes for the things multisets of which stand in the relation of logical consequence, while I use 'premise*', 'conclusion*', and their likes for the inputs and outputs of the metarules of the systems in question.

26 Of course, given monotonicity but not contraction (as is the case for $\mathbf{IK}^{\omega}$), it is the second conjunct that really matters, while, given contraction but not monotonicity (as would be the case, e.g., for certain presentations of certain relevant logics), it is the first conjunct that really matters.

27 Following standard usage, I call ¬, ⊗, ⊕, → '*connectives*', while ∀ and ∃ (possibly together with the immediately following variable) '*quantifiers*'. I use '*operator*' for both kinds of logical expressions.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

While such metarules and the operators governed by them are unexceptionable from a purely logical and technical point of view, and have indeed proven to be very interesting and useful additions to a logic, they are multiply problematic from the philosophical point of view adopted in this paper. From this point of view, they are in general problematic because premise and conclusion combination in general should go by the pairwise- and countable-sum operations given in Definition 3.1. This is in turn so because of at least two reasons. First (and foremost), focusing without loss of generality on the case of premise combination, one of the guiding ideas of the approach pursued in this paper is that the *joint logical strength* of Γ and Γ can be different from the logical strength of Γ alone. That is what failure of contraction in the sense explained in Section §2 requires: although $T\wr$ in itself does not entail a contradiction, $T\wr$ taken twice does. Hence, assuming that premise combination is supposed to give us the joint logical strength of the multisets of premises to be combined,[28] it should result in a combining multiset that preserves the distinctness of the premises' occurrences in the combined multisets, so that the logical strength of each such occurrence is not obliterated in the combining multiset. And this is exactly what is achieved by summing the premises' distinct occurrences. Second (and less important), short of appealing to monotonicity (which is present in $\mathbf{IK}^{\omega}$ but may not be present in similarly motivated logics), premise and conclusion combination going by the pairwise- and countable-sum operations given in Definition 3.1 seems to be the only natural option in the very common case where *different* premises or conclusions occur in different multisets that have to be combined. But then it would be arbitrary to deviate from that general method of combination in the very special case where it is exactly the *same* premises or conclusions that occur in each multiset that has to be combined.

From the philosophical point of view adopted in this paper, the metarules for the additive operators are also problematic on more specific grounds. For it would follow from an easy extension of the cut-elimination Theorem 4.9 to a deductive system including the metarules for the additive operators that, for example, $\oslash \vdash \varphi \overset{A}{\vee} \psi$ holds iff either $\oslash \vdash \varphi$ holds or $\oslash \vdash \psi$ holds (and that $\oslash \vdash \overset{A}{\exists} \xi\varphi$ holds iff, for some τ, $\oslash \vdash \varphi_{\tau/\xi}$ holds). This is

[28] I think this assumption is hardly questionable with respect to what premise combination is typically supposed to do in reasoning: *pool together pieces of information and make them interact*. But I'm very open to investigating, for certain purposes, alternative notions of premise combination. According to one such notion, for example, premise combination is supposed to give us, rather than the *joint* logical strength of the multisets of premises to be combined, the logical strength of *any* of them. (This is something like the contrast between "You can have both English and continental breakfast" and the less generous but more sensible "You can have either English or continental breakfast.") Indeed, that is arguably the notion underlying the metarules for the additive operators, and we can in effect see the main feature of that notion reflected precisely in the logical behavior of those operators. Thus, for example, speaking a bit roughly, while from $\varphi \overset{A}{\wedge} \psi$ one can infer φ and one can infer ψ, one cannot infer both φ and ψ. Speaking a bit less roughly, even if $\varphi, \psi \vdash \chi$ holds, it does not follow (even in the presence of S) that $\varphi \overset{A}{\wedge} \psi \vdash \chi$ holds (while of course, by $\otimes$-L, it does follow that $\varphi \otimes \psi \vdash \chi$ holds). As I've said, I'm very open to investigating, for certain purposes, this alternative notion of premise combination (and others as well). But I think that, insofar as one is concerned with developing a logic for the purpose of using it in reasoning, and so typically for the purpose of pooling together pieces of information and making them interact, the relevant notion of premise combination is the one according to which it gives us the joint logical strength of the multisets of premises to be combined.

so because of the more general fact that, in a standard substructural deductive (sequent-calculus) system $\Rightarrow$ with only I, K-L, and K-R as structural rules and metarules (with S possibly being admissible), the sequent $\oslash \Rightarrow \varphi \overset{A}{\vee} \psi$ must be got from either $\overset{A}{\vee}$-R^0 or $\overset{A}{\vee}$-R^1, which clearly can only happen if either $\oslash \Rightarrow \varphi$ holds or $\oslash \Rightarrow \psi$ holds (and the sequent $\oslash \Rightarrow \overset{A}{\exists} \xi\varphi$ must be got from $\overset{A}{\exists}$-R, which clearly can only happen if, for some τ, $\oslash \Rightarrow \varphi_{\tau/\xi}$ holds).[29,30] That imposes on additive operators strong *constructivist* features that few of us would expect to be present in our informal notions of conjunction, disjunction, and quantification. Of course the issues arising at this juncture go far beyond the scope of this paper. Let me just register the nonconstructivist character of the philosophical point of view adopted in this paper and its consequent rejection of additive operators as giving adequate expression to our informal notions of conjunction, disjunction, and quantification.[31,32]

Conversely, no doubt the shift to multiplicative quantifiers has here been achieved at what are usually regarded as significant costs. First, $\forall$-L and $\exists$-R are *infinitary* in the sense of having infinitely many explicitly occurring premises or conclusions, while $\forall$-R and $\exists$-L are infinitary in the sense of having infinitely many premises* (with overall infinitely many explicitly occurring premises or conclusions). Second, $\forall$-R and $\exists$-L clearly only make sense with respect to the intended meaning of $\forall$ and $\exists$ as *objectual* quantifiers if every object is denoted by a closed term of $\mathscr{L}^1$, which in turn implies, given standard assumptions about $\mathscr{L}^1$, that the objects these quantifiers range over not only form a set,

---

29 Note that an analogous fact does not hold for $\oplus$, since $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \oplus \neg\varphi$ holds for every φ (see Theorem 3.3), while it is a consequence of Corollary 4.11 to Theorem 4.9 that, for many choices of φ, neither $\oslash \vdash_{\mathbf{IK}^\omega} \varphi$ nor $\oslash \vdash_{\mathbf{IK}^\omega} \neg\varphi$ hold. Note also that, since $\oslash \vdash_{\mathbf{IK}^\omega} \varphi, \neg\varphi$ also holds (see again Theorem 3.3), in a suitable extension of $\mathbf{IK}^\omega$, by $\overset{A}{\vee}$-R^0 and $\overset{A}{\vee}$-R^1, $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \overset{A}{\vee} \neg\varphi, \varphi \overset{A}{\vee} \neg\varphi$ would also hold, but, given the absence of W-R, we could not go from that to $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \overset{A}{\vee} \neg\varphi$.

30 Indeed, even stronger results encompassing arguments with premises would be available: for example, if the only members of Γ are *Rasiowa–Harrop* wffs (i.e., wffs where no disjunction or particular quantification has a "strictly positive" occurrence), $\Gamma \vdash \varphi \overset{A}{\vee} \psi$ holds iff either $\Gamma \vdash \varphi$ holds or $\Gamma \vdash \psi$ holds (and $\Gamma \vdash \overset{A}{\exists} \xi\varphi$ holds iff, for some τ, $\Gamma \vdash \varphi_{\tau/\xi}$ holds).

31 Again, the last point is compatible with the recognition of the fact that additive operators are unexceptionable from a purely logical and technical point of view, and that they have indeed proven to be very interesting and useful additions to a logic. Even more strongly, it is compatible with such operators expressing interesting notions (although different from our informal notions of conjunction, disjunction, and quantification). I take no stand on this last specific issue, while recording a certain scepticism as to whether laws, rules, and metarules are ever sufficient to determine a meaning. Be that as it may, the issue is to a large extent tangential to the purposes of this paper: as I have already observed in the text, Theorem 4.9 is easily extendable to a deductive system including the metarules for the additive operators, if one wishes to include them (however, in order not to raise complexity beyond necessity, I won't spell out the details of such extension).

32 Having said all this, I believe that the metarules for the additive operators do capture some important features of our informal notions of conjunction, disjunction, and quantification: as can easily be seen from Theorems 3.9 and 3.10, the theory of conjunction, disjunction, and quantification developed in this paper validates analogues of the unipremise* metarules $\overset{A}{\wedge}$-L^0, $\overset{A}{\wedge}$-L^1, $\overset{A}{\vee}$-R^0, $\overset{A}{\vee}$-R^1, $\overset{A}{\forall}$-L, and $\overset{A}{\exists}$-R *unrestrictedly*, and, as can easily be checked, it also validates analogues of the multipremise* (or unipremise* but with the usual no-free-occurrence restriction) metarules $\overset{A}{\wedge}$-R, $\overset{A}{\vee}$-L, $\overset{A}{\forall}$-R, and $\overset{A}{\exists}$-L *for the special case in which* $\Gamma = \Delta = \oslash$.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

but are indeed *countably many*. While I must confess to not being particularly moved by the first worry (at least without further elaboration and detail), and will say no more about it, the second worry does point to a very severe limitation of the systems developed in this paper. It is however, I think, a limitation worth incurring in order to keep complexity within necessity, and I trust that it will be easy to see that and how the systems developed in this paper can be so generalized as to lift the restriction to the countable. Moreover, overall, the naturalness and coherence of the systems developed in this paper does show that there are at least some intuitive and well-behaved notions of multiplicative quantifiers, as opposed to a certain scepticism to be found in the literature ("no good context-free versions are known," Troelstra & Schwichtenberg, 2000, p. 295). This is even more important if I am right in my contention that additive quantifiers have little or no place in the development of the kind of noncontractive solution to the semantic paradoxes presented in this paper.

Whereas, in comparison with other deviations from classical logic proposed in order to deal with the semantic paradoxes while preserving the naive theory of truth, $\mathbf{IK}^{\omega}$ is obviously weak in the specific respect constituted by the absence from it of W-L and W-R, it is *surprisingly strong* in many other respects, approximating the simplicity and symmetry of classical logic to an extent unmatched by its naive rivals. I will now go through an indicative survey of principles valid in $\mathbf{IK}^{\omega}$, for most of the time letting the self-evidence of such principles speak for itself, but commenting a bit on certain consequences and comparisons.

To begin with, in $\mathbf{IK}^{\omega}$ $\neg\varphi$ is *complete* over φ:

THEOREM 3.3. *The law of excluded middle and the attendant rule of exhaustion hold in* $\mathbf{IK}^{\omega}$*, that is:*

(LEM) $\oslash \vdash_{\mathbf{IK}^{\omega}} \varphi \oplus \neg\varphi$
(EXH) $\oslash \vdash_{\mathbf{IK}^{\omega}} \varphi, \neg\varphi$

hold.[33]

This is in sharp contrast to analetheic naive theories of truth (see some references in Section 2.1), in which both the law of excluded middle and the rule of exhaustion fail.

Given the dualities of the logic, it is no surprise that in $\mathbf{IK}^{\omega}$ $\neg\varphi$ is also *inconsistent* with φ:

THEOREM 3.4. *The law of noncontradiction and the attendant rule of explosion hold in* $\mathbf{IK}^{\omega}$*, that is:*

(LNC) $\varphi \otimes \neg\varphi \vdash_{\mathbf{IK}^{\omega}} \oslash$
(EXP) $\varphi, \neg\varphi \vdash_{\mathbf{IK}^{\omega}} \oslash$

hold.

This is in sharp contrast to dialetheic naive theories of truth (see some references in Section 2.1), in which both the law of noncontradiction and the rule of explosion fail.[34] In fact, by

[33] Throughout, for clarity I use more or less traditional names (like 'the law of excluded middle') for principles seen not as specific to a certain system, but as able to hold or fail to hold relative to different systems (as when we say, e.g., that the law of excluded middle holds in classical logic but fails to hold in intuitionist logic), while I use the corresponding acronyms (like '(LEM)') for the $\mathbf{IK}^{\omega}$- or $\mathbf{IKT}^{\omega}$-specific principles, which can only either absolutely hold or absolutely fail to hold, depending on the properties of $\mathbf{IK}^{\omega}$ and $\mathbf{IKT}^{\omega}$ (as when I say that (LEM) holds).

