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On the full periodicity kernel for one-dimensional maps

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Abstract. Let $\propto$ be the topological space obtained by identifying the points 1 and 2 of the segment $[0, 3]$ to a point. Let ∞ be the topological space obtained by identifying the points 0, 1 and 2 of the segment $[0, 2]$ to a point. An $\propto$ (respectively ∞) map is a continuous self-map of $\propto$ (respectively ∞) having the branching point fixed. Set $E \in \{\propto, \infty\}$. Let f be an E map. We denote by $\text{Per}(f)$ the set of periods of all periodic points of f. The set $K \subset \mathbb{N}$ is the *full periodicity kernel* of E if it satisfies the following two conditions: (1) if f is an E map and $K \subset \text{Per}(f)$, then $\text{Per}(f) = \mathbb{N}$; (2) for each $k \in K$ there exists an E map f such that $\text{Per}(f) = \mathbb{N} \setminus \{k\}$. In this paper we compute the full periodicity kernel of $\propto$ and ∞.

1. *Introduction and main results*

Let E be a topological space. We shall study some properties of the set of periods for a class of continuous maps from E into itself. We need some notation.

The sets of natural numbers, real numbers and complex numbers will be denoted by $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ respectively. For a map $f : E \to E$ we use the symbol f^n to denote $f \circ f \circ \cdots \circ f$ ($n \in \mathbb{N}$ times), f^0 denotes the identity map of E. Then, for a point $x \in E$ we define the *orbit* of x, denoted by $\text{Orb}_f(x)$, as the set $\{f^n(x) : n = 0, 1, 2, \ldots\}$. We say x is a *fixed point* of f if $f(x) = x$. We say x is a *periodic point of* f *of period* $k \in \mathbb{N}$ (or simply a k-*point*) if $f^k(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < k$. In this case we say the orbit of x is a *periodic orbit of period* k (or simply a k-*orbit*). Note that if x is a periodic point of period k, then $\text{Orb}_f(x)$ has exactly k elements, each of which is a periodic point of period k. We denote by $\text{Per}(f)$ the set of periods of all periodic points of f.

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A *connected finite regular graph* (or just a *graph* for short) is a pair consisting of a connected *Hausdorff* space E and a finite subspace V, whose elements are called *vertices*, such that the following conditions hold:

(1) $E \setminus V$ is the disjoint union of a finite number of open subsets $e_1, \ldots, e_k$, called *edges*, where each e_i is homeomorphic to an open interval of the real line;

(2) the boundary, $\mathrm{cl}(e_i) \setminus e_i$, of the edge e_i consists of two distinct vertices, and the pair $(\mathrm{cl}(e_i), e_i)$ is homeomorphic to the pair $([0, 1], (0, 1))$.

A vertex which belongs to the boundary of at least three different edges is called a *branching point* of E. A vertex which belongs to a unique edge is called an *endpoint*.

An E *map* is a continuous self-map of E having fixed all the branching points of E.

We say an E map f has *full periodicity* if $\mathrm{Per}(f) = \mathbb{N}$. The set $K \subset \mathbb{N}$ is a *full periodicity kernel* of E if it satisfies the following two conditions:

(1) if f is an E map and $K \subset \mathrm{Per}(f)$, then $\mathrm{Per}(f) = \mathbb{N}$;

(2) for each $k \in K$ there exists an E map f such that $\mathrm{Per}(f) = \mathbb{N} \setminus \{k\}$.

The above condition (1) says that the set K is sufficient to force full periodicity. Condition (2) means that K is necessary to have full periodicity. Of course, the set K is the minimal set which forces periodic points of all periods. Note that, for a given E, if there is a full periodicity kernel, then it is unique.

Blokh proved in [**8**] that for every graph E, there exists a natural number $L(E)$, such that for any continuous self-map f of E, $\{1, 2, \ldots, L(E)\} \subset \mathrm{Per}(f)$ implies $\mathrm{Per}(f) = \mathbb{N}$. This result shows that if there exists the full periodicity kernel of E, then it is a finite set. In fact, then the set $\{1, 2, \ldots, L(E)\}$ contains the full periodicity kernel of E.

On the other hand, one of the most important questions in one-dimensional combinatorial dynamics is the problem of describing all possible sets of periods for E maps. There is a kind of conjecture saying that with finitely many different orderings of the set $\mathbb{N}$ it will be possible to control all possible sets of periods for E maps, see for instance [**2**]. If this conjecture is true, then the full periodicity kernel would contain the first elements of the different orderings controlling the periodic structure of E maps. Thus, the full periodicity kernel contains interesting information about the new orderings which can appear in the periodic dynamics of E maps.

The *topological entropy* of a continuous self-map f on a graph E is a non-negative real number $h(f)$ associated to f which increases with the complexity of f. For a definition and main properties see [**2**]. Llibre and Misiurewicz [**12**] obtained the next result. If f is a continuous map on a graph into itself, then the following two statements are equivalent:

(1) $h(f) > 0$;

(2) there is $m \in \mathbb{N}$ such that $\{m \cdot n : n \in \mathbb{N}\} \subset \mathrm{Per}(f)$.

Another proof of this equivalence can be found in [**8**]. From this result it follows that if K_E is the full periodicity kernel of E and f is an E map, then $K_E \subset \mathrm{Per}(f)$ implies $h(f) > 0$. In other words, if a map has the periods of its full periodicity kernel, then it has positive topological entropy.

From now on, the topological space E will denote one of the following seven spaces:

$$\mathbf{I} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } y = 0\},$$
$$\mathbf{Y} = \{z \in \mathbb{C} : z^3 \in [0, 1]\},$$

$$\mathbf{X} = \{z \in \mathbb{C} : z^4 \in [0, 1]\},$$
$$\mathbf{O} = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 1)^2 = 1\},$$
$$\boldsymbol{\sigma} = \mathbf{I} \cup \mathbf{O},$$
$$\propto = \boldsymbol{\sigma} \cup \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 0 \text{ and } y = 0\},$$
$$\infty = \mathbf{O} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 = 1\}.$$

The spaces $\mathbf{I}$, $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{O}$, $\boldsymbol{\sigma}$, $\propto$ and ∞ are called the *interval*, the 3-*od* or 3-*star*, the 4-*od* or the 4-*star*, the *circle*, the *sigma*, the *alpha* and the *eight* respectively.

The spaces $\mathbf{Y}$, $\mathbf{X}$, $\boldsymbol{\sigma}$, $\propto$ and ∞ have exactly one branching point, namely $\mathbf{0} \in \mathbf{Y}$, $\mathbf{0} \in \mathbf{X}$, $\mathbf{0} = (0, 0) \in \boldsymbol{\sigma}$, $\mathbf{0} = (0, 0) \in \propto$ and $\mathbf{0} = (0, 0) \in \infty$. We also denote by $\mathbf{0}$ the $(0, 0) \in \mathbf{O}$.

The full periodicity kernel for the interval, the 3-star, the 4-star, the circle and the sigma are known and presented in the following five theorems.

THEOREM 1.1. *The set* $\{3\}$ *is the full periodicity kernel of* $\mathbf{I}$.

THEOREM 1.2. *The set* $\{2, 3, 4, 5, 7\}$ *is the full periodicity kernel of* $\mathbf{Y}$.

THEOREM 1.3. *The set* $\{2, 3, 4, 5, 6, 7, 10, 11\}$ *is the full periodicity kernel of* $\mathbf{X}$.

THEOREM 1.4. *The set* $\{1, 2, 3\}$ *is the full periodicity kernel of* $\mathbf{O}$.

THEOREM 1.5. *The set* $\{2, 3, 4, 5, 7\}$ *is the full periodicity kernel of* $\boldsymbol{\sigma}$.

Theorem 1.1 is due to Sharkovskii [**18**] (see also [**2, 7, 11, 19**]), Theorem 1.2 was shown by Mumbrú [**17**] (see also [**1**]), Theorem 1.3 has been proved by Alsedà and Moreno [**3**], Theorem 1.4 is due to Block [**5**] (see also [**16**]), Theorem 1.5 has been proved by Llibre *et al* [**13**].

Our goal in this paper is to characterize the full periodicity kernel of the alpha and eight spaces. Thus, our main results are the following.

THEOREM 1.6. *The set* $\{2, 3, 4, 5, 6, 7, 10, 11\}$ *is the full periodicity kernel of* $\propto$.

THEOREM 1.7. *The set* $\{2, 3, 4, 5, 6, 7, 8, 10, 11\}$ *is the full periodicity kernel of* ∞.

Leseduarte and Llibre compute in [**10**] the full periodicity kernel for other spaces: the circle with three whiskers, the circle with four whiskers, the eight with one whiskers, the eight with two whiskers and the trefoil. Also they compare the upper bound of the full periodicity kernel of Blokh, $L(E)$, with the best upper bound for all proper subgraphs of the trefoil.

Theorems 1.6 and 1.7 are proved in §§8 and 11 respectively. All the other sections present preliminary definitions and results that are necessary for proving these two main theorems.

2. *Preliminary results*

Sharkovskii proved his famous theorem in the 1960s. It characterizes the set $\mathrm{Per}(f)$ for continuous maps on the interval.

The *Sharkovskii ordering* $>_s$ on the set $\mathbb{N}_s = \mathbb{N} \cup \{2^\infty\}$ is given by:

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$3 >_s 5 >_s 7 >_s \cdots >_s 2{\cdot}3 >_s 2{\cdot}5 >_s 2{\cdot}7 >_s \cdots >_s 2^2{\cdot}3 >_s 2^2{\cdot}5 >_s 2^2{\cdot}7 >_s \cdots >_s 2^n{\cdot}3 >_s 2^n{\cdot}5 >_s 2^n{\cdot}7 >_s \cdots >_s 2^\infty >_s \cdots >_s 2^n >_s \cdots >_s 2^4 >_s 2^3 >_s 2^2 >_s 2 >_s 1.$

We shall use the symbol $\geq_s$ in the natural way. We have to include the symbol 2^∞ to ensure the existence of supremum of every subset with respect to the ordering $>_s$. For $n \in \mathbb{N}_s$ we denote $S(n) = \{k \in \mathbb{N} : n \geq_s k\}$. So $S(2^\infty) = \{2^i : i = 0, 1, 2, \ldots\}$.

THEOREM 2.1. (Interval theorem)
(a) *If f is an interval map, then* $\mathrm{Per}(f) = S(n)$ *for some* $n \in \mathbb{N}_s$.
(b) *If* $n \in \mathbb{N}_s$ *then there exists an interval map* f *such that* $\mathrm{Per}(f) = S(n)$.

If we want to get a similar result for the space **Y**, we need two new orderings.

The *green ordering* $>_g$ on $\mathbb{N} \setminus \{2\}$ is given by:

$5 >_g 8 >_g 4 >_g 11 >_g 14 >_g 7 >_g 17 >_g 20 >_g 10 >_g \cdots >_g 3\cdot 3 >_g 3\cdot 5 >_g 3\cdot 7 >_g \cdots >_g 3\cdot 2\cdot 3 >_g 3\cdot 2\cdot 5 >_g 3\cdot 2\cdot 7 >_g \cdots >_g 3\cdot 2^2\cdot 3 >_g 3\cdot 2^2\cdot 5 >_g 3\cdot 2^2\cdot 7 >_g \cdots >_g 3\cdot 2^3 >_g 3\cdot 2^2 >_g 3\cdot 2 >_g 3\cdot 1 >_g 1.$

The *red ordering* $>_r$ on $\mathbb{N} \setminus \{2, 4\}$ is given by:

$7 >_r 10 >_r 5 >_r 13 >_r 16 >_r 8 >_r 19 >_r 22 >_r 11 >_r \cdots >_r 3\cdot 3 >_r 3\cdot 5 >_r 3\cdot 7 >_r \cdots >_r 3\cdot 2\cdot 3 >_r 3\cdot 2\cdot 5 >_r 3\cdot 2\cdot 7 >_r \cdots >_r 3\cdot 2^2\cdot 3 >_r 3\cdot 2^2\cdot 5 >_r 3\cdot 2^2\cdot 7 >_r \cdots >_r 3\cdot 2^3 >_r 3\cdot 2^2 >_r 3\cdot 2 >_r 3\cdot 1 >_r 1.$

For $n \in \mathbb{N} \setminus \{2\}$ denote $G(n) = \{k \in \mathbb{N} : n \geq_g k\}$, for $n \in \mathbb{N} \setminus \{2, 4\}$ denote $R(n) = \{k \in \mathbb{N} : n \geq_r k\}$ and additionally $G(3{\cdot}2^\infty) = R(3{\cdot}2^\infty) = \{1\} \cup \{3n : n \in S(2^\infty)\}$. We also denote $\mathbb{N}_g = (\mathbb{N} \setminus \{2\}) \cup \{3\cdot 2^\infty\}$ and $\mathbb{N}_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3\cdot 2^\infty\}$.