[34] Most naive theories of truth, even those that do not validate the law of excluded middle or the law of noncontradiction, have a minimally decent conditional $\rightarrow$ validating the law of *reflexivity*

$\neg$-R, (LNC) yields $\oslash \vdash_{\mathbf{IK}^{\omega}} \neg(\varphi \otimes \neg\varphi)$. This latter, extremely plausible law is upheld in the best dialetheic theories (such as those referenced in Section 2.1) but disappointingly fails even in the best analetheic theories (such as those referenced in Section 2.1), which are thus not in a position to assert extremely plausible claims like the one that it is not the case that [the strengthened Liar sentence is true and the strengthened Liar sentence is not true].[35,36]

Indeed, the joint strength of $\neg$-L and $\neg$-R ensures that negation enjoys many of its habitual interactions with *suppositional reasoning*:

THEOREM 3.5. *The metarules of weak reductio ad absurdum hold in in* $\mathbf{IK}^{\omega}$, *that is:*

(WRA$_0$) *If* $\Gamma \vdash_{\mathbf{IK}^{\omega}} \varphi$ *holds,* $\Gamma, \neg\varphi \vdash_{\mathbf{IK}^{\omega}} \oslash$ *holds*
(WRA$_1$) *If* $\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}} \oslash$ *holds,* $\Gamma \vdash_{\mathbf{IK}^{\omega}} \neg\varphi$ *holds*

hold.

Thus, although (the intuitionistically acceptable version of) full *reductio ad absurdum* has to fail in the background logic of the naive theory of truth presented in this paper,[37] we still preserve the core of the connections between negation and inconsistency in suppositional reasoning, contrary to most naive theories of truth (certainly all those referenced in this

---

$\varphi \rightarrow \varphi$. Letting $\varphi \vee \psi$ be $\neg\varphi \rightarrow \psi$ and $\varphi \wedge \psi$ be $\neg(\varphi \rightarrow \neg\psi)$, under minimal assumptions such theories could thus define a disjunctive and a conjunctive connective validating the law of excluded middle and the law of noncontradiction, respectively. Can this theft really replace the honest toil with which such laws are validated by the naive theory of truth presented in this paper? For several reasons, no. First, it is typically not an unwanted by-product of a naive theory of truth that it rejects those laws, but, for better or worse, *one of the most immediate consequences of its informal philosophical motivation*, and so there is no question of trying to recover those laws in the first place. Second, as I have indicated in the text, a major reason in favor of those laws is that they would seem to reveal that $\neg\varphi$ is complete over and inconsistent with φ. That is only the case, however, if the attendant rules of exhaustion and explosion *also* hold, and the strategy in question, focusing on $\vee$ and $\wedge$, can do nothing in that respect. Third, it is very unclear how a similar strategy could be applied to secure analogous principles for the *quantifiers*. Fourth, the resulting connectives would anyhow behave in ways highly aberrant with respect to our informal notions of disjunction and conjunction, variously failing the rules of addition, simplification, finite abjunction, and finite adjunction (see Theorems 3.9 and 3.11) that arguably encode the fundamental *extensionality* of those notions (exactly which of these rules and other principles will fail will depend on the details of the system). Thanks to Greg Restall for discussion of this issue.

35 Henceforth, I use square brackets to disambiguate constituent structure in English.

36 It will probably be rejoined that the alleged extremely plausible law entails, given the relevant De Morgan principle, the law of excluded middle, and hence that, since the latter is resistible, so is the former. But this way of reasoning strikes me as deeply misguided: given that the alleged extremely plausible law is indeed such, it is not resistible, and hence one should either also accept the law of excluded middle or else reject the relevant De Morgan principle (as intuitionists do). Thanks to Kevin Scharp for impressing upon me the importance of this point.

37 (The intuitionistically acceptable version of) full *reductio ad absurdum* says that, if $\Gamma, \varphi \vdash \Delta, \neg\varphi$ holds, $\Gamma \vdash \Delta, \neg\varphi$ holds. This metarule has to fail in the background logic of the naive theory of truth presented in this paper because, as I have already observed in Section 2.1, it would reinstate paradox, for example, with the strengthened Liar sentence $\neg T\wr$ (for then, since $T\wr$ entails $\neg T\wr$, $\neg T\wr$ would be a law, which is easily seen to lead to inconsistency in the naive theory of truth presented in this paper). Notice that, applying $\neg$-R to $\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}} \Delta, \neg\varphi$, we do get $\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \neg\varphi, \neg\varphi$, but, given the absence of W-R, we cannot go from that to $\Gamma \vdash_{\mathbf{IK}^{\omega}} \Delta, \neg\varphi$.

paper), where either of the metarules fails.[38] Negation in $\mathbf{IK}^{\omega}$ has thus all the hallmarks of *Boolean* negation. Hence, it is such as to cover *all and only those ways in which the negated sentence fails to hold*—a very intuitive and theoretically central notion, which cannot however be expressed not only in analetheic and dialetheic theories, but in most naive theories of truth (certainly in all those referenced in this paper; see Priest, 2006b, for a good critical discussion of Boolean negation in the context of the semantic paradoxes).[39]

Much less distinctively, negation in $\mathbf{IK}^{\omega}$ also has all the hallmarks of *De Morgan* negation:

THEOREM 3.6. *The sentential De Morgan rules hold in* $\mathbf{IK}^{\omega}$*, that is:*

(SDM$_0$) $\varphi \otimes \psi \vdash_{\mathbf{IK}^{\omega}} \neg(\neg\varphi \oplus \neg\psi)$
(SDM$_1$) $\varphi \oplus \psi \vdash_{\mathbf{IK}^{\omega}} \neg(\neg\varphi \otimes \neg\psi)$
(SDM$_2$) $\neg(\varphi \otimes \psi) \vdash_{\mathbf{IK}^{\omega}} \neg\varphi \oplus \neg\psi$
(SDM$_3$) $\neg(\varphi \oplus \psi) \vdash_{\mathbf{IK}^{\omega}} \neg\varphi \otimes \neg\psi$

hold.

THEOREM 3.7. *The quantificational De Morgan rules hold in* $\mathbf{IK}^{\omega}$*, that is:*

(QDM$_0$) $\forall\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \neg\exists\xi\neg\varphi$
(QDM$_1$) $\exists\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \neg\forall\xi\neg\varphi$
(QDM$_2$) $\neg\forall\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \exists\xi\neg\varphi$
(QDM$_3$) $\neg\exists\xi\varphi \vdash_{\mathbf{IK}^{\omega}} \forall\xi\neg\varphi$

hold.

THEOREM 3.8. *The rules of double-negation introduction and double-negation elimination hold in* $\mathbf{IK}^{\omega}$*, that is:*

(DNI) $\varphi \vdash_{\mathbf{IK}^{\omega}} \neg\neg\varphi$
(DNE) $\neg\neg\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi$

hold.

---

[38] Indeed, it is very much arguable that, once the distinction has been drawn between (WRA$_1$) and the metarule standardly called '*reductio ad absurdum*' formulated in Footnote 37, the name '*reductio ad absurdum*' for the latter is an egregious misnomer. For a metarule properly called '*reductio ad absurdum*' should have an antecedent saying that certain premises lead to an absurdity (i.e., are inconsistent, i.e., entail $\oslash$), which is clearly the case for (WRA$_1$) and clearly not the case for the metarule standardly called '*reductio ad absurdum*' formulated in Footnote 37, which would more properly be called '*reductio ad ipsius negationem*'.

[39] Unfortunately, that book completely ignores the kind of proposal presented in this paper, and in so doing overstates its case when it says (p. 94, where contraction is blatantly presupposed in the "proof" of the claim): "[...] if we have Boolean negation and the truth predicate (together with self-reference), triviality ensues" and when it says (p. 99, where again the only way envisaged of keeping both Boolean negation and naive truth is to give up self-reference, which, as Priest says, is "clearly no ground for smugness either"): "[n]or has a classical logician any reason to feel smug about this. As we have seen, if Boolean negation is meaningful, then a predicate satisfying the unrestricted T-schema cannot be." The naive theory of truth presented in this paper has both Boolean negation and naive truth (indeed, as Corollary 3.24 to Theorem 3.22 will show, has naive truth in a much stronger form than the one Priest himself envisages and endorses), but, as Corollary 4.12 to Theorem 4.9 will show, is not trivial. This does give me some grounds for smugness.

Given the other standard features of $\mathbf{IK}^{\omega}$, Theorems 3.6, 3.7, and 3.8 clearly suffice to establish the classical equivalences, in the presence of negation, between conjunction and disjunction on the one hand and between universal quantification and particular quantification on the other hand, so that, in the presence of negation, each of these two pairs consists of interdefinable operations.[40]

We can also verify that, thanks to K-L and K-R, these operators obey some other rules that are typically uncontroversial for naive theories of truth:

THEOREM 3.9. *The rules of simplification and addition hold in* $\mathbf{IK}^{\omega}$, *that is:*

(SIMP$_0$) $\varphi \otimes \psi \vdash_{\mathbf{IK}^{\omega}} \varphi$
(SIMP$_1$) $\varphi \otimes \psi \vdash_{\mathbf{IK}^{\omega}} \psi$
(ADD$_0$) $\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi \oplus \psi$
(ADD$_1$) $\psi \vdash_{\mathbf{IK}^{\omega}} \varphi \oplus \psi$

hold.

THEOREM 3.10. *The rules of universal instantiation and particular generalization hold in* $\mathbf{IK}^{\omega}$, *that is:*

(UI) $\forall \xi \varphi \vdash_{\mathbf{IK}^{\omega}} \varphi_{\tau/\xi}$
(PG) $\varphi_{\tau/\xi} \vdash_{\mathbf{IK}^{\omega}} \exists \xi \varphi$

hold.

Indeed, even more straightforwardly, $\mathbf{IK}^{\omega}$ validates the converse rules:

THEOREM 3.11. *The rules of finite adjunction and finite abjunction hold in* $\mathbf{IK}^{\omega}$, *that is:*

(ADJ^f) $\varphi, \psi \vdash_{\mathbf{IK}^{\omega}} \varphi \otimes \psi$
(ABJ^f) $\varphi \oplus \psi \vdash_{\mathbf{IK}^{\omega}} \varphi, \psi$

hold.

THEOREM 3.12. *The rules of denumerable adjunction and denumerable abjunction hold in* $\mathbf{IK}^{\omega}$, *that is:*

(ADJ^d) $\varphi_{\upsilon_0/\xi}, \varphi_{\upsilon_1/\xi}, \varphi_{\upsilon_2/\xi} \ldots \vdash_{\mathbf{IK}^{\omega}} \forall \xi \varphi$
(ABJ^d) $\exists \xi \varphi \vdash_{\mathbf{IK}^{\omega}} \varphi_{\upsilon_0/\xi}, \varphi_{\upsilon_1/\xi}, \varphi_{\upsilon_2/\xi} \ldots$

hold.

Together with $\otimes$-L and $\oplus$-R, Theorem 3.11 ensures that the operation of conjunction denoted by $\otimes$ and the operation of disjunction denoted by $\oplus$ perfectly capture the operation of finite combination of premises and finite combination of conclusions, respectively. And together with $\forall$-L and $\exists$-R, Theorem 3.12 ensures that the operation of universal quantification denoted by $\forall$ and the operation of particular quantification denoted by $\exists$ perfectly capture the operation of denumerable combination of premises (all of the form $\varphi_{\upsilon_i/\xi}$) and denumerable combination of conclusions (all of the form $\varphi_{\upsilon_i/\xi}$), respectively. And, as I have variously anticipated in Footnotes 17, 32, and 34, all this together with Theorems 3.9 and 3.10 in turn ensures that such operations exhibit the necessary degree

[40] Strictly speaking, equivalence guarantees definability in the usual sense (which requires full intersubstitutability) only in the presence of Theorem 3.22.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

of *extensionality* that is arguably fundamental in our informal notions of conjunction, disjunction, and quantification.

Indeed, the joint strength of $\oplus$-L and $\oplus$-R ensures that disjunction enjoys many of its habitual interactions with *suppositional reasoning*:

THEOREM 3.13. *The metarule of weak reasoning by cases holds in* $\mathbf{IK}^{\omega}$*, that is:*

(WRC) *If* $\Gamma_0, \varphi_0 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \psi_0$ *and* $\Gamma_1, \varphi_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_1, \psi_1$ *hold,* $\Gamma_0, \Gamma_1, \varphi_0 \oplus \varphi_1 \vdash_{\mathbf{IK}^{\omega}} \Delta_0, \Delta_1, \psi_0 \oplus \psi_1$ *holds*

holds.