The following theorem is due to Alsedà *et al* [**1**] for **Y** maps and to Baldwin for arbitrary continuous self-maps of **Y** [**4**].

THEOREM 2.2. (**Y** theorem)
(a) *If f is a* **Y** *map, then* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ *for some* $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$ *and* $n_r \in \mathbb{N}_r$.
(b) *If* $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$ *and* $n_r \in \mathbb{N}_r$, *then there exists a* **Y** *map* f *such that* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$.

The n-*od* space $\mathbf{I}_n$ is defined as the set of all complex numbers z such that z^n is in the interval $[0, 1]$ and the branching point is $\mathbf{0} = 0$.

Baldwin [**4**] extends Sharkovskii's result to the n-od. Thus, he establishes a conjecture presented by Alsedà *et al* [**1**] in the affirmative. The set of periods of a continuous self-map of the n-od can be described as a non-empty union of initial segments of some orderings $\geq_n$ which we are going to state.

We define the partial orderings $\geq_n$ for $n \geq 1$. The ordering $\geq_1$ is the ordering $\geq_s$. If $n > 1$, then the ordering $\geq_n$ is defined as follows. Let m, k be positive integers.

Case 1: $k = 1$. Then $k \geq_n m$ if and only if $m = 1$.

Case 2: k is divisible by n. Then $k \geq_n m$ if and only if either $m = 1$ or m is divisible by n and $k/n \geq_s m/n$.

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Case 3: $k > 1$, k not divisible by n. Then $k \geq_n m$ if and only if either $m = 1$, $m = k$, or $m = ik + jn$ for some integers $i \geq 0$, $j \geq 1$.

From the definition we have that $>_2$ is the Sharkovskii ordering. A set Z is an *initial segment* of $\geq_p$ for $p \geq 0$, if whenever k is an element of Z and $k \geq_p m$, then m also belongs to Z.

THEOREM 2.3. (n-od theorem)

(a) *Let f be a continuous self-map of $\mathbf{I}_n$. Then* $\mathrm{Per}(f)$ *is a non-empty union of initial segments of* $\{\geq_p : 1 \leq p \leq n\}$.

(b) *If Z is a non-empty finite union of initial segments of* $\{\geq_p : 1 \leq p \leq n\}$, *then there is a continuous map* $f : \mathbf{I}_n \longrightarrow \mathbf{I}_n$ *such that* $f(\mathbf{0}) = \mathbf{0}$ *and* $\mathrm{Per}(f) = Z$.

While Baldwin works with the partial orderings $\geq_n$, Alsedà and Moreno in [**3**] show that the set of periods of a continuous self-map of the n-star can be expressed as the union of 'initial segments' of the linear orderings associated to all rationals in the interval $(0, 1)$ with denominator smaller than or equal to n defined in certain subsets of the natural numbers. Two of these ordering are exactly the green and the red ordering appearing in the characterization of maps of $\mathbf{Y}$ in [**1**]. Moreover, in [**3**] the authors show that the full periodicity kernel of continuous self-maps of $\mathbf{I}_n$ exist and they provide an algorithm for computing them.

We define the *Block ordering* $>_b$ on $\mathbb{N}_b = \mathbb{N}$ as the converse of the usual ordering on $\mathbb{N} \setminus \{1\}$ and we add the 1 as the smallest element; i.e. $2 >_b 3 >_b 4 >_b \cdots >_b 1$. For $n \in \mathbb{N}_b$, we denote $B(n) = \{k \in \mathbb{N} : n \geq_b k\}$. Sharkovskii Theorem has been generalized by Block to the circle maps having fixed points in [**6**].

THEOREM 2.4. (Circle theorem)

(a) *If f is a circle map having fixed points, then* $\mathrm{Per}(f) = S(n_s) \cup B(n_b)$ *for some* $n_s \in \mathbb{N}_s$ *and* $n_b \in \mathbb{N}_b$.

(b) *If* $n_s \in \mathbb{N}_s$ *and* $n_b \in \mathbb{N}_b$, *then there exists a circle map f having fixed points such that* $\mathrm{Per}(f) = S(n_s) \cup B(n_b)$.

The following theorem describes the set of periods for $\boldsymbol{\sigma}$ maps. It was proved by Llibre *et al* [**14**].

THEOREM 2.5. ($\boldsymbol{\sigma}$ theorem)

(a) *If f is a $\boldsymbol{\sigma}$ map, then* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$ *for some* $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g, n_r \in \mathbb{N}_r$ *and* $n_b \in \mathbb{N}_b$.

(b) *If* $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g, n_r \in \mathbb{N}_r$ *and* $n_b \in \mathbb{N}_b$, *then there exists a $\boldsymbol{\sigma}$ map f such that* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$.

Furthermore, Leseduarte and Llibre [**9**] obtained the $\boldsymbol{\sigma}$ theorem for a class of continuous self-maps of the $\boldsymbol{\sigma}$ more general than the continuous self-maps having the branching point fixed.

3. *Intervals and basic intervals*

From now on we shall talk about the whiskers or the circles of E. We define these sets as follows: the *circle of* $\boldsymbol{\sigma}$ is $\mathbf{O}$, the *whiskers of* $\boldsymbol{\sigma}$ is $\mathbf{I}$, the *circle of* $\propto$ is $\mathbf{O}$, the *whiskers of* $\propto$

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are the sets whiskers$(A) = \mathbf{I}$ and whiskers$(B) = \{(x, y) \in \propto : -1 \leq x \leq 0$ and $y = 0\}$, and finally the circles of ∞ are the sets circle$(A) = \mathbf{O}$ and circle$(B) = \{(x, y) \in \infty : x^2 + (y-1)^2 = 1\}$. Notice that all the above whiskers are homeomorphic to $\mathbf{I}$ and all the above circles are homeomorphic to $\mathbf{O}$.

A *closed* (respectively *open*, *half-open* or *half-closed*) interval J of E is a subset of E homeomorphic to the closed interval $[0, 1]$ (respectively $(0, 1)$, $[0, 1)$). Notice that an interval cannot be a single point.

Let J be a closed interval of E, and let $h : [0, 1] \longrightarrow J$ be a homeomorphism. Then $h(0) = a$ and $h(1) = b$ are called the *endpoints* of J. If a and b belong to $\mathbf{I}$, $\mathbf{Y}$, $\mathbf{X}$ or a whiskers of E, then J will be denoted by $[a, b]$ or $[b, a]$. If a and b belong to a circle of E then we write $[a, b]$ to denote the closed interval from a counter-clockwise to b.

Notice that it is possible that two different intervals of a circle of E have the same endpoints. But two different points of $\mathbf{I}$, $\mathbf{Y}$, $\mathbf{X}$ or the whiskers of E always determine a unique interval.

Now we define a special class of subintervals of E. Let $Q = \{q_1, q_2, \ldots, q_n\}$ be a finite subset of E containing $\mathbf{0}$. For each pair q_i, q_j such that $q_i \neq q_j$ we say that the interval $[q_i, q_j]$ (respectively $[q_j, q_i]$) is *basic* if, and only if, $(q_i, q_j) \cap Q = \emptyset$ (respectively $(q_j, q_i) \cap Q = \emptyset$). The set of all these basic intervals is called the *set of basic intervals associated to* Q.

4. *Loops and f-graphs*

Let $f : E \to E$ be an E map. If K and J are intervals of E, then we say that K *f-covers* J or $K \to J$ (or $J \leftarrow K$), if there is a closed subinterval M of K such that $f(M) = J$. If K does not f-cover J we write $K \nrightarrow J$.

A *path of length m* is any sequence $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m$, where $J_0, J_1, \ldots, J_m$ are closed subintervals of E (in general, basic intervals). Furthermore, if $J_0 = J_m$, then this path is called a *loop of length m*. Such a loop will be called *non-repetitive* if there is no integer i, $0 < i < m$, such that i divides m and $J_{j+i} = J_j$ for all j, $0 \leq j \leq m - i$. We say that we *add* or we *concatenate* the loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$ to the loop $K_0 \to K_1 \to \cdots \to K_{n-1} \to K_0$ if they have a common vertex $J_0 = K_0$ and we form the new loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to K_0 \to K_1 \to \cdots \to J_0$. A loop which cannot be formed by adding two loops will be called *elementary*.

Let Q be a finite subset of E containing $\mathbf{0}$. An *f-graph of Q* is a graph with the basic intervals associated to Q as vertices, and such that if K and J are basic intervals and K f-covers J, then there is an arrow from K to J. Note that the f-graph of Q is unique up to labeling of the basic intervals. Hence, from now on we shall talk about *the f-graph* of Q (or just *the f-graph* for short). The next three lemmas are well-known in one dimensional dynamics, see for instance [**2**]. We leave the proofs to the reader.

LEMMA 4.1. *Let f be an E map and let K, J, L be closed subintervals of E. If $L \subset J$ and K f-covers J, then K f-covers L.*

LEMMA 4.2. *Let f be an E map and let J be a subinterval of E such that J f-covers J. Then f has a fixed point in J.*

LEMMA 4.3. *Let f be an E map and let $J_0, J_1, \ldots, J_{n-1}$ be closed subintervals of E such that $J_i \to J_{i+1}$ for $i = 0, 1, \ldots, n-2$ and $J_{n-1} \to J_0$. Then there exists a fixed point x of f^n in J_0 such that $f^i(x) \in J_i$ for $i = 1, 2, \ldots, n-1$.*

Let J be a subset of E. As usual $\mathrm{Int}(J)$ and $\mathrm{Cl}(J)$ denote the interior and the closure of J respectively.

PROPOSITION 4.4. *Let $E \in \{\mathbf{I}, \mathbf{Y}, \mathbf{X}, \propto, \infty\}$ and let f be an E map having r periodic orbits of periods $k_1, k_2, \ldots, k_r$. Let Q be the set formed by the union of the above r periodic orbits with the branching point. Consider the set of basic intervals associated to Q. Suppose that there are points of Q in each connected component of $E \setminus \{\mathbf{0}\}$. Let $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ be a non-repetitive loop of length m of the f-graph of Q such that at least one J_i does not contain $\mathbf{0}$. If $m \notin \{2k_1, 2k_2, \ldots, 2k_r\}$, then $m \in \mathrm{Per}(f)$.*

Proof. By Lemma 4.1 J_0 f^m-covers J_0. Then by Lemma 4.2 there exists $x \in J_0$ such that $f^m(x) = x$. If x has period m we are done. So suppose that x has period s, $0 < s < m$. Thus s divides m.

It is not possible that $x = \mathbf{0}$ because $\mathbf{0}$ is a fixed point and some $f^i(x) \in J_i$ with $J_i \cap \{\mathbf{0}\} = \emptyset$.

If $x \in \mathrm{Int}(J_0)$, then $\mathrm{Orb}_f(x) \cap Q = \emptyset$. So each $f^i(x)$ is exactly in one basic interval, and consequently the loop is repetitive (because $s < m$ and s divides m). Hence, x must be a point of Q. So $\mathrm{Orb}_f(x) \subset Q$. Without loss of generality we can assume that $s = k_1$.

From Lemma 4.3 it is easy to construct a closed interval $K_0 \subset J_0$ such that $x \in K_0$ and $f^i(x) \in f^i(K_0) \subset J_i$ for $i = 0, 1, \ldots, m$. Since $x = f^s(x) \in f^s(K_0) \subset J_s$ it follows that J_0 and J_s have a common endpoint x.

Assume that $J_0 = J_s$. Both sets K_0 and $f^s(K_0)$ are contained in J_0 and contain x, an endpoint of J_0. Therefore $L = K_0 \cap f^s(K_0)$ is an interval [in fact it is either K_0 or $f^s(K_0)$]. Clearly $f^i(L) \subset f^i(K_0) \subset J_i$, $f^i(L) \subset f^{s+i}(K_0) \subset J_{s+i}$, and $f^i(L)$ is an interval for $0 \le i \le s$. Thus $J_i = J_{s+i}$ for $i = 0, 1, \ldots, s-1$.

Repeating this process we get that $J_i = J_{s+i}$ for $i = 0, 1, \ldots, m-s$. Hence, the loop is repetitive because s divides m, in contradiction with the assumptions. So $J_0 \neq J_s$.

If $J_q = J_{q+s}$ for some $0 < q < m-s$, then the above arguments prove that $J_{q+i} = J_{q+s+i}$ for $i = 0, 1, \ldots, s-1$. Repeating this process we obtain that $J_i = J_{s+i}$ for $i = 0, 1, \ldots, m-s$ and so the loop is repetitive, a contradiction with the assumptions. Therefore, we can assume that $J_q \neq J_{q+s}$ for $0 \le q < m-s$.