THEOREM 3.14. *The metarule of inconsistency transmission from disjuncts to disjunction holds in* $\mathbf{IK}^{\omega}$*, that is:*

(ITDD) *If* $\Gamma_0, \varphi_0 \vdash_{\mathbf{IK}^{\omega}} \oslash$ *and* $\Gamma_1, \varphi_1 \vdash_{\mathbf{IK}^{\omega}} \oslash$ *hold,* $\Gamma_0, \Gamma_1, \varphi_0 \oplus \varphi_1 \vdash_{\mathbf{IK}^{\omega}} \oslash$ *holds*

holds.

This is in sharp contrast to supervaluationist and revision naive theories of truth (see some references in Section 2.1), in (possibly an appropriate extension to a multiple-conclusion framework of) which the rules of finite and denumerable abjunction and the metarules of weak reasoning by cases and inconsistency transmission from disjuncts to disjunction all fail. Thus, although full reasoning by cases has to fail in the background logic of the naive theory of truth presented in this paper,[41] we still preserve the core of the function of disjunction in suppositional reasoning and the core of the connections between disjunction and inconsistency in suppositional reasoning. Analogous considerations apply for particular quantification.

Turning to implication, it exhibits a pair of mutually converse features traditionally thought to be of the essence of this operation. On the one hand:

THEOREM 3.15. *The rule of modus ponens holds in* $\mathbf{IK}^{\omega}$*, that is:*

(MP) $\varphi, \varphi \rightarrow \psi \vdash_{\mathbf{IK}^{\omega}} \psi$

holds.

On the other hand, a strong *side-premise* version of the metarule represented by the *deduction theorem* is of course nothing more than $\rightarrow$-R. This is in sharp contrast to most naive theories of truth (certainly to all those referenced in this paper), as in such theories no connective that satisfies *modus ponens* also satisfies even the *single-premise* version of the deduction theorem. For consider again the Curry sentence denoted by $\mathfrak{c}$ and the paradoxical argument presented in Section 2.2. That argument relies only on the naive theory of truth, on the structural rules and metarules of reflexivity, transitivity and contraction, on *modus*

[41] Full reasoning by cases for a disjunctive connective is in effect the analogue for that connective of what $\overset{A}{\vee}$-L is for $\overset{A}{\vee}$. This metarule has to fail in the background logic of the naive theory of truth presented in this paper because it would reinstate W-R (for then, since $\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi$ holds, $\varphi \oplus \varphi \vdash_{\mathbf{IK}^{\omega}} \varphi$ would hold, which, by $\oplus$-R and S, allows one to derive W-R in $\mathbf{IK}^{\omega}$). In turn, W-R would reinstate paradox, for example, with the strengthened Liar sentence $\neg T\mathfrak{l}$ (for then, since according to the naive theory of truth presented in this paper $\oslash \vdash T\mathfrak{l}, T\mathfrak{l}$ holds, $\oslash \vdash T\mathfrak{l}$ would hold, which is easily seen to lead to inconsistency in the naive theory of truth presented in this paper). Notice that all this does not imply that $\overset{A}{\vee}$ cannot consistently be added to $\mathscr{L}^1$ (it can), for it does not obey the analogue for it of what $\oplus$-R is for $\oplus$.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

ponens, and on the single-premise version of the deduction theorem. As most naive theories of truth (certainly all those referenced in this paper) validate the first four, most naive theories of truth (certainly all those referenced in this paper) have to give up the fifth.

Although this is not the place to elaborate on the deep issues involved here, I do want to emphasize that this is a *prima facie* very serious (and strangely underestimated)[42] problem for such theories: for, if φ entails ψ, it would seem eminently reasonable to infer 'If φ, ψ'. If that eminently reasonable inference is blocked, an explanation should certainly be given of what it is about logical consequence, or what it is about any *modus*-*ponens* satisfying kind of implication, that blocks the inference—an explanation should certainly be given of how it can be that φ *entails* ψ while failing to be in any interesting sense a *sufficient condition* for it (although one should be open to the idea that there are kinds of *modus-ponens* satisfying implications for which that inference fails, we would seem to have the notion of at least one such kind for which that inference is valid). I myself don't know of any remotely persuasive explanation for this to date, and, in the absence of such an explanation, the question must arise as to why that treatment of this version of Curry's paradox is anything more than a piece of adhocery.

We should note that, in the presence of K-L and K-R, the deduction theorem makes implication very similar to classical material implication:

THEOREM 3.16. *The rules of positive sufficiency and negative sufficiency hold in* $\mathbf{IK}^{\omega}$*, that is:*

(PS) $\varphi \vdash_{\mathbf{IK}^{\omega}} \psi \rightarrow \varphi$

(NP) $\neg\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi \rightarrow \psi$

hold.

Notice that, although the similarity is very high indeed, we still do not have the collapse of implication on classical material implication, as, for example, the classically valid contraction law mentioned in Footnote 42 is not valid in $\mathbf{IK}^{\omega}$. What we have is nevertheless enough to establish the classical equivalences, in the presence of negation, between implication and conjunction on the one hand and implication and disjunction on the other hand, so that, in the presence of negation, each of these two pairs consists of interdefinable operations (with, as we have already seen, conjunction and disjunction also being directly interdefinable in the presence of negation, see Footnote 40 and the text to which it is appended for further details):

[42] For what's worth, I conjecture that at least part of the explanation for this underestimation of the problem is due to the fact that, traditionally, people have worked on the semantic paradoxes in an *axiomatic* framework. In such a framework, Curry's paradox is most naturally presented in a version that, instead of appealing to the *structural metarule* of contraction, appeals to the *law* $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$ (which is sometimes also confusingly called 'contraction' and which, using I, $\rightarrow$-L and $\rightarrow$-R, one could derive in $\mathbf{IK}^{\omega}$ iff one could contract on φ). Such a version does *not* appeal to the deduction theorem (unsurprisingly so, since the deduction theorem is typically absent from the defining principles of an axiomatic system, given that such systems are typically formulated in such a way that the deduction theorem is rather merely admissible in them). This is certainly another respect in which the shift of focus from the axioms of the Fregean and Hilbertian tradition to the consequence relations of the Tarskian tradition helps to improve our understanding of certain issues in logic and its philosophy.

THEOREM 3.17. *The rules of no-counterexample and of false-antecedent-or-true-consequent hold in* $\mathbf{IK}^\omega$*, that is:*

(NC) $\varphi \to \psi \vdash_{\mathbf{IK}^\omega} \neg(\varphi \otimes \neg\psi)$ *and* $\neg(\varphi \otimes \neg\psi) \vdash_{\mathbf{IK}^\omega} \varphi \to \psi$
(FATC) $\varphi \to \psi \vdash_{\mathbf{IK}^\omega} \neg\varphi \oplus \psi$ *and* $\neg\varphi \oplus \psi \vdash_{\mathbf{IK}^\omega} \varphi \to \psi$

hold.

Although it may not immediately be obvious what the big advantage is of having a conditional obeying (NC) (or (FATC)) along with other standard principles like (MP), it is actually arguable that the availability of some such operator is crucial for the expression of thoughts involving *restricted quantification* (I plan to develop this point at length in future work).

I hope I've said enough to give a good and concrete sense of the strength of $\mathbf{IK}^\omega$, especially when compared with other deviations from classical logic proposed in order to deal with the semantic paradoxes while preserving the naive theory of truth. Of course, $\mathbf{IK}^\omega$ is obviously weak in the specific respect constituted by the absence from it of contraction in the premises and in the conclusions, which on the contrary holds in those other logics. Notice again that it is not only that W-L and W-R are absent from the defining principles of $\mathbf{IK}^\omega$ (and $\mathbf{IKT}^\omega$): as should be expected and as will be shown in Corollary 4.14 to Theorem 4.9, W-L and W-R are simply not admissible in $\mathbf{IK}^\omega$ (or in $\mathbf{IKT}^\omega$)—they simply *fail to hold* in $\mathbf{IK}^\omega$ (or in $\mathbf{IKT}^\omega$).

However, quite generally, the fact that a certain principle is not valid in a certain logic in a way does not mean that much even from the standpoint of an adherent of that logic. For that the principle is not valid simply means that it is not *formally valid*, and for that it suffices that *at least one instance* of the principle be unacceptable either from the point of view of the logic itself or from the more general outlook (philosophical or otherwise) informing the logic (of course, if the logic is at least sound with respect to the outlook, the latter property entails the former). And, clearly, that sufficient condition is perfectly compatible with *other instances* of the principle being acceptable either from the point of view of the logic itself or from the more general outlook (philosophical or otherwise) informing the logic (of course, if the logic is at least sound with respect to the outlook, the former property entails the latter). The trite example of the status of the law of excluded middle in intuitionism illustrates well both cases. Although the law is formally invalid in intuitionist logic, some instances of it are acceptable from the point of view of the logic itself and are indeed deemed to be logical truths—for example, 'Either, if Goldbach's Conjecture is true, Goldbach's Conjecture is true, or it is not the case that, if Goldbach's Conjecture is true, Goldbach's Conjecture is true'. And although the law is formally invalid in intuitionist logic, some instances of the law, although themselves not acceptable from the point of view of the logic itself and indeed not deemed to be logical truths, are acceptable from the more general philosophical outlook informing the logic (at least the usual one) and are deemed to be true—for example, 'Either $341, 785, 601, 361, 360+703, 217, 609, 131, 275 = 1, 045, 003, 210, 492, 635$ or it is not the case that $341, 785, 601, 361, 360 + 703, 217, 609, 131, 275 = 1, 045, 003, 210, 492, 635$'. The first case simply reflects the familiar point about the importance of the right level of fineness of grain in logical form; the second case is more interesting and demonstrates how a particular instance of a principle can fail to be acceptable in a logic (by failing to be an instance of a principle valid in the logic, which is the only kind of acceptability envisioned in standard formal logics) while being acceptable from the more general outlook (philosophical or otherwise) informing the logic.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

There are of course various ways in which one could fill the second gap: mainly, one could envision a nonstandard logic where the relevant instances of the law of excluded middle do get counted as logical truths, or, since one is anyways in the business of using the logic as a background for a theory, one could simply add those instances to the relevant theory. Setting aside the interesting question of what the substantial differences are between these two courses (and others), the latter is the one usually taken. Now, it is a very beautiful fact about intuitionist logic that, very roughly speaking, the addition to a theory of the instance of the law of excluded middle for φ has the consequence that φ behaves in effect in the theory as it would behave in classical logic, and that, more generally, adding all instances of the law of excluded middle to every set of premises yields in effect classical logic.

It is a very beautiful fact about $\mathbf{IK}^{\omega}$ that it enjoys a similar property, where the relevant instances are not instances of the law of excluded middle but instances of the law of *superidempotency of self-conjunction* $\varphi \to \varphi \otimes \varphi$ and of the law of *subidempotency of self-disjunction* $\varphi \oplus \varphi \to \varphi$:

DEFINITION 3.18. *Let* $\mathbf{IK}^{\omega}\mathbf{W}^{X}$ *be the system got by adding to* $\mathbf{IK}^{\omega}$ *the instances of the laws of superidempotency of self-conjunction and of subidempotency of self-disjunction for every* $\varphi \in X$*:*

(SPSC) $\oslash \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \varphi \to \varphi \otimes \varphi$
(SBSD) $\oslash \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \varphi \oplus \varphi \to \varphi$

THEOREM 3.19. *For every* $\varphi \in X$*, contraction on* φ *in the premises holds in* $\mathbf{IK}^{\omega}\mathbf{W}^{X}$*: if* $\Gamma, \varphi, \varphi \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \Delta$ *holds, so does* $\Gamma, \varphi \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \Delta$*. For every* $\varphi \in X$*, contraction on* φ *in the conclusions also holds in* $\mathbf{IK}^{\omega}\mathbf{W}^{X}$*: if* $\Gamma \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \Delta, \varphi, \varphi$ *holds, so does* $\Gamma \vdash_{\mathbf{IK}^{\omega}\mathbf{W}^{X}} \Delta, \varphi$.

COROLLARY 3.20. $\mathbf{IK}^{\omega}\mathbf{W}^{WFF_{\mathscr{L}^1}}$ *is classical logic with the ω-rule over* $\mathscr{L}^1$.

3.2. The theory of truth. So much for the background logic $\mathbf{IK}^{\omega}$. We now add to it the following metarules for T:

$$\frac{\Gamma, \varphi \vdash_{\mathbf{IKT}^{\omega}} \Delta}{\Gamma, T\ulcorner\varphi\urcorner \vdash_{\mathbf{IKT}^{\omega}} \Delta}\ T\text{-L} \qquad\qquad \frac{\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, \varphi}{\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\varphi\urcorner}\ T\text{-R}$$

thereby obtaining the theory of truth $\mathbf{IKT}^{\omega}$.

As with $\mathbf{IK}^{\omega}$, I will now go through an indicative survey of principles holding in $\mathbf{IKT}^{\omega}$, for most of the time letting the self-evidence of such principles speak for itself, but commenting a bit on certain consequences and comparisons (given the properties of $\mathbf{IKT}^{\omega}$, some of the points to be made will of course be analogous to points made about $\mathbf{IK}^{\omega}$). We start by noting that $\mathbf{IKT}^{\omega}$ has the required minimal strength:

THEOREM 3.21. $\mathbf{IKT}^{\omega}$ *is a naive theory of truth.*

But $\mathbf{IKT}^{\omega}$ is much stronger than any old naive theory of truth. To appreciate this, the following general intersubstitutability fact will be useful:

THEOREM 3.22. *If* φ_1 *is the result of replacing* ψ *by* χ *in* φ_0*, with* $\psi \vdash_{\mathbf{IKT}^{\omega}} \chi$ *and* $\chi \vdash_{\mathbf{IKT}^{\omega}} \psi$ *holding, then* [$\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^{\omega}} \Delta$ *holds iff* $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^{\omega}} \Delta$ *holds*] *and* [$\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, \varphi_0$ *holds iff* $\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, \varphi_1$ *holds*].

Proof. It clearly suffices to prove the result for the following cases:

- $\varphi_0 = \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.
- $\varphi_0 = \neg\psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\psi \vdash_{\mathbf{IKT}^\omega} \chi$ holds, by $\neg$-L, $\psi, \neg\chi \vdash_{\mathbf{IKT}^\omega} \oslash$ holds, and hence, by $\neg$-R, $\neg\chi \vdash_{\mathbf{IKT}^\omega} \neg\psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.
- $\varphi_0 = \psi_0 \otimes \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds and since, by (ADJ^f), $\psi_0, \psi \vdash_{\mathbf{IKT}^\omega} \psi_0 \otimes \psi$ holds, by S, $\psi_0, \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \otimes \psi$ holds, and hence, by $\otimes$-L, $\psi_0 \otimes \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \otimes \psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.
- $\varphi_0 = \psi_0 \oplus \psi$. Dual arguments hold.
- $\varphi_0 = \psi_0 \rightarrow \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds and since, by (MP), $\psi_0, \psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \chi$ holds, by S, $\psi_0, \psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, and hence, by $\rightarrow$-R, $\psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \rightarrow \psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.
- $\varphi_0 = \psi \rightarrow \psi_0$. Dual arguments hold.
- $\varphi_0 = \forall\xi\psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, clearly, for every i, $\chi_{\upsilon_i/\xi} \vdash_{\mathbf{IKT}^\omega} \psi_{\upsilon_i/\xi}$ holds, and hence, by $\forall$-R, $\chi_{\upsilon_0/\xi}, \chi_{\upsilon_1/\xi}, \chi_{\upsilon_2/\xi} \ldots \vdash_{\mathbf{IKT}^\omega} \forall\xi\psi$ holds, and so, by $\forall$-L, $\forall\xi\chi \vdash_{\mathbf{IKT}^\omega} \forall\xi\psi$ holds, whence, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.
- $\varphi_0 = \exists\xi\psi$. Dual arguments hold.

□

The extreme strength of $\mathbf{IKT}^\omega$ should now be manifest:

DEFINITION 3.23. *A theory of truth $\vdash$ is transparent iff, if φ_1 is the result of replacing ψ by $T\ulcorner\psi\urcorner$ in φ_0, then* $[\Gamma, \varphi_0 \vdash \Delta$ *holds iff* $\Gamma, \varphi_1 \vdash \Delta$ *holds*$]$ *and* $[\Gamma \vdash \Delta, \varphi_0$ *holds iff* $\Gamma \vdash \Delta, \varphi_1$ *holds*$]$.

COROLLARY 3.24. $\mathbf{IKT}^\omega$ *is transparent.*

Coupled with the previous background logic, transparency yields very intuitive, simple, and elegant principles governing truth. To begin with, in addition to naivety, we also have ingenuousness:

COROLLARY 3.25. $\mathbf{IKT}^\omega$ *is ingenuous.*

Bearing in mind that the *falsity* of φ is very plausibly understood to consist in the truth of $\neg\varphi$, in addition to the logical law of excluded middle, we also have its semantic counterpart reflecting the fact that, in $\mathbf{IKT}^\omega$, falsity is *complete* over truth:

COROLLARY 3.26. *The law of bivalence holds in* $\mathbf{IKT}^\omega$*, that is:*

(BIV) $\oslash \vdash_{\mathbf{IKT}^\omega} T\ulcorner\varphi\urcorner \oplus T\ulcorner\neg\varphi\urcorner$

holds.

This is in sharp contrast to analetheic naive theories of truth (see some references in Section 2.1), in which the law of bivalence fails.

Given the dualities of the theory, it is no surprise that, in addition to the logical law of noncontradiction, we also have its semantic counterpart reflecting the fact that, in $\mathbf{IKT}^{\omega}$, falsity is *inconsistent* with truth:

COROLLARY 3.27. *The law of contravalence holds in* $\mathbf{IKT}^{\omega}$*, that is:*

(CONTRAV) $T\ulcorner\varphi\urcorner \otimes T\ulcorner\neg\varphi\urcorner \vdash_{\mathbf{IKT}^{\omega}} \oslash$

holds.

In fact, by ¬-R, (CONTRAV) yields $\oslash \vdash_{\mathbf{IKT}^{\omega}} \neg(T\ulcorner\varphi\urcorner \otimes T\ulcorner\neg\varphi\urcorner)$. This latter, extremely plausible law is upheld in at least some dialetheic theories (such as the transparent dialetheic theory advocated by Beall, 2009) but disappointingly fails even in the best analetheic theories (such as those referenced in Section 2.1), which are thus not in a position to assert extremely plausible claims like the one that it is not the case that [the simple Liar sentence is true and the simple Liar sentence is false]. (Very roughly, a *simple Liar sentence* is a sentence saying of itself that it is false.) Thus, the traditional conception according to which there are neither *gaps* nor *gluts* between the true and the false is fully upheld in $\mathbf{IKT}^{\omega}$.

Other traditional semantic laws familiar from truth-functional semantics also hold in $\mathbf{IKT}^{\omega}$:

COROLLARY 3.28. *The truth-functional laws:*

(NEG$^{\Rightarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\neg\varphi\urcorner \to \neg T\ulcorner\varphi\urcorner$
(NEG$^{\Leftarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} \neg T\ulcorner\varphi\urcorner \to T\ulcorner\neg\varphi\urcorner$
(CONJ$^{\Rightarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi \otimes \psi\urcorner \to T\ulcorner\varphi\urcorner \otimes T\ulcorner\psi\urcorner$
(CONJ$^{\Leftarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi\urcorner \otimes T\ulcorner\psi\urcorner \to T\ulcorner\varphi \otimes \psi\urcorner$
(DISJ$^{\Rightarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi \oplus \psi\urcorner \to T\ulcorner\varphi\urcorner \oplus T\ulcorner\psi\urcorner$
(DISJ$^{\Leftarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi\urcorner \oplus T\ulcorner\psi\urcorner \to T\ulcorner\varphi \oplus \psi\urcorner$
(COND$^{\Rightarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi \to \psi\urcorner \to (T\ulcorner\varphi\urcorner \to T\ulcorner\psi\urcorner)$
(COND$^{\Leftarrow}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} (T\ulcorner\varphi\urcorner \to T\ulcorner\psi\urcorner) \to T\ulcorner\varphi \to \psi\urcorner$

hold.

Corresponding laws could easily be proven for the quantifiers, but doing so would at least require building more syntax into $\mathbf{IKT}^{\omega}$ (and, for certain statements of such laws, extending the theory of truth to a theory of satisfaction), which would go beyond the purposes of this paper.

I would like to stress the presence in $\mathbf{IKT}^{\omega}$ of (CONJ$^{\Leftarrow}$) and (DISJ$^{\Rightarrow}$) on the one hand and of (COND$^{\Rightarrow}$) on the other hand. One is easily tempted to understand failure of contraction, especially if one focuses on the ensuing failures of (SPSC) and (SBSD) (see Theorem 3.19), as a failure of the truth of certain conjuncts to imply the truth of their conjunction, and as a failure of the truth of a certain disjunction to imply the truth of either of its disjuncts. As made manifest by the presence in $\mathbf{IKT}^{\omega}$ of (CONJ$^{\Leftarrow}$) and (DISJ$^{\Rightarrow}$), that temptation should be resisted: failure of contraction is perfectly compatible with the very plausible semantic law that the truth of certain conjuncts implies the truth of their conjunction, and perfectly compatible with the dual very plausible semantic law that the truth of a certain disjunction implies the truth of either of its disjuncts. What, for example, generates or at least is connected with the failures of (SPSC) are not the alleged failures of the former law, which on the contrary continues to hold in $\mathbf{IKT}^{\omega}$; it is rather the fact that φ's being true cannot always be assumed to imply that [φ is true and φ is true] (which alone is the proper statement of the fact that *both* conjuncts of $\varphi \otimes \varphi$ are true, which, by

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

(CONJ$^{\Leftarrow}$), would then imply the truth of $\varphi \otimes \varphi$; dual considerations hold for the failures of (SBSD)). Granted, at first blush, that may seem completely baffling. But if it is, it should be *no additional* bafflement to the original one generated by the failure of (SPSC), as the failure of the implication from 'φ is true' to 'φ is true and φ is true' just is the failure of (yet another) instance of (SPSC). Moreover, I think it should start to look *less* baffling once one realizes that that implication would give one the license to infer from φ's truth not just what intuitively follows from it, but also what intuitively follows from it *together with φ's truth being kept fixed* (for one could use one conjunct of 'φ is true and φ is true' to infer what intuitively follows from φ's truth and use the other conjunct to keep φ's truth fixed). Once one pays heed to the fact that paradoxical sentences arguably exhibit, under naive-truth conditions, a certain dynamicity, that license should start to seem problematic (see Section 2.3).

The presence in $\mathbf{IKT}^{\omega}$ of (COND$^{\Rightarrow}$) is also very welcome. Not only is that law very intuitive in itself, but it is also easily seen to imply, given the properties of $\mathbf{IKT}^{\omega}$, that the law of *truth preservingness of modus ponens* holds in $\mathbf{IKT}^{\omega}$, that is that:

$$(\text{TP}^{\text{MP}})\ \ \oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner\varphi\urcorner \otimes T\ulcorner\varphi \to \psi\urcorner \to T\ulcorner\psi\urcorner$$

holds. I submit that (TP$^{\text{MP}}$) is very plausible for the same reasons as (COND$^{\Rightarrow}$) is. Very interestingly, it is a surprising feature of most transparent theories of truth (certainly of all those referenced in this paper) that they cannot vindicate the law of truth preservingness of *modus ponens*, in the sense that adding that law to them generates inconsistency (the best of them do vindicate *modus ponens* and the analogues of (COND$^{\Rightarrow}$), but I think that validating *modus ponens* and the analogues of (COND$^{\Rightarrow}$) while being inconsistent with the law of truth preservingness of *modus ponens* leaves a rather bitter taste in mouth). The reason for this is relatively straightforward and, in its essence, has first been pointed out by Meyer *et al.*, 1979. Take a Curry sentence $T\mathfrak{d} \to Q$, with $\mathfrak{d}$ being the designated individual constant for that sentence, and instantiate the law of truth preservingness of *modus ponens* with $T\mathfrak{d}$ for the antecedent and Q for the consequent. By transparency and definition of $\mathfrak{d}$, that instance is equivalent with $T\mathfrak{d} \wedge T\mathfrak{d} \to Q$,[43] which, on most transparent theories of truth, is in turn equivalent with $T\mathfrak{d} \to Q$ (*since, on most transparent theories of truth, φ is equivalent and fully intersubstitutable with $\varphi \wedge \varphi$). And that in turn entails Q on most transparent theories of truth.* I think that such inconsistency with the law of truth preservingness of *modus ponens* is extremely problematic, especially when coupled with acceptance of unrestricted *modus ponens*.