Since x is a periodic point of period s, if follows that $J_0 = J_{2s}$ and $J_s = J_{3s}$. By the above arguments we get $J_m = J_0 = J_{2s} = J_{4s} = \cdots$ and $J_s = J_{3s} = J_{5s} = \cdots$. In particular m must be even. Furthermore, $J_i = J_{2s+i}$ for $0 \le i \le 2s-1$. Hence, $2s = 2k_1$ divides m. Since $m \neq 2k_1$ the loop is repetitive, in contradiction with the hypotheses. □

5. *Q-linear maps*

Let $T \in \{\mathbf{I}, \mathbf{Y}, \mathbf{X}\}$. It is easy to see that any tree T has a metric m such that if $x, y \in T$ and $z \in [x, y]$, then $m(x, y) = m(x, z) + m(z, y)$, this metric is called the *taxicab metric*.

Let f be an E map and let $Q = \{q_1, q_2, \dots, q_m\}$ be an invariant subset of E under f such that $\mathbf{0} \in Q$. We assume that there are points of Q in each connected component of $E \setminus \{\mathbf{0}\}$. Let E_Q be the minimal connected subgraph of E containing all the basic intervals associated to Q. Clearly E_Q is homeomorphic to E. We say that f is *Q-linear* if the following conditions hold:

(1) $E_Q = E$, in particular the endpoints of E are points of Q;

(2) for any basic interval J associated to Q, $f(J)$ is a tree formed by the union of basic intervals of Q;

(3) $f|_J : J \longrightarrow f(J)$ is linear with respect to the taxicab metric, i.e. for any $x, y, z \in J$ such that $m(x, y) = m(x, z) + m(z, y)$ we have that $m(f(x), f(y)) = m(f(x), f(z)) + m(f(z), f(y))$.

We say that an E map g is a *Q-linearization* of f if the following conditions hold:

(1) $g|_Q = f|_Q$;

(2) g is Q-linear;

(3) the g-graph of Q is a subgraph of the f-graph of Q.

In particular, if f is an E map having a periodic orbit P such that P has points in each connected component of $E \setminus \{\mathbf{0}\}$ we will talk about the *P'-linearization of f* in the above way, where $P' = P \cup \{\mathbf{0}\}$.

Let J be a basic interval. If $\mathbf{0} \in J$, then J will be called a *branching interval*; otherwise J will be called a *non-branching interval*.

In the next lemma and proposition we assume that $E \in \{\propto, \infty\}$ and f is an E map having r periodic orbits of periods $k_1, k_2, \dots, k_r$. Let Q be the set formed by the union of the above r periodic orbits with the branching point. Consider the set of basic intervals associated to Q. Suppose that there are points of Q in each connected component of $E \setminus \{\mathbf{0}\}$.

LEMMA 5.1. *Let K and J be basic intervals and let g be a Q-linearization of f. If $x \in \operatorname{Int}(J)$, $g(x) \neq \mathbf{0}$ and $g(x) \in K$, then J g-covers K.*

We leave the proof of the above lemma to the reader. The following proposition is the converse result of Proposition 4.4 for Q-linear maps.

PROPOSITION 5.2. *Let g be a Q-linearization of f. If g has a periodic point of period m for $m \notin \{1, 2, 3, 4, k_1, k_2, \dots, k_r\}$, then there exists a non-repetitive loop of length m through the g-graph such that at least one basic interval of the loop does not contain $\mathbf{0}$.*

Proof. Let x be a periodic point of period m for g. Then $\operatorname{Orb}_g(x) \cap Q = \emptyset$, so for each i, $0 \le i < m$, there exists a unique basic interval J_i containing $g^i(x)$. Since g is Q-linear, by Lemma 5.1, $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ is a loop of the g-graph. First we shall show that this loop is non-repetitive.

Since g is Q-linear, we can define by backward induction on i, a collection of subintervals K_i of J_i such that $g : K_i \longrightarrow K_{i+1}$ is one-to-one and onto, where $K_m = J_m = J_0$. Suppose now the loop is repetitive, then there exists s, $0 < s < m$, such that s divides m and $J_i = J_{i+s}$ for $0 \le i \le m - s$. We claim that $K_i \subset K_{i+s}$ for $0 \le i \le m - s$. To prove the claim consider $K_{m-s} \subset J_{m-s} = J_m = K_m$ and by backward induction, suppose $K_{i+1} \subset K_{i+s+1}$ and $K_i \not\subseteq K_{i+s}$. So, there is $a \in K_i$ such that $a \notin K_{i+s}$, and $g(a) \in K_{i+1} \subset K_{i+s+1}$. Since $K_{i+s} \to K_{i+s+1}$, there exists $b \in K_{i+s}$

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(and so $b \neq a$) such that $g(b) = g(a)$. This is a contradiction to the fact that g is Q-linear and $g|_{J_i}$ is one-to-one. Hence, the claim is proved.

Thus, $g^s(K_0) = K_s \supset K_0$ and by Lemma 4.2, g^s has a fixed point $y \in K_0$. Since m is divisible by s, $g^m(y) = y$. Note that $x \neq y$ because x has period m, and y has period $s < m$. Hence the map $g^m : K_0 \longrightarrow K_m$ is linear and has at least two fixed points. Therefore $g^m|_{K_0}$ must be the identity map and so $K_0 = K_m = J_m = J_0$. Then we get $K_0 = K_s = K_{2s} = \cdots = K_m$ because $K_0 \subset K_s \subset K_{2s} \subset \cdots \subset K_m = K_0$. Now consider the linear map $g^s : K_0 \longrightarrow K_s = K_0$ which has a fixed point. Denote by id the identity map. Since $g^s|_{K_0}$ is one-to-one and onto, we have two possibilities.

Case 1: $g^s|_{K_0} = \mathrm{id}$. Then $g^s(x) = x$ but x has period $m > s$, a contradiction.

Case 2: $g^s|_{K_0} \neq \mathrm{id}$ and $g^{2s}|_{K_0} = \mathrm{id}$. Let $x_0 \in K_0 = J_0$ be a periodic point of period k_i for f for some $k_i \in \{k_1, k_2, \ldots, k_r\}$ such that $\mathrm{Orb}_f(x_0) \subset Q$. Then $g^{2s}(x_0) = x_0$. Moreover x_0 is an endpoint of K_0 and so $k_i = 2s$. On the other hand, since $g^{2s}(x) = x$ and x has period $m > s$ we have $2s = m$. So $k_i = m$, a contradiction with the hypotheses. In short we have proved that the loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ is non-repetitive.

Suppose that all the basic intervals of the non-repetitive loop of length m contain the branching point $\mathbf{0}$. Therefore $\mathrm{Orb}_g(x)$ is contained in the branching intervals. Since $m > 4$, there is a basic interval J_i containing at least two points of $\mathrm{Orb}_g(x)$. Let $u, v \in \mathrm{Orb}_g(x) \cap J_i$ such that $(\mathbf{0}, v) \cap \mathrm{Orb}_g(x) = \emptyset$, and $(u, v) \cap \mathrm{Orb}_g(x) = \emptyset$. Since $m > 1$ and the loop is non-repetitive, there is $J_j \neq J_i$ such that $J_j \cap \mathrm{Orb}_g(x) \neq \emptyset$. Let $z \in J_j \cap \mathrm{Orb}_f(x)$ such that $(z, \mathbf{0}) \cap Q = \emptyset$. Therefore, there is r, $0 < r < m$ such that $g^r(u) = z$ and $g^{m-r}(z) = u$. On the other hand $g^r|_{[u,\mathbf{0}]}$ is lineal and so $g^r|_{[u,\mathbf{0}]} = [z, \mathbf{0}]$. Furthermore, $v \in (u, \mathbf{0})$ and so $g^r(v) \in (z, \mathbf{0})$ in contradiction with the fact that $(z, \mathbf{0}) \cap \mathrm{Orb}_g(x) = \emptyset$. □

COROLLARY 5.3. *Let $E \in \{\propto, \infty\}$. Let f be an E map having a periodic orbit P of period k, such that P has points in each connected component of $E \setminus \{\mathbf{0}\}$. Let g be a P'-linearization of f. If $m \in \mathrm{Per}(g)$ and $m \notin \{2, 3, 4, 2k\}$, then $m \in \mathrm{Per}(f)$.*

Proof. Both E maps f and g have points of periods 1 and k. If $m \notin \{1, 2, 3, 4, k\}$, then by Proposition 5.2 there exists a non-repetitive loop in the g-graph of length m such that at least one of its basic intervals does not contain $\mathbf{0}$. Therefore, since the g-graph of P' is a subgraph of the f-graph of P', by Proposition 4.4, f has a periodic point of period m. □

Remark 5.4. Suppose that f is P'-linear. Then each branching interval f-covers exactly one branching interval, and perhaps some non-branching intervals. Moreover each non-branching interval f-covers either zero or two branching intervals.

Now we add a proposition for P'-linear maps which we will use for the computation of the full periodicity kernel of $\propto$ and ∞.

PROPOSITION 5.4. *Let $E \in \{\mathbf{I}, \mathbf{Y}, \mathbf{X}, \propto, \infty\}$. Let f be an E map having a periodic orbit P of period k. Suppose that P has points in each component of $E \setminus \{\mathbf{0}\}$ and that f is P'-linear. Assume that each basic interval is f-covered by some basic interval different from itself and that there is a basic interval J_0 such that $J_0 \to J_0$. Then $\{n \in \mathbb{N} : n \geq k\} \setminus \{2k\} \subset \mathrm{Per}(f)$.*

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Proof. We denote by S the set of basic intervals associated to P'. Notice that $\text{Card}(S) = k$ if $E \in \{\mathbf{I}, \mathbf{Y}, \mathbf{X}\}$, $\text{Card}(S) = k + 1$ if $E = \propto$ and $\text{Card}(S) = k + 2$ if $E = \infty$. Since each basic interval is f-covered by some basic interval we get that $f(E) = E$.

Set $K_i = f^i(J_0)$ for $i \geq 0$. Note that each K_i is a connected set and $\text{Card}(K_1 \cap P) \geq 2$.

Case 1: $E \in \{\mathbf{I}, \mathbf{Y}, \mathbf{X}, \propto\}$. From the fact that P is a periodic orbit and $f(E) = E$, it follows that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_r = E$ and $\text{Card}(K_i \cap P) \geq i+1$ for $i < r$. Since P has period k we have that $r \leq \text{Card}(K_{r-1} \cap P) \leq k$. Since each basic interval is f-covered by some basic interval different from itself, for each $J_i \in S$, $J_i \subset K_i \setminus K_{i-1}$ there exists $J_{i-1} \in S$, $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$ such that $J_{i-1} \to J_i$. By hypotheses there exists $M \in S$, $M \neq J_0$ such that $M \to J_0$. Hence, there is a loop of length $l \leq r + 1 \leq k + 1$ containing J_0. By construction, this loop is formed by pairwise different basic intervals and so is non-repetitive. The above loop of length l together with the loop $J_0 \to J_0$ give us a non-repetitive loop of length n for each $n \geq k+1$ containing J_0.

We claim that the above loop contains some non-branching interval. If $\mathbf{0} \notin J_0$, then we are done. So suppose that $\mathbf{0} \in J_0$. Since $J_0 \to J_0$, $f(\mathbf{0}) = \mathbf{0}$ and f is P'-linear we get that the basic intervals different from J_0 of K_1 do not contain $\mathbf{0}$. So the claim is proved. Hence by Proposition 4.4 the result follows.

Case 2: $E = \infty$. From the facts that P is a periodic orbit, K_i is connected and $f(E) = E$, we have that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_r = E'$, where either $E' = \infty$ or $E' = \infty \setminus \{J_1, J_2\}$, with J_1 and J_2 basic intervals contained in different circles of ∞ and such that $J_1 \rightleftarrows J_2$.

First we assume that $E' = \infty \setminus \{J_1, J_2\}$. Then E' is homeomorphic to some space of $\{\mathbf{I}, \mathbf{Y}, \mathbf{X}\}$. Of course, $P \subset E'$. Consider the E' map $g = f|_{E'}$. Clearly g is well-defined because f is P'-linear. Thus g is either an $\mathbf{I}$ map, a $\mathbf{Y}$ map or an $\mathbf{X}$ map. Moreover, $\text{Per}(g) \subset \text{Per}(f)$. Then the result follows as in Case 1.

Finally, we suppose that $E' = \infty$. We remark that if $r \leq k$, then the result follows as in Case 1. So, since $\text{Card}(S) = k + 2$, from now on, we can assume that $r = k + 1$.

Subcase 2.1. Suppose that J_0 is a non-branching interval. Let $s > 0$ be the smaller integer such that K_s containg a branching interval J_s. Let $J_{s-1} \in K_{s-1}$ be such that $J_{s-1} \to J_s$. By the minimality of s, J_{s-1} is non-branching and from Remark 5.4 J_{s-1} f-covers two different branching intervals. Again by the minimality of s we get that $r \leq k$ in contradiction with the assumptions.