The previous argument against the consistency of the law of truth preservingness of *modus ponens* fails for the naive theory of truth presented in this paper, for the simple reason that, in $\mathbf{IKT}^{\omega}$, φ is not equivalent with $\varphi \otimes \varphi$, as the former does not entail the latter (which is the crucial direction employed in the previous argument; the converse direction does indeed hold in $\mathbf{IKT}^{\omega}$, as it is just a particular case of (SIMP$_0$) or (SIMP$_1$)). That entailment does not hold because otherwise, by $\to$-R, (SPSC) would hold, which contradicts Theorem 3.19 taken together with Corollary 4.14 to Theorem 4.9. I also note that, very interestingly, even once one somehow manages to get up to $T\mathfrak{d} \to Q$, in

[43] Notice that transparency is sufficient but not necessary for validating the relevant entailment here, which would equally hold, for example, in the *naive and ingenuous but nontransparent* theory of truth advocated by Priest (2006a). It is on these grounds that also that theory cannot vindicate the law of truth preservingness of *modus ponens*.

$\mathbf{IKT}^{\omega}$ that does not entail Q. That entailment does not hold because otherwise, by (EXH), transparency, (NS) and S, $\oslash \vdash_{\mathbf{IKT}^{\omega}} Q, Q$ would hold, which contradicts Corollary 4.11 to Theorem 4.9. While this last feature of $\mathbf{IKT}^{\omega}$ is certainly very interesting (I plan to expand on it in future work) it does not however constitute an additional reason for why the previous argument against the consistency of the law of truth preservingness of *modus ponens* fails in $\mathbf{IKT}^{\omega}$. For, while, as I have just shown, $T\mathfrak{d} \rightarrow Q \vdash_{\mathbf{IKT}^{\omega}} Q$ does not hold, $T\mathfrak{d} \rightarrow Q, T\mathfrak{d} \rightarrow Q \vdash_{\mathbf{IKT}^{\omega}} Q$ clearly holds. But, if $T\mathfrak{d} \wedge T\mathfrak{d} \rightarrow Q \vdash_{\mathbf{IKT}^{\omega}} T\mathfrak{d} \rightarrow Q$ held, since, by ($\mathrm{TP}^{\mathrm{MP}}$), $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\mathfrak{d} \wedge T\mathfrak{d} \rightarrow Q$ holds, ($\mathrm{TP}^{\mathrm{MP}}$) would suffice to yield the problematic premise multiset $[T\mathfrak{d} \rightarrow Q, T\mathfrak{d} \rightarrow Q]$.

The point generalizes beyond the particular case of *modus ponens* and the particular case of transparent theories of truth: in many other cases too, it is a surprising feature of most naive theories of truth (certainly of all those referenced in this paper), whether transparent or not, that they validate some rules while being inconsistent with the corresponding semantic laws stating that such rules are truth preserving (see Field, 2006, for an interesting survey and for a penetrating discussion of the problem).[44] Now, I myself would actually be most wary of the postulation of any simple and straightforward connection between validity and truth preservation: not only would I be wary of many nontrivial principles stating that certain versions of truth preservation are *sufficient* for validity, but, because of various considerations relating to semantic context dependence and to higher-order indeterminacy which would lead us too far afield to rehearse here, I would also be wary of many principles stating that certain versions of truth preservation are *necessary* for validity (I plan to discuss these issues in future work), which is the direction of implication that fails on such naive theories of truth. Still, none of the considerations underlying that caution target the necessity for the validity of such a most basic rule as *modus ponens* of such a minimal version of truth preservation as the law of truth preservingness of *modus ponens* (or an analogous necessary condition for the valid rules that fail to be truth preserving in other naive theories), so that I think that the above-mentioned theories' inability to vindicate the truth preservingness of the rules they validate remains extremely problematic.

In the presence of this important failure of most naive theories of truth, the interesting question arises whether the naive theory of truth presented in this paper can do any better. Well, we've already seen that it can do better *to a certain interesting extent* not only in *being consistent with* ($\mathrm{TP}^{\mathrm{MP}}$), but indeed in *entailing* it on the sole strength of the background logic represented by $\mathbf{IK}^{\omega}$ and the naive theory of truth encapsulated in $\mathbf{IKT}^{\omega}$. I will now show, with a beautifully simple argument, that the theory does *much* better than even that—roughly, that whenever, according to $\mathbf{IKT}^{\omega}$, certain finitely many conclusions follow from

---

44 Unfortunately, that paper completely ignores the kind of proposal presented in this paper, and in so doing overstates its findings when it says (pp. 601–602): "Curry's Paradox [...] shows that any logic that accepts the standard introduction and elimination rules for the conditional and the introduction and elimination rules for truth is completely trivial: it implies anything whatsoever. Thus however compelling the argument that validity coincides with necessary truth-preservation may have seemed, it relies on assumptions that cannot be jointly accepted [...] The divergence between the rules one employs and the rules one regards as unrestrictedly truth preserving is virtually inescapable." The naive theory of truth presented in this paper validates all of $\rightarrow$-L, $\rightarrow$-R, T-L, and T-R, but, as Corollary 4.12 to Theorem 4.9 will show, is not trivial and, as Theorem 3.29 will show, entails that truth preservation is necessary for validity (which, of the various connections between validity and truth preservation, is the real target in this passage).

certain finitely many premises, it is also the case that, according to $\mathbf{IKT}^{\omega}$, if all the premises are true, so is some of the conclusions. In fact, this follows from a yet stronger result that allows for countably many side premises and conclusions:

THEOREM 3.29. *If* $\Gamma, \varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_i \vdash_{\mathbf{IKT}^{\omega}} \Delta, \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ *holds,* $\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\varphi_0\urcorner \otimes T\ulcorner\varphi_1\urcorner \otimes T\ulcorner\varphi_2\urcorner \ldots \otimes T\ulcorner\varphi_i\urcorner \rightarrow T\ulcorner\psi_0\urcorner \oplus T\ulcorner\psi_1\urcorner \oplus T\ulcorner\psi_2\urcorner \ldots \oplus T\ulcorner\psi_j\urcorner$ *holds.*

Proof. Suppose that $\Gamma, \varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_i \vdash_{\mathbf{IKT}^{\omega}} \Delta, \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ holds. Then, by (i applications of) T-L, $\Gamma, T\ulcorner\varphi_0\urcorner, T\ulcorner\varphi_1\urcorner, T\ulcorner\varphi_2\urcorner \ldots, T\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^{\omega}} \Delta, \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ holds, and so, by (j applications of) T-R, $\Gamma, T\ulcorner\varphi_0\urcorner, T\ulcorner\varphi_1\urcorner, T\ulcorner\varphi_2\urcorner \ldots, T\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\psi_0\urcorner, T\ulcorner\psi_1\urcorner, T\ulcorner\psi_2\urcorner \ldots, T\ulcorner\psi_j\urcorner$ holds. Thus, by (i applications of) $\otimes$-L, $\Gamma, T\ulcorner\varphi_0\urcorner \otimes T\ulcorner\varphi_1\urcorner \otimes T\ulcorner\varphi_2\urcorner \ldots \otimes T\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\psi_0\urcorner, T\ulcorner\psi_1\urcorner, T\ulcorner\psi_2\urcorner \ldots, T\ulcorner\psi_j\urcorner$ holds, and so, by (j applications of) $\oplus$-R, $\Gamma, T\ulcorner\varphi_0\urcorner \otimes T\ulcorner\varphi_1\urcorner \otimes T\ulcorner\varphi_2\urcorner \ldots \otimes T\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\psi_0\urcorner \oplus T\ulcorner\psi_1\urcorner \oplus T\ulcorner\psi_2\urcorner \ldots \oplus T\ulcorner\psi_j\urcorner$ holds, whence, by $\rightarrow$-R, $\Gamma \vdash_{\mathbf{IKT}^{\omega}} \Delta, T\ulcorner\varphi_0\urcorner \otimes T\ulcorner\varphi_1\urcorner \otimes T\ulcorner\varphi_2\urcorner \ldots \otimes T\ulcorner\varphi_i\urcorner \rightarrow T\ulcorner\psi_0\urcorner \oplus T\ulcorner\psi_1\urcorner \oplus T\ulcorner\psi_2\urcorner \ldots \oplus T\ulcorner\psi_j\urcorner$ holds. □

A corresponding theorem could easily be proven for the case of denumerably many premises or conclusions, but, again, doing so would require building more syntax into $\mathbf{IKT}^{\omega}$, which would go beyond the purposes of this paper. Clearly, as the above proof demonstrates, the unique availability of Theorem 3.29 for $\mathbf{IKT}^{\omega}$ is tightly connected with the unique availability of $\rightarrow$-R for $\mathbf{IKT}^{\omega}$. $\rightarrow$-R is a highly intuitive and compelling principle about implication, which in turn yields with Theorem 3.29 a highly intuitive and compelling principle about validity and truth preservation. I regard the availability of these highly intuitive and compelling principles as one of the main advantages of the naive theory of truth presented in this paper over its naive rivals, which would seem to sacrifice much of their innocent simplicity to the sophistries involved in trying to uphold their rejection of these two principles.

§4. The consistency of a noncontractive naive theory of truth.

4.1. Hauptsatz. The informal notion of *consistency* comes in different interesting strengths, ranging from mere *nontriviality* to *conservativeness* with respect to various background theories (of course, the informal notion of consistency does not require conservativeness with respect to any background theory whatsoever—while I guess we want a theory of truth to be conservative over fundamental physics, I suppose we don't want to require that it be conservative over pure logic). For the purposes of this paper, we will rest content here with proving a central result of intermediate strength: taking a *deductive system* $\Rightarrow_{\mathbf{I^K T}^{\omega}}$ that is sound and complete with respect to $\mathbf{IKT}^{\omega}$, we will prove that the rule of inference corresponding to S (and commonly known as '*cut*') is *eliminable* in $\Rightarrow_{\mathbf{I^K T}^{\omega}}$.

More specifically:

DEFINITION 4.1. $\Rightarrow_{\mathbf{I^K T}^{\omega}}$ *is the deductive (sequent-calculus) system obtained by taking:*

$$\frac{}{\Gamma, \varphi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta, \varphi}\ \mathrm{I^K}$$

as its axiom and the closure principles in Sections 3.1 and 3.2 (minus K-L *and* K-R*) as its rules of inference (with the structure and terminology assumed for the rules of* $\mathbf{IKT}^{\omega}$ *carrying over to the structure and terminology for the sequents of* $\Rightarrow_{\mathbf{I^K T}^{\omega}}$*).*

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

DEFINITION 4.2. *A $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction is any upwards branching tree with countably many branches of finite length, where each node n is occupied by a sequent of $\Rightarrow_{\mathbf{I^K T^\omega}}$ such that:*[45]

- *If n is a leaf, n is occupied by the conclusion* of an instance of* $\mathrm{I^K}$;
- *If n is immediately below all and only the Xs, n is occupied by the conclusion* and the Xs are occupied by the premises* of one of the rules of inference of $\Rightarrow_{\mathbf{I^K T^\omega}}$.*

Clearly, $\Gamma \vdash_{\mathbf{IKT^\omega}} \Delta$ holds iff $\Gamma \Rightarrow_{\mathbf{I^K T^\omega}} \Delta$ holds.[46]

Some additional standard definitions will be useful:

DEFINITION 4.3. *In an instance of* S:[47]

$$\frac{\begin{array}{c}\mathcal{D}_0\\ \Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

φ is the cut formula.