Subcase 2.2. Suppose that J_0 is a branching interval. From Subcase 2.1 we can assume that each non-branching interval does not f-cover itself. Therefore there exists a non-branching interval J_s in the same circle as J_0 such that J_s f-covers two different branching intervals, one of each circle of ∞ and $J_s \not\to J_0$. Thus $\text{Card}(K_s \cap P) \geq s+2$ in contradiction with the assumptions. So the result is proved. □

6. *The graph of an $\propto$ map*

If we identify the points 1 and 2 of the segment $[0, 3]$, then we obtain a space homeomorphic to $\propto$. The segments $[0, 1]$ and $[2, 3]$ represents the two whiskers of $\propto$ and the segment $[1, 2]$ with the points 1 and 2 identified to the branching point $\mathbf{0}$ represents $\mathbf{O}$.

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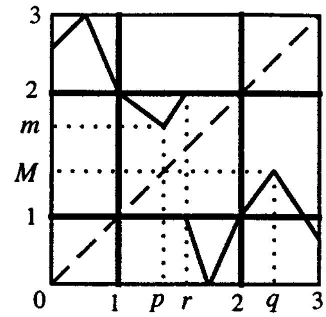

FIGURE 6.1. The graph of an $\propto$ map f.

We represent the cartesian product $\propto \times \propto$ as the square $[0, 3] \times [0, 3]$ identifying the points $(1, y)$ and $(2, y)$ for all $y \in [0, 3]$, and the points $(x, 1)$ and $(x, 2)$ for all $x \in [0, 3]$. Thus the graph of an $\propto$ map f is the subset $\{(x, f(x)) : x \in \propto\}$ of $\propto \times \propto$, and it can be represented as in Figure 6.1. Roughly speaking, we think in the graph of an $\propto$ map like the graph of an interval map g from $[0, 3]$ into itself with the above identifications. This allows us to talk about local or absolute maximum or minimum for an $\propto$ map in the same way as for interval maps. Thus, for instance in the points p and q the $\propto$ map f represented in Figure 6.1 has a local minimum and maximum with values m and M respectively.

Let f be a P'-linear $\propto$ map such that each basic interval associated to P' does not f-cover itself. Therefore the graph of f does not touch the diagonal except at the branching point. Let $V = [a, b]$ be a closed interval contained in whiskers(A), whiskers(B) or $\mathbf{O}$ such that $g(a) = g(b) \in \{1, 2\}$, $g(c) \neq g(a)$ for all $c \in (a, b)$ and $g(V)$ is strictly contained in $\mathbf{O}$. Then we say that V is an *upper* (respectively *lower*) *subinterval* according to whether it contains more local minima (respectively maxima) than local maxima (respectively minima) of g. Since f is P'-linear these upper and lower subintervals are well-defined. Thus, for instance, the subinterval $[1, r]$ is an upper subinterval of the map f of Figure 6.1.

7. *The unfolding of* $\propto$

In this section f will be a P'-linear $\propto$ map such that each basic interval associated to P' does not f-cover itself. Let k be the period of P. Let $K = [a, b]$ be a closed subinterval of $\propto$ such that $f(a) = f(b) = \mathbf{0}$, and $f(c) \neq \mathbf{0}$ for all $c \in (a, b)$, then we say that K *f-covers* $\mathbf{O}$, or $K \to \mathbf{O}$ (or $\mathbf{O} \leftarrow K$). We say that such a K is a *crossing subinterval*.

In this section we also assume that $\propto$ has no crossing subintervals. Then following ideas of [**14**] and [**15**] we define the *unfolding* of $\propto$ as follows. Define the graph $\propto^* = G_1 \cup G_2 \cup G_3$ where:

$$G_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, y = 0, -1 \leq x \leq 1\};$$
$$G_2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + (y + 1)^2 = 1\};$$
$$G_3 = \{(x, y, z) \in \mathbb{R}^3 : y = z, x^2 + (y + 1)^2 = 1\}.$$

(See Figure 7.1.)

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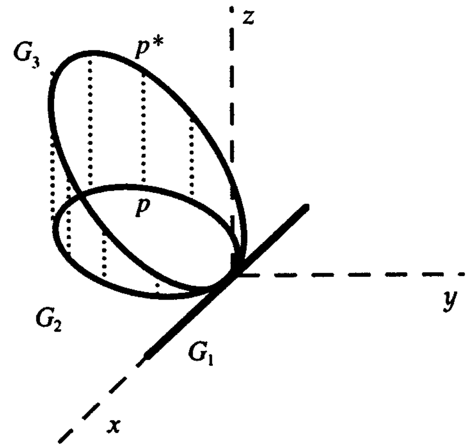

FIGURE 7.1. The unfolding of $\propto$.

Clearly $G_1 \cup G_2$ is homeomorphic to $\propto$, so we identify $\propto$ with $G_1 \cup G_2$. Consider the projection $\pi : \propto^* \longrightarrow \propto$ defined by $\pi(x, y, z) = (x, y, 0)$. We denote by p^* the unique point of G_3 such that $\pi(p^*) = p$.

Since f is P'-linear, f has finitely many local extrema; and consequently finitely many upper and lower subintervals. Moreover from the fact that there are no crossing subintervals of $\propto$, it follows that there exists a finite 'partition' of $\propto$ into upper and lower subintervals. Now for the given $\propto$ map f we define $f^* : \propto \longrightarrow \propto^*$ as follows. If $p \in \propto$ then $f^*(p)$ is either $f(p)^*$ if $f(p) \in \mathbf{O}$ and p belongs to an upper subinterval; or $f(p)$ otherwise. Clearly f^* is well-defined. We remark that $f = \pi \circ f^* : \propto \longrightarrow \propto$. Define $F = f^* \circ \pi : \propto^* \longrightarrow \propto^*$. In the rest of this section we shall study the relationship between the periods of f and F.

LEMMA 7.1. *Assume that there are no crossing subintervals. If $q \in \propto^*$ is a periodic point of F of period n, then $p = \pi(q)$ is a fixed point of f^n.*

Proof. Since $q = F^n(q) = (f^* \circ \pi)^n(q) = f^* \circ (\pi \circ f^*)^{n-1} \circ \pi(q) = f^*(f^{n-1}(p))$, we get that $p = \pi(q) = f^n(p)$. □

LEMMA 7.2. *Assume that there are no crossing subintervals. Then the following statements hold:*
(a) *if $p = \pi(q)$ is an n-point for f, then $p = \pi(F^n(q))$;*
(b) *if $p \in G_1$ is an n-point for f, then p is a fixed point of F^n.*

Proof. Statement (a) follows from the equalities

$$p = \pi(q) = f^n(\pi(q)) = (\pi \circ f^*)^n(\pi(q)) = \pi \circ (f^* \circ \pi)^n(q) = (\pi \circ F^n)(q).$$

If p is a periodic point of f of period n, we have that

$$p = f^n(p) = f^n(\pi(p)) = (\pi \circ f^*)^n(\pi(p)) = \pi \circ (f^* \circ \pi)^n(p) = (\pi \circ F^n)(p).$$

Since $p \in G_1$, we get that $F^n(p) = p$, and statement (b) is proved. □

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PROPOSITION 7.3. *Suppose that there are no crossing subintervals. Then the following statements hold:*

(a) *if q is an n-point for F, then $p = \pi(q)$ is an n-point for f;*

(b) *if p is an n-point for f and $p \in G_1$, then p is an n-point for F.*

Proof. We prove (a). Let q be an n-point for F. By Lemma 7.1, $p = \pi(q)$ is a fixed point of f^n. Therefore, there is a divisor s of n such that p is an s-point for f. If $s = n$, then we are done. So, assume that $s < n$. By Lemma 7.2(a), $p = \pi(F^s(q))$. Since $s < n$, $F^s(q) = p'$ with $p' \neq q$, and of course p' belongs to the F-periodic orbit of q. Then $q = F^n(q) = (f^* \circ \pi)^n(q) = (f^* \circ \pi)^{n-1} \circ f^*(\pi(q)) = (f^* \circ \pi)^{n-1} \circ f^*(p) = (f^* \circ \pi)^{n-1} \circ f^*(\pi(F^s(q))) = (f^* \circ \pi)^n(F^s(q)) = F^n(p') = p'$, which is a contradiction. Hence $s = n$ and (a) is proved.

Now we show (b). Let p be an n-point for f and $p \in G_1$. By Lemma 7.2(b), $p = F^n(p)$. Again, there is a divisor s of n such that p is an s-point for F. If $s = n$, then we are done. So, assume that $s < n$. Then $F^s(p) = p$. By Lemma 7.1, since $p \in G_1$ we get that $p = f^s(p)$, a contradiction. Then the lemma follows. □

PROPOSITION 7.4. *Assume that there are no crossing subintervals. If* $\{5, 7\} \subset \mathrm{Per}(f)$ *then* $\mathbb{N} \setminus \{2, 3, 4, 6, 10, 11\} \subset \mathrm{Per}(f)$.

Proof. Since P has elements on each component of $\propto \setminus \{\mathbf{0}\}$, by Proposition 7.3(b) we have that $k \in \mathrm{Per}(F)$. Again, from the facts that P has elements on each component of $\propto \setminus \{\mathbf{0}\}$ and there are no crossing subintervals, we get that $F(\propto^*)$ is homeomorphic to $\mathbf{Y}$ or $\mathbf{X}$. So, from the $\mathbf{Y}$ theorem and the n-od theorem we obtain that if $k = 5$, then $\mathbb{N} \setminus \{2, 3, 4, 6, 7, 10, 11, 15\} \subset \mathrm{Per}(F)$ and if $k = 7$, then $15 \in \mathrm{Per}(F)$. Now from Proposition 7.3(a) the result follows. □

8. *The full periodicity kernel of* $\propto$

The goal of this section is to prove Theorem 1.6. Since $\mathbf{X}$ is homeomorphic to $\{(x, y) \in \propto : y \geq -1\}$, in this section we shall consider $\mathbf{X} = \{(x, y) \in \propto : y \geq -1\}$. Let f be an $\mathbf{X}$ map, we shall extend f to an $\propto$ map $\overline{f}$ as follows. We define $\overline{f}(z) = f(z)$ if $z \in \mathbf{X}$ and $\overline{f}|_{\mathrm{Cl}(\propto \setminus \mathbf{X})}$ is any homeomorphism between $\mathrm{Cl}(\propto \setminus \mathbf{X})$ and the unique closed interval in $\mathbf{X}$ having $f(1, -1)$ and $f(-1, -1)$ as endpoints such that $\overline{f}(1, -1) = f(1, -1)$ and $\overline{f}(-1, -1) = f(-1, -1)$. Of course $\mathrm{Per}(f) = \mathrm{Per}(\overline{f})$. By Theorem 1.3, $\{2, 3, 4, 5, 6, 7, 10, 11\}$ is a subset of the full periodicity kernel of $\propto$. Then, to prove Theorem 1.6 it is sufficient to show the following proposition.

PROPOSITION 8.1. *Let f be an $\propto$ map. If* $\{5, 7\} \subset \mathrm{Per}(f)$ *then* $\mathbb{N} \setminus \{2, 3, 4, 6, 10, 11\} \subset \mathrm{Per}(f)$.

In the rest of this section we fix the $\propto$ map f having a periodic orbit P of period $k \in \{5, 7\}$ *and the set of the basic intervals associated to P'*. This fixed $\propto$ map will be called the *standard $\propto$ map*.

LEMMA 8.2. *Let f be the standard $\propto$ map. If the periodic orbit P has no points into each connected component of $\propto \setminus \{\mathbf{0}\}$ then Proposition 8.1 holds.*

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Proof. Let $E' \subset \propto$ be a union of connected components of $\propto \setminus \{\mathbf{0}\}$. Suppose that $P \subset E'$. Then we define the map $g : E' \longrightarrow E'$ as follows. For $z \in E'$, $g(z) = f(z)$ if $f(z) \in E'$; and $g(z) = \mathbf{0}$ if $f(z) \in E \setminus E'$. Notice that g is either an **I**, $\boldsymbol{\sigma}$ or **O** map. Clearly $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence, from the Interval theorem, the $\boldsymbol{\sigma}$ theorem and the Circle theorem the result follows. □

Remark 8.3. From Lemma 8.2 we can assume that the periodic orbit P has points into each connected component of $\propto \setminus \{\mathbf{0}\}$. Furthermore, by Corollary 5.3 in what follows we can suppose that the standard map f will be P'-linear.

LEMMA 8.4. *Let f be the standard $\propto$ map. Suppose that there is a basic interval J such that there are no basic intervals f-covering J different from itself. Then Proposition 8.1 holds.*

Proof. We claim that each basic interval L contained in the whiskers of $\propto$ is f-covered by some basic interval different from itself. To see this, without loss of generality we can assume that $L \subset \mathrm{whiskers}(A)$. Let p be the endpoint of whiskers(A), $p \neq \mathbf{0}$. Since f is P'-linear, we have that $p \in P$. Moreover, from the fact that $\mathbf{0}$ is a fixed point, f is P'-linear and $f(E)$ is connected, it follows that each basic interval contained in the whiskers of $\propto$ is covered by some basic interval. Suppose that $L \to L$, otherwise we are done. Since $[p, \mathbf{0}] = \mathrm{whiskers}(A) \subset \propto$, we can consider a total ordering $<$ on whiskers(A) such that $\mathbf{0}$ is the largest element and p the smallest one. Set $L = [p_j, p_k]$, with $p \leq p_j < p_k \leq \mathbf{0}$. Now, since f is P'-linear we can consider two cases.