DEFINITION 4.4. *For every operator $\star$, the wff with an explicit occurrence of $\star$ in* $\star$-L ($\star$-R) *is the principal wff of* $\star$-L ($\star$-R)*.*

The specifics of our proof-theoretical framework will also require some nonstandard definitions. In particular, $\forall$-R and $\exists$-L determine that the number of nodes of a $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction may be *denumerable*. Thus, a measure for our purposes adequate of the "size" of a $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction will have to discern in each such deduction a subtree with a *finite* number of nodes:

DEFINITION 4.5. *Given a $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$, a pertinent subtree of $\mathcal{D}$ is any subtree T of $\mathcal{D}$ such that:*

- *T has the same root as $\mathcal{D}$;*
- *If n is a node of T and is immediately below all and only the Xs in an instance of a rule of inference other than* $\forall$-R *and* $\exists$-L, *then all of the Xs are nodes of T;*
- *If n is a node of T and is immediately below all and only the Xs in an instance of either* $\forall$-R *or* $\exists$-L*, then, for some i, i-many Xs are nodes of T.*

DEFINITION 4.6. *Given an i-long sequence S of $\Rightarrow_{\mathbf{I^K T^\omega}}$-deductions, a route R through S is an i-long sequence of trees such that, for every $j < i$, $R(\mathcal{D}_j)$ is a pertinent subtree of $\mathcal{D}_j$.*

---

[45] Throughout, we understand identity of nodes as requiring identity of the sequents occupying the nodes.

[46] Why did I then introduce the background logic of the naive theory of truth presented in this paper using I, K-L, and K-R rather than, more simply, $\mathrm{I^K}$? Because I on the one side and K-L and K-R on the other side clearly seem to concern two quite different, logically natural properties that a consequence relation might have or fail to have, and which in a *philosophical* discussion is of the utmost importance to keep distinct. $\mathrm{I^K}$, on the contrary, clearly seems to do nothing more than rather unilluminatingly conflating these two properties. That being said, it is also the case that the simplicity of a deductive system based on $\mathrm{I^K}$ makes it a far better tool for the *proof-theoretic* investigation of this section.

[47] Throughout, '$\mathcal{D}_0$' and its likes as they occur in proof-display mode are understood to denote the relevant $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction that ends with the node immediately below '$\mathcal{D}_0$' (*including* such a node).

DEFINITION 4.7. *If a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has a unique bottommost instance of* S, *the cut depth of $\mathcal{D}$ relative to a route R through a sequence of which $\mathcal{D}$ (or a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}'$ of which $\mathcal{D}$ is a subdeduction) is an element ($\mathrm{cd}_R(\mathcal{D})$) is the number of nodes in the maximal subtree of $R(\mathcal{D})$ (or of $R(\mathcal{D}')$) whose root is the conclusion* of $\mathcal{D}$'s bottommost instance of* S. *If, on the contrary, $\mathcal{D}$ is* S*-free, the cut depth of $\mathcal{D}$ relative to a route through a sequence of which $\mathcal{D}$ (or a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction of which $\mathcal{D}$ is a subdeduction) is an element is 0.*

DEFINITION 4.8. *A route R through a sequence S of $\Rightarrow_{\mathbf{I^K T}^\omega}$-deductions is antimonotonic over an i-long subsequence of S whose elements are $\Rightarrow_{\mathbf{I^K T}^\omega}$-deductions that have a unique bottommost instance of* S *iff, for every $j < i-1$, every node of the maximal subtree of $R(\mathcal{D}_{j+1})$ whose root is the conclusion* of the bottommost instance of* S *(minus those nodes that are not members of $\mathcal{D}_j$) is also a node of the maximal subtree of $R(\mathcal{D}_j)$ whose root is the conclusion* of the bottommost instance of* S.

THEOREM 4.9. S *is eliminable in* $\mathbf{I^K T}^\omega$.

Proof. We follow the broad outlines of the standard strategy essentially going back to Gentzen, 1934. It clearly suffices to prove that topmost instances of S in a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction can be eliminated. In turn, for that it clearly suffices to prove that a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step can be transformed into a S-free $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction with the same root as $\mathcal{D}$. And in turn, for that it clearly suffices to prove that, given a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, there are both a $\mathcal{D}$-initial sequence S of $\Rightarrow_{\mathbf{I^K T}^\omega}$-deductions with the same root as $\mathcal{D}$ and a route R through S such that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$. We will prove this last claim in two parts. In the first part of the proof, we will prove that, for every $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, it can be transformed into a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$ with the same root as $\mathcal{D}$ for which there is a route R such that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. In the second and last part of the proof, it will then suffice to observe that the defined transformations have the property that, for any sequence S obtained by repeatedly applying them, there is at least one route R through S that can be used as a constant witness for the claim that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$.

We start with the first part of the proof, distinguishing main cases, subcases, subsubcases, and subsubsubcases, and presupposing the general structure and notation displayed in Definition 4.3 for the instance of S:

Main case 1. Either premise* is the conclusion* of an instance of $\mathrm{I^K}$.

Subcase 1a. The left premise* is the conclusion* of an instance of $\mathrm{I^K}$.

Subsubcase 1aa. The cut formula only occurs as a conclusion. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{}{\Gamma_0 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \varphi}\,\mathrm{I^K} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1}\,\mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\dfrac{}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1}\,\mathrm{I^K}$$

where, for every R, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

Subsubcase 1ab. The cut formula occurs both as a premise and as a conclusion. Then the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{}{\Gamma_0', \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \varphi}\,\mathrm{I^K} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{matrix}}{\Gamma_0', \varphi, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{E}$:

$$\begin{matrix}\mathcal{D}_1^{\$} \\ \Gamma_0', \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1\end{matrix}$$

where $\mathcal{D}_1^{\$}$ is a suitable variant of $\mathcal{D}_1$ (obtained by fiddling with $\mathrm{I^K}$) and where, for every R, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 1b. A symmetrical argument holds if the right premise* is an instance of $\mathrm{I^K}$.

Main case 2. The cut formula is not principal in at least one of the premises*.

Subcase 2a. The cut formula is not principal in the left premise*. Then the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$ has the form:[48]

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0' \\ \Gamma_0' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0', \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_0'' \\ \Gamma_0'' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0''\end{matrix} \qquad \begin{matrix}\mathcal{D}_0''' \\ \Gamma_0''' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0''' \ldots\end{matrix}}{\Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \varphi}\,\star\text{-L/R} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{E}$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0' \\ \Gamma_0' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0', \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{matrix}}{\Gamma_0', \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0', \Delta_1}\,\mathrm{S} \qquad \begin{matrix}\mathcal{D}_0'' \\ \Gamma_0'' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0''\end{matrix} \qquad \begin{matrix}\mathcal{D}_0''' \\ \Gamma_0''' \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0''' \ldots\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\star\text{-L/R}$$

where, for every R antimonotonic over $\langle \mathcal{D}, \mathcal{E} \rangle$ and such that $R(\mathcal{E})$ includes a branch of $\mathcal{D}_0'$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 2b. A symmetrical argument holds if the cut formula is not principal in the right premise*.

Main case 3. The cut formula is principal in both premises*.

Subcase 3a. The cut formula is of the form $\neg\psi$. Then the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0' \\ \Gamma_0, \psi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0\end{matrix}}{\Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \neg\psi}\,\neg\text{-R} \qquad \dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1, \psi\end{matrix}}{\Gamma_1, \neg\psi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1}\,\neg\text{-L}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{E}$:

---

48 Throughout, I use '$\star$-L/R' (with $\star$ being an operator) when I mean to talk indiscriminatingly about $\star$-L and $\star$-R (with context disambiguating between the 'both'-reading and the 'either'-reading).

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

$$\dfrac{\overset{\mathcal{D}_1'}{\Gamma_1 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1, \psi} \qquad \overset{\mathcal{D}_0'}{\Gamma_0, \psi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0}}{\Gamma_1, \Gamma_0 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1, \Delta_0}\,\text{S}$$

where, for every R antimonotonic over $\langle \mathcal{D}, \mathcal{E} \rangle$, $\text{cd}_R(\mathcal{E}) < \text{cd}_R(\mathcal{D})$.
Subcase 3b. The cut formula is of the form $\psi \otimes \chi$. Then the $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\overset{\mathcal{D}_0'}{\Gamma_0' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0', \psi} \qquad \overset{\mathcal{D}_0''}{\Gamma_0'' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \chi}}{\Gamma_0 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \psi \otimes \chi}\,\otimes\text{-R} \qquad \dfrac{\overset{\mathcal{D}_1'}{\Gamma_1, \psi, \chi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1}}{\Gamma_1, \psi \otimes \chi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1}\,\otimes\text{-L}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \Delta_1}\,\text{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction $\mathcal{E}$:

$$\dfrac{\overset{\mathcal{D}_0'}{\Gamma_0' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0', \psi} \qquad \dfrac{\overset{\mathcal{D}_0''}{\Gamma_0'' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \chi} \qquad \overset{\mathcal{D}_1'}{\Gamma_1, \psi, \chi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1}}{\Gamma_0'', \Gamma_1, \psi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \Delta_1}\,\text{S}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \Delta_1}\,\text{S}$$

Our strategy will be that of eliminating the upper instance of S and in so doing showing that, as for the remaining lower instance of S, the resulting $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction already exhibits a reduction of the cut depth (relative to some routes). As for the elimination of the upper instance of S, observe that, for every R antimonotonic over $\langle \mathcal{D}, \mathcal{E} \rangle$, $\text{cd}_R(\mathcal{E}_1) < \text{cd}_R(\mathcal{D})$ (with $\mathcal{E}_1$ being the maximal subtree of $\mathcal{E}$ whose root is the displayed occurrence of $\Gamma_0'', \Gamma_1, \psi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \Delta_1$). Thus, by the induction hypothesis, the upper instance of S in $\mathcal{E}$ can be eliminated producing a S-free $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction $\mathcal{E}_1^*$ of $\Gamma_0'', \Gamma_1, \psi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \Delta_1$ and a corresponding $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction $\mathcal{E}^*$ of $\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \Delta_1$ having only one instance of S at its last step. As for $\mathcal{E}^*$'s reduction of the cut depth (relative to some routes), it is easy to verify that the transformations defined in this proof have the property that, for some R_0 suitable for the elimination of the upper instance of S, $|\text{fld}(R_0(\mathcal{E}_1^*))| \leq |\text{fld}(\mathcal{D}_1) \cap \text{fld}(R_0(\mathcal{D}))| + |\text{fld}(\mathcal{D}_0'') \cap \text{fld}(R_0(\mathcal{D}))|$ (with $\text{fld}(X)$ being the field of the relation X and with $\mathcal{D}_1$ being the maximal subtree of $\mathcal{D}$ whose root is the displayed occurrence of $\Gamma_1, \psi \otimes \chi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1$). From this, it follows that, for some R_1, $\text{cd}_{R_1}(\mathcal{E}^*) < \text{cd}_{R_1}(\mathcal{D})$.
Subcase 3c. A dual argument holds if the cut formula is of the form $\psi \oplus \chi$.
Subcase 3d. A similar argument holds if the cut formula is of the form $\psi \rightarrow \chi$.
Subcase 3e. The cut formula is of the form $\forall \xi \psi$. Then the $\Rightarrow_{\mathbf{I^K T}^{\omega}}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\overset{\mathcal{D}_0'}{\Gamma_0' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0', \psi_{\upsilon_0/\xi}} \qquad \overset{\mathcal{D}_0''}{\Gamma_0'' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0'', \psi_{\upsilon_1/\xi}} \qquad \overset{\mathcal{D}_0'''}{\Gamma_0''' \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0''', \psi_{\upsilon_2/\xi} \ldots}}{\Gamma_0 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \forall \xi \psi}\,\forall\text{-R} \qquad \dfrac{\overset{\mathcal{D}_1'}{\Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1}}{\Gamma_1, \forall \xi \psi \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_1}\,\forall\text{-L}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^{\omega}} \Delta_0, \Delta_1}\,\text{S}$$

Clearly, a denumerable submultiset of the displayed occurrence of $[\psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots]$ must be introduced in $\mathcal{D}$ either by an instance of $\forall$-R or $\exists$-L or by an instance of I^{K}. In the following, we assume without loss of generality that such submultiset has the form $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ (see Footnote 49).
Subsubcase 3ea. $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ is introduced by an instance of $\forall$-R or $\exists$-L.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