Case 1: $p \leq f(p_j) < p_j < p_k$ and $f(p_k) \notin [p, p_k)$. If there are no basic intervals $K \neq L$ such that $K \to L$, then $f(P \cap [p, p_j]) \subset P \cap [p, p_j]$ with $P \cap [p, p_j] \neq \emptyset$. This is a contradiction because P is a periodic orbit not contained into whiskers(A).

Case 2: $p \leq f(p_k) \leq p_j < p_k$ and $f(p_j) \notin [p, p_k)$. Then $p_k < \mathbf{0}$, and clearly $f([p_k, \mathbf{0}]) \supset [f(p_k), f(\mathbf{0})] \supset [f(p_k), \mathbf{0}] \supset [p_j, \mathbf{0}] \supset L$. Therefore, there is a basic interval $J_1 \subset [p_k, \mathbf{0}]$ which f-covers L and $J_1 \neq L$. Therefore, the claim is proved and so $J \subset \mathbf{O}$.

Consider the following map $g = f|_{\propto \setminus \mathrm{Int}(J)} : \propto \setminus \mathrm{Int}(J) \longrightarrow \propto \setminus \mathrm{Int}(J)$. Clearly g is well-defined because f is P'-linear. Moreover g is either a **Y** map or an **X** map such that $\mathrm{Per}(g) = \mathrm{Per}(f)$. Hence, from the **Y** theorem and the n-od theorem, Proposition 8.1 holds and so the lemma follows. □

Remark 8.5. From Lemma 8.4 we can assume that each basic interval is f-covered by some different basic interval.

Remark 8.6. Proposition 5.5 shows that if there exists some basic interval which f-covers itself, then Proposition 8.1 holds. So, from now on we suppose that each basic interval does not f-cover itself.

Remark 8.7. If there are no closed subintervals of $\propto$ f-covering **O**, from Proposition 7.4, Proposition 8.1 holds. So, from now on, we can assume that there is a crossing subinterval $K_1 \subset \propto$ such that $K_1 \to \mathbf{O}$.

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Remark 8.8. In a similar way as in Lemma 4.1, if K and L are closed subintervals of $\propto$ such that $L \subset \mathbf{O}$, $K \to \mathbf{O}$ and $\mathbf{0} \in \mathrm{Cl}(\mathbf{O} \setminus L)$, then $K \to L$.

LEMMA 8.9. *Let f be an $\propto$ map having a k-orbit P. Suppose that f is P'-linear. If J_0 is a subinterval of $\propto$ with endpoints which are elements of P' and are contained in one of the whiskers of $\propto$, then there is a loop of length k in the f-graph containing J_0 formed by intervals of $\propto$.*

Proof. Let $J_0 = [x, y]$ with $x, y \in P'$ and $[x, y]$ contained in one of the whiskers of $\propto$. For each i, $0 < i \le k$, we define J_i recursively as the interval with endpoints $f^i(x)$ and $f^i(y)$ and such that $J_{i-1} \to J_i$. Then $J_k = J_0$ because J_0 is contained in a whiskers. Define the intervals K_i for $0 \le i \le k$ by backward induction on i as follows. Let $K_k = J_k$, and if K_{i+1} has been defined and is a subset of J_{i+1}, then let K_i be a subset of J_i such that $K_i \to K_{i+1}$. Then we have the loop $J_0 \supset K_0 \to K_1 \to \cdots \to K_k = J_0$ of length k. Note that in general, the intervals J_i are not basic and the loop can be repetitive or non-repetitive. □

Since the periodic orbit P has points into each connected component of $\propto$, there are exactly four branching intervals denoted by A, B, C and D. We shall assume that $A \subset \mathrm{whiskers}(A)$, $B \subset \mathrm{whiskers}(B)$ and $\{C, D\} \subset \mathbf{O}$. Moreover, since f is P'-linear and each basic interval does not f-cover itself, it follows that $\mathbf{0}$ is the unique fixed point of f and from Remark 5.4 each basic interval of $\{A, B, C, D\}$ f-covers a unique basic interval of $\{A, B, C, D\}$ different from itself.

Proof of Proposition 8.1. Take $k = 5$. Let f be the standard $\propto$ map. Since there is a crossing subinterval K_1 (see Remark 8.7) and there are no fixed points different from $\mathbf{0}$, we get that $K_1 \subset \mathrm{whiskers}(A)$ or $K_1 \subset \mathrm{whiskers}(B)$. Without loss of generality we can assume that $K_1 \subset \mathrm{whiskers}(A)$ and that K_1 has endpoints which are elements of P'. Denote by p_a and p_b the endpoints of whiskers(A) and whiskers(B) respectively different from $\mathbf{0}$. If $f^{k-1}(p_a) \in \mathrm{whiskers}(A)$, then the interval $[p_a, f^{k-1}(p_a)] \subset \mathrm{whiskers}(A)$ has a fixed point, in contradiction with the assumptions. So we can suppose that $f^{k-1}(p_a) \notin \mathrm{whiskers}(A)$ and in the same way $f^{k-1}(p_b) \notin \mathrm{whiskers}(B)$. Therefore we consider two cases.

Case 1: $f^{k-1}(p_a) \in \mathbf{O}$. Since $\mathbf{0}$ is a fixed point and $f^{k-1}(p_a) \in \mathbf{O}$, we have that there are two closed subintervals $K_2, K_3 \subset \mathbf{O}$ f-covering whiskers(A), such that $\mathbf{0} \in \mathrm{Cl}(\mathbf{O} \setminus K_2)$ and $\mathbf{0} \in \mathrm{Cl}(\mathbf{O} \setminus K_3)$. From Lemma 4.1 and Remark 8.8 we get that $K_2 \rightleftarrows K_1 \rightleftarrows K_3$.

First suppose that at least one of these three K_i does not contain $\mathbf{0}$. Then from the subgraph $K_2 \rightleftarrows K_1 \rightleftarrows K_3$ we can construct a non-repetitive loop $J_0 = K_1 \to J_1 \to \cdots \to J_n = K_1$ of length n for each n even containing the interval K_i such that $\mathbf{0} \notin K_i$. By Lemma 4.3 there exists $x \in K_1$ such that $f^n(x) = x$ and $f^i(x) \in J_i$ for $1 \le i \le n-1$. Since $\mathbf{0}$ does not belong to some interval of $\{K_1, K_2, K_3\}$ and the loop is non-repetitive, x has period n. So $\{2n : n \in \mathbb{N}\} \subset \mathrm{Per}(f)$. On the other hand, by Lemma 8.9 there is a loop of length k containing K_1 and formed by closed subintervals of $\propto$. This loop together with the loops $K_2 \rightleftarrows K_1 \rightleftarrows K_3$ give us a non-repetitive loop of length n for each $n > k$ odd ($k = 5$). This loop can be chosen in such a way that at least one of its intervals does

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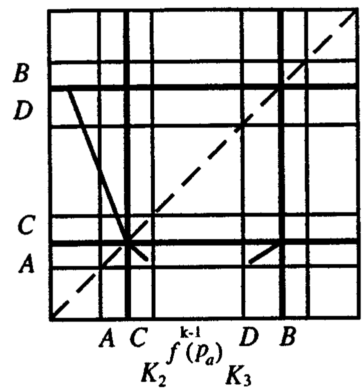

FIGURE 8.1. The graph of f when $C \rightleftarrows A \leftarrow D$ and $B \rightarrow A$.

not contain $\mathbf{0}$. Then $\{n \geq k : n \text{ odd}\} \subset \mathrm{Per}(f)$. Hence we get that $S(k) \subset \mathrm{Per}(f)$ and the result follows.

Finally, suppose that $\mathbf{0} \in K_1 \cap K_2 \cap K_3$. Therefore we have one of the following subgraphs: $C \rightleftarrows A \leftarrow D$ or $C \rightarrow A \rightleftarrows D$. Furthermore we can assume that $f(\mathbf{O}) \subset \text{whiskers}(A)$; otherwise we can find a subinterval K_i of $\mathbf{O}$ such that $\mathbf{0} \notin K_i$. By symmetry we can assume that the graph of f contains $C \rightleftarrows A \leftarrow D$. Since each basic interval of $\{A, B, C, D\}$ f-covers a unique interval of $\{A, B, C, D\}$ we only need consider two subcases.

Subcase 1.1: $C \rightleftarrows A \leftarrow D$ and $B \rightarrow A$. Suppose that there are no basic intervals different from B, C and D f-covering A. So, by P'-linearity we have that $f(\mathbf{O} \cup \text{whiskers}(B)) \subset \text{whiskers}(A)$ and $f(\text{whiskers}(A)) \subset \mathbf{O} \cup \text{whiskers}(B)$ (see Figure 8.1). Consequently k must be even, in contradiction with $k = 5$. Hence, we can assume that there is a basic interval $J \notin \{A, B, C, D\}$ such that $J \rightarrow A$. So $J \rightarrow A \cup M$ for some $M \in \{B, C, D\}$. We claim that there exists a path γ of length $l \leq k-1$ starting at one of the intervals A or C and ending at J such that at least one of the intervals of γ does not contain $\mathbf{0}$. Now we prove the claim. Denote by S the set of the basic intervals associated to P'. Since each basic interval is f-covered by some basic interval we get that $f(\propto) = \propto$. Set $K_i = f^i(A \cup C)$ for $i \geq 0$. Notice that K_i is a connected set for all i and $\mathrm{Card}(K_1 \cap P) \geq 3$. Since P is a periodic orbit and $f(\propto) = \propto$, it follows that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_r = \propto$ and $\mathrm{Card}(K_i \cap P) \geq i+2$ for $i < r$. From the fact that P has period k we get that $r+1 \leq \mathrm{Card}(K_{r-1} \cap P) \leq k$, and so $r \leq k-1$. From the assumptions, for each basic interval $J_i \in S$, $J_i \subset K_i \setminus K_{i-1}$ there is $J_{i-1} \in S$, $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$ such that $J_{i-1} \rightarrow J_i$. Hence, given $J \in S$, $J \notin \{A, C\}$ there exists a path γ of length $l \leq r \leq k-1$ starting at one of the intervals A or C and ending at J. Moreover, since $A \cap C = \{\mathbf{0}\}$, $A \rightleftarrows C$ and f is P'-linear, we obtain that the basic intervals of $K_1 \setminus (A \cup C)$ do not contain $\mathbf{0}$. Then the claim is proved. The path γ together with the paths $J \rightarrow A$, $J \rightarrow M \rightarrow A$ and $C \rightleftarrows A$ give us a loop of length n for each $n \geq k+1$. By construction this loop is non-repetitive and at least one of its intervals does not contain $\mathbf{0}$. Therefore, by Proposition 4.4 $\{n \in \mathbb{N} : n \geq k\} \setminus \{2k\} \subset \mathrm{Per}(f)$. So the result follows.

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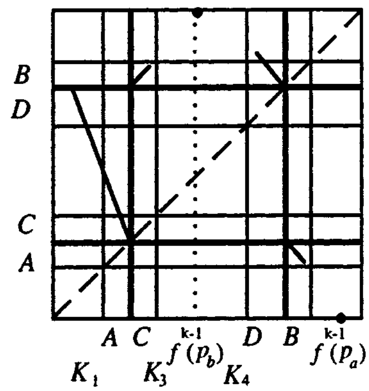

FIGURE 8.2. The graph of f when $\mathbf{0} \in K_1 \cap K_2 \cap K_3 \cap K_4$.

Subcase 1.2: $C \rightleftarrows A \leftarrow D \leftarrow B$ or $B \to C \rightleftarrows A \leftarrow D$. From the facts that $\mathbf{0}$ is the unique fixed point of f and $f(\mathbf{O}) \subset \text{whiskers}(A)$, we get that $f^{k-1}(p_b) \in \text{whiskers}(A)$. Then, since $K_1 \subset \text{whiskers}(A)$ and $K_1 \to \mathbf{O}$ it follows that there is a basic interval J such that either $D \leftarrow J \to A$, or $D \leftarrow J \to B$, or $C \leftarrow J \to A$, or $C \leftarrow J \to B$ (see Figure 8.1). In a similar way as in Subcase 1.1 we obtain a non-repetitive loop of length n for each $n \geq k+2$ in the hypotheses of Proposition 4.4. Then the result holds.

Case 2: $f^{k-1}(p_a) \in \text{whiskers}(B)$. Set $n_a = \text{Card}(\text{whiskers}(A) \cap P)$, $n_b = \text{Card}(\text{whiskers}(B) \cap P)$ and $n_o = \text{Card}(\mathbf{O} \cap P)$. Since $f(\mathbf{0}) = \mathbf{0}$ and $f^{k-1}(p_a) \in \text{whiskers}(B)$, we have that $\text{whiskers}(B) \to \text{whiskers}(A)$. By Lemma 4.1 there is a closed subinterval $K_2 \subset \text{whiskers}(B)$ such that $K_2 \to K_1$. Since $f^{k-1}(p_b) \notin \text{whiskers}(B)$, we consider two subcases.

Subcase 2.1: $f^{k-1}(p_b) \in \mathbf{O}$. Since $f(\mathbf{0}) = \mathbf{0}$, there are two closed subintervals $K_3, K_4 \subset \mathbf{O}$ f-covering whiskers(B). From Lemma 4.1 we get $K_3 \to K_2 \leftarrow K_4$. By Lemma 8.9 there exists a loop of length k containing K_1.

First assume that at least one of the intervals $\{K_1, K_2, K_3, K_4\}$ does not contain $\mathbf{0}$. Therefore the loop of length $k = 5$ together with the loops $K_1 \to K_3 \to K_2 \to K_1$ and $K_1 \to K_4 \to K_2 \to K_1$ give us a non-repetitive loop of length n for $n \in \mathbb{N} \setminus \{2, 4, 10\}$. By Lemma 4.3, there is $x \in K_1$ such that $f^n(x) = x$. Since $\mathbf{0}$ does not belong to some K_i and the loop is non-repetitive we obtain that x has period n. Thus the result follows.

Finally assume that $\mathbf{0} \in K_1 \cap K_2 \cap K_3 \cap K_4$, there are no closed subintervals K_i of $\mathbf{O}$ such that $f(K_i) = \text{whiskers}(B)$ and $\mathbf{0} \notin K_i$, and there are no closed subintervals K_j of whiskers(B) such that $f(K_j) = \text{whiskers}(A)$ and $\mathbf{0} \notin K_j$. Then we have that $f(\mathbf{O}) \subset \text{whiskers}(B)$, $f(\text{whiskers}(B)) \subset \text{whiskers}(A)$ and the f-graph contains the paths $C \to B \leftarrow D$, $B \to A$ and either $A \to C$ or $A \to D$ (see Figure 8.2). By symmetry we can suppose that $B \to A \to C \to B \leftarrow D$. Since whiskers$(A) \to \mathbf{O}$ and f is P'-linear, we get that $n_a \geq 2$. On the other hand, since C and D f-cover B, if $n_o \geq 2$ then $n_b \geq 2$. Therefore for $k = 5$ there are two possibilities: $n_a = 2$, $n_b = 2$ and $n_o = 1$; or $n_a = 3$, $n_b = 1$ and $n_o = 1$. Set $\{a_i : i = 1, 2, \ldots, 5\}$ the 5-orbit P. The basic intervals associated to P' are A, B, C, D, E and F.

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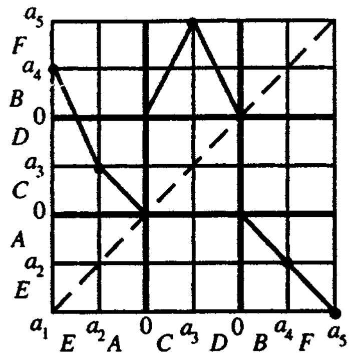

FIGURE 8.3. The graph of f when $n_a = 2, n_b = 2, n_o = 1$.

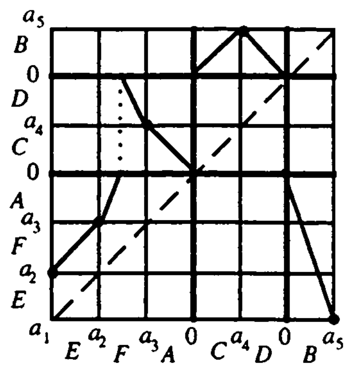

FIGURE 8.4. The graph of f when $n_a = 3, n_b = 1, n_o = 1$.

Possibility (i): $n_a = 2, n_b = 2$ and $n_o = 1$. Then we take $A = [a_2, \mathbf{0}]$, $B = [\mathbf{0}, a_4]$, $C = [\mathbf{0}, a_3]$, $D = [a_3, \mathbf{0}]$, $E = [a_1, a_2]$ and $F = [a_4, a_5]$ (see Figure 8.3). Since $f^{k-1}(p_b) \in \mathbf{O}$, we have that $f(a_3) = a_5$. Moreover $f(a_2) = a_3$ because whiskers$(A) \to \mathbf{O}$. Since $f(\text{whiskers}(B)) \subset \text{whiskers}(A)$, $f(\{a_4, a_5\}) = \{a_1, a_2\}$. Since P has period 5, it follows that $f(a_5) = a_1$, $f(a_4) = a_2$ and $f(a_1) = a_4$. So we obtain the loops $E \to D \to F \to E$, $E \to B \to A \to C \to F \to E$, and $E \to D \to B \to A \to C \to F \to E$ of lengths 3, 5 and 6 respectively (see again Figure 8.3). Then $\text{Per}(f) \supset \mathbb{N} \setminus \{2, 4, 10\}$ and the result follows.

Possibility (ii): $n_a = 3$, $n_b = 1$ and $n_o = 1$. We take $A = [a_3, \mathbf{0}]$, $B = [\mathbf{0}, a_5]$, $C = [\mathbf{0}, a_4]$, $D = [a_4, \mathbf{0}]$, $E = [a_1, a_2]$ and $F = [a_2, a_3]$ (see Figure 8.4). Since $f^{k-1}(p_b) \in \mathbf{O}$ we have that $f(a_4) = a_5$. Moreover $f(a_3) = a_4$ because $K_1 \to \mathbf{O}$ and f is P'-linear. Since $f^{k-1}(p_a) \in \text{whiskers}(B)$, $f(a_5) = a_1$. By periodicity $f(a_1) = a_2$ and $f(a_2) = a_3$. Clearly we obtain the loops $B \to A \to C \to B$, $B \to F \to D \to B$, $B \to E \to F \to D \to B$ and $B \to F \to A \to C \to B$ of lengths 3, 3, 4 and 4 respectively. Consequently $\text{Per}(f) \supset \mathbb{N} \setminus \{2, 10\}$ and the result follows.

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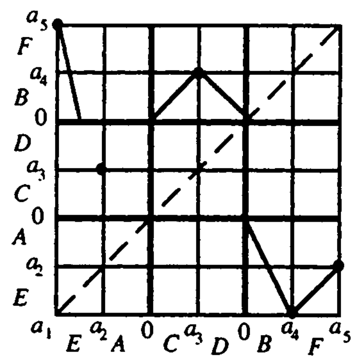

FIGURE 8.5. The graph of f when $n_a = 2$, $n_b = 2$ and $n_o = 1$.

Subcase 2.2: $f^{k-1}(p_b) \in \text{whiskers}(A)$. Since $f(\mathbf{0}) = \mathbf{0}$ we have that $\text{whiskers}(A) \to \text{whiskers}(B)$. Thus there exist $K_3 \subset \text{whiskers}(A)$ and $K_4 \subset \text{whiskers}(B)$ such that $K_3 \rightleftarrows K_4$. Furthermore, since $\text{whiskers}(A) \to \mathbf{O}$ we can suppose that K_3 does not contain $\mathbf{0}$. Moreover we can take K_3 such that its endpoints are elements of P. By Lemma 8.9 there is a loop of length $k = 5$ containing K_3. This loop together with the loop $K_3 \rightleftarrows K_4$ give us a non-repetitive loop of length n for each $n = 5i + 2j$ with $i \geq 1$, $j \geq 1$. By Lemma 4.3 there is $x \in K_3$ such that $f^n(x) = x$. Since the loop is non-repetitive and $\mathbf{0} \notin K_3$, x has period n. Hence $\text{Per}(f) \supset \{n \in \mathbb{N} : n \geq 9\} \setminus \{10\}$. Now we need to show that $8 \in \text{Per}(f)$.

First, suppose that $f(p_a) \notin \text{whiskers}(B)$ or $f(p_b) \notin \text{whiskers}(A)$. Therefore from the facts that $f^{k-1}(p_b) \in \text{whiskers}(A)$ and $f^{k-1}(p_a) \in \text{whiskers}(B)$ it follows that there are two closed subintervals of $\text{whiskers}(B)$ f-covering $\text{whiskers}(A)$ or two closed subintervals of $\text{whiskers}(A)$ f-covering $\text{whiskers}(B)$. Furthermore, since $\text{whiskers}(A) \rightleftarrows \text{whiskers}(B)$ we have that there are three closed subintervals $K_3' \in \text{whiskers}(A)$, $K_4' \in \text{whiskers}(B)$ and $K_5 \in \text{whiskers}(A)$ or $K_5 \in \text{whiskers}(B)$ such that $\mathbf{0} \notin K_5 \cap K_3' \cap K_4'$ and either $K_3' \rightleftarrows K_4' \rightleftarrows K_5$ or $K_5 \rightleftarrows K_3' \rightleftarrows K_4'$. Hence, we obtain a non-repetitive loop of length 8 such that at least one of its intervals does not contain $\mathbf{0}$. In a similar way as above we get that $8 \in \text{Per}(f)$.

Finally, suppose that $f(p_a) \in \text{whiskers}(B)$ and $f(p_b) \in \text{whiskers}(A)$. Since $\text{whiskers}(A) \to \mathbf{O}$, $\text{whiskers}(A) \to \text{whiskers}(B)$ and f is P'-linear we have that $n_a \geq 2$. Since $\{f(p_a), f^{k-1}(p_a)\} \subset \text{whiskers}(B)$ and $k \neq 2$ we have $n_b \geq 2$. Furthermore, $n_o \geq 1$ because $P \cap \mathbf{O} \neq \emptyset$. Then for $k = 5$ the only possibility is $n_a = 2$, $n_b = 2$ and $n_o = 1$. Set a_i for $i = 1, 2, \ldots, 5$ the 5-orbit P. The basic intervals associated to P' are A, B, C, D, E and F. Take $A = [a_2, \mathbf{0}]$, $B = [\mathbf{0}, a_4]$, $C = [\mathbf{0}, a_3]$, $D = [a_3, \mathbf{0}]$, $E = [a_1, a_2]$ and $F = [a_4, a_5]$ (see Figure 8.5). From the facts that $\text{whiskers}(A) \to \mathbf{O}$ and f is P'-linear, we get that $f(a_2) = a_3$. Since $f^{k-1}(p_b) \in \text{whiskers}(A)$, we have that $f(a_1) = a_5$. Notice that $f(a_5) \neq a_1$ because $k \neq 2$. Moreover $f(p_b) \in \text{whiskers}(A)$ and so $f(a_5) = a_2$. Since $f^{k-1}(p_a) \in \text{whiskers}(B)$, $f(a_4) = a_1$ and then $f(a_3) = a_4$. Therefore we obtain the non-repetitive loop $E \to B \to E \to B \to E \to B \to E \to F \to E$ of length 8 in the hypotheses of Proposition 4.4 (see Figure 8.5). Consequently $8 \in \text{Per}(f)$ and the proposition is proved. □

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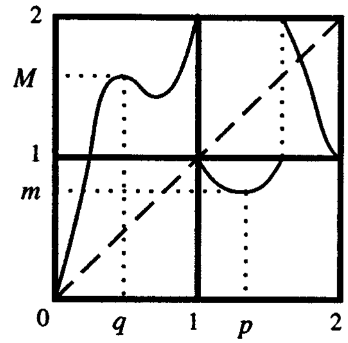

FIGURE 9.1. The graph of an ∞ map f.

9. *The graph of an* ∞ *map*

If we identify the points 0, 1 and 2 of the segment $[0, 2]$, then we obtain a space homeomorphic to ∞. The segments $[0, 1]$ and $[1, 2]$ with the endpoints identified to the branching point represent the two circles of ∞.

We represent the cartesian product $\infty \times \infty$ as the square $[0, 2] \times [0, 2]$ identifying the points $(0, y)$, $(1, y)$ and $(2, y)$ for all $y \in [0, 2]$, and the points $(x, 0)$, $(x, 1)$ and $(x, 2)$ for all $x \in [0, 2]$. Thus, the graph of an ∞ map f is the subset $\{(x, f(x)) : x \in \infty\}$ of $\infty \times \infty$, and it can be represented as in Figure 9.1. Roughly speaking, we think in the graph of an ∞ map like the graph of an interval map g from $[0, 2]$ into itself with the above identifications. This allows us to talk about local or absolute maximum or minimum for an ∞ map in the same way as for interval maps. Thus, for instance in the points p and q the ∞ map f represented in Figure 9.1 has a local minimum and maximum with values m and M respectively.