Subsubsubcase 3eaa. $[\psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi}\ldots]$ is introduced by an instance of $\forall$-R. Then $\mathcal{D}'_1$ has the form:

$$\begin{array}{c}
\dfrac{\begin{matrix}\mathcal{D}_1^{\#}\\ \Gamma_1^{\#}, \psi_{v_i/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1^{\#}\end{matrix} \qquad \begin{matrix}\mathcal{D}_1^{\#\#}\\ \Gamma_1^{\#\#}, \psi_{v_{i+1}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1^{\#\#}\end{matrix} \qquad \begin{matrix}\mathcal{D}_1^{\#\#\#}\\ \Gamma_1^{\#\#\#}, \psi_{v_{i+2}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1^{\#\#\#}\ldots\end{matrix}}{\Gamma_1^{*}, \psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi}\ldots \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1^{*}}\ \forall\text{-R}\\
\mathcal{D}_1^{*}\\
\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi}\ldots \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1
\end{array}$$

so that, for instance, we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}}$-deduction $\mathcal{E}$:

$$\begin{array}{c}
\dfrac{\begin{matrix}\mathcal{D}_0^{\prime i-1}\\ \Gamma_0^{\prime i-1} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i-1}, \psi_{v_{i-1}/\xi}\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_0^{\prime i\$}\\ \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2}\ldots \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2}\ldots\psi_{v_i/\xi}\end{matrix} \qquad \begin{matrix}\mathcal{D}_1^{\#\$}\\ \Gamma_1^{*}, \psi_{v_i/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1^{*}\\ \mathcal{D}_1^{*\pounds}\\ \Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_i/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_1\end{matrix}}{\Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2}\ldots\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_{i-1}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2}\ldots\Delta_1}\ \mathrm{S}}{\Gamma_0^{\prime i-1}, \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}\ldots\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_{i-2}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i-1}, \Delta_0^{\prime i}, \Delta_0^{\prime i+1}\ldots\Delta_1}\ \mathrm{S}\\
\vdots\\
\dfrac{\begin{matrix}\mathcal{D}'_0\\ \Gamma'_0 \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta'_0, \psi_{v_0/\xi}\end{matrix} \qquad \dfrac{}{\Gamma''_0, \Gamma'''_0, \Gamma''''_0\ldots\Gamma_1, \psi_{v_0/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta''_0, \Delta'''_0, \Delta''''_0\ldots\Delta_1}\ \mathrm{S}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0, \Delta_1}\ \mathrm{S}
\end{array}$$

where '$'^i$' is a string of $i+1$ occurrences of '$'$', where $\mathcal{D}_0^{\prime i\$}$ and $\mathcal{D}_1^{\#\$}$ are suitable variants of $\mathcal{D}_0^{\prime i}$ and $\mathcal{D}_1^{\#}$, respectively (both obtained by fiddling with I^{K}), and where $\mathcal{D}_1^{*\pounds}$ is a suitable variant of $\mathcal{D}_1^{*}$ (obtained by substituting $\psi_{v_i/\xi}$ for $\psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi}\ldots$ in $\mathcal{D}_1^{*}$). Our strategy will be that of eliminating the topmost instance of S, in so doing showing that, as for the immediately lower instance of S, the resulting $\Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}}$-deduction already exhibits a reduction of the cut depth (relative to some routes) and pointing at the fact that a reduction of the cut depth (relative to some routes) is already exhibited also by all lower instance of S (so that all the finitely many instances of S in $\mathcal{E}$ can be eliminated one after the other from top to bottom). As for the elimination of the topmost instance of S, observe that, for every R antimonotonic over $\langle\mathcal{D}, \mathcal{E}\rangle$ (identifying a node in $\mathcal{D}$ with its \$- or £-variant in $\mathcal{E}$) and such that $R(\mathcal{E})$ includes a subtree of $\mathcal{D}_0^{\prime i\$}$ and a subtree of $\mathcal{D}_1^{\#\$}$, $\mathrm{cd}_R(\mathcal{E}_1) < \mathrm{cd}_R(\mathcal{D})$ (with $\mathcal{E}_1$ being the maximal subtree of $\mathcal{E}$ whose root is the displayed occurrence of $\Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2}\ldots\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_{i-1}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2}\ldots\Delta_1$). (I said 'for instance' because, of course, the choice of taking $\psi_{v_i/\xi}$ is arbitrary—for every $j > i$, taking $\psi_{v_j/\xi}$ instead would produce an equally acceptable transformation.) Thus, by the induction hypothesis, the topmost instance of S in $\mathcal{E}$ can be eliminated producing a S-free $\Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}}$-deduction $\mathcal{E}_1^{*}$ of $\Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2}\ldots\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_{i-1}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2}\ldots\Delta_1$ and a corresponding $\Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}}$-deduction $\mathcal{E}^{*}$ of $\Gamma_0^{\prime i-1}, \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}\ldots\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots, \psi_{v_{i-2}/\xi} \Rightarrow_{\mathbf{I}^{\mathbf{K}}\mathbf{T}^{\omega}} \Delta_0^{\prime i-1}, \Delta_0^{\prime i}, \Delta_0^{\prime i+1}\ldots\Delta_1$ having only one instance of S at its last step.[49] As for $\mathcal{E}^{*}$'s reduction of the cut depth (relative to some

[49] If the denumerable submultiset does not have the form $[\psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi}\ldots]$ and other (j-many) denumerable submultisets of $[\psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\ldots]$ are introduced independently, j-many relevant instances of S are required which can be eliminated in an analogous fashion.

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

routes), it is easy to verify that the transformations defined in this proof have the property that, for some R_0 suitable for the elimination of the topmost instance of S, $|\text{fld}(R_0(\mathcal{E}_1^*))| < |\text{fld}(\mathcal{D}_1) \cap \text{fld}(R_0(\mathcal{D}))| + |\text{fld}(\mathcal{D}_0^{\prime i}) \cap \text{fld}(R_0(\mathcal{D}))|$ (with $\mathcal{D}_1$ being the maximal subtree of $\mathcal{D}$ whose root is the displayed occurrence of $\Gamma_1, \forall\xi\psi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1$). From this, it follows that, for some R_1 such that $R_1(\mathcal{D})$ includes a subtree of $\mathcal{D}_0^{\prime i-1}$, $\text{cd}_{R_1}(\mathcal{E}^*) < \text{cd}_{R_1}(\mathcal{D})$. Analogous considerations allow the elimination of all lower instances of S.

Subsubsubcase 3eab. The same argument holds if $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ is introduced by an instance of $\exists$-L.

Subsubcase 3eb. $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ is introduced by an instance of $\mathrm{I^K}$.

Subsubsubcase 3eba. A wff which is not a member of $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ occurs both as a premise and as a conclusion of the relevant instance of $\mathrm{I^K}$. Then $\mathcal{D}_1'$ has the form:

$$\begin{array}{c}
\dfrac{}{\Gamma_1^*, \psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1^*}\,\mathrm{I^K} \\
\mathcal{D}_1^* \\
\Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1
\end{array}$$

so that we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\dfrac{\begin{array}{c}\mathcal{D}_0^{\prime i-1} \\ \Gamma_0^{\prime i-1} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i-1}, \psi_{\upsilon_{i-1}/\xi}\end{array} \qquad \begin{array}{c}\dfrac{}{\Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2} \ldots \Gamma_1^* \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2} \ldots \Delta_1^*}\,\mathrm{I^K} \\ \mathcal{D}_1^{*\$£} \\ \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2} \ldots \Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_{i-1}/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2} \ldots \Delta_1\end{array}}{\Gamma_0^{\prime i-1}, \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1} \ldots \Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_{i-2}/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i-1}, \Delta_0^{\prime i}, \Delta_0^{\prime i+1} \ldots \Delta_1}\,\mathrm{S}$$

$$\vdots$$

$$\dfrac{\begin{array}{c}\mathcal{D}_0' \\ \Gamma_0' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0', \psi_{\upsilon_0/\xi}\end{array} \qquad \dfrac{}{\Gamma_0'', \Gamma_0''', \Gamma_0'''' \ldots \Gamma_1, \psi_{\upsilon_0/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0'', \Delta_0''', \Delta_0'''' \ldots \Delta_1}\,\mathrm{S}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1}\,\mathrm{S}$$

where $\mathcal{D}_1^{*\$£}$ is a suitable variant of $\mathcal{D}_1^*$ (obtained by fiddling with $\mathrm{I^K}$ and by substituting $\varnothing$ for $\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots$ in $\mathcal{D}_1^*$) and where an argument similar to that of subsubsubcase 3eaa holds.

Subsubsubcase 3ebb. No wff which is not a member of $[\psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots]$ occurs both as a premise and as a conclusion of the relevant instance of $\mathrm{I^K}$. Then, for some $j \geq i$, $\mathcal{D}_1'$ has the form:

$$\begin{array}{c}
\dfrac{}{\Gamma_1^*, \psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots \psi_{\upsilon_j/\xi} \ldots \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1^*, \psi_{\upsilon_j/\xi}}\,\mathrm{I^K} \\
\mathcal{D}_1^* \\
\Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_i/\xi}, \psi_{\upsilon_{i+1}/\xi}, \psi_{\upsilon_{i+2}/\xi} \ldots \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1
\end{array}$$

so that we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\dfrac{\begin{array}{c}\mathcal{D}_0^{\prime i-1} \\ \Gamma_0^{\prime i-1} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i-1}, \psi_{\upsilon_{i-1}/\xi}\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_0^{\prime j\$} \\ \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2} \ldots \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2} \ldots \psi_{\upsilon_j/\xi}\end{array} \qquad \begin{array}{c}\dfrac{}{\Gamma_1^*, \psi_{\upsilon_j/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1^*, \psi_{\upsilon_j/\xi}}\,\mathrm{I^K} \\ \mathcal{D}_1^{*£} \\ \Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_{i-1}/\xi}, \psi_{\upsilon_j/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1\end{array}}{\Gamma_0^{\prime i}, \Gamma_0^{\prime i+1}, \Gamma_0^{\prime i+2} \ldots \Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_{i-1}/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i}, \Delta_0^{\prime i+1}, \Delta_0^{\prime i+2} \ldots \Delta_1}\,\mathrm{S}}{\Gamma_0^{\prime i-1}, \Gamma_0^{\prime i}, \Gamma_0^{\prime i+1} \ldots \Gamma_1, \psi_{\upsilon_0/\xi}, \psi_{\upsilon_1/\xi}, \psi_{\upsilon_2/\xi} \ldots, \psi_{\upsilon_{i-2}/\xi} \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0^{\prime i-1}, \Delta_0^{\prime i}, \Delta_0^{\prime i+1} \ldots \Delta_1}\,\mathrm{S}$$

https://doi.org/10.1017/S1755020311000177 Published online by Cambridge University Press

$$\vdots$$

$$\dfrac{\begin{array}{c}\mathcal{D}'_0\\ \Gamma'_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta'_0, \psi_{v_0/\xi}\end{array} \quad \dfrac{}{\Gamma''_0, \Gamma'''_0, \Gamma''''_0 \ldots \Gamma_1, \psi_{v_0/\xi} \Rightarrow_{\mathbf{I^K T^\omega}} \Delta''_0, \Delta'''_0, \Delta''''_0 \ldots \Delta_1}\,\mathrm{S}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

where $\mathcal{D}_0^{\prime j\$}$ is a suitable variant of $\mathcal{D}_0^{\prime j}$ (obtained by fiddling with $\mathrm{I^K}$), where $\mathcal{D}_1^{*\pounds}$ is a suitable variant of $\mathcal{D}_1^*$ (obtained by substituting $\psi_{v_j/\xi}$ for $\psi_{v_i/\xi}, \psi_{v_{i+1}/\xi}, \psi_{v_{i+2}/\xi} \ldots \psi_{v_j/\xi} \ldots$ in $\mathcal{D}_1^*$) and where an argument similar to that of subsubsubcase 3eaa holds.

Subcase 3f. A dual argument holds if the cut formula is of the form $\exists\xi\psi$.

Subcase 3g. The cut formula is of the form $T^\ulcorner\psi^\urcorner$. Then the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'_0\\ \Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \psi\end{array}}{\Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, T^\ulcorner\psi^\urcorner}\,T\text{-R} \quad \dfrac{\begin{array}{c}\mathcal{D}'_1\\ \Gamma_1, \psi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{array}}{\Gamma_1, T^\ulcorner\psi^\urcorner \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1}\,T\text{-L}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{E}$:

$$\dfrac{\begin{array}{c}\mathcal{D}'_0\\ \Gamma_0 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \psi\end{array} \quad \begin{array}{c}\mathcal{D}'_1\\ \Gamma_1, \psi \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_1\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T^\omega}} \Delta_0, \Delta_1}\,\mathrm{S}$$

where, for every R antimonotonic over $\langle\mathcal{D}, \mathcal{E}\rangle$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

This proves that, for every $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, it can be transformed into a $\Rightarrow_{\mathbf{I^K T^\omega}}$-deduction $\mathcal{E}$ with the same root as $\mathcal{D}$ for which there is a route R such that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. This is so because clearly, for each of the defined transformations, there is at least one route R that satisfies the conditions which are imposed in the transformation and whose satisfaction suffices for its being the case that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. The first part of the proof is thus completed.

As for the second and last part of the proof, it suffices to observe that the defined transformations have the property that, for any sequence S obtained by repeatedly applying them, there is at least one route R through S that can be used as a constant witness for the claim that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$. Keeping the antimonotonicity condition fixed, this is so because, for any i-long sequence S of $\Rightarrow_{\mathbf{I^K T^\omega}}$-deductions obtained by repeatedly applying the defined transformations, if, on the one hand, none of subcases 3b, 3c, 3d, 3e, and 3f occurs, for every $j < i - 2$, the conditions imposed on the transformation from $\mathcal{D}_{j+1}$ to $\mathcal{D}_{j+2}$ are in effect at worst (i.e., in main case 2) simply a further satisfiable specification of the conditions imposed on the transformation from $\mathcal{D}_j$ to $\mathcal{D}_{j+1}$ (somewhat pictorially, a further satisfiable specification of how the pertinent subtrees in R should behave in some parts that, being "further up," were *not* concerned by the conditions imposed on the transformation from $\mathcal{D}_j$ to $\mathcal{D}_{j+1}$). If, on the other hand, either of subcases 3b, 3c, 3d, 3e, or 3f occurs, and in the subinduction for one of these cases either main case 2 or subcases 3e or 3f occur, it might be worried that the conditions imposed on the transformation defined for the latter cases can enter into conflict with the conditions later imposed on one of the transformations in the superinduction, in particular if in the superinduction either main case 2 or subcases 3e or 3f occur (somewhat pictorially, it is possible that the conditions imposed on the transformations defined for these cases in the superinduction are a further specification of how the pertinent subtrees in R should behave *in the same parts* that were concerned by the conditions imposed on the transformations defined for these very same cases in the subinduction, so that it might be worried that there would no longer be a guarantee that this further specification is satisfiable). However, the clause of Definition 4.5 regarding ∀-R or ∃-L (which are in effect the rules of inference creating this worry) clearly does guarantee that, in any instance of ∀-R or ∃-L, there are enough subtrees included in at least some pertinent subtrees as to avoid any conflict between the conditions imposed on one of the transformations in a subinduction and the conditions later imposed on one of the transformations in the corresponding superinduction. □

4.2. Applications of the Hauptsatz. From Theorem 4.9 and the properties of S-free $\Rightarrow_{\mathbf{IKT}^{\omega}}$-deductions we easily obtain a series of interesting consistency properties for $\mathbf{IKT}^{\omega}$.

COROLLARY 4.10. $\mathbf{IKT}^{\omega}$ *is minimally consistent, in the sense that* $\oslash \vdash_{\mathbf{IKT}^{\omega}} \oslash$ *does not hold.*

Thus, the logical truths of $\mathbf{IKT}^{\omega}$ are not logical falsities of $\mathbf{IKT}^{\omega}$.

COROLLARY 4.11. $\mathbf{IKT}^{\omega}$ *is positively conservative, in the sense that, for every wff* φ *with no occurrence of* $\neg$, $\rightarrow$ *and* T, *neither* $\oslash \vdash_{\mathbf{IKT}^{\omega}} \varphi$ *nor* $\varphi \vdash_{\mathbf{IKT}^{\omega}} \oslash$ *holds.*

Thus, no intuitively nonlogical wff is either a logical truth or a logical falsity of $\mathbf{IKT}^{\omega}$.

COROLLARY 4.12. $\mathbf{IKT}^{\omega}$ *is nontrivial, in the sense that, for some* φ *and* ψ, $\varphi \vdash_{\mathbf{IKT}^{\omega}} \psi$ *does not hold.*

Thus, entailment in $\mathbf{IKT}^{\omega}$ is not vacuous.

COROLLARY 4.13. $\mathbf{IKT}^{\omega}$ *is nonanaletheic and nondialetheic, in the sense that either* $\varphi \vdash_{\mathbf{IKT}^{\omega}} \oslash$ *or* $\neg\varphi \vdash_{\mathbf{IKT}^{\omega}} \oslash$ *does not hold and either* $\oslash \vdash_{\mathbf{IKT}^{\omega}} \varphi$ *or* $\oslash \vdash_{\mathbf{IKT}^{\omega}} \neg\varphi$ *does not hold.*

Thus, a wff and its negation cannot both be logical falsities of $\mathbf{IKT}^{\omega}$ and cannot both be logical truths of $\mathbf{IKT}^{\omega}$.

COROLLARY 4.14. *W-L and W-R are not admissible in either* $\mathbf{IK}^{\omega}$ *or* $\mathbf{IKT}^{\omega}$.

Thus, the semantic paradoxes are indeed blocked by $\mathbf{IKT}^{\omega}$ in the way suggested in Section §2.

§5. Conclusion and glimpses beyond. This paper has presented a novel naive theory of truth. It has done so by, after briefly motivating philosophically the failure of contraction, developing a formal noncontractive naive theory of truth and proving its consistency. The focus has been on demonstrating the logical and truth-theoretic strength and the coherence of the theory, especially in the surprisingly many respects of philosophically interesting strength in which it outperforms its naive rivals.

The success, however, as the reader will certainly have noticed, has only been partial in many important respects. Let me conclude by listing and briefly commenting on what I regard as the most pressing open problems (clearly, there are important connections among many of them—the order is not fortuitous):

- The metaphysical picture sketched in Section 2.3, although appealing, is certainly in need of further elaboration and defence;
- The theory I have offered has for simplicity presupposed a countable domain, and should be extended to domains of larger cardinalities;
- The theory also does not include a theory of identity or of higher-order quantification, and should be so extended;
- The theory, which I have developed in a broadly proof-theoretical fashion, should be provided with a suitable semantics;
- Even if I have proven in Section §4 that the theory is in itself consistent, I have not proven that it does not impose any additional constraint on other theories we might also wish to endorse. Stronger conservativeness results are highly desirable;
- Even if I have indicated in Sections 2.1 and 2.2 where some of the most venerable semantic paradoxes are blocked by the theory, I have not done so for each and every semantic paradox that has been discussed in the literature, in spite of the fact that, for some of these, to do so would shed significant additional light into the inner workings of the theory (the consistency proof I've offered in Section §4 shows of course that all such paradoxes expressible in the system developed in Section §3 are effectively blocked one way or another);
- I have not tried to discuss the important question of whether and to what extent the theory is subject to "revenge" worries;
- I have not discussed at all semantic paradoxes generated using intuitively correct principles governing other semantic notions (such as the notions of being true of, satisfying, denoting, etc). I think that it is a requirement on the kind of theory I've presented that it be smoothly extendible to other semantic notions;
- Nor have I discussed broadly "self-referential" paradoxes generated using intuitively correct principles governing other nonsemantic notions (such as the notions of set, property, knowledge, etc). I am open to the idea that at least some of these paradoxes require differential treatments.

Many important problems remain thus open. I hope that this paper has done enough to justify the idea that the theory giving rise to such problems is a new worthy candidate in the debate among naive theories of truth.

§6. Acknowledgements. Earlier versions of the material in this paper have been presented in 2010 at a workshop on *Vagueness and Self-Reference* (New University of Lisbon/University of Évora), at a conference on *Language, Culture and Civilisation: East and West* (Jawaharlal Nehru University) and at a LOGOS Colloquium (University of Barcelona); in 2011, at a research seminar (UNAM), where Javier García Salcedo gave a valuable response, in a graduate course on *Formal Theories of Truth* at UNAM, where in particular Carlos Romero Castillo provided me with helpful feedback, at a NIP Philosophy of Logic Seminar (University of Aberdeen) and at the sixth COGITO Day on *Philosophy of Mathematics and Philosophy of Logic* (University of Bologna). I would like to thank all these audiences for very stimulating comments and discussions. Special thanks go to J. C. Beall, Aaron Cotnoir, Mario Gómez Torrente, Øystein Linnebo, Dan López de Sa, José Martínez Fernández, Marco Panza, Andreas Pietz, Graham Priest, Stephen Read, David Ripley, Sven Rosenkranz, Ricardo Santos, Kevin Scharp, Crispin Wright, and Shunsuke Yatabe.

Laura Delgado provided much needed bibliographical help. Greg Restall carefully went through different drafts of the paper, making very useful suggestions for improving both its content and presentation. I started to take seriously the ideas presented here when, to my surprise, my former supervisor Hartry Field started to take them seriously. I'm extremely grateful to Hartry for the encouragement and support he has given me for this project throughout the past years, and I owe to him much of my interest in and knowledge of the topics discussed here. In writing this paper, I have benefited, at different stages, from an AHRC Postdoctoral Research Fellowship and a UNAM Postdoctoral Research Fellowship, as well as from partial funds from the project FFI2008-06153 of the Spanish Ministry of Science and Innovation on *Vagueness and Physics, Metaphysics, and Metametaphysics*, from the project CONSOLIDER-INGENIO 2010 CSD2009-00056 of the Spanish Ministry of Science and Innovation on *Philosophy of Perspectival Thoughts and Facts* (PERSP) and from the European Commission's seventh Framework Programme FP7/2007-2013 under grant FP7-238128 for the European Philosophy Network on *Perspectival Thoughts and Facts* (PETAF).

BIBLIOGRAPHY

Beall, J. C. (2009). *Spandrels of Truth*. Oxford, UK: Oxford University Press.

Blizard, W. (1989). Multiset theory. *Notre Dame Journal of Formal Logic*, **30**, 36–66.

Brady, R. (2006). *Universal Logic*. Stanford, CA: CSLI Publications.

Curry, H. (1942). The inconsistency of certain formal logics. *The Journal of Symbolic Logic*, **7**, 115–117.

Field, H. (2006). Truth and the unprovability of consistency. *Mind*, **115**, 567–605.

Field, H. (2008). *Saving Truth from Paradox*. Oxford, UK: Oxford University Press.

Gabbay, D. (1985). Theoretical foundations for non-monotonic reasoning in expert systems. In Apt, K., editor. *Logics and Models of Concurrent Systems*. Berlin: Springer, pp. 439–457.

Gentzen, G. (1934). Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, **39**, 176–210.

Gupta, A., & Belnap, N. (1993). *The Revision Theory of Truth*. Cambridge, MA: MIT Press.

Herzberger, H. (1982). Naive semantics and the liar paradox. *The Journal of Philosophy*, **79**, 479–497.

Kripke, S. (1975). Outline of a theory of truth. *The Journal of Philosophy*, **72**, 690–716.

Laertius, D. *De vitis, dogmatibus et apophthegmatibus clarorum philosophorum*.

López de Sa, D., & Zardini, E. (2007). Truthmakers, knowledge and paradox. *Analysis*, **67**, 242–250.

López de Sa, D., & Zardini, E. (2011). No-no. Paradox and consistency. *Analysis*, **71**, 472–478.

Maudlin, T. (2004). *Truth and Paradox*. Oxford, UK: Oxford University Press.

McGee, V. (1991). *Truth, Vagueness, and Paradox*. Indianapolis, IN: Hackett.

Meyer, R., Routley, R., & Dunn, M. (1979). Curry's paradox. *Analysis*, **39**, 124–128.

Priest, G. (2006a). *In Contradiction*, second edition. Oxford, UK: Oxford University Press.

Priest, G. (2006b). *Doubt Truth to Be a Liar*. Oxford, UK: Oxford University Press.

Troelstra, A., & Schwichtenberg, H. (2000). *Basic Proof Theory*, second edition. Cambridge, UK: Cambridge University Press.

Zardini, E. (2008). Truth and what is said. *Philosophical Perspectives*, **22**, 545–574.

Zardini, E. (2011a). The role of utterances in Bradwardine's theory of truth. *Recherches de théologie et philosophie médiévales*. Forthcoming.

INSTITUTO DE INVESTIGACIONES FILOSÓFICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO,
MEXICO CITY, 04510, MEXICO
NORTHERN INSTITUTE OF PHILOSOPHY
UNIVERSITY OF ABERDEEN,
ABERDEEN, AB24 3UB, UK
E-mail: elia.zardini@abdn.ac.uk