Let f be a P'-linear ∞ map such that each basic interval associated to P' does not f-cover itself. Therefore, the graph of f does not touch the diagonal except at the branching point. Let $V = [a, b]$ be a closed interval contained in $\mathbf{O}$ or in circle(B) such that $g(a) = g(b) \in \{0, 1, 2\}$, $g(c) \neq g(a)$ for all $c \in (a, b)$ and $g(V)$ is strictly contained in $\mathbf{O}$ or in circle(B). Then we say that V is an *upper* (respectively *lower*) *subinterval* according to whether it contains more local minima (respectively maxima) than local maxima (respectively minima) of g.

10. *The unfolding of* ∞

In this section f will be a P'-linear ∞ map such that each basic interval associated to P' does not f-cover itself, and k will be the period of P. Let $K = [a, b]$ be a closed subinterval of ∞ such that $f([a, b]) = \mathbf{O}$, (respectively $f([a, b]) =$ circle(B)), $f(a) = f(b) = \mathbf{0}$, and $f(c) \neq \mathbf{0}$ for all $c \in (a, b)$, then we say that K *f-covers* $\mathbf{O}$ (respectively circle(B)), or $K \to \mathbf{O}$ (or $\mathbf{O} \leftarrow K$). Moreover, K will be called an $\mathbf{O}$-*crossing* (respectively circle(B)-*crossing*) *subinterval.*

From now on in this section, we also assume that there are no $\mathbf{O}$*-crossing subintervals.* Again following ideas of [**14**] and [**15**] we define the *unfolding* of ∞ as follows. Define

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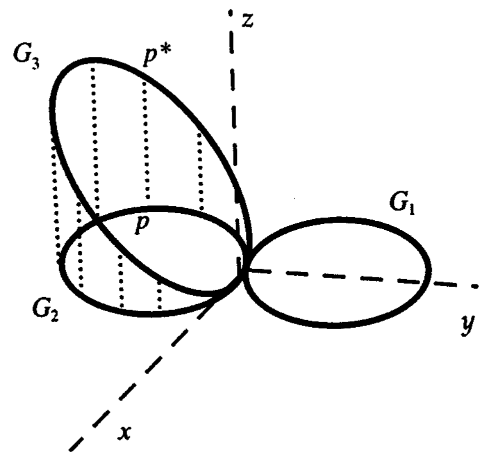

FIGURE 10.1. The unfolding of ∞.

the graph $\infty^* = G_1 \cup G_2 \cup G_3$ where:

$$G_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + (y-1)^2 = 1\};$$
$$G_2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + (y+1)^2 = 1\};$$
$$G_3 = \{(x, y, z) \in \mathbb{R}^3 : y = z, x^2 + (y+1)^2 = 1\}.$$

(See Figure 10.1.)

Clearly $G_1 \cup G_2$ is homeomorphic to ∞, so we identify ∞ with $G_1 \cup G_2$. Consider the projection $\pi : \infty^* \longrightarrow \infty$ defined by $\pi(x, y, z) = (x, y, 0)$. We denote by p^* the unique point of G_3 such that $\pi(p^*) = p$.

Since f is P'-linear, f has finitely many local extrema; and consequently finitely many upper and lower subintervals. Moreover, from the fact that there are no **O**-crossing subintervals, it follows that there exists a finite 'partition' of ∞ into upper and lower subintervals. Now for the given ∞ map f we define $f^* : \infty \longrightarrow \infty^*$ as follows. If $p \in \infty$ then $f^*(p)$ is either $f(p)^*$ if $f(p) \in \mathbf{O}$ and p belongs to an upper subinterval; or $f(p)$ otherwise. Clearly f^* is well-defined. Since f has no **O**-crossing subintervals and P has elements on each component of $\infty \setminus \{\mathbf{0}\}$, $f^*(\infty)$ is homeomorphic to $\boldsymbol{\sigma}$ or $\propto$. We remark that $f = \pi \circ f^* : \infty \longrightarrow \infty$. Define $F = f^* \circ \pi : \infty^* \longrightarrow \infty^*$.

PROPOSITION 10.1. *Suppose that there are no* **O***-crossing subintervals. Then the following statements hold:*

(a) *if q is an n-point for F, then $p = \pi(q)$ is an n-point for f;*
(b) *if p is an n-point for f and $p \in G_1$, then p is an n-point for F.*

Proof. The proof follows as Proposition 7.3. □

PROPOSITION 10.2. *Suppose that there are no* **O***-crossing subintervals. If* $\{5, 7\} \subset \mathrm{Per}(f)$ *then* $\mathbb{N} \setminus \{2, 3, 4, 6, 8, 10, 11\} \subset \mathrm{Per}(f)$.

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Proof. By Proposition 10.1 we have that $k \in \text{Per}(F)$. Since P has elements into each circle of ∞ and there are no **O**-crossing intervals, we get that $F(\infty^*)$ is homeomorphic to $\boldsymbol{\sigma}$ or $\propto$. So, from the $\boldsymbol{\sigma}$ theorem and Proposition 8.1 the result follows. □

11. *The full periodicity kernel of* ∞

The goal in this section is to prove Theorem 1.7. Since $\propto$ is homeomorphic to $\{(x, y) \in \infty : y \leq 1\}$, in this section we shall consider $\propto = \{(x, y) \in \infty : y \leq 1\} \subset \infty$. Let f be an $\propto$ map, we shall extend f to a ∞ map $\overline{f}$ as follows. We define $\overline{f}(z) = f(z)$ if $z \in \propto$ and $\overline{f}|_{\text{Cl}(\infty \setminus \propto)}$ is any homeomorphism between $\text{Cl}(\infty \setminus \propto)$ and the unique interval in $\propto$ having $f(1, 1)$ and $f(-1, 1)$ as endpoints such that $\overline{f}(1, 1) = f(1, 1)$ and $\overline{f}(-1, 1) = f(-1, 1)$. Of course $\text{Per}(f) = \text{Per}(\overline{f})$. By Theorem 1.6, $\{2, 3, 4, 5, 6, 7, 10, 11\}$ is a subset of the full periodicity kernel of ∞. Then, to prove Theorem 1.7 it is sufficient to show the following two propositions.

PROPOSITION 11.1. *Let f be an ∞ map. If* $\{5, 7\} \in \text{Per}(f)$ *then* $\mathbb{N} \setminus \{2, 3, 4, 6, 8, 10, 11\} \subset \text{Per}(f)$.

PROPOSITION 11.2. *There exists an ∞ map g such that* $\text{Per}(g) = \mathbb{N} \setminus \{8\}$.

In the rest of this section we fix the ∞ map f having a periodic orbit P of period $k \in \{5, 7\}$ and the set S of basic intervals associated to P'. This fixed ∞ map will be called the *standard ∞ map*.

LEMMA 11.3. *Let f be the standard ∞ map. If $P \subset$ **O** or $P \subset$* $\text{circle}(B)$, *then Proposition 11.1 holds.*

Proof. Without loss of generality we can assume that $P \subset \mathbf{O}$. Then we define the **O** map $g : \mathbf{O} \longrightarrow \mathbf{O}$ as follows. For $z \in \mathbf{O}$, $g(z) = f(z)$ if $f(z) \in \mathbf{O}$; and $g(z) = \mathbf{0}$ if $f(z) \notin \mathbf{O}$. Clearly $\text{Per}(g) \subset \text{Per}(f)$. Hence, from the Circle theorem the result holds. □

Remark 11.4. From Lemma 11.3 we can assume that P has points into each circle of ∞. Furthermore, by Corollary 5.3 we can suppose that the standard ∞ map f is P'-linear.

LEMMA 11.5. *Let f the standard ∞ map. Suppose that there is $J \in S$ such that there are no basic intervals of $S \setminus \{J\}$ f-covering J. Then Proposition 11.1 holds.*

Proof. Consider the map $g = f|_{\infty \setminus \text{Int}(J)} : \infty \setminus \text{Int}(J) \to \infty \setminus \text{Int}(J)$. From the assumptions it follows that g is well-defined. Clearly g is either a $\boldsymbol{\sigma}$ map or an $\propto$ map such that $\text{Per}(g) = \text{Per}(f)$. Hence, from the $\boldsymbol{\sigma}$ theorem and Proposition 8.1 the result follows. □

Remark 11.6. From Lemma 11.5 we can assume that each $J \in S$ is f-covered by some basic interval of $S \setminus \{J\}$.

LEMMA 11.7. *Let f be the standard ∞ map. Let J and K be basic intervals such that J f^m-covers K, for some $m \geq 1$. Eventually $J = K$. Then there is a path of length m starting at J and ending at K.*

Proof. If $m = 1$ it is trivial. So suppose that $m > 1$. For $1 < i \leq m$, given $J_i \in S$, $J_i \subset f^i(J)$, since f is P'-linear, we can select $J_{i-1} \in S$ such that $J_{i-1} \subset f^{i-1}(J)$ and

$J_{i-1} \to J_i$. Then, by induction assumption, the path $J_0 = J \to J_1 \to \cdots \to J_{m-1} \to J_m = K$ proves the lemma. □

LEMMA 11.8. *Let f be the standard ∞ map. Let $J, K \in S$. Then at least one of the following statements holds:*

(a) $\mathbb{N} \setminus \{2, 3, 4, 6, 8, 10, 11\} \subset \mathrm{Per}(f)$;

(b) *there is a path of length m for some $1 \leq m \leq k+1$ starting at J and ending at K.*

Proof. From Remark 11.6 each basic interval is f-covered by some basic interval different from itself. Then we get that $f(\infty) = \infty$. Set $K_i = f^i(J)$ for $i \geq 0$. Moreover, since P is a periodic orbit, there is an integer $r \geq 1$ such that $\cup_{i=0}^{r} K_i = \cup_{i=0}^{r+1} K_i = E' \neq \cup_{i=0}^{r-1} K_i$. Notice that $E \setminus E'$ is either $\emptyset$, or formed by exactly two basic intervals J_1 and J_2 such that $J_1 \subset \mathbf{O}$, $J_2 \subset \mathrm{circle}(B)$ and $J_1 \rightleftarrows J_2$. Therefore, either $E' = \infty$ or E' is homeomorphic to some space of $\{\mathbf{I}, \mathbf{Y}, \mathbf{X}\}$. Since $\mathrm{Card}(S) = k+2$, we get that $r \leq k+1$. Notice that $P \subset E'$.

First, suppose that E' is homeomorphic to $\mathbf{I}$, $\mathbf{Y}$ or $\mathbf{X}$. Then we define the E' map $g = f|_{E'} : E' \longrightarrow E'$. Since $\cup_{i=0}^{r} K_i = \cup_{i=0}^{r+1} K_i = E'$, g is well-defined. Of course P is a periodic orbit of period k for g. Then from the $\mathbf{I}$ theorem, $\mathbf{Y}$ theorem and n-od theorem, statement (a) holds.

Finally, suppose that $E' = \infty$. Therefore, J f^m-covers K, for some $1 \leq m \leq k+1$. Thus by Lemma 11.7 there is a path of length m starting at J and ending at K. □

Proof of Proposition 11.1. From Proposition 5.5 with $k \in \{5, 7\}$, if there exists some basic interval which f-covers itself, then Proposition 11.1 holds. So from now on we can suppose that $J \not\to J$ for any $J \in S$.

If there are no $\mathbf{O}$-crossing subintervals of ∞, then from Proposition 10.2, Proposition 11.1 holds. By using similar arguments, if there are no circle(B)-crossing subintervals, then Proposition 11.1 follows. So from now on we can assume that there are two crossing subintervals $K_1, K_2 \subset \infty$ such that $K_1 \to \mathbf{O}$ and $K_2 \to \mathrm{circle}(B)$. Since f is P'-linear and $J \not\to J$ for any $J \in S$, we get that $\mathbf{0}$ is the unique fixed point of f. Therefore $K_1 \subset \mathrm{circle}(B)$ and $K_2 \subset \mathbf{O}$. Moreover, from the fact that $K_1 \rightleftarrows K_2$, there is $x \in K_1$ such that $f(x) \in K_2$ and $f^2(x) = x$. Let $L \subset K_2$ and $M \subset K_1$ be the basic intervals containing $f(x)$ and x respectively. By the linearity of f we get that $L \rightleftarrows M$.

First we suppose that L or M f^k-covers itself. Without loss of generality we can assume that L f^k-covers L. Then, by Lemma 11.7 there exists a loop of length k containing L. Therefore the above loop of length k together with the loop $L \rightleftarrows M$ give us a loop of length n for each $n > k$ odd and each $n \geq 2k+2$ even. Notice that the loop of length n is non-repetitive because k is not multiple of 2. We claim that we can construct the above loop of length n in such a way that at least one of its basic intervals does not contain $\mathbf{0}$. Now we prove the claim. If $\mathbf{0} \notin L$ or $\mathbf{0} \notin M$, then we are done. So suppose that $\mathbf{0} \in L \cap M$. Since k is not multiple of 2, the loop of length n is not a repetition of $L \rightleftarrows M$. Furthermore, since $\mathbf{0} \in L \cap M$ it follows that the only branching intervals f-covered by L and M are L and M (see Remark 5.4). Hence the loop of length n contains some $J \in S$ with $\mathbf{0} \notin J$ and the claim is proved. By Proposition 4.4 we get that $\mathbb{N} \setminus \{2, 3, 4, 6, 8, 10\} \subset \mathrm{Per}(f)$ and Proposition 11.1 holds.

Now we can assume that L and M do not f^k-cover themselves. Thus, since P has period k, we get that L f^k-covers J for each $J \in S$ with $J \subset \mathbf{O}$ and M f^k-covers J for each $J \in S$ with $J \subset \text{circle}(B)$.

Without loss of generality we have three possibilities for the basic intervals L and M: either $\mathbf{0} \in L \cap M$; or $\mathbf{0} \in L$ and $\mathbf{0} \notin M$; or $\mathbf{0} \notin L \cup M$. If $\mathbf{0} \in L \cap M$, then without loss of generality we can assume that there is a basic interval $M_1 \subset \text{circle}(B) \setminus \text{Int}(M)$ such that $L \to M_1$. Moreover, since f is P'-linear, $\mathbf{0} \notin M_1$. If $\mathbf{0} \in L$ and $\mathbf{0} \notin M$, then since $f(\mathbf{0}) = \mathbf{0}$ and $L \to M$, we have that there is a basic interval $M_1 \subset \text{circle}(B) \setminus \text{Int}(M)$ such that $L \to M_1$. Finally, since $k > 4$, if $\mathbf{0} \notin L \cup M$, then again we can suppose that there is a basic interval $M_1 \subset \text{circle}(B) \setminus \text{Int}(M)$ such that $L \to M_1$.

In short, we get that there is $M_1 \in S$ such that $M_1 \subset \text{circle}(B) \setminus \text{Int}(M)$, $L \to M_1$ and $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$. Therefore, M f^k-covers M_1. From Lemma 11.7 there is a path $M \to \cdots \to M_1$ of length k. From Lemma 11.8, if statement (a) holds, then Proposition 11.1 follows; otherwise, from statement (b) we can assume that there is a path $M_1 \to \cdots \to M$ of length $m \le k+1$ and m is the shortest length of all the paths from M_1 to M. Concatenating the path of length m together with the paths $M \to L \to M_1$ and the path $M \to \cdots \to M_1$ of length k we obtain two loops of lengths $m+2$ and $k+m$. Notice that both loops contain M and M_1. We take $k = 5$.

First suppose that m is odd. So $m \le k$. The loop $M \to L \to M_1 \to \cdots \to M$ of length $m+2$ and the loop $M \rightleftarrows L$ allow us to construct a non-repetitive loop of length n for each $n \ge k+2$ odd, containing M and M_1. On the other hand, the loops $M \to \cdots \to M_1 \to \cdots \to M$ of length $k+m$ and the loop $M \rightleftarrows L$ allow us to construct a non-repetitive loop of length n for each $n \ge 2k+2$ even containing M and M_1. Since $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$, by Proposition 4.4 we have that $\mathbb{N} \setminus \{2, 3, 4, 6, 8, 10\} \subset \text{Per}(f)$.

Finally, suppose that m is even. The loop $M \to L \to M_1 \to \cdots \to M$ of length $m+2$ and the loop $M \rightleftarrows L$ give us a non-repetitive loop of length n for each $n \ge k+3$ even containing M and M_1. Moreover, the loop $M \to \cdots \to M_1 \to \cdots \to M$ of length $k+m \le 2k+1$ and the loop $M \rightleftarrows L$ give us a non-repetitive loop of length n for each $n \ge 2k+1$ odd containing M and M_1. Since $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$, from Proposition 4.4 we have that $\mathbb{N} \setminus \{2, 3, 4, 6, 7, 9\} \subset \text{Per}(f)$. Now we will prove that $9 \in \text{Per}(f)$. Notice that if $m < k+1$, then $m \le k-1$ because m is even. Hence, the loop $M \to \cdots \to M_1 \to \cdots \to M$ of length $k+m \le 2k-1$ odd and the loop $M \rightleftarrows L$ give us a non-repetitive loop of length $2k-1 = 9$, and we are done. So, from now on, we can assume that $m = k+1$ and that there are no non-repetitive loops of length 9 containing some non-branching interval; and we will to obtain a contradiction.

Let $J_0 = M_1 \to J_1 \to \cdots \to J_{m-1} \to J_m = M$ be the above path of length $m = 6$. By the minimality of m, all basic intervals of this path are different. So, $J_{m-1} = L$. From the facts that M f^k-covers J, for each $J \in S$ with $J \subset \text{circle}(B)$ and L f^k-covers J, for each $J \in S$ with $J \subset \mathbf{O}$ and since there are no non-repetitive loops of length 9 containing some non-branching interval, it follows that $\{J_1, J_3\} \subset \text{circle}(B)$ and $\{J_2, J_4\} \subset \mathbf{O}$.

If there is a unique $\mathbf{O}$-crossing subinterval, since $J \nrightarrow J$ for any $J \in S$ it follows that L is f-covered by an odd number of basic intervals. This is a contradiction with the facts that $J_4 \to L \leftarrow M$ and the minimality of m. Otherwise, there are at least two $\mathbf{O}$-crossing subintervals. Therefore, there exists $N \in S$ with $N \subset \text{circle}(B)$ and

$N \neq M$ such that $N \to L$. Since M f^k-covers N, we obtain a non-repetitive loop $M \to \cdots \to N \to L \to M \to L \to M$ of length 9 containing some non-branching interval, in contradiction with the assumptions. So the proposition is proved. □

Proof of Proposition 11.2. We need to construct an ∞ map g such that $\mathrm{Per}(g) = \mathbb{N} \setminus \{8\}$.

Let $\{a_1, a_2, \ldots, a_4\}$ and $\{b_1, b_2, \ldots, b_5\}$ be periodic orbits of periods 4 and 5 respectively such that $g(a_i) = a_{i+1}$ for $i = 1, 2, 3$ and $g(a_4) = a_1$; $g(b_i) = b_{i+1}$ for $i = 1, 2, 3, 4$ and $g(b_5) = b_1$. Let Q be the union of two above periodic orbits with the branching point. Define the basic intervals J_i associated to Q for $i = 1, 2, \ldots, 11$ as follows. The intervals $J_i \subset \mathbf{O}$ for $i \in \{1, 3, 4, 5, 8, 10\}$ and $J_i \subset \mathrm{circle}(B)$ for $i \in \{2, 6, 7, 9, 11\}$ where $J_1 = [a_1, \mathbf{0}]$, $J_2 = [\mathbf{0}, a_2]$, $J_3 = [\mathbf{0}, a_3]$ $J_4 = [a_4, a_1]$, $J_5 = [b_1, a_4]$, $J_6 = [b_2, \mathbf{0}]$, $J_7 = [a_2, b_3]$, $J_8 = [a_3, b_4]$, $J_9 = [b_3, b_5]$, $J_{10} = [b_4, b_1]$ and $J_{11} = [b_5, b_2]$.

Now define a Q-linear ∞ map g such that the only elementary loops in the g-graph are the following: $J_1 \to J_2 \to J_3 \to J_1$, $J_1 \to J_2 \to J_3 \to J_4 \to J_1$, $J_2 \to J_3 \to J_4 \to J_2$, $J_5 \to J_6 \to J_7 \to J_8 \to J_9 \to J_{10} \to J_{11} \to J_5$, $J_7 \rightleftarrows J_8$ and $J_7 \to J_8 \to J_9 \to J_{10} \to J_{11} \to J_7$.

By construction $\{4, 5\} \subset \mathrm{Per}(g)$. The loops $J_7 \rightleftarrows J_8$, $J_2 \to J_3 \to J_4 \to J_2$ and $J_2 \to J_3 \to J_1 \to J_2 \to J_3 \to J_4 \to J_2$, have lengths 2, 3 and 6 respectively, are non-repetitive and at least one interval in each loop does not contain $\mathbf{0}$. Since $\{2, 3, 6\} \cap \{8, 10\} = \emptyset$, from Proposition 4.4 we have that $\{2, 3, 6\} \subset \mathrm{Per}(g)$.

In a analogous way, concatenating the loops $J_7 \rightleftarrows J_8$ and $J_7 \to J_8 \to J_9 \to J_{10} \to J_{11} \to J_7$ of lengths 2 and 5 respectively, we obtain that $\mathrm{Per}(f) \supset \{n \in \mathbb{N} : n \geq 5, n \text{ odd}\}$ and $\mathrm{Per}(f) \supset \{n \in \mathbb{N} : n \geq 12, n \text{ even}\}$.

Now we prove that $10 \in \mathrm{Per}(f)$. We note that $\mathbf{Y}$ is homeomorphic to $J_1 \cup J_2 \cup J_3 \cup J_4 \subset \infty$ and so we shall identify $\mathbf{Y} = J_1 \cup J_2 \cup J_3 \cup J_4$. Since $\mathbf{Y}$ is an invariant set for the map g, we can consider the $\mathbf{Y}$ map $h = g|_{\mathbf{Y}}$. Of course, $\{a_1, a_2, a_3, a_4\}$ is a periodic orbit of period 4 for h and the h-graph is a subgraph of the g-graph. Using the loop $J_1 \to J_2 \to J_3 \to J_4 \to J_1 \to J_2 \to J_3 \to J_1 \to J_2 \to J_3 \to J_1$ of length 10 in Proposition 4.4, we get $10 \in \mathrm{Per}(f)$.

On the other hand, since there are no non-repetitive loops of length 8 in the g-graph, from Proposition 5.2 we get that $8 \notin \mathrm{Per}(g)$. □

REFERENCES

[1] L. Alsedà, J. Llibre and M. Misiurewicz. Periodic orbits of maps of **Y**. *Trans. Amer. Math. Soc.* **313** (1989), 475–538.

[2] L. Alsedà, J. Llibre and M. Misiurewicz. *Combinatorial Dynamics in Dimension One (Advanced Series in Nonlinear Dynamics 5)* World Scientific, Singapore, 1993.

[3] L. Alsedà and J. M. Moreno. Linear orderings and the full periodicity kernel for the n-star. *J. Math. Anal. Appl.* **180** (1993), 599–616.

[4] S. Baldwin. An extension of Sharkovskii's theorem to the n-od. *Ergod. Th. & Dynam. Sys.* **11** (1991), 249–271.

[5] L. Block. Periodic orbits of continuous maps of the circle. *Trans. Amer. Math. Soc.* **260** (1980), 553–562.

[6] L. Block. Periods of periodic points of maps of the circle which have a fixed point. *Proc. Amer. Math. Soc.* **82** (1981), 481–486.

[7] L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young. *Periodic Points and Topological Entropy of One Dimensional Maps (Lecture Notes in Mathematics, 819).* Springer, 1980, pp. 18–34.

[8] A. Blokh. *The Spectral Decomposition, Periods of Cycles and Misiurewicz Conjecture for Graph Maps (Lecture Notes in Mathematics, 1514).* Springer, 1991, pp. 24–31.

[9] M. C. Leseduarte and J. Llibre. On the set of periods for σ maps. *Trans. Amer. Math. Soc.* **347** (1995), 4899–4942.

[10] C. Leseduarte and J. Llibre. The full periodicity kernel of the trefoil. *Ann. Inst. Fourier* **46** (1996), 219–262.

[11] T. Li and J. York. Period three implies chaos. *Amer. Math. Monthly* **82** (1975), 985–992.

[12] J. Llibre and M. Misiurewicz. Horseshoes, entropy and periods for graph maps. *Topology* **32** (1993), 649–664.

[13] J. Llibre, J. Paraños and J. A. Rodríguez. The full periodicity kernel for σ maps. *J. Math. Anal. Appl.* **183** (1994), 639–651.

[14] J. Llibre, J. Paraños and J. A. Rodríguez. On the extension of Sharkovskii's theorem to connected graphs with non-positive Euler characteristic. *Int. J. Bifurcation Chaos* **5** (1995), 1395–1405.

[15] J. Llibre, J. Paraños and J. A. Rodríguez. Sets of periods for maps on connected graphs with zero Euler characteristic having all branching points fixed. *J. Math. Anal. Appl.* to appear.

[16] J. Llibre and R. Reventós. Sur le nombre d'orbites périodiques d'une application continue du cercle en lui-même. *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), 52–54.

[17] P. Mumbrú. Periodes 1,2,3,4,5,7 equivalen a caos. *Master Thesis,* Universitat Autònoma de Barcelona, 1982.

[18] A. N. Sharkovskii. Co-existence of the cycles of a continuous mapping of the line into itself. *Ukrain. Math. Zh.* **16** (1964), 61–71 (in Russian).

[19] P. D. Straffin. Periodic points of continuous functions. *Math. Mag.* **51** (1978), 99–105